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Geometric Objects in the Final Grade of Middle Schools in B&H, Croatia, Macedonia, Montenegro, Serbia, and Turkey and Students' Progress of Geometrical Thinking which their Mathematical Curricula Enable

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Abstract. The powerful development of geometric thinking among the students of the final grade of Middle School enables them to progress better through mathematical contents in the upcoming High School. In order to gain insight into possible levels of development of students' geometric skills, we have examined the mathematics curricula of the final grades of Middle Schools in B&H, Croatia, Macedonia, Montenegro, Serbia and Turkey. Our intention was to gain insight into geometric contents planned in these teaching programs of the aforementioned socio-political communities. As a connecting thread of our observations, we offer the following conclusions regarding the previous research. All observed models have a significant common part that relates to the teaching and learning of geometric solids. In the case where this part of the curricula would be complemented by geometric transformations, it would then be possible to estimate that these curricula open the possibility of a full development of robust abilities among pupils. The authors think that although the quality of mathematics teachers' work has a significant impact on the formation of geometric proficiencies and abilities of pupils in schools, the influence of policy makers of mathematics education and mathematical curricula designers is more significant. The authors are also very close to the suggestion that each of these societies should form a permanent team of experts for monitoring applications and further development of mathematical curricula.

Keywords and phrases: curriculum, geometry, teaching, Middle School, B&H, Croatia, Macedonia, Montenegro, Serbia, Turkey.

1. Introduction

The observed societies are not only geographically close to each other but they have a significant part of interrelated history, generally, and in mathematical education, in particular. The elementary school system in the observed societies is comprised of 8 (or 9) grades. Formally and declaratively they are divided into three triads, but the practice is realized in two forms: the Primary part (for children from 6/7 to 10 years of age) and the Middle part (for children from 10 to 15 years of age). In what follows we will use the term 'Middle School' to designate this second part of the Elementary school.

One of the fundamental problems in the designing of the geometry components of the mathematics curriculum for the 8th grade in the Middle School is simply that the necessary geometric knowledge of students who complete the Elementary School should be rounded up and prepared them for the learning, understanding, and acceptance of significantly more cognitively demanding geometric knowledge in high school. This paper does not seek to resolve disagreements over the geometry curricula. Rather, the intention of the authors is to identify and review similarities and diversities concerning the design of the geometry curricula in the 8th grade in Middle Schools of the mentioned socio-political communities. These issues include the nature of geometry, the aims of geometry teaching, and the relative merits of different approaches to school geometry. The authors are close to the conclusion that the choice of geometric contents for the Middle School is extremely demanding since this geometric knowledge must have well-built links between geometric proficiencies acquired in the Primary school and geometric knowledge that should be mastered in the Secondary schools. Some of the most important goals of learning geometry at the Elementary School are the development of observational abilities, and skills to capture detail and analytics proficiency. In doing so, students begin to accept the existence of abstract concepts and develop skills of contemplative manipulations with abstract geometric objects. Add to this that students are becoming aware of the existence and importance of a number of logical principles they encounter when learning geometry such as, for example, 'the principle of excluded third' and 'the principle of non-contradiction'. Also, students become more aware of the notions of some important mathematical concepts such as the 'concept of unlimited' and the 'concept of infinity' as opposed to their earlier intuitive understanding of these abstract issues. Designing suitable geometry curriculum for Middle School grades are probably the most difficult task for those who are charged with constructing mathematics curricula. It is also probably the most enduring dilemma in mathematics curricula design and has probably been the subject of more inquiries and commentaries than any other area of the mathematics curricula.

The purpose of this paper is not to try to resolve the range of disagreements about the observed geometry curricula of the abovementioned countries. Given the range of issues, such an endeavor is unlikely to be successful. The authors’ goal is more modest (and hopefully achievable). It is to identify and review some of the critical issues in the design of the geometry curriculum that relates to the potential for the development of a pupil's geometric thinking provided by such designed curricula.

2. Literature Review

According to Jones [21], in the language of designing a geometry curriculum in the Middle School, for those who want to reduce the complexity of arguments, the question could be put like this. Is it better that student learners grasp in depth one approach to geometry or experience a wide range of approaches to geometry? For some, like for the authors, the answer to this question is obvious. It means a restoration of geometry in the Euclidean tradition as the dominant or even sole form of geometry in schools (e.g., [47]). The studies of the phenomenon 'geometrical thinking' and the associated 'development of the ability of spatial orientation' are the subject of continuing observations in the domain 'Research in Mathematical Education’. At each ERME conference (starting from the third), there is a working group interested in Geometric Thinking (e.g., [20], [24], [25], [33], [35], [46]). The authors estimate that the report [25] is very important in perceiving and overcoming the problems encountered by middle school teachers and researchers in geometric education at that grade level. Thus, for example, the text [26] provides an overview of the geometric surveys presented at ERME conferences (CERME 3 - CERME 10). In article [46], possible roots of students' wrong geometric reasoning were investigated with the aim of investigating the nature of the gap between teaching and learning geometry. Also, research texts related to geometry teaching to Middle school students and their geometry learning are often part of the annual reports of the ‘International Group for the Psychology of Mathematics Education' (PME) as well as many others (e.g., [1], [2], [3], [27], [31], [32]). As an illustration, the report [2] can well be used. In that study, the authors investigated the relationship between students' cognitive styles (verbal deductive, spatial imagery, and object imagery) and performance on geometry problems. A very inspiring text is the article by [22] in which the authors sublimately exhibit the main trends in the research of the school geometry presented at PME conferences in the period 2005-2015. In relation to the aforementioned trends in research on the teaching of geometry, in the study [32], student's weaknesses in understanding geometric issues of geometric measurements is disclosed, angles and shapes, transformations and construction of shapes; ideas which can help in balancing the perceived obstacles have been offered. 'How should be proofs of theorems look like in Middle School geometry' - is the question that the study [31] attempted to answer.

Observation of the problems encountered by students in the Balkans when learning geometric content is the subject of many studies (e.g., [13], [24], [40], [41]). What does this mean for teaching geometry in the Middle School if teachers insist that students recognize, understand, accept, and can reproduce some evidence in geometry? - is one of the crucial questions that researchers in geometric education have to ask themselves. The aim of the study [24] was to examine Middle School upper grades students' performances when they were asked to identify, name, and draw geometrical objects. In [40], [41], the fourth author of the present paper exhibited some of his thoughts about the interdependence of mathematical education of future teachers of school geometry and components of the official school mathematics curricula.

In the attempts to look at the general picture of contemporary view on the design of mathematical curricula by reviewing the available literature, the authors have formed a belief that this phenomenon is in the focus of the special interest of the academic community of researchers in mathematics education. Lately, many researchers are actively considering the form, desirable characteristics, and the applicability of mathematical curricula (e.g., [8], [11], [21], [23], [27], [34]). For example, in [8], it has been conceptualized curriculum ergonomics as a field that studies the interactions between users and curriculum materials. The authors of [8] identified some of the themes that should be in the focus of the mathematical curricula theoreticians. According to the introduction part of the report [28], its purpose was to investigate how middle school mathematics teachers use curriculum resources to plan lessons and what teachers notice in the curriculum. Furthermore, the article [45] reports on the development of mathematical curricula in Sweden. Also, the article [23] provides information on national efforts in Singapore in the harmonization of mathematical curriculum and school practices.

Since societies whose curricula are analyzed are of an authoritarian type, the organization and realization of teaching is centralized. Therefore, the association of teachers in these social communities does not have a significant influence on the principled-philosophical orientation incorporated into mathematical curricula. Thus, we estimate, there are differences (some visible and many more invisible) in the commitment of the academic core in the design of the Middle School mathematical curriculum. In most of these observed societies (this is a specific reference to B&H, Montenegro and Serbia), mathematics teachers are not creators, but only realizers of mathematics education. Comparison of conceptual aspects in designing mathematical curricula of the observed societies, on the one hand, and mathematical curricula in many European countries, on the other hand, should include significantly more explicit socio-political dimensions of analysis than is the case in this paper. The authors seem to think that the available research reports from other socio-political communities are not compatible with the content displayed in this paper. Some of these societies introduced significant changes to the Middle School mathematical curriculum in the last ten to fifteen years. Incorporating an analysis of these changes into this paper would significantly change its original design.

3. Theoretical and methodological aspects of research in geometry

In the observation of the phenomenon, researchers in mathematics education have the possibility: observation and data collection, experimentation and theoretical analysis. This has methodological implications. The collection of data is very important for both aspects of research. Our intention with this research is that it has substantial and longstanding effects to the policy-makers in mathematics education and curriculum designers in mentioned communities, on collected data, their using and interpretation.

The most important theories in geometry education that were identified and discussed are the following: Van Hiele's levels [44], Geometrical Working Space and Geometrical paradigms [20] and Duval's semiotic approach [12]. Houdement and Kuzniak [20] proposed that elementary geometry has to be split into three various paradigms, characterizing different forms of geometry: Geometry 1 (natural geometry), Geometry 2 (natural axiomatic geometry) and Geometry 3 (formal axiomatic geometry). The theoretical framework developed specifies the nature of the geometrical objects, the use of different techniques and the validation mode accepted in each of the three paradigms. The findings of the Panaoura and Gagatsis [33] indicate that students working in the paradigm of Natural Geometry (mostly Middle school students) tend to consider geometrical objects as material objects and specific images rather than as theoretical idealized concepts that bear specific properties. These difficulties result in the phenomenon that students try to solve geometrical problems almost anytime relying on the visual perception of the given geometric figure rather on mathematical deduction based on the properties of the geometrical objects involved. According Fischbein [14], this phenomenon is related to the students' difficulty working with geometrical figures as' figural concepts'. This difficulty is often referred to as "from geometrical figure to figural concept difficulty" [14].

3.1. Van Hiele's levels of geometric thinking

There are five developmental levels of geometric reasoning based on a study by Dina van Hiele-Geldof and her husband, Pierre Marie van Hiele. They are ([1], [4], [24], [37], [44]):

- Level 0 (Basic Level): Visualization

At this level students view objects as entire entities, not noticing individual components or properties. The focus is on the whole object, not on its parts.

- Level 1: Analysis

Students begin to recognize that geometric shapes have parts and special properties. However, they are not able to describe how these properties are related, nor are they able to understand definitions.

- Level 2: Informal Deduction

At this level students comprehend the connection between properties within geometric figures and from one set of figures to another. Students are able to follow proofs, but are not able to construct proofs independently.

- Level 3: Formal Deduction

At this level students can construct a geometric proof and understand the connection between postulates, theorems, and undefined terms.

- Level 4: Rigor

At this level students see geometry abstractly. Students can move between different geometric systems and can compare and contrast them.

3.2. Spatial reasoning

Students' spatial reasoning involves their ability to think and to reason by comparing, manipulating, and transforming mental pictures. Spatial sense comprises two components: spatial visualization and spatial orientation. According to Gutiérrez [15], visualization refers to the kind of reasoning activity based on their use of visual or spatial elements, either mental of real. This is integrated into four elements: mental images, external representations, the process of visualization and ability to visualize. Due to the specific nature of mathematics, every mathematical concept might have special representation that corresponds to a specific single mental image. At this point of view, a mental image is a cognitive visual or spatial vivid description of a mathematical concept and it is the basic element for visualization [15]. External representations are any kind of graphical pictures of concepts that assist in creating mental images. So, they open a window onto visual reasoning. The process of visualization is associated with mental or physical actions with mental images, and it is composed of two main processes; the visual interpretation of information and the interpretation of mental images. The first phase is to create mental images. The second is to generate information about the created mental images. As a result of these processes, as Angel Gutiérrez summarizes in his well-known article [15], individuals should use one or more of the following abilities of visualization to carry out the necessary steps to create mental images and further manipulations: (i) Figure-ground perception, (ii) Perceptual constancy, (iii) Mental rotation, (iv) Perception of spatial positions, (v) Perception of spatial relationships, (vi) Visual discrimination. On the development of mathematical thinking through visualization, the authors suggest to the readers to look at the texts [19] and [30].

Spatial orientation ability is defined by McGee [29] as involving the comprehension of the arrangement of elements within a visual stimulus pattern and the aptitude to remain unconfused by the changing orientation in which a spatial configuration may be presented. This ability has an important place in mathematics education [9]. On the other hand, spatial orientation skill is the ability to visualize an object's view from a different perspective. Spatial orientation is a skill to know where you are and how to get around in the world. So, it is understanding and operating on relationships between different positions in space, at first with respect to your own position and your movement through it, and eventually from a more abstract perspective [10]. School geometry is the study of those spatial objects, relationships, and transformations that have been formalized and the axiomatic mathematical systems that were constructed to represent them [10]. The development of geometric thinking is the result of students' interaction with abstract concepts in their efforts to understand, accept and use such objects. Spatial reasoning, on the other hand, consists of a set of cognitive processes by which the mental representations of space objects, relationships, and transformations are constructed and manipulated. Therefore, the development of students' spatial ability is the result of their interaction with real objects. Clearly, geometry and spatial reasoning are strongly interrelated. Most mathematics educators consider spatial reasoning as part of the geometry curriculum. Usiskin [43], for example, has described four levels of geometry: (a) visualization, drawing, and construction of figures; (b) study of the spatial aspects of the physical world; (c) its use as a means of representing non-visual mathematical concepts and relationships; and (d) representation as a formal mathematical system. According to Clements and Battista [9], the first three of these dimensions require the use of developed student spatial reasoning.

3.3. The SOLO Taxonomy

Mathematics is a hierarchically organized subject because every concept in mathematics is continuous and consecutive. In other words, the more complex a given problem is, the more connected concepts are needed. To solve a given problem, a student must be able to select and determine the elements that can be used in problem solving. The problem solving process is related to the cognitive domain of the students. Therefore, problem solving is also related to students' mathematics learning achievement. Each element of mathematical problem solving has different characteristics, so the response given by each student to problem solving can also different. Biggs [6], Biggs and Collis [7] explain that each stage of cognitive response is the same and increasing from the simple to the abstract. This theory is known as Structure of the Observed Learning Outcome [SOLO].

The SOLO Taxonomy is a taxonomy that classifies how students' thinking levels fall into five categories: pre-structural, uni-structural, multi structural, relational, and extended abstract levels. Criteria for each level based on SOLO taxonomy can be seen in Table 1.

Table 1 Criteria Level SOLO Taxonomy

No	The level thinking	Criteria
1.	SOLO 1: "The Pre-Structural Level"	- Here the student does not have any kind of understanding but uses irrelevant Information and/or misses the point altogether. - Scattered pieces of information may have been acquired, but they are unorganized, unstructured, and essentially void of actual content or relation to a topic or problem.
2.	SOLO 2: "The Uni-Structural Level"	- The student can deal with one single aspect and make obvious connections.

- The student can use terminology, recite (remember things), perform simple instructions / algorithms, paraphrase, identify, name, count, etc.

3. SOLO 3: "The Multi-Structural Level"
- At this level the student can deal with several aspects but these are considered independently and not in connection.
- Te student is able to enumerate, describe, classify, combine, apply methods, structure, execute procedures, etc.

4. SOLO 4: "The Relational Level"
- At level four, the student may understand relations between several aspects and how they might fit together to form a whole.
- The student may thus have the competence to compare, relate, analyze, apply theory, explain in terms of cause and effect, etc.

5. SOLO 5: "The Extended Abstract Level"
- At this level, which is the highest, the student may generalize structure beyond what was given, may perceive structure from many different perspectives, and transfer ideas to new areas.
- The student may have the competence to generalize, hypothesize, criticize, theorize, etc

Table 1 shows the level of thinking criteria held by students based on SOLO taxonomy. This should enable teachers to assess the quality of student level of thinking in solving a problem.

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3.4. MATH taxonomy

In 1956, Bloom and his team[1] provided guidelines for an assessment framework in his now classic work Taxonomy of Educational Objectives: Cognitive Domain. Bloom's classification system, commonly known as Bloom's taxonomy, defines and categorizes processes that teachers might use to estimate students' content knowledge into a hierarchical system. Smith et al. in [42] suggested that Bloom's taxonomy is quite good for structuring assessment tasks, but does have some limitations in the mathematical context, and proposed a modification of Bloom's taxonomy, the MATH taxonomy (Mathematical Assessment Task Hierarchy) for the structuring of assessment tasks. In this taxonomy eight categories of Mathematical knowledge and skills serve to recognize, register and assess the quality and level of students' mathematical literacy.

Table 2 Categories of the MATH Taxonomy ([42])

Group A	Group B	Group C
(A1) Factual knowledge (A2) Comprehension (A3) Routine procedures	(B1) Information transfer (B2) Application to new situation	(C1) Justifying and interpreting (C2) Implications, conjectures and comparisons (C3) Evaluation

4. Comparing of the Curricula

By comparing mathematics curricula in Middle School, the authors try to show to international academic community some of the interesting developments in mathematical education in the Balkans ([5], [17]). The observed mathematics curriculum models are the result of the political orientation of the creators of educational policies in selected countries. Their design is the result of the work of ad hoc committees compiled mostly by the Middle School mathematics teachers. The authors, in their intention to analyze these

[1] Since the Bloom taxonomy is widely known, the authors have estimated that in this case there is no need to refer to sources.

mathematics curricula models, have found that there is a lot of difficulties in reaching the intended goal. Since the models constructed are so far away from what the teachers of Middle school mathematics would like to see and use in their work in the classroom, the examination of the characteristics of these curricula should include the assessment of the principally-philosophical attitudes of their creators and designers. Remillard and Heck [34] noted that this form of model of mathematics curriculum is difficult to analyze and document since a significant part of it exists only in the mind of a teacher. It seems, and so most teachers of mathematics experience it, as instructions to achieve the highest level of procedural knowledge among students. In the first approximation, with a lot of restraint in making the next conviction, it can be said that such models of mathematics curricula enable students to learn, understand and accept a large number of categorical terms within a planed mathematical vocabulary [23]. The authors estimate that policy makers of education and designers of these mathematics curricula models transfer a significant part of their responsibilities to teachers of mathematics. In order to gain insight into the complexity of this phenomenon, we suggest to the reader to look at the article [34] in which the process of constructing the operational model of the curriculum is exposed for understanding these interrelations. There are researches that confirm the emergence of difficulties and even tensions with construction of the operational curriculum since the feasibility of operational plans is fundamentally dependent on the complexity and completeness of the official curriculum [11]. These difficulties are exposed through the following dimensions:

- volume and depth of entry into mathematical contents;
- the method of assessing a proportion of students' responsibilities according to the previous necessary mathematical knowledge;
- concrete cognitive and affective goals of teaching mathematics
- tools by which the planned goals can be achieved, and
- which and what kind of teaching material can be used in the classroom.

In this part of the article we show the curricula of mathematics (the number of classes) for the VIII grade Middle School in B&H, Croatia, Macedonia, Montenegro, Serbia and Turkey. Information on mathematics teaching plans for the observed grade in Middle schools in the mentioned countries is presented in Table 3.

State	Number of Weekly Classes	Number of Annually Classes	Croatia = 1	Grades
B&H [FB&H]	4	132	0.923	IX
B&H [RS]	4	136	0.971	IX
Croatia	4	140	1.000	VIII
Macedonia	4	144	1.029	IX
Montenegro	4	136	0.971	IX
Serbia	4	136	0.971	VIII
Turkey	5	180	1.286	VIII

Table 3: Teaching plan of mathematics for VIII grade

The teaching contents of mathematics for the observed grade are the following

Thematic content	**B&H** RS	**B&H** FB&H	**[CRO]**	**[MK]**	**[MNE]**	**[SRB]**	**[TR]**
Polygons;				+(10)	+(16)		+(18)
Triangles			+				
Points, lines, planes	+(8)	+(10)		+(13)	+(13)	+(12)	
Geometric figures and							

Geometric solids		+(35)	+	+(25)			+(15)
prisms, pyramids	+(35)				+(15)	+(30)	
cylinder, cone, sphere	+(30)				+(18)	+(28)	
Measure and measurement				+(7)			
Congruence and Similarity				+(12)		+(8)	+(8)
Geometric transformations			+				
Percentage	47.80	34.09		46.53	45.59	57.35	22.78

Table 4: Geometric contents in the curricula for 8th classes of Middle Scholl

Legend: Numbers in brackets indicate the suggestions of the Ministry of Education about the necessary hours for teaching and learning of noted themes

5. Our finding

It is known to authors that, except in Croatia, some of the official standards of mathematics education do not precede the observed mathematical curricula.

The most important open question relates to the methodical preparation of teachers for realization of the planned activities in the mathematical classroom of the 8th grade of the Middle School. To this question, the authors of this text cannot offer even a partially valid answer, since there is no objective and complete research of this problem. This statement fully refers to the countries that had originated from the former Yugoslavia.

5.1. Geometric teaching contents

Geometric teaching materials in RS are: Basic geometric concepts (point, line and plane) and their interrelations. Normality of a line on a plane; The angle between two planes – diedar; Geometrical solids (prisms, pyramids, cylinder, cone, sphere).

Geometric teaching materials in FB&H are: Basic geometric concepts (point, line and plane) and their interrelations. Normality of a line on a plane; The angle between two planes – diedar; Geometrical solids (prisms, pyramids, cylinder, cone, sphere).

Geometric teaching contents in Montenegro are: Polygonal line and Polygons (length of polygonal line and measure of area of a polygon, sum of inner angles and the number of diagonals); Point, line and plane and their interrelations (Normality of a line on a plane, angle between two planes); Geometrical solids (prisms, pyramids, cylinder, cone, sphere).

Geometric teaching contents in the 8th grade Middle School in Turkey are: Polygons (Triangles, Congruence and Similarity of Triangles), Geometric Transformations and Geometric Solids.

Geometrical contents in Croatia in the 8th grade of Middle School are the Pythagorean theorem and its application, Geometric transformations (Translation, Axis of a symmetry, Central symmetry, Rotation) and geometric solid.

Geometric contents in Serbia in the 8th grade of the Middle School are the Congruence and Similarity of the Triangles; Point, line and plane, and their mutual relations; Geometric solids.

5.2. Declarative geometric goals of teaching mathematics

The geometric goals of the teaching of mathematics in the 8th grade of the Middle School, among other things, are: Cognitive goals (Adoption of concepts: definition, axiom and theorem; Acquiring knowledge about mutual relations of point and line, point and plane, two lines and two planes; Enriching the mathematical vocabulary by terms related to geometric solids; Understanding, acceptance and use of the concept of measurement of the area and volume.), and Affective goals (Developing socio-affective goals, value orientations and positive attitudes toward science; Acquiring mathematical knowledge and abilities necessary for understanding quantitative and spatial relations and laws in nature and society.) which includes the development, acceptance and use of students' spatial abilities.

6. Comments and Observations

Four (B&H, MK, MNE and SRB) of the analyzed curriculum models for the 8th grade in the Middle School include the theme 'the Basic Geometric Object (Point, Line and Plane and Their Interactions)'. This topic is, in fact, a recapitulation and systematization some of very important previously adopted geometric knowledge. It makes it possible for students to recall geometric concepts such as line segment, position of a point on the line and on a segment, the point-to-line ratio and the point-to-plane relationship, the relationship between the lines, the relationship between two planes, and so on. It is important that the students are reminded the term 'border point':

> For point A of segment AB we say that it is the *starting point* because it is located in front of all other points of AB.
>
> For point B of segment AB we say that it is the *end point* because all points of AB are located in front of this point.

As it can be seen, in both cases we used the term 'located in front'. This last term should be understood by students intuitively although it may be described with some of the more commonly used notions. Encouraging students to think about these concepts and their interactions enable the development of their geometric thinking. However, when a teacher insists that students consider the relationship of a point to a line, he or she strongly stimulates the development of their logical thinking. If a student accepts by common sense the existence of relationships such as: 'The point is on the line or the point is not on the line' and 'The point cannot simultaneously be and not to be on the line', the teacher attains one of the affective goals of teaching mathematics: the development of the logical thinking components.

With a teacher insists that students adopt knowledge about polygons, a 'level 2' student's understanding of geometry can be achieved. Apart from recognizing the types of polygons (triangles, quadrilaterals, etc.), by spotting the edges, the apexes, and the inner angles of these polygons, students are enabled to develop their ability to analyze geometric figures. Identifying connections between the number of edges and the number of diagonals in a single polygon and streaming that connection into an algebraic form, the student exposes their understanding of geometry within 'level 2'.

On the other hand, when a teacher expects students not only to recognize basic geometric objects but also to know their names, this is assessed as 'level 0' of students' understanding of geometry (according to van Hiele's classification). Furthermore, suppose that a student accepts that the line (or segment) consists of unlimited numbers of points, that there are unlimited number lines at the plane, possesses the ability to see, understand, and name the constituent parts of geometric objects. It is then estimated that the student's understanding of geometry is within ‘level 1'. So, a student from the 8th grade of the Middle School should

- Recognize (and know the names of) the following geometric objects: prisms pyramid, cylinder, cone and sphere;

- Since these geometric objects are subsets of the geometric space (in set-theoretic sense), a student should identify the constituent elements of these objects, know the names of the constituent parts, and perceive their interactions in the observed geometric objects.

For example, a student should distinguish a prism from a pyramid, a cube from a square, a cylinder from a cone. The student should notice the difference between the edges of the geometric figures and its faces. Likewise, edge is a part of line, but a face is a part of the two-dimensional plane.

The Croatian curriculum of mathematics envisages the teaching of students about geometric transformations (Translation, Axial and Central symmetry, Rotation). It is the only model that allows fir the development of students’ spatial ability using the fully-needed tools for spatial visualization and spatial orientation.

Measurement of surfaces and measurement of the volume of the observed geometric figures are present in all models as a stand alone topic or, most often, included in a theme that teaches solids.

Since the 1980s, mathematics educators have agreed that developing problem-solving skills should be the focus of geometrics learning. Geometrical problem solving has a prominent role in a curriculum for several reasons:

(1) To build new geometrical knowledge - to supplement the geometric vocabulary, i.e., to increase students' cognitive level with knowledge of geometric concepts.

(2) To acquire skill of mathematical problems solving that arises in the geometry of solids or in other contexts - develop students' procedural proficiencies.

(3) To implement and adapt various steps in the problems-solving processes, i.e. develop problem - solving strategies, develop students' strategic proficiencies.

(4) To understand necessity to monitor and reflect on any problem solving process.

The processes of solving geometric problems require students to have mastered a significant foundation of mathematical knowledge; i.e., the process of problem solving requires students to understand and know how to use the hierarchical structure of mathematics. For this purpose, the ideas of SOLO taxonomy can be applied very successfully ([6], [7]). The SOLO taxonomy is used to classify students' ability to respond to a problem at five different levels: pre-structural, uni-structural, multi-structural, relational, and extended abstract levels. The task of the type 'Calculate the surface (or volume) of this geometric figure' is classified into a uni-structural level. Forms of tasks that require at least one previous process to find an answer to the question asked in these tasks are sorted into multi-structural level.

Also, solving tasks and problems in spatial geometry can be viewed from the aspect of MATH taxonomy. One of the most relevant modifications of Bloom's Taxonomy to Middle and High schools mathematics was conducted by Geoffrey Charles Smith and associates [42], where their focus was on the skills required to complete a particular mathematical task. Their Mathematical Assessment Task Hierarchy (MATH) was designed to assist the development and construction of advanced mathematics estimation in order to ensure that students are assessed on a variety of abilities and skills.

In spite of the best intent, the authors failed to understand the policy-makers of mathematics education and the designers of the covered mathematical curricula, why they do not assume that the teachers of mathematics in the Middle school are familiar with and can use the components of Bloom's (appeared in 1953), van Hiele's (appeared in 1957), SOLO (appeared in 1996) and MATH (appeared in 1996) taxonomies. A more comfortable use by teachers of the ideas and components of these theoretical models of classifying the attainment of students' abilities would provide more accuracy in the process of assessing the success of teaching and learning mathematics.

Related to the constructivist view of learning mathematics, it seems that the construction of knowledge is the result of understanding mathematical concepts and associated processes developed through the formation of mental objects and the connections among them in the minds of students. Thus, when mathematical teaching and learning is perceived from the constructivist learning theory perspective, it is necessary to pay attention to the process of formation of new knowledge and its main elements and to support the learning process that is expected. Theories about constructing new mathematical knowledge that is part of the constructivist view begins with what Piaget stated about the process of reflective abstraction. Reflective abstraction as a method of knowledge construction is the core theory of APOS (Action - Process Object Schema). As the last conclusion to this article, the next consideration seems to be acceptable: In the belief that students of the 8th grade of the Middle School are sufficiently mathematically literate, the authors are convinced that the use of the ideas of APOS theory in mathematical classrooms may greatly improve the quality of students' mathematics education at this grade level.

Since the first three authors of this paper are candidates for doctoral degrees in the field of "Research in Mathematis Education" with topics that are very close to the topic discussed in this paper, it is quite acceptable that their future research, among other things, will be with full attention on the development of mathematical curricula.

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