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Marvelous Medians

An example of connections made in a graduate course for teachers

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Abstract. Open-ended questions can be used as a tool to spark interest and start productive discussions for students with different levels of confidence. This paper presents investigations that evolved in a graduate course on Geometry for teachers from a question concerning areas of four polygons produced by two medians of a triangle. This gives an example of an approach that may help students feel like a team where every person is valued and people with different levels of expertise and experience work together and learn from each other.

Key words: teaching and learning geometry, posing questions, open-ended approach.

Introduction

We learn mathematics by solving problems. In fact, in the year 2000, the largest professional teachers' organization in the world, National Council of Teachers of Mathematics [1] published Principles and Standards for School Mathematics emphasizing problem solving (a process standard), and then, ten years later, this theme was emphasized in Common Core State Standards

[2] - *Make sense of problems and persevere in solving them*.

This emphasizes the importance of problems that teachers choose to offer their students. Needless to say, the 2010 standards were only one step on the long path of worldwide discussion concerning problem solving in mathematical classrooms. The term "open-ended problem" became popular in 1990s. Chapter 1 in [3] starts with a discussion of two meanings of this term: "the problems that … have multiple correct answers" and "… when students are asked … to develop different methods, ways, or approaches to getting an answer" (p.1).

By now, many mathematics educators view the open-ended approach as a convenient tool that may help involve struggling students in a meaningful problem solving process as well as challenge the most advanced students. (e.g., [4], [5], [6]).

Every homework assignment in my graduate classes includes a few purposefully ambiguous questions, and every subsequent class starts with discussion around those questions. At the beginning of the course people sometimes come back with hesitation, saying something along the lines: "I did not know what you wanted me to do. So I did this…" My answer is always the same: "Great! Every idea enriches our discussion." Luckily, in our classroom two walls are covered with whiteboards. This gives ample space for everybody to show their solutions. At the beginning of every class, every student writes his or her solutions on the white board. Often this results in everybody's amazement at the variety of methods and approaches.

Here I would like to share some investigations that evolved in the course titled "Geometry for Middle and High School Teachers." Of course, all participants in this course were well aware that:

- A median is a line segment that connects a vertex of a triangle with the midpoint of the opposite side.
- All three medians intersect at one point that is called a centroid.
- The distance from a centroid to a vertex is twice the distance from a centroid to the midpoint of the opposite side.

The following question was posed for homework:

Two medians of a triangle divide it into four polygons: three triangles and a quadrilateral. What can be stated concerning the areas of these polygons?

The next class started, as always, with students drawing their diagrams and calculations on the white boards. Some ideas were similar and we combined them into a single investigation. Four different approaches were formulated as a result of this work.

Investigation 1. Special case: Equilateral triangle

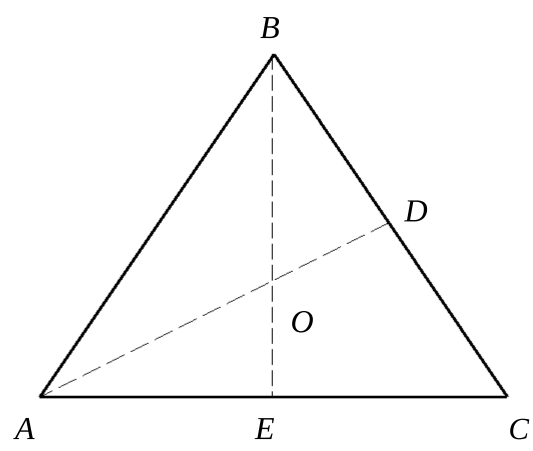


Figure 1. Equilateral triangle

Figure 1 shows an equilateral triangle *ABC* with medians *BE* and *AD*. In equilateral triangles the medians are also heights of the triangle (meaning they are perpendicular to the side they bisect.) Consider right triangle *AEB*. Let's say, the length of the hypotenuse *AB* is *s*. Then the length of *AE* is $\frac{1}{2}s$. Using the Pythagorean Theorem, $BE = \frac{\sqrt{3}}{2}s$. Therefore, the area of triangle *AEB* is $\frac{1}{2} \times AE \times BE = \frac{\sqrt{3}}{8}s^2$. Consider triangle *AOE*. The length of leg *OE* is $\frac{1}{3} \times \frac{\sqrt{3}}{2}s$. Then the area of the triangle *AOE* is

$$\frac{1}{2} \times AE \times OE = \frac{1}{2} \times \frac{1}{2}s \times \frac{1}{3} \times \frac{\sqrt{3}}{2}s = \frac{1}{3} \times \frac{\sqrt{3}}{8}s^2.$$

Thus the area of triangle *AOE* is one-third of the area of *AEB*. This means that the area of the triangle *AOB* is two-thirds of the area of *AEB*.

Similarly, we can consider *ADB*, which is congruent to *AEB*, and conclude that area of triangle *DOB* equals the area of triangle *AOE*. Looking at triangle *ADC*, we can conclude that the area of the triangle *AOB* equals the area of the quadrilateral *EODC*. Now we are ready to state that if two medians are drawn in an equilateral triangle, then the two smaller triangles have equal areas that can be expressed as $\frac{\sqrt{3}}{24}s^2$, where s is the length of the side of the triangle; the area of the larger triangle is twice as big and equals the area of the quadrilateral.

Investigation 2. Special case: Right triangle.

Area *S* is equal to area *T*

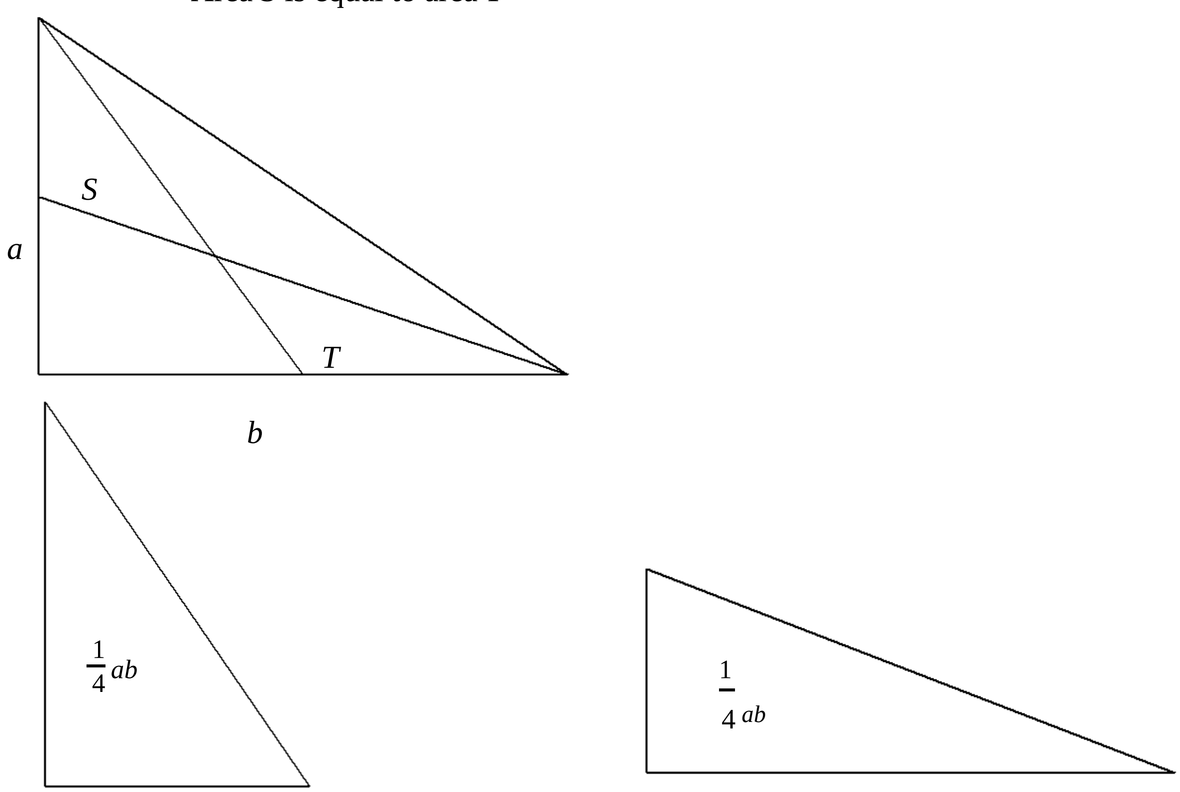


Figure 2. In any right triangle, two medians drawn to the legs cut out two triangles of equal areas

Figure 2 was drawn on the white board with a short comment that two triangles have equal areas. The quadrilateral is a part of both triangles. Therefore, the remaining areas (triangles *S* and *T*) have equal areas.

Figure 3 provided a great opportunity to continue our discussion. This is a wonderful example of a detailed diagram which leads to correct calculations.

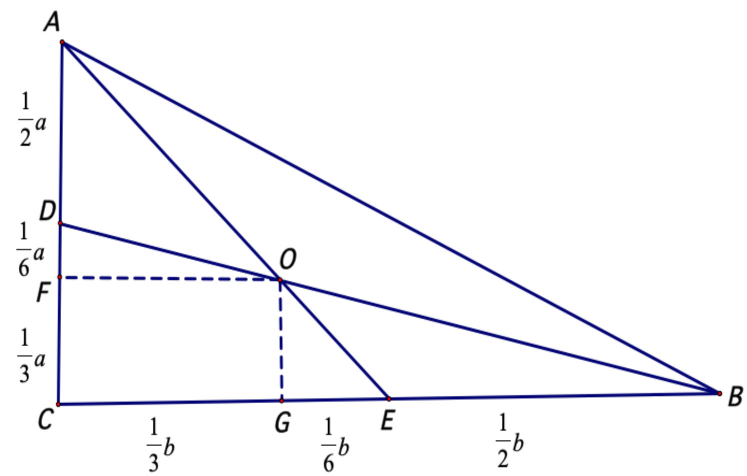


Figure 3. Detailed diagram

Auxillary lines: *OF* is perpendicular to *CA* and *OG* is perpendicular to *CB*. Triangle *ACE* is similar to triangle *AFO*. It follows from here $FC = \frac{1}{3}a$ and $FO = \frac{2}{3}$ of $\frac{1}{2}b$. The quadrilateral *FCGO* is a rectangle whose area is $FC \times FO = \frac{1}{3}a \times \frac{1}{3}b = \frac{1}{9}ab$. The area of triangle *ODF* can be expressed as $\frac{1}{36}ab$ ($\frac{1}{2} \times DF \times FO = \frac{1}{2} \times \frac{1}{6}a \times \frac{1}{3}b = \frac{1}{36}ab$) as well as the area of triangle *OEG* ($\frac{1}{2} \times EG \times FC = \frac{1}{2} \times \frac{1}{6}b \times \frac{1}{3}a = \frac{1}{36}$ab). Thus, area of the triangle *ODF* equals to the area of the triangle *OEG*. The area of triangle *ODA* can be expressed as $\frac{1}{12}ab$ ($\frac{1}{2} \times AD \times FO = \frac{1}{2} \times \frac{1}{2}a \times \frac{1}{3}b = \frac{1}{12}$ab) as well as the area of triangle *OBE* ($\frac{1}{2} \times EB \times$ FC $= \frac{1}{2} \times \frac{1}{2}b \times \frac{1}{3}a = \frac{1}{12}$ab). Thus, the area of quadrilateral *DCEO* equals $\frac{1}{6}ab$. The area of triangle *AOB* equals $\frac{1}{6}ab$, which can be found, for example, as the difference of the areas of triangles *ABE* ($\frac{1}{2} \times AC \times BE = \frac{1}{2} \times a \times \frac{1}{2}b = \frac{1}{4}ab$) and *OBE* ($\frac{1}{2} \times CF \times BE = \frac{1}{2} \times \frac{1}{3}a \times \frac{1}{2}b = \frac{1}{12}ab$).

Conclusion: if two medians are drawn in a right triangle, then the two smaller triangles have equal areas that can be expressed as $\frac{1}{12}ab$, where *a* and *b* are the shorter sides (the legs) of the triangle; the area of the larger triangle is twice as big and equals the area of the quadrilateral. If the right triangle is also an isosceles triangle, then the area of each of the two smaller triangles is $\frac{1}{12}a^2$ and the area of the larger triangle as well as the area of the quadrilateral is is $\frac{1}{6}a^2$.

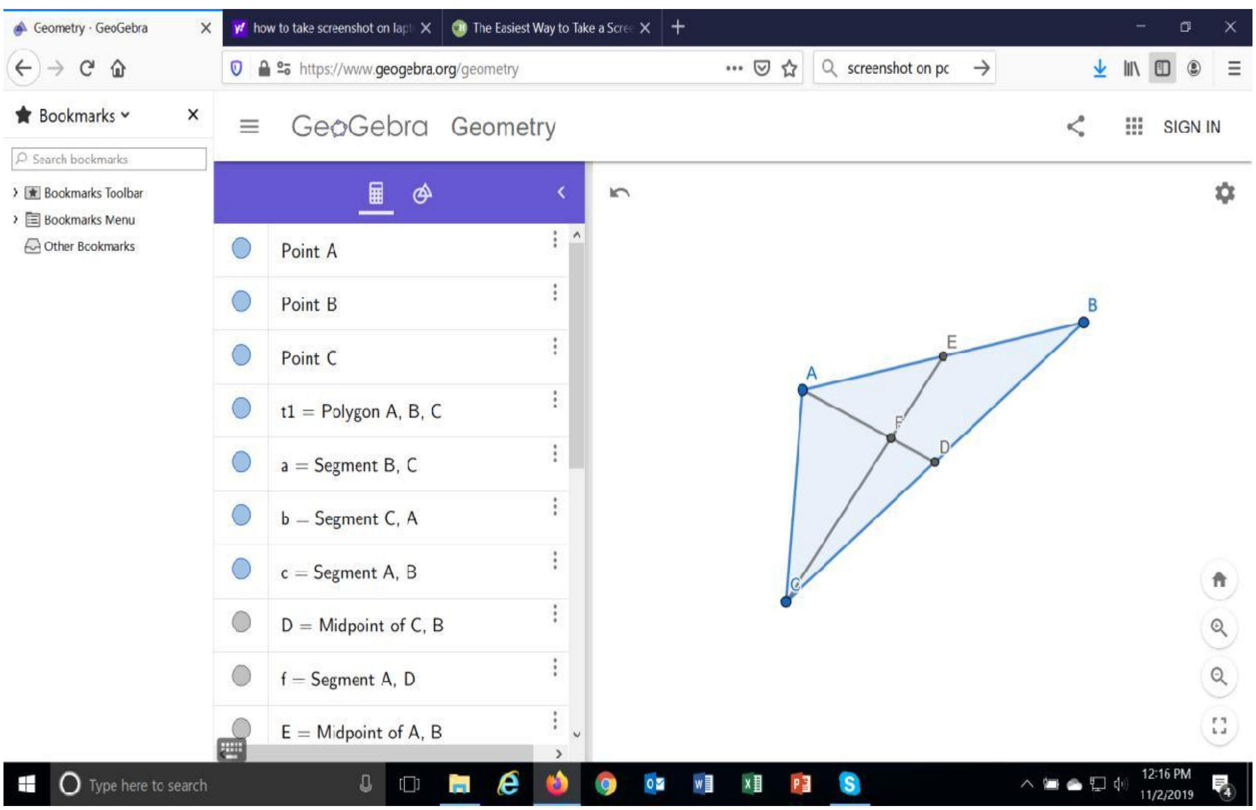
Investigation 3. Multiple examples constructed using GeoGebra

Having experience of teaching geometry to ninth graders who are not very confident with geometrical concepts, I have noticed that sometimes students try creating a proof for a statement that is not true. To avoid such situations, I used to encourage my students to take a three-step approach. Step one: formulate a hypothesis. Step two: create an example to verify that the statement might be true. Step three: create multiple examples to verify that the statement is likely to be true. Then you are ready to think about a proof. I shared this approach with the students in my graduate class and suggested that GeoGebra may be a convenient tool for creating multiple examples.

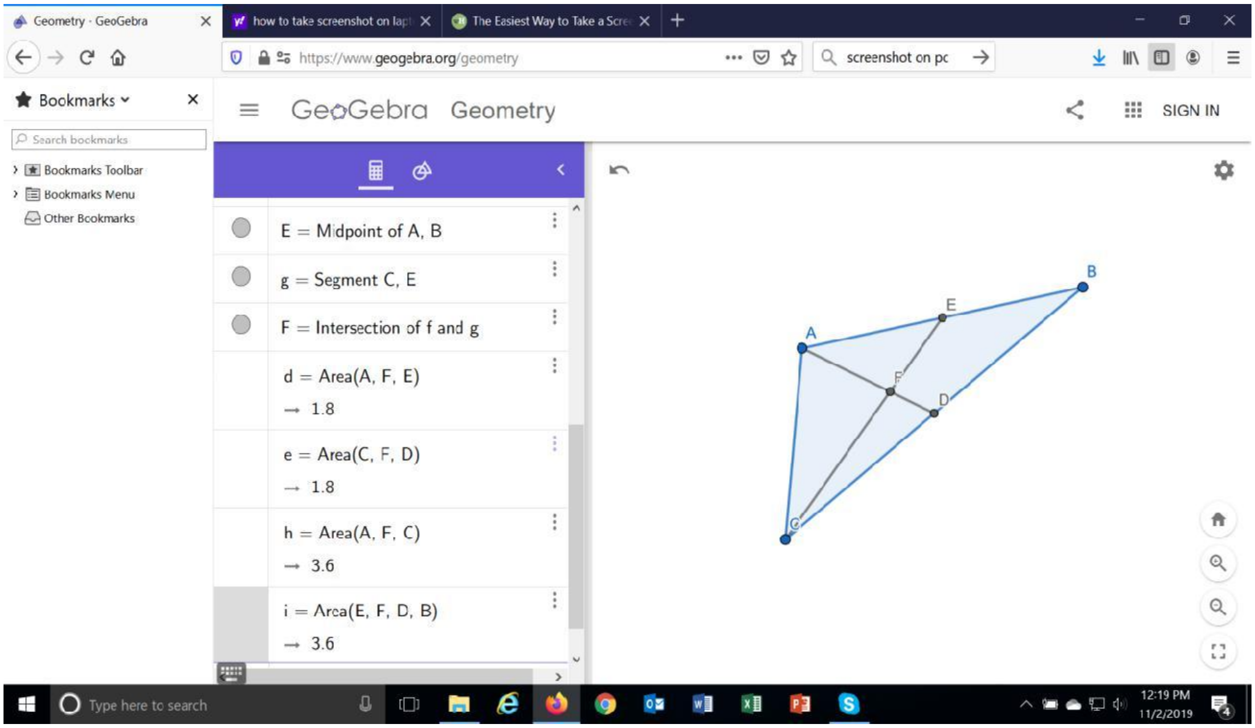
If I had to justify my choice of GeoGebra, I would start by saying that the basic version of it is available free to anyone with Internet access. At the same time, the benefits for students are widely discussed in mathematics education publications (e.g., [7], [8]).

I always emphasize that even very many examples do not yet make a proof, but surely give a rather convincing justification for looking for a proof. I shared this approach with my students in this graduate course. Many of them liked it and used it in their classes. For this investigation a couple of students created examples similar to those shown in Figure 4.

Screenshot 1 – a diagram and the instructions on how to build it



Screenshot 2 – the same diagram as above; area calculations are shown in the left column



Screenshot 3 – different diagram and different calculations are shown in the left column

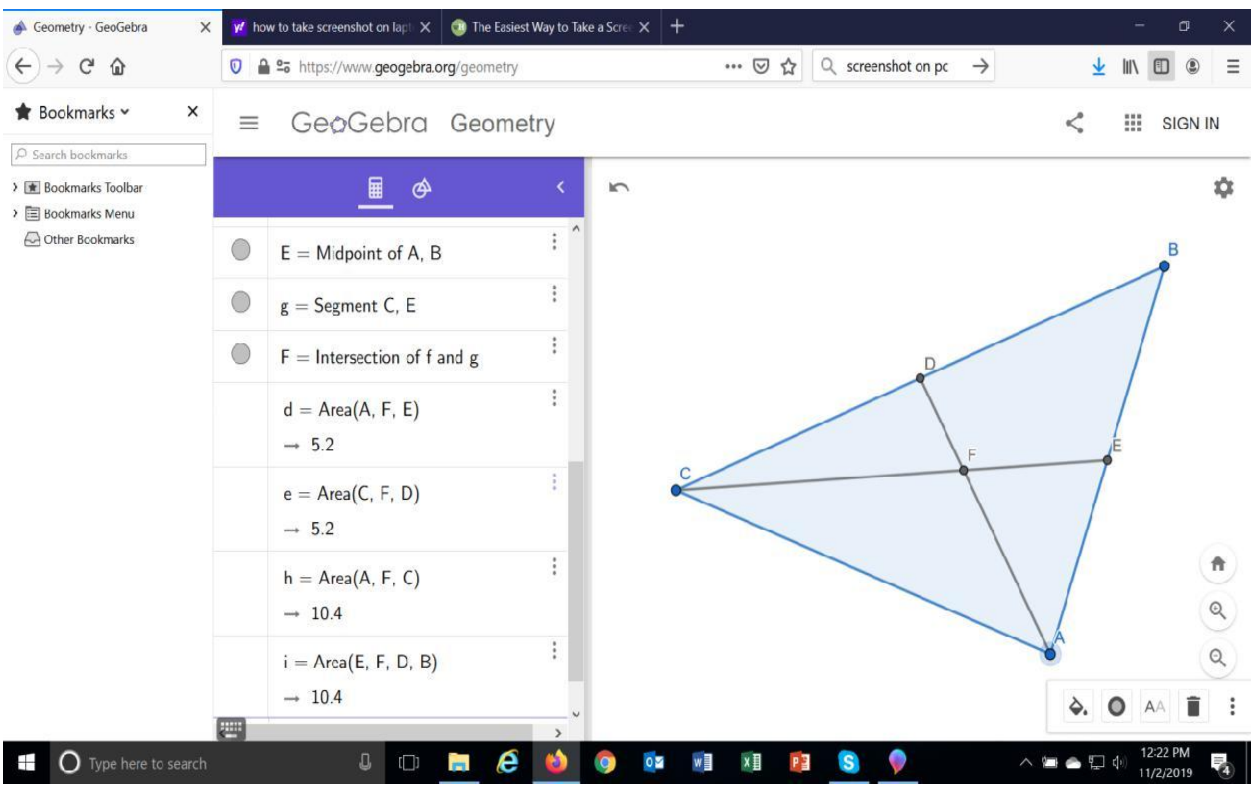


Figure 4. Three screenshots of examples created with GeoGebra

Moving vertices *A*, *B*, and *C*, we could see many examples, but every time Area (*A*,*F*,*E*) equals Area(*C*,*F*,*D*) and is exactly half of Area(*A*,*F*,*C*), which equals Area(*E*,*F*,*D*,*B*.) In other words: two smaller triangles have equal areas while the area of the larger triangle is twice as big and equals the area of the quadrilateral.

Investigation 4. General case: scalene triangle

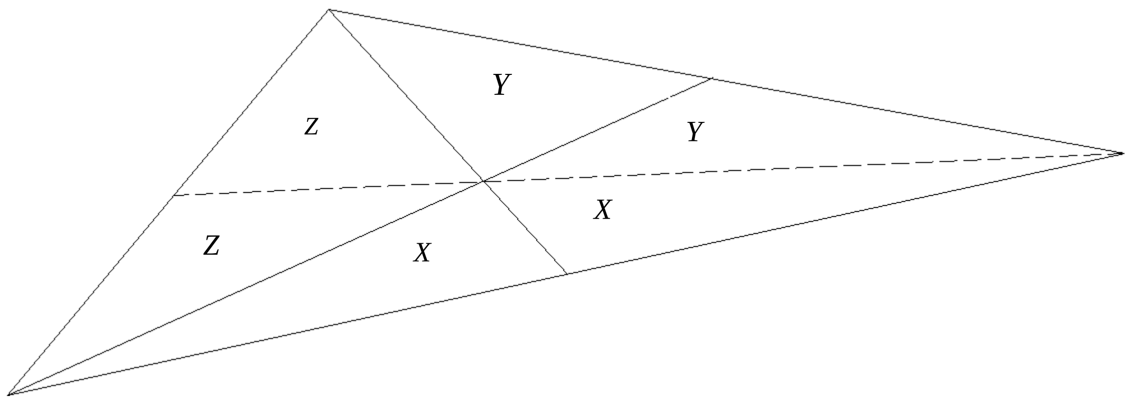


Figure 5. Equal areas are marked by the same letters

Auxiliary line – the third median. Triangles marked by the same letters have the same area, because their bases and heights are equal. By the same logic, $Z + Z + X = Y + Y + X$. This means that $Z = Y$. Analogously, $Z + Z + Y = X + X + Y$. This means that $Z = X$. Conclusion: Three medians divide a triangle into 6 triangles with equal areas. Therefore, the two medians divide a triangle into two smaller triangles, where each one has the area equal to one-sixth of the original triangle, and a larger triangle and a quadrilateral whose areas are equal to one-third of the area of the original triangle.

Looking at this elegant solution offered by one of their peers, somebody exclaimed "Why did I calculate my three special cases, when it is so easy to prove it once and for all?" Another person said: "Actually looking at those $\frac{1}{12}ab$ and $\frac{1}{6}ab$ for the right triangle, I seemed to remember some connections to one-sixth, but did not remember exactly." Indeed, the area of the right triangle is $\frac{1}{2}ab$. Therefore, $\frac{1}{12}ab$ is one-sixth of the area of the original triangle and, $\frac{1}{6}ab$ is one-third of the area of the original triangle. Someone else added: "now I will always remember the fact that medians divide any triangle into six triangles with equal areas."

I had to remind them that in our classes the journey matters as much as the destination. Giving this question for homework, I knew that the majority of the students teach 7th and 8th grade. This means they work actively with equilateral and right triangles, with similarity and calculations of areas. At the same time, special properties of the medians are not the focus in most of their classes. Rather than just remind them of the fact (which would be soon again forgotten). I formulated the question in such a way that everybody could do something. Combining our ideas, we achieved much more than just a proof of one particular fact: we discussed how to express the area of an equilateral triangle using its side, we considered decomposition of shapes in Figure 2, we were reminded about ideas concerning similarity of triangles in Figure 3, and we created multiple examples using GeoGebra. Calculations in special cases required careful work with algebraic expressions. All this was a solid preparation for deep understanding of the general case.

Every student participated in the discussion, sharing diagrams and calculations they completed at home. My role was to facilitate the discussion. Seeing diagrams and calculations presented on the white boards helped me choose the most effective order, moving in a logical manner from one investigation to the next. This methodology may help students feel like a team where every person is valued and people with different levels of expertise and experience work together and learn from each other.

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