IMVI **ISSN 1840-4383**
Open Mathematical Education Notes
Vol. 3 (2013), 1-15

On the Significance of History-Based Research Projects in Undergraduate Mathematics

Ilhan M. Izmirli

George Mason University, USA
Iizmirl2@gmu.edu

Abstract

If one were to sum up the major premise embodying this paper in one paradigmatic sentence, it would have to be the following: integrating the historic development of mathematical concepts in the educational practice of mathematics plays a vital role in the cognitive development of students. For, by following the path of these developments, some of which have instigated bona fide *transformations in the history of mathematics and even of sciences in general, the students are not only exposed to creative problem solving techniques, but also to many false starts, incorrect solutions, and flawed and even erroneous theories, and consequently, to the essentially non-absolutist nature of mathematical research. As a matter of fact, decades ago Bradis, Minkovskii, and Kharcheva (1963) showed that a unique and effective system for teaching mathematical reasoning was to lead students toward a clearly false conclusion and then make them analyze the lapses.*

In this paper we will present the results of an extensive study conducted on two hundred nineteen college students to show that integrating history-based research projects in the undergraduate curriculum significantly improves students' understanding of and attitudes towards mathematics.

1. Introduction

The idea of using history of mathematics in the educational praxis is certainly not new; in fact, as early as 1913 Barwell was advocating the value of introducing "a little of the story of mathematical growth" (Barwell, 1913, p. 72) into mathematics instruction. Some more modern papers advocating a similar point of view include Freudenthal (1981); Wolfson (1981); Morley (1982); Pimm (1982); Katz (1986); Ernest (1988); Torkil (1992); Laubenbacher, Pengelley and Siddoway (1994); Furinghetti (1997); Tzanakis (1995); Barbin (1996); and Burns (1997). The emergence of non-absolutist (fallibilist) approaches to the connection between history and pedagogy of mathematics is given in Ernest (1989, 1991, 1998), Glas (1998), and Furinghetti and Somaglia (1998).

At the opposite end of the spectrum are the papers which argue that the pedagogy and the history of a discipline should be kept completely apart: Monk and Osborne (1997) and Fried (2001) are best examples of these. Siu (2004) lists a host of objections – sixteen in all, as a sarcastic reproduction of some of the more common arguments put forward by people in the radical separation side of the issue, most of which can be easily refuted (Izmirli, 2008).

One common overlapping part of the arguments against the use of history in mathematics classes is that it is usually not very "natural" to do so. However, we advocate the integration of history and mathematics only when it is efficient, convenient, and natural to do so and claim that in the case of instantaneous rate of change, such moments do arise frequently in the classroom, provided that the teachers have the necessary background and preparation to seize them. It is only then that the history of mathematics will become a useful pedagogical tool in constructing new mathematical ideas and objects, and will convey the idea that mathematics is a socio-cultural process.

It will be argued that educational outcomes in a mathematics course should be achieved through a coherent sequence of dynamic learning experiences that should reflect the historical developments of the concepts introduced in that particular course. It is our contention that these experiences can be presented as organic components of the course. We will also claim that the research proficiency and competence gained from such experiences would go well beyond the realm of mathematics and would be transformative not only to other topics but also to further graduate and postgraduate studies. Consequently, we will conclude that this particular pedagogical approach is too important to be pursued only marginally, especially since its basic ancillary benefits, namely, team work, improved intellectual posture, and a well-developed academic frame of mind are major steps in providing full preparedness to learners to thrive in a world of increasing technological complexity.

We will use the concept of instantaneous rate of change, a concept which is aptly situated at the conjunction of mathematics, physics, and history of science, as means of testing our hypothesis on the role of history of mathematics in pedagogy of mathematics. There are four major reasons for this choice. The most compelling one is its peculiarly unique historical status. Indeed, when compared to all the other problems that have led to the advancement of modern mathematics, in particular calculus, such as the area problem or the tangent problem, determining the instantaneous rate of change of an object in motion holds a rather incongruous position. The area problem was one of the oldest problems in mathematics (Mankiewicz, 2004); in the Rhind papyrus the Egyptian scribe Ahmes recorded problems whose solutions required the computation of areas of rectangles and triangles. Certainly, the Greek mathematicians knew how to determine the areas of most standard geometric figures. As a matter of fact, Archimedes (287 B.C. – 212 B.C.) developed the method of exhaustion of Eudoxus (408 B.C. – 355 B.C.), and applied it to problems of finding areas of non-polygonal regions. Albeit in the special case of a circle, the Greeks also dealt with the tangent problem. Yet, only in Europe, and only several centuries later, did mathematicians try to express motion and instantaneous velocity in mathematical terms, and started an arduous trek with Galilean kinematics and the mean speed rule as its first milestones, a trek that seemed to reach its final destination in the classical mechanics of Newton, only to continue its ascent via the theories of general relativity and quantum mechanics a few centuries later (Izmirli 2008).

Secondly, the concept is very versatile and is applicable at a variety of levels (Yerushalmy and Gilead, 1999).

Thirdly, this choice would also help students develop correct ideas about motion, by no means a trivial achievement. In a research conducted with 478 college students (Halloun and Hestenes, 1985) several misconceptions students had about motion, force, and velocity were mentioned, showing most college students taking university physics had very Aristotelian or impetus theory based views of these physical concepts.

Lastly, despite its historically unique status, applicational versatility and flexibility, and exceptional position of integrating physical concepts and mathematics, research involving instantaneous rate of change as the conduit for integrating history of mathematics to pedagogy of mathematics seems to be rather scarce and sporadic: Tzanakis (1996), Steuwer (1998), Tzanakis (1999), and Lombardi (1999) are some of the better known exceptions.

2. Methodology

To efficiently collect the data required to establish our contention that integrating the history and pedagogy of mathematics is a beneficial learning tool, we decided to conduct our research in several parts.

First, we used extensive questionnaires to assess students' attitudes. As is well known, questionnaires, probably the most common way of implementing observational studies, are widely used to expedite the gathering of scientific information on a population's ideas, beliefs, and opinions on particular issues; to identify possible cause and effect relationships; and to determine if a procedural change has afforded a difference in attitudes, beliefs or opinions of a population (McMillan and Schumacher, 2006).

Most of our questions were close-ended and were aimed at restricting the respondent to selecting an answer from the specified response options, and hence to provide us with some consistency. Some of the close-ended questions were constructed as *scaled-response* questions where the answer was to be selected from a list of alternative responses that increased or decreased in intensity in an ordered fashion. These were intended to measure students' feelings on projects as learning tools. Yet some others were designed as *binary* questions where the information obtained could best be analyzed in the yes-no categories. The final few questions were open-ended, i.e., they gave respondents an opportunity to interpret and answer the questions in their own words and imposed no structure on their responses.

Initially, we developed more questions than the number we intended to ask, and through pilot studies chose the most appropriate and useful ones for the questionnaire. These studies were conducted, in parts, in small pilot groups selected from all area universities that we expected to participate in our survey. As some questions that formed a somewhat coherent whole were formulated, they were sent out to a predetermined pilot group. Eventually, the entire questionnaire was tested. The testing process helped us discover poor or ambiguous wording, eliminate flawed ordering of questions, identify errors in the questionnaire layout and instructions, and determine problems caused by the respondents' inability or unwillingness to answer the questions. For a complete list of the final questions see (Izmirli, 2008).

To establish a survey population, names of instructors teaching precalculus and calculus classes in local universities were selected at random from course schedules. These instructors, as well as an acquaintance teaching at an overseas university, were contacted by a letter explaining the purpose of our survey and asking if they would be willing to participate in it by assigning some projects incorporating historical development of a concept (rate of change in case of precalculus courses and instantaneous rate of change in case of calculus courses), and using the outcomes as part of their evaluation and assessment process. This letter also contained a suggested list of sample project topics. However, the instructors were encouraged to come up with their own projects.

The surveys were hand-delivered to the participants in area institutions (at the end of the Spring Semester of 2007, if their institutions were on a semester system, and at the end of the Winter and Spring Quarters of 2007 if their institutions were on a quarter system) and were collected back after completion. The hand-delivery method was preferred over any other because of its enhancement of better response rates (Fowler, 1995).

As for the non-US university, in agreement with the instructor, the surveys were mailed to the instructor. Although in this university the instruction is in English, to create an additional level of comfort, the questionnaire was also translated to the native language and respondents were given a choice of completing the questionnaire in either language.

With those who agreed to participate, face-to-face interviews were conducted after they completed their next level mathematics course. These respondents were visited at a location of their choice, and the extent to which the techniques they acquired while working on these projects contributed to the depth of their mathematical understanding and carried over to the next level were investigated.

At this stage of our research, in order to provide a comparison, we used students selected from some mathematics classes in which the factor whose effect was being estimated was absent (i.e., classes where no such projects were assigned) as our control group.

As a second step, in order to gauge students' progressive development we decided to survey a group of students on their attitudes towards mathematics at the beginning of the semester and then at the end of the semester (same questions) after they have been exposed to history of mathematics and research projects involving history of mathematics.

Lastly, to get some input from the teachers whose classes were involved in this research, we asked them to provide us with some information on to what extent they covered history of mathematics in their classes prior to assigning projects, what sources they used, what handouts they distributed, and what were some of their project topics.

3. Description and Analysis of Data

3.1 The Survey

We conducted two different tests of significance. First, to establish if there were statistically significant differences in the answers to the survey questions based on a single category, we used the test of significance with two proportions. We also used a χ^2 test to determine overall association between the classifying variables. For both tests we used significance level of $\alpha = 0.05$. The reference period for the survey was the Spring Semester of 2007 for institutions using a semester system, and the Winter and Spring Quarters of 2007 for institutions using a quarter system.

Although our questionnaire was designed to test whether incorporating historical development of a mathematical concept, in this particular case, the concept of instantaneous rate of change, into the assessment process through some research activities, would enrich students' understanding and appreciation of mathematics and alter the common misperception that mathematics is merely an ossified list of facts, we also explored possible associations between students' responses to questions related to their conception of mathematics and their academic backgrounds, academic affiliations, and future career choices. For detailed tables see (Izmirli, 2008).

3.2 Correspondences with Professors

Next, to get some input from the instructors whose classes were involved in this research, we asked them to provide us with some information on to what extent they covered history of mathematics in their classes prior to assigning projects, what sources they used, what handouts they distributed, and what were some of their project topics.

Overall, of the fourteen professors who took part in our survey nine answered. Some gave the reference sites they used, some talked about how they covered history in their classes, and some sent a list of project assignments.

Here are some samples of what they said in reference to using history of mathematics in their classes:

> "I make history a part of all my courses, but the emphasis varies quite a lot depending on the course.
>
> In the number theory part of my Introduction to Mathematical Thought class, I mention Euclid in connection with prime numbers and perfect numbers. I also talk about Mersenne primes. In my Calculus class I always mention Newton and his development of calculus. In my Mathematical Logic class, which focuses on the development of first-order logic, which occurred mostly in 20th century, I give the mathematical details along with their historical context and the people involved, principally Gödel and Turing. In my analysis class I emphasize an axiomatic approach starting with Peano Postulates, with a mention of Giuseppe Peano. I use the same approach when I introduce some main theorems named after people: Dedekind, Cantor, Borel, Heine, Weierstrass, Cauchy, Bolzano, Riemann, etc."

> "I spend a good deal of time on graph theory. I start with the history of Seven Bridges of Konigsberg and its solution by Euler. I assign a recent article by Brian Hopkins and Robin J. Wilson 'The Truth about Konigsberg' (Coll. Math. Jour., vol 35, no 3, May 2004). I also talk about the Four Color Theorem, and I enjoy recounting the origins of the conjectures and some of

the early attempts to prove it. I assign as reading the book 'Four Colors Suffice' by Robin J. Wilson (Princeton Univ. Press, 2002)"

"I insisted on having them use the Internet as a resource. At the beginning of the year I gave the following online resources and each chapter I covered I asked them to find some relevant historical information from these resources and have a short class presentation
http://mathforum.org/kb/forum.jspa?forumID=193
History Topics Index from the MacTutor History of Mathematics: An extensive list of essays on math history.
Math history timeline: Timeline from 600 B.C.E. to present.
Living Math History Course[1]: Biographies, resources, links to other history of math sites."

"In the past I was able to cover more, but now some of these sections have been deleted. When I covered infinite sets and cardinals in this course, I could discuss Zeno's paradoxes and assign papers on this topic. I would definitely have them write an essay on ancient Greek thoughts about the infinite and its conundrums. Of course, in my upper level classes I still talk about Cantor and the Continuum Hypothesis and discuss its independence from the other axioms of sets as established by Gödel in 1940 and Cohen in 1963."

"When I teach my graduate level geometry course, it is like teaching history. I review the axiomatic approach by Euclid and discuss its revision by David Hilbert in the early 20th century. I base the course on Klein's Erlanger Program with its emphasis on groups of transformations. We study hyperbolic geometry from that viewpoint and I mention the effect of the "existence" (i.e. consistency and presence of models) of hyperbolic geometry on ideas of the geometry of physical space."

"I do not assign readings, but in this course since I was going to assign some projects, I did give some references. I researched some sites on the Internet. My favorite was http://www-groups.dcs.st-and.ac.uk/%7Ehistory/ "

"Regardless of the purpose for your research, the topic is quite interesting to me – I studied the factors which influence attitudes toward mathematics in college students at Ohio University, and never considered HISTORICAL development as one of these.

When I teach my classes I mostly provide historical notes in terms of "oral antidotes [sic]" I never hold them responsible for these facts, or require (until last semester when you contacted me) them to do outside research. So the number pi, approximately equal to 3.14, was historically discovered as the fixed ratio of ANY circle's circumference to its diameter?"

"I think I will use this type of assignments more in the future. These research projects seem to be excellent conduits for the emblematic numeric – graphic – algebraic approach to learning of mathematics. Did you see the Robert and Speer (2001) paper?"

Some gave the collection of references they asked their students to read as part of the course work. For a detailed list see (Izmirli, 2008).

Most professors used our suggestions and similar ideas to assign project topics involving the historical development of instantaneous rate of change and its implications in precalculus and calculus. Some did assign some different topics as well. Some kept the idea and continued with assigning these types of projects in the following semesters as well, interjecting some very interesting ideas (Izmirli, 2008).

[1] There were no addresses given for these websites.

3.3 The Interviews

Finally, with those survey participants who had agreed to be included in the interviewing process, face-to-face sessions were conducted following the completion of the next level mathematics course they took in the Fall 2007 term. We were expecting that a significant portion of the students who completed the surveys would agree to an interview as well. However, most students declined our request for an interview; and overall fifty-two interviews were conducted.

The interviews took place mostly during the first three weeks of December, 2007. The respondents were visited at a location of their choice. In most cases, a classroom or an office space was provided so as we could conduct the interviews in private. In one school where there was not enough space, the meetings were held at the school cafeteria. Although the initial plan was to record the interviews by a tape recorder, faced with several remarks from the participants implying that this would make them uncomfortable, we decided to take notes instead. Each interview started out by proper introductions, an explanation of how the responses would be used, and a reminder that the participant could change his/her mind at any time or choose not to answer some of the questions. The total duration of an interview was about fifteen minutes. The foreign students were given the choice of conducting the interviews in their native tongue or in English, or a mixture of the two.

The interview questions were aimed at determining the extent to which the techniques the participants acquired while working on their projects contributed to their faculties of analysis, especially in reference to problems of mathematics education; affected their perception of mathematics; and carried over to the next level of mathematics classes.

3.4 Results

It will now be shown that the statistical outcomes and data analyses of this research, as summarized in the previous section, strongly support our hypothesis that integrating the historic development of mathematical concepts in the educational practice of mathematics via some research projects plays a vital role in the cognitive development of students and helps resolve problems associated with students' mostly negative perception of mathematics.

The survey sample consisted of 219 students selected at random from some postsecondary institutions (including major universities, a community college, and a for-profit institution) in the metropolitan Washington D.C. area and a non-U.S. university. These participants, in general, displayed fairly typical characteristics: most of them had no major breaks in their academic pursuits, had the appropriate degrees and background, and did not have a work schedule that would interfere with their studies. Their classes were carefully monitored through quizzes and homework assignments, though not necessarily through attendance. Most of them enjoyed being a part of their institutions, but had no interest in becoming educators. Almost none had any previous experiences with research projects.

We were also able to establish that the project assignments were not going to place any non-mathematical burdens on the participants. A vast majority of them was technically adept and almost all had access to a computer. Thus, they could easily exchange information via e-mails and did not have to actually meet several times to complete a group project. Except for the for-profit institution, library facilities were more than adequate, and in all cases students did have access to on-line sources. Moreover, the participants from the non-U.S. university, though not native speakers, were quite fluent in English. Consequently, neither lack of technology nor of linguistic skills would engender a significant obstacle in the completion of the projects.

3.4.1 The Survey Group Results

(i) **Relevance of Projects**

As the first measure of the effectiveness of the research projects, we wanted to see if the participants appreciated the relevance of the research projects in their class work, in other words, if they were able to establish meaningful ties between the research they were conducting for their projects and their course work. When we compared the proportion of the respondents who categorized the midterm project as being either "Very Relevant" or "Somewhat Relevant" to those who categorized the first project to be so, we observed a significant positive change (*P*-value: < 0.0001); the same observation held true when these respective categories were compared between the final and the midterm projects (P-value = 0.0271). Also, by applying the χ^2 test of independence, we reached the same conclusion across all categories with *P*-values of 4.693775×10^{-6} and 2.33071×10^{-15}, respectively. In other words, the more the students worked on projects, the more they appreciated the relevance of research in classroom mathematics.

Thus, research projects based on history of mathematics provided an excellent medium to instill in the students one of the most vital tenets of mathematics: the topics routinely taught as part of an undergraduate mathematics curriculum today were once the subjects of research mathematics.

(ii) Change in Attitude

Although the participants' appreciation of the relevance of the projects showed a steady increase, the same was not the case in their attitudes towards this type of research. Since we were mostly interested in the "Enjoyable and Informative" category, we tested those ratios. No significant difference was observed between the first assignment and the midterm assignment (P-value = 0.0637). The same was the case for all categories as verified by the χ^2 test of independence (*P*-value = 0.220452) indicating no association between students' attitude towards research between the midterm and the first assignment.

But when we applied the two sample proportion test to the "Enjoyable and Informative" category in the midterm and the final assignments, we observed a significant positive change (*P*-value = 0.0333). The χ^2 test of independence also indicated a strong association (*P*-value = 1.565339×10^{-6}) between students' attitudes after the completion of all three research projects.

These outcomes lead to two possible conclusions

- The occurrence of the change in attitude may need more time than the development of an appreciation of the relevance of the research, in other words, it may take more than one research project to infuse such a change
- The students in the for-profit institution may have skewed the results (These students were not assigned a final project, and consequently did not answer this question in reference to the final project). It is possible that the students in the for-profit institution, who are mostly working adults, may have already established firmer (possibly negative) attitudes on mathematics, and consequently, attitude change would be more likely to take place among students of other institutions

(iii) Importance of Particular Disciplines

For each individual topic[2], we observed a very strong association between one's institutional affiliation and one's perception of importance of that topic in terms of gaining a deeper understanding of one's major area; in fact, the *P*-values were almost zero (for Geometry *P*-value = 4.377094×10^{-7}, for Algebra 7.501677×10^{-4}, for Precalculus 1.674902×10^{-22}, for Precalculus with Trigonometry 4.234720×10^{-42}, for Calculus for Business 4.998387×10^{-53}, and for Calculus 1.584912×10^{-45}.)

This, of course, was to be expected. For instance, an engineering student might rate calculus as being extremely important and only certain types of institutions would have engineering majors. Similarly, business majors, who, in our case were usually the community college or the for-profit students, would rate calculus for business as being extremely important.

[2] Namely Geometry, Algebra, Precalculus, Precalculus with Trigonometry, Calcuklus for Business, and Calculus.

(iv) Comparisons Based on Individual Disciplines

Subject by subject, we compared the proportion of students who thought their understanding of that particular subject got better after the completion of a project to the proportion of those who believed it got better after the completion of the previous project. We also conducted two χ^2 tests of independence for each subject. With the first one we looked for an association between the row variable with categories "Project 1" and "Midterm", and the column variable with categories "Better", "Same", and "Worse"; and with the second one we looked for an association between the row variable with categories "Project 1", "Midterm", and "Final", and the column variable with categories "Better", "Same", and "Worse".

The first subject was geometry. Comparing the "Better" category for the first and the midterm assignments, we observed a significant change (*P*-value = 0.012). The same was the case when we compared the identical category for the midterm and the final assignments (*P*-value = 0.0255). Also, the χ^2 test of independence indicated that there was an association between students' understanding of geometry and the completion of a particular assignment (*P*-value = 0.048171 when their level of understanding after the first assignment was compared to their level of understanding after the midterm assignment, and was 1.614696×10^{-4} after the completion of all three assignments). Thus, there was a sequential, consistent improvement in students' understanding of geometry.

The next topic was algebra. Comparing the "Better" category for the first and the midterm, we observed a significant change (*P*-value = 0.0277). Same was the case when we compared the identical category for the midterm and the final assignments (*P*-value = 0.0038). Also, the χ^2 test of independence indicated that there was an association between students' understanding of algebra and the completion of a particular assignment (*P*-value = 0.045762 when their level of understanding after the first assignment was compared to their level of understanding after the midterm assignment, and was after the 1.639858 $\times 10^{-4}$ completion of all three assignments). Thus, there was a sequential, consistent improvement in students' understanding of algebra as well.

There were two exceptions to sequential and consistent betterment of students' understanding of a subject, namely, precalculus and calculus for business.

In case of precalculus, comparing the "Better" category for first and midterm, we did not observe a significant change (*P*-value = 0.0918). But, same was not the case between midterm and final (*P*-value = 0.0367). The χ^2 test of independence indicated that there was no association between students' understanding of precalculus and the completion of an assignment in case of the first and the midterm assignments (*P*-value = 0.355259). However, *P*-value = 0.036113 when the same comparison was made after the completion of all three assignments.

Calculus for business proved to be the other exception. Comparing the "Better" category for the first and the midterm assignment, we did not observe a significant change (*P*-value = 0.2683). Contrary to the other exception, precalculus, there was no significant change in this category for the midterm and the final either (*P*-value = 0.2011). The χ^2 test of independence indicated that there was no association between students' understanding of calculus for business and the assignment (*P*-value = 0.506617 first to midterm, 0.110308 for all three assignments).

A possible explanation might be that these courses were mostly taken by community college and for-profit students. These students might not have been as mathematically prepared as their four-year university counterparts to benefit from these projects. Another possible explanation might be the inflexibility of the community college and the for-profit institution instructors in assigning the projects. They took our proposal to formulate their assignments in terms of instantaneous velocity, a concept that would make more sense to a physics or engineering major than to a business major, literally. It would have been better had these instructors, most of whose students were not science or engineering majors, adapted the concept of the historic development of instantaneous velocity to the historic development of comparable notions involving instantaneous change.[3]

[3] In fact, in the letters we sent to the instructors, we clearly gave them this latitude.

In case of precalculus with trigonometry, when we compared the "Better" category for the first and the midterm assignments, we observed a significant change (*P*-value = 0.034). The same was the case when we compared the identical category for the midterm and the final assignments (*P*-value = 0.0343). Also, the χ^2 test of independence indicated that there was an association between students' understanding of precalculus with trigonometry and the completion of a particular assignment (*P*-value = 0.041906 when their level of understanding after the first assignment was compared to their level of understanding after the midterm assignment, and was 0.003314 after the completion of all three assignments). Thus, there was a sequential, consistent betterment in students' understanding of precalculus with trigonometry.

Our last subject was calculus. Comparing the "Better" category for the first and the midterm, we observed a significant change (*P*-value = 0.0328). The same was the case when we compared the identical category for the midterm and the final assignments (*P*-value = 0.0354). Also, the χ^2 test of independence indicated that there was an association between students' understanding of calculus and the completion of a particular assignment (*P*-value = 0.043559 when their level of understanding after the first assignment was compared to their level of understanding after the midterm assignment, and was after the 0.001296 completion of all three assignments). Thus, there was a sequential, consistent betterment in students' understanding of calculus as well.

Clearly, research assignments involving history of mathematics increase students' understanding of mathematics. However, to obtain the optimum benefit, the instructors should not restrict themselves to a preset, inflexible list of assignments; they should adapt the research topics to the classes they teach and make sure that these topics make sense to the participants in terms of their majors and/or everyday experiences.

3.4.2 The Control Group vs. the Survey Group

Our first major observation was that there was a statistically significant increase in the number of students (from 6 out of 34 in the control group to 83 out of 219 in the survey group) who would take a mathematics class as an elective (*P*-value = 0.0107008).

Next, we compared the control and survey group responses as to what extent they agreed with the Harvard Calculus Consortium's "Rule of Three"[4] that most ideas in mathematics should be introduced via a combination of graphical, numerical, and analytical approaches, for we wanted to see if the students had a chance to observe that such a combined approach proved to be a useful tool in their research projects. There was a significant increase in the proportion of students in the survey group who strongly agreed to that of control group students who strongly agreed (*P*-value = 0.0016). Also, applying the χ^2 test of independence, we were able to establish a strong association between level of agreement and being a member of either the survey or the control group (*P*-value = 1.343652×10^{-8}).

We then wanted to see if the participants' conception of mathematics as "a science based on eternal and unchanging truths" changed after their involvement in research projects. First, we asked if they believed mathematics to be eternal and unchanging. It is interesting to note that after delving a bit into the history of mathematics, the proportion of students who regarded mathematics as a static, non-evolving science decreased significantly (P-value = 0.0084). Similarly, the χ^2 test of independence indicated a strong association (*P*-value = 0.021437).

There was also a significant increase in the proportion of students in the survey group over those in the control group who agreed that mathematical truths were subject to change (*P*-value = 0.0061), as well as an overall association between the students' understanding of the nature of mathematical truths and working on research projects (*P*-value = 0.024818). A similar significant increase was noted in the proportion of students who realized that a mathematical statement could be proved in more than one way

[4] At the time the survey was conducted Harvard Calculus Consortium was promoting The Rule of Three in teaching mathematic: using graphical, numerical, and analytic approaches. This rule is now extended to The Rule of Four, the fourth approach being the use of the Internet resources.

(*P*-value = 0.0095), as well as an overall association between the nature of mathematical proofs and whether the student was in the experimental or control group (*P*-value = 0.018938).

Next, we offered the following hypothetical situation. If the participants were grading a student's proof, and the proof differed from the one given in the book, what might the participant conclude? The choices were "the book must be wrong", "the student must be wrong", "both may be wrong", and "both may be correct". There was a significant increase in the "most reasonable" choice that both the student and the text may be correct (*P*-value = 0.0003), as well as an overall association between choice and whether the student was in the experimental or control group (*P*-value = 1.589205×10^{-4}).

Would one need to take a mathematics class to become a problem solver? There were significant increases in the proportion of students who strongly agreed (*P*-value = 0.071), and agreed (*P*-value = 0.0434) as well as an overall association between choice and whether the student was in the experimental or control group (*P*-value = 0.001622).

How about to become a critical thinker? Again we observed significant increase in the proportion of those who strongly agreed (*P*-value = 0.0006) and agreed (*P*-value = 0.0007), as well as an overall association between choice and whether the student was in the experimental or control group (*P*-value = 6.747233×10^{-11}).

Comparing Answers to Open-Ended Questions: Survey Group vs. Control Group

The advantage of asking open-ended questions is that we get to view the participants' own opinions in their own words. Unfortunately this comes with a price, for it is, understandably, harder to compare the answers to open-ended questions on a numerical scale, as these tend to be vaguer than the answers to close-ended questions. In what follows, the counts and *ergo* the proportions are based on answers that, in our opinion, were "close enough" to "the right" or "the expected" answers.

When asked how they felt about education, 19 out of 23 participants (82.6%) in the control group and 153 out of 166 (92.2%) in the survey group had positive things to say, indicating no significant difference (*P*-value = 0.06657). Also, 16 out of 21 responses (76.2%) in the control group and 113 out of 127 (89%) responses in the survey group drew clear distinctions between education and training, again demonstrating no significant difference (P-value = 0.052341). Thus, both groups had positive feelings about education and could clearly distinguish between education and training.

However, when they were asked to elaborate on whether they thought mathematics should be an integral part of an overall education, three out of 26 answers (11.5%) in the control group and 105 out of 219 (47.9%) in the survey group were affirmative, which indicated a statistically significant increase (P-value = 0.000204)

When they were asked if they believed using research projects that emphasize the historical development of a concept improves one's understanding of mathematics, only one out of seven responses (14.3%) in the control group was affirmative, compared to 115 out of 219 responses (52.5%) in the survey group, again a statistically significant increase (*P*-value = 0.02319).

3.4.3 Pre and Post Project Responses

When we compared the reasons students mentioned for taking a mathematics class, significant increase was observed in the proportion of those that would take mathematics as an elective or for personal interest in the post project responses (*P*-value = 0.026), as well as an overall association between working on projects and reasons for taking a mathematics course (*P*-value = 0.044739).

Comparing Answers to Open-Ended Questions: Pre and Post Project Responses

When asked how they felt about education, 24 out of 30 participants' responses (80%) in the pre-project stage and 28 out of 30 (93.3%) in the post project stage had something positive to say, showing no significant change (*P*-value = 0.06437). Again, in both stages participants were able to distinguish between education and training (26 out of 30 (86.7%) in the pre project stage and 29 out of 30 (96.7%) in the post project stage), with no significant change (*P*-value = 0.08056).

As in the case of control group vs. survey group comparisons, the significant changes occurred in responses to questions related specifically to mathematics. For instance, when we asked the participants what was interesting about mathematics and why (here, we were interested in answers that used affective terminology such as “fun”, “love”, “fascinating”, etc., or mentioned a connection to logic and/or critical thinking), there were three such “positive” responses out of a total of 30 (10%) in the pre project stage and 26 out of 30 (86.7%) in the post project stage, a significant increase (*P*-value = 1.413736×10^{-9}).

To the question “Is mathematics important?” 19 out of 30 (63.3%) in the pre project stage and 23 out of 30 (76.7%) in the post project stage answered affirmatively. There does not seem to be a significant difference on the basis of these proportions (*P*-value = 0.1299), but we must also take into consideration the qualitative characteristics of the answers as well. In the pre project stage most “Yes” answers mentioned simple skills involving the use of four basic operations in everyday activities. The post project responses indicated a much deeper understanding of the “applicability” of mathematics, mentioning “logical development,” “increased analytical thinking abilities,” and “improved problem solving techniques.”

Another good measure of in depth understanding of a concept is the rate at which the learners gain the ability to describe it to someone who does not know much about it. When we asked the participants how they would describe the course they are taking to a friend who has not taken it yet, there were no good descriptions in the pre project stage. However, as documented in (Izmirli , 2008), we got 15 meaningful descriptions out of the 30 responses given in the post project stage (*P*-value = $3.875557736 \times 10^{-6}$).

3.4.4 Interviews

The face-to-face interviews were very useful in refining our pedagogical model. For instance, we got a good insight into what the students perceived as problems with the way mathematics is taught. Indeed, there were three recurring themes.

The first one was the ineptitude of the secondary education teacher candidates. Most answers in this category expressed the belief that the teachers in the secondary system do not have a solid understanding of mathematics. Most of the interviewees had the impression that these teachers learned one and only one way of solving a problem from a single source, and were not open to alternate methods. Some even explicitly stated that it was quite obvious that when the instructors got stuck and could not come up with a satifactory solution, they chose to hide behind the cover of the Socratic method[5].

The second major theme was the perceived indifference of some of professors. The students were under the impression that some of their professors did not have time for them, and stressed that they were made to feel like “unwelcome diversions.” Some students were discouraged from asking questions in class, for fear of being made to feel mentally inferior.

The third recurring issue was the apathy and disinterest exhibited by the student body itself. Some participants said that most of their classmates either did not attend classes or when they did, they text-messaged their friends, though they never admitted to similar behavior themselves.

[5] The Socratic method also known as ***method of elenchus***, (from Greek: ελεγχος meaning an argument for refutation, see (Vlastos, 1983)) is a method based on asking and answering questions to stimulate critical thinking.
As an example, say a student makes an assertion “Mathematics is the science of working with numbers.” The teacher then, by a series of questions, leads the student to a branch of mathematics that does not deal with numbers at all, say topology, and thus, refutes the assertion.
Of course, when used properly, the method is a great educational tool. The comment here was intended to imply that the teacher kept asking questions instead of giving an answer simply because he/she could not.

Through the next few questions, we were able to conclude that our interviewees gained a new appreciation of mathematics and carried their research techniques and study habits they gained while working on the research projects to the next level of classes. For instance, when asked why they would take a mathematics course, predictably, most brought up prerequisites, and better preparedness for upper level classes. But what was remarkable here was the fact that a majority of the interviewees also mentioned taking a mathematics course as an elective or for the development of logical thinking. Even more significant were the affective responses. Confronting fear of mathematics, having fun with mathematics, fascination with certain branches of mathematics, enjoyment of mathematics, becoming conscious of the cultural aspects of mathematics, and learning a "universal language" were all brought up. Interestingly enough, some students mentioned their desire to teach mathematics, now that they understood that it was not just a series of simple calculations but was a worthwhile endeavor.

When asked to elucidate what they found interesting about mathematics, the interviewees had interesting answers ranging from putting simple ideas together to make a complex whole, to using irrational numbers like π whose exact value would never be known, to the concept of infinity, and to the use of mathematical concepts in developing computer games. They also valued the importance of mathematics and most described this as "mathematics being the foundation of everything else, from economics, to physics, to engineering, to philosophy of science."

Then, they were asked to comment on how the research projects affected their perception of mathematics. The most common answers indicated a new found level of comfort in realizing that that even professional mathematicians could make mistakes, or have false starts. Several interviewees described mathematical discoveries as well-educated guesses verified by deductive methods. Almost all seemed to understand the importance of social interaction and cooperation in sciences, the significance of gaining the ability to express mathematical ideas using proper language, and the fact that mathematical ideas were open to discussion.

Finally, they were asked to describe how they use the techniques they acquired while they were working on their research projects in the previous classes in their current classes. Answers included heightened level of confidence, a new interest in the historic development of a concept, appreciation of group work, coming up with innovative solutions, improved presentation techniques, learning about library and Internet research, and communicating mathematical ideas in a precise manner.

4. Conclusions

The statistical outcomes and data analyses of this research, as summarized in the previous section, strongly supported the hypotheses that incorporating a coherent sequence of dynamic learning experiences on the history of mathematics into mathematics classes in terms of projects

1. *Is a useful pedagogical tool not only in constructing new mathematical ideas but also in emphasizing the socio-cultural aspects of mathematics.* As for the pedagogical benefits the most commonly mentioned outcomes were developing a new interest in the historic progress of a concept, valuing different and innovative solutions of existing problems, learning about library and Internet research, and acquiring a heightened level of confidence.
   As for socio-cultural aspects of mathematics, all answers (those given in surveys and those given in interviews) indicated that participants quickly became cognizant of the social aspects of mathematics and the means of successfully exchanging ideas with each other. With a very few exceptions, the responses also revealed the enhancement of a strong sense of cooperation and the understanding of the importance of equitable work sharing among the participants. Indeed, when asked to describe the benefits of working on their research projects, most of the answers accentuated appreciation of group work, improved presentation techniques, and communicating mathematical ideas in a precise manner.

2. *Helps corroborate the fallibilist aspects of mathematics*. For instance, after delving a bit into the history of mathematics, the proportion of students who regarded mathematics as a static, non-

evolving science decreased significantly. There was also a significant increase in the proportion of students in the survey group over those in the control group who agreed that mathematical truths were subject to change. A similar significant increase was noted in the proportion of students who realized that a mathematical statement could be proved in more than one way.
Indeed, when asked to comment on how the research projects affected their perception of mathematics, the most common answers indicated a new found level of comfort in realizing that that even professional mathematicians could make mistakes, or have false starts. Several interviewees described mathematical discoveries as well-educated guesses verified by deductive methods. Almost all seemed to understand the importance of social interaction and cooperation in sciences, the significance of gaining the ability to express mathematical ideas using proper language, and the fact that mathematical ideas were open to discussion.

3. *Plays a vital role in the cognitive development of students*. Subject by subject, we compared the proportion of students who thought their understanding of that particular subject got better after the completion of a project to the proportion of those who believed it got better after the completion of the previous project. In a huge majority of topics we observed a sequential, consistent improvement in students' understanding of that particular topic.

4. *Helps resolve problems associated with students' mostly negative perception of mathematics*. Over the course of the semester there was a notable change in attitude towards finding mathematical projects "Enjoyable and Informative". Moreover, there was a statistically significant increase in the number of students who would take a mathematics class as an elective, as well as in the number of students who deemed the necessity of mathematical thinking to be a problem solver and a critical thinker. There was also a significant increase in the number students who thought mathematics should be an integral part of an overall education, and in the number of those who would take a mathematics class as an elective or for personal interest.
Significant changes were also observed in responses to questions related to the use of adjectives defining mathematics. For instance, when we asked the participants what was interesting about mathematics and why (here, we were interested in answers that used affective terminology such as "fun", "love", "fascinating", etc., or mentioned a connection to logic and/or critical thinking), there was a significant increase.

5. *Instills in the students one of the most vital tenets of mathematics, namely, the fact that the topics routinely taught as part of an undergraduate mathematics curriculum today were once the subjects of research mathematics.*

6. *A rather fortuitous outcome was the positive effect these projects generated among the instructors.* Most instructors, who later responded to me, indicated that they were very excited to see the renewed level of interest these projects created in their students, and that they will from now on make these assignments a permanent part of their assessment process. Indeed, several reported new project ideas involving the history of mathematics.

Acknowledgements. The author would like to express his deepest appreciation to the referees. Without their invaluable guidance and constructive recommendations the final form of this paper would not have been possible.

References

Barbin, E. (1996). The role of problems in the history and teaching of mathematics. In R. Calinger (Ed.), *Vita Mathematica: Historical Research and Integration with Teaching:* 17-25. Washington, D.C.: The Mathematical Association of America.

Barwell, M. E. (1913). The advisability of including some instruction in the school course on the history of mathematics. *The Mathematical Gazette*, **7**(104): 72 – 79.

Bradis, V. M., Minkovskii, V. L., and Kharcheva, A. K. (1963). *Lapses in Mathematical Reasoning.* Dover Edition (1999). Mineola, N.Y.: Dover Books.

Burns, S. (1997). The Babylonian clay tablet. *Mathematics Teaching* **158**: 44-45.

Ernest, P. (1988). The attitudes and practices of student teachers of primary school mathematics. In A. Borbas (Ed.), *Proceedings of PME-**12*** (1): 288-295. Veszprem, Hungary.

Ernest, P. (1989). The impact of beliefs on the teaching of mathematics. In P. Ernest (Ed.) *Mathematics Teaching: The State of the Art:* 249-254. London: Falmer Press.

Ernest, P. (1991). *The Philosophy of Mathematics Education.* London: Falmer Press.

Ernest, P. (1998). *Social Constructivism as a Philosophy of Mathematics.* Albany, NY: State University of New York Press.

Fowler, F. J., Jr. (1995). *Improving Survey Questions: Design and Evaluation.* Thousand Oaks, CA.: Sage Publications.

Freudenthal, H. (1981). Should a mathematics teacher know something about the history of mathematics? *For the Learning of Mathematics,* **2**(1): 30-33.

Fried, M. N. (2001). Can mathematics education and history of mathematics coexist? *Science and Education*, **10**(4): 391-408.

Furinghetti, F. (1997). History of mathematics, mathematics education, school practice: case studies in linking different domains. *For the Learning of Mathematics*, **17** (1): 55-61.

Furinghetti, F., and Somaglia, A. (1998). History of mathematics in school across disciplines. *Mathematics in School*, **27**(4): 48-51.

Glas, E. (1998). Fallibilism and the use of history in mathematics education. *Science and Education*, **7**(4): 361-379.

Halloun-Abou, I., and Hestenes. D. (1985). The initial knowledge state of college physics students. *American Journal of Physics*, **53**(11): 1043 – 1055.

Izmirli, I. M. (2008). *Study of Instantaneous Rate of Change in a Historical Context.* Ph.D. Dissertation.

Katz, V. J. (1986). Using history in teaching mathematics. *For the Learning of Mathematics*, **6**(3): 13-19.

Laubenbacher, R., Pengelley, D., and Siddoway, M. (1994). Recovering motivation in mathematics: teaching with original sources. *UME Trends*, **6**(4): 1-2.

Lombardi, O. (1999). Aristotelian physics in the context of teaching science: a historical-philosophical approach. *Science and Education*, **8**(3): 217-239.

Mankiewicz, R. (2004). *The Story of Mathematics.* Princeton: Princeton University Press.

McMillan, J. H., and Schumacher, S. (2006). *Research in Education: Evidence Based Inquiry*. Boston: Allyn & Bacon.

Monk, M. and Osborne, J. (1997). Placing the history and philosophy of science on the curriculum: a model for the development of pedagogy. *Science Education*, **81**(4): 405-424.

Morley, A. (1982). Should a mathematics teacher know something about the history of mathematics? *For the Learning of Mathematics,* **2**(3): 46.

Pimm, D. (1982). Why the history and philosophy of mathematics should not be rated X? *For the Learning of Mathematics,* **3**(1): 12-15.

Siu, M. K. (1993). Proof and pedagogy in ancient China: Examples from Liu Hui's commentary on Jiu Zhang Suan Shu. *Educational Studies in Mathematics,* **24**(4): 345-357.

Steuwer, R. (1998). History and physics. *Science and Education* **7**(1): 13-30.

Torkil, H. (1992). Why teach history of mathematics? *Mathematical Gazette*, **76**(475): 151-157.

Tzanakis, C. (1995). Reversing the customary deductive teaching of mathematics by using its history: The case of abstract algebraic concepts. In *Proceedings of the first European Summer University on the history and epistemology in mathematics education*: 271-273. IREM de Montpellier.

Tzanakis, C. (1996). The history of the relation between mathematics and physics as an essential ingredient of their presentation. In Eduardo Veloso (Ed.). *Proceedings of HEM Meeting*: 96-104. Braga, Portugal.

Tzanakis, C. (1999). Unfolding relations between mathematics and physics, in a presentation motivated by history: two examples. *International Journal of Mathematical Education in Science and Technology,* **30**(1): 103-118.

Vlastos, G. (1983). The Socratic Elenchus. In J. Annas (Ed.), *Oxford Studies in Ancient Philosophy **I***: 27-58. Oxford: Oxford University Press.

Wolfson, P. (1981). Philosophy enters the mathematics classroom. *For the Learning of Mathematics,* **2**(1): 22-26.

Yerushalmy. M., and Gilead, S. (1999). Structures of constant rate word problems: A functional approach analysis. *Educational Studies in Mathematics,* **39**(1): 185 – 203.