IMVI ***ISSN (p) 2303-4882, ISSN 1840-4383***
Open Mathematical Education Notes
Vol. 10(2020), 15-32
www.imvibl.org / JOURNALS / IMVI OMEN
DOI: 10.7251/OMEN2001015A

A NOTE ON TEACHING MATHEMATICS TO ELEMENTARY TEACHER CANDIDATES: COOKIES, CREATURES, FIBONACCI NUMBERS AND SPREADSHEETS

Sergei Abramovich

State University of New York at Potsdam, United States

abramovs@potsdam.edu

Abstract: The paper shares several teaching ideas that may be used in a mathematics course for elementary teacher candidates. The meaning of tacit assumptions in word problems and ensuing methods of problem solving that various interpretations imply are discussed. A value of insight in recognizing a connection between two seemingly unrelated contexts as avenues converging to the same mathematical concept is acknowledged. The notion of conceptual shortcut as a trial and error solution informed by conceptual understanding of the corresponding concrete situation is applied to finding two unknowns forming a meaningful bond. It is demonstrated how the use of technology in computational experiments can be presented as an instrumental act within which technology, placed between a problem to solve and a problem solver, serves as a bi-directional mediator of the act. The paper includes excerpts from teacher candidates' solicited reflections on the pedagogy of the courses taught by the author.

Key words: mathematics, problem solving, teacher education, concrete materials, early algebra, spreadsheets, Fibonacci numbers.

1 Introduction

A typical problem for an early elementary classroom used around the world is to decompose a positive integer into a sum of two like integers (Becker and Selter, 1996; Serrazina and Rodrigues, 2015; Van de Walle, 2001; Young-Loveridge, 2002). This decomposition problem can be presented in context: *How many ways can ten cookies be put on two plates in all possible orders with at least one cookie on a plate?* One can draw a picture of plates and cookies and then describe the images using uncomplicated addition and its commutative property, thereby providing an answer to this question in terms of the following nine equalities (each describing one of the nine ways):

$$10 = 1+9 = 9+1 = 2+8 = 8+2 = 3+7 = 7+3 = 4+6 = 6+4 = 5+5 \ .$$

A tacit assumption that one takes for granted when decomposing the number 10 into a sum of two positive integers as a way of answering the above question is the absence of any difference between the cookies; in other words, the cookies are *assumed* to be identical. However, if all ten cookies are different, a more complicated counting technique has to be used. The first two sums, 1 + 9 and 9 + 1, each will be replaced by the number C_{10}^{1} which represents the number of ways a cookie can be selected from 10 cookies to be put on a plate with one cookie. For each such selection, the remaining 9 cookies will be put on the second plate. Therefore, the plate with one cookie can be created in $2C_{10}^{1}$ ways. The plate with two cookies (alternatively a plate with 8 cookies) can be created Type equation here.in $2C_{10}^{2}$ ways. The plate with three cookies (alternatively a plate with 7 cookies) can be created in $2C_{10}^{3}$ ways. The plate with four cookies (alternatively a plate with 6 cookies) can be created in $2C_{10}^{4}$ ways. The plate with five cookies can be created in C_{10}^{5} ways. Because the symbol C_{10}^{6} would represent the number of ways a plate with 6 cookies can be created, the computations do not need to continue as plates with 6, 7, 8, and 9 cookies were already created. Therefore, noting that due to the classic summation formula for binomial coefficients $\sum_{i=1}^{n} C_n^i = 2^n$ (Cuoco, 2005), the sum

$$\sum_{i=1}^{9} C_{10}^{i} = \sum_{i=1}^{10} C_{10}^{i} - C_{10}^{0} - C_{10}^{10} = 2^{10} - 2 = 1022$$

represents the number of ways 10 *different* cookies can be put on two plates with regard to the order of plates. In general, *n* different cookies can be put on two plates with regard to the order of plates in $2^n - 2$ ways.

This extension into cookies (or any kind of objects) being all different was used to illustrate an expectation by the Association of Mathematics Teacher Educators (2017) for elementary teachers to "hold deep conceptual understanding of the mathematics they teach as well as knowledge of how these foundational mathematical ideas connect to subsequent learning in the mathematical horizon" (p. 48). Furthermore, this extension was used to make elementary teacher candidates aware of how mathematics they are to teach depends on context and how the assumption of cookies being identical simplifies the solution of the problem. In what follows, although identical cookies will be used only, the paper intends to demonstrate how mathematical ideas appropriate for exploring with elementary teacher candidates can "reveal a surprising intricacy and complexity when they are examined in depth" (ibid, p. 48).

By extending the cookies context in a different direction, one can introduce the third plate filled with ten cookies and interpret the above nine decompositions of the number 10 as having three plates filled with cookies so that one of the plates has as many cookies as the other two plates combined. Adding a simplifying condition that plates are arranged in order (that is, ranked by ordinal numbers and, thereby, not subject to commutation), one can still intricate the situation by extending it to four plates where the third plate has as many cookies as the first and the second plates combined, and the fourth plate has as many cookies as the second and the third plates combined. Now the following question can be posed: *How many cookies can one put on the first and the second plates, so that the third plate has as many cookies as the first two plates and the second and the third plates have ten cookies combined?*

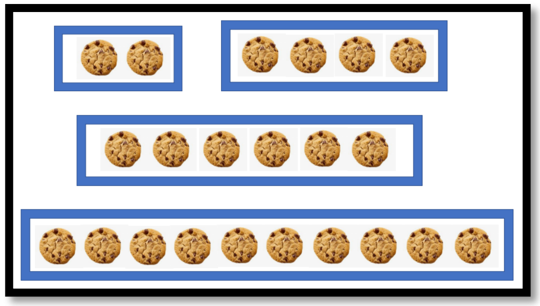

Figure 1. One way to get 10 cookies on the fourth plate: (2, 4, 6, 10).

Figure 1 shows a possible solution with two cookies on the first plate, four – on the second plate, six – on the third plate, and ten – on the fourth plate. In other words, the quadruple (2, 4, 6, 10) is such that 6 = 2 + 4 and 10 = 4 + 6. As was already mentioned above, adding the fourth plate makes the situation more intricate in one sense and less complex in another sense for ranking the plates leaves out the possibility of swapping the first two plates if the number of cookies on the fourth plate is specified. Indeed, swapping the first two numbers yields the quadruple (4, 2, 6, 8) which provides an erroneous response to the question posed. Below, we will refer to the process according to which cookies have been (and will be) put on plates as the ATLT (*add the last two*) rule. By reflecting on the solution shown in Figure 1, a teacher can ask students whether this solution is unique and if not, how all solutions to the problem about ten cookies on the fourth plate can be found. As an elementary teacher candidate noted, "*I like the idea of prompting the students with a difficult question that has many answers ... some students that I have met would thrive with this type of activity ... [expecting] the students to present their thinking and problem solving that went into the problem, why they got that answer, etc.*" First, in the environment encouraging the discussion of difficult questions, the students could note that the above quadruple of integers can be extended to the quintuple (2, 2, 4, 6, 10), thereby, prompting the extension to the fifth plate having ten cookies on it so that, according to the ATLT rule, 4 = 2 + 2, 6 = 2 + 4 and 10 = 4 + 6. Second, by changing the number of cookies and the number of plates, new problems structured by the ATLT rule can be formulated. Such changes should not be considered trivial – for example, one can check to see that whereas there is only one way to put nine cookies on the fifth plate following the ATLT rule, there is no solution for nine cookies and six plates. This is where technology may come into play under the umbrella of problem posing and solving, allowing for arithmetic, concrete materials and digital computation to meet.

However, the introduction of technology as an enhancement of arithmetic involved in hands-on explorations is not straightforward and it requires one's ability to decontextualize from the concreteness of cookies by taking advantage of the abstractness of mathematics. An important task for a teacher towards developing such an ability is to appreciate this relation between problem posing and problem solving, to learn how to recognize the emergence of a new problem through a teacher-student interaction, and to "be expert in ... the craft of task design ... and the mining of student ideas" (Conference Board of the Mathematical Sciences, 2012, p. 65). In the words of another elementary teacher candidate, "*Mathematics has moved away from pure memorization in arithmetic and progressed into manipulation, association and reasoning. These skills enhance a student's ability to think abstractly and apply one idea to many ideas in mathematics.*"

The teacher candidate's view of the modern-day mathematics classroom confirms the importance of students' learning to explore various seemingly unrelated contexts using experience with one idea, even if this idea is not yet fully developed. For example, the idea of counting cookies on plates can be associated with using the context of pet store mathematics (Abramovich, 2005) as follows: *A pet store sells exotic creatures, drimps and grimps. It is known that among a drimp and two grimps there are ten legs. How many legs does a drimp have and how many legs does a grimp have?* It follows from Figure 2, which shows four triples of one drimp and two grimps, that because the total number of legs is 10, a drimp may only have an even number of legs as, regardless of the number of legs a grimp has, two grimps have an even number of legs. If a drimp has two legs, a grimp has four legs; if a drimp has four legs, a grimp has three legs; if a drimp has six legs, a grimp has two legs; and if a drimp has eight legs, a grimp has one leg.

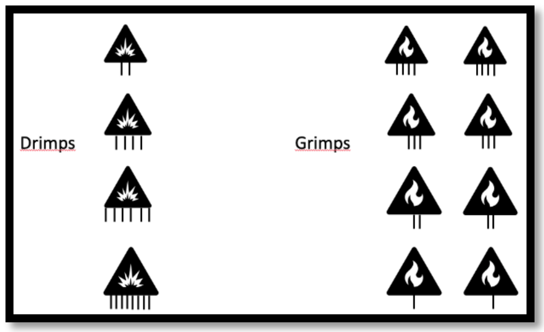

Figure 2. Solving a problem with drimps and grimps.

Counting legs can go as follows: drimp and grimp have six legs; adding another grimp to the mix yields ten legs. Alternatively, adding cookies on the first two plates, results in cookies on the third plate and adding cookies on the third and the second plates yields cookies on the fourth plate. Therefore, in addition to the quadruple (2, 4, 6, 10), three more quadruples, (4, 3, 7, 10), (6, 2, 8, 10) and (8, 1, 9, 10), define solutions to the original question about ten cookies on the fourth plate. Connecting cookies to creatures may be seen by psychologists as an insight (Dunker, 1945) or productive thinking (Wertheimer, 1959) when solution of one problem is recognized to be stemming from that of another, seemingly unrelated, problem. It is interesting to note that in the 4th century B.C., Aristotle was referring to these modern-day psychological concepts as *sagacity* – "a hitting by guess upon the essential connection in an inappreciable time" (cited in Pólya, 1945, p. 58). What is important for a teacher in explaining the meaning of insight (or sagacity) is to develop a conceptual bridge between two problem situations. A picture can play the role of such a bridge. Regarding the use of pictures, an elementary teacher candidate who was afraid of students' questions due to the traditional experience with mathematics as a never ending application of memorized rules, admitted that "*by being able to break the problem down into pictures that students can understand, I can have more confidence in my mathematical conceptual skills which will help me explain it better ... this makes math more enjoyable and fun for me, and I hope to pass this on to my future students.*"

One can note that Figure 2 is not only a visual representation of the statement *one drimp and two grimps have ten legs* but also is kind of a forebear of an algebraic equation $D+2G=10$, in which the right-hand side and the coefficient in the unknown *G*, both being even numbers, define (conceptually rather than procedurally) possible values of the unknown *D*. In other words, an equation as a mathematical model of a word problem emerges from one's thinking about its pictorial solution rather than being communicated by a 'more knowledgeable other' to be used automatically as the only means of obtaining a numeric solution. Put another way, a word problem of the above type can be solved by partitioning a number of objects in two groups of sets of different cardinality when the cardinality of a set in one group can be determined from the relationship between the number of objects and the number of sets in another group.

Such an approach to teaching mathematics, when a solution of a word problem serves as a kind of a forebear for an equation it satisfies, follows the framework of action learning – a pedagogy focusing on reflection and critical thinking as means of improving problem-solving performance (Dilworth, 1998; Abramovich, Grinshpan and Milligan, 2019). In the context of elementary mathematics teacher education, this approach reflects a position by Vygotsky (1987) regarding instruction which "*is only useful when it moves ahead of development … [as through this move it] impels or wakens a whole series of functions that are in a stage of maturation lying in the zone of proximal development*" (p. 212, italics in the original). Working within this zone under the teacher's guidance, a student can develop skills in solving an algebraic equation using what may be called a conceptual shortcut (Canobi, 2005; Kuo, Hull, Gupta and Elby, 2013). For example, the equation $D+2G=10$ can be solved through a conceptual shortcut: because 2 and 10 are multiples of 2, the possible values for D are 2, 4, 6 and 8, thereby, the corresponding values for G are 4, 3, 2 and 1 (see Figure 2). This approach to word problems is also in agreement with one of the tenets of Gestalt psychology asserting that for many children "it makes a big difference whether or not there is some real sense in putting the problem at all" (Wertheimer, 1959, p. 273). To support this assertion, Wertheimer, one of the founders of Gestalt psychology, gave an example of a 9-year old girl who was not successful in her studies at school. In particular, she was unable to solve simple problems requiring the use of basic arithmetic. However, when given a problem which grew out of a concrete situation with which she was familiar and the solution of which "was required by the situation, she encountered no unusual difficulty, frequently showing excellent sense" (*ibid*, pp. 273-274). That is, as a teacher candidate shared with the author, "*It is not about how much mathematical knowledge one knows to give their class, it is about how you teach it and how it impacts the students. We should think about math in terms of application to the real world problems that students learn as they develop their quantitative reasoning*." The next sections will continue presenting ideas along these lines motivating concepts through action learning.

2 Mapping cookies/legs to plates/creatures

George Pólya, known for his truly fundamental contributions that made mathematics education a university subject and problem solving an object of disciplined inquiry, advised mathematics educators that "observation and analogy may lead to discoveries" (Pólya, 1981, p. 158). Following this advice, note that due to the ATLT rule, the following relations in each quadruple (confirmed by the images of Figure 2) can be observed:

$$6 + 4 = 2 + 4 + 4,\ 7 + 3 = 4 + 3 + 3,\ 8 + 2 = 6 + 2 + 2,\ 9 + 1 = 8 + 1 + 1.$$

This combination of numeric and visual observations may prompt one to contextualize the above four relations using cookies as shown in the diagram of Figure 3. That is, the contextualization suggests that ten cookies can be arranged in three groups of cookies put either on the first or on the second plate in the problem explored in the previous section. But how can one explain that cookies on the second and the third plates can be put on three plates in four different ways as shown in Figure 3? Indeed, six cookies can be put in two groups: two cookies from the first plate and four cookies from the second plate in the original arrangement. Likewise, in the case of the quadruple (4, 3, 7, 10) the number 7 can be decomposed into 3 and 4, so that 10 = 4 + (3 + 3); for the quadruple (8, 1, 9, 10) the number 9 can be decomposed into 1 and 8, so that 10 = 8 + (1 + 1); for the quadruple (6, 2, 8, 10) the number 8 can be decomposed into 6 and 2, so that 10 = 6 + (2 + 2).

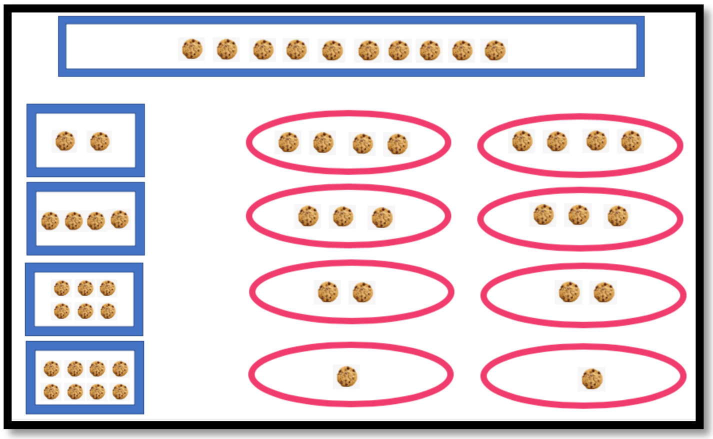

Figure 3. Ten cookies on three plates of two types.

One can describe numerically what is seen in Figures 2 and 3 as follows:

$$10 = 2 + (4 + 4), \quad 10 = 4 + (3 + 3), \quad 10 = 6 + (2 + 2), \quad 10 = 8 + (1 + 1).$$

In the context of Figure 3, all cookies on the fourth plate (i.e., ten cookies in the top rectangle) can be put in three groups the cardinalities of which are either the number of cookies on the first plate or the number of cookies on the second plate and such arrangement of cookies can be done in four different ways.

Acknowledging the value of open-ended questions as a means of learning mathematics, an elementary teacher candidate, looking forward to her own classroom, admitted being prepared to "*encourage students to ask similar kind of questions by coming up with word problems that can be expanded*." With this in mind, one can slightly change the numbers for cookies and plates to pose a new question: *How many ways can one put cookies on the first two plates so that, when using the ATLT rule, the fifth plate would have 20 cookies?* In answering this question, the quintuple (1, 6, 7, 13, 20) can be found through trial and error (e.g., by keeping a single cookie on the first plate and changing the number of cookies on the second plate to see if three consecutive applications of the ATLT rule would yield 20 cookies; if this does not work, start with two cookies on the first plate, and so on). Now, by analogy, having experience with numeric interpretation of Figure 3, one can note that the number of cookies on the fifth plate can be represented as the following combinations of such numbers on the first two plates:

$$(1+1)+(6+6+6)=20 \text{ or } 2\cdot 1 + 3\cdot 6 = 20.$$

In order to find all ways to put 20 cookies on the plates of two types (Figure 4) – two plates of one type (rectangle) and three plates of another type (oval) – one can use the following systematic approach. If a cookie is put on each of the two plates of the first type, then the remaining 18 cookies can be put evenly on each of the three plates of another type (because 18 is divisible by 3). If two or three cookies are put on each of the two plates of the first type, then the remaining 16 or 14 cookies, respectively, cannot be put evenly on three plates of another type (because neither 16 nor 14 is divisible by 3). If four cookies are put on each of the two plates of the first type, then the remaining 12 cookies can be put evenly on three plates of another type (because 12 is divisible by 3). Continuing in the same vein, one can find that when seven cookies are put on each of the two plates of the first type, the remaining six cookies can be put evenly on the three plates of another type (because 6 is divisible by 3). The arrangement of cookies proves to be completed by noting that neither eight nor nine cookies put on the two plates of the first type allow for a solution in the context of the remaining three plates (because neither 4 nor 2 is divisible by 3). Alternatively, if there are 20 legs among two drimps and three grimps, the equation $2D+3G=20$ has three integer solutions (1, 6), (4, 4) and (7, 2) from where the corresponding quintuples follow, respectively, (1, 6, 7, 13, 20), (4, 4, 8, 12, 20) and (7, 2, 9, 11, 20). These quintuples can help one comprehend the meaning of the coefficients in the equation $2D+3G=20$ and their origin, by presenting its left-hand side in the form

$$(D)+(G)+(D+G)+(G+D+G)+(D+G+G+D+G),$$

where the five addends, grouped in parentheses, show the rule according to which the left-hand side develops and the values of D and G in each such group correspond to the three pairs of integers satisfying the equation.

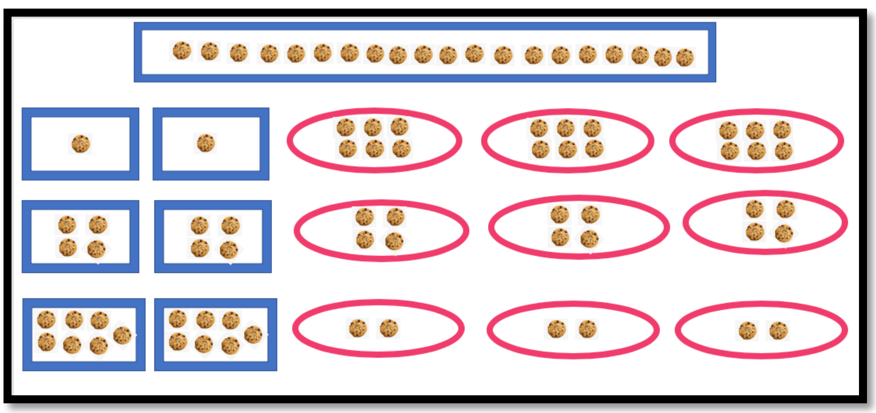

Figure 4. There are three ways to put twenty cookies on five plates of two types.

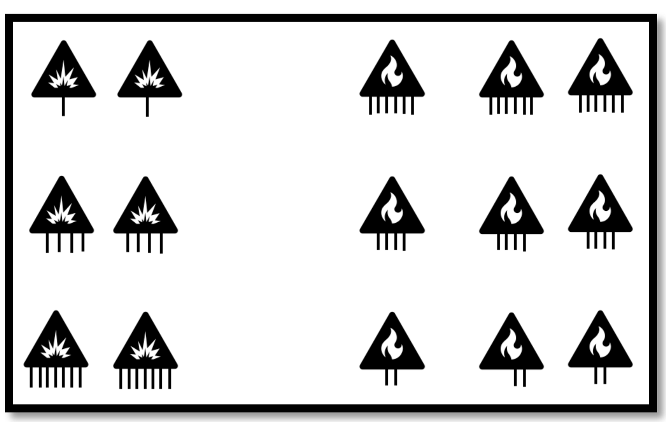

Figure 5. There are three ways to distribute 20 legs among two drimps and three grimps

3 Connecting the ATLT rule to Fibonacci numbers

One may discover that not all problems about cookies or creatures are easily solvable, let alone when their numbers are large. For example, whereas a problem with seven cookies and four plates or with seven legs among one drimp and two grimps can be solved by trial and error (resulting in three solutions for both contexts), the problem with 25 cookies and seven plates requires more than trial and error. Indeed, using the sequence

$$D, G, D + G, D + 2G, 2D + 3G, 3D + 5G, 5D + 8G, \ldots, \qquad (1)$$

the first seven terms of which show five consecutive applications of the ATLT rule when counting either cookies or legs, one can first prove that 5 drimps and 8 grimps may not have 25 legs. Indeed, 5 drimps may have either 5, or 10, or 15, or 20 legs, thereby, leaving to 8 grimps either 20, or 15, or 10, or 5 legs, the quantities not divisible by 8. Likewise, if D and G represent the number of cookies on the first two plates, respectively, no matter what the numbers are, repeating the first and second plates, respectively, 5 and 8 times may not yield 25 cookies. It is interesting to recognize the duality of the two contexts, cookies on plates and legs on creatures. Indeed, a case with a small number of cookies, e.g., seven, is difficult to resolve in terms of creatures and a case with a large number of cookies, e.g., 25, is easier to resolve in terms of creatures. Notwithstanding, when the number of plates and the number of cookies on the last plate (alternatively, the number of consecutive applications of the ATLT rule in counting legs among drimps and grimps) increase, the above reasoning may be difficult to use in either case. That is where educational computing can support students' conceptual understanding and elicit their epistemic advancement. But conceptualization requires generalization.

With this in mind note that the coefficients in D and G in sequence (1) develop through the ATLT rule as well. Indeed, in the sum $5D + 8G$ we have $5 = 3 + 2$ and $8 = 3 + 5$ where, likewise, $3 = 2 + 1$ and $2 = 1 + 1$. Mathematically, the sequence of coefficients can be defined as follows: its first two terms are equal to the number 1 and every term beginning from the third is the sum of the previous two terms. As is well known, these numbers are called Fibonacci numbers[1], named after Leonardo Fibonacci (1270-1350, Italy) – the most prominent mathematician of his time, credited with the introduction of Arabic numerals into the Western mathematics. Changing the first two Fibonacci numbers and keeping the rule according to which all other terms develop, result in a sequence called Fibonacci-like sequence. A famous example of a Fibonacci-like sequence is represented by Lucas numbers 2, 1, 3, 4, 7, 11, 18, 29, ... , named after a mathematician Edouard Lucas (1842-1891, France), the author of the famous Tower of Hanoi puzzle (Hofstadter, 1985). Coincidentally, it was Lucas who in 1876 gave the sequence 1, 1, 2, 3, 5, 8, 13, 21, ... its modern name (Koshy, 2001, p. 5).

4 Reflection as a path to problem posing

In the spirit of action learning, one can reflect on problem-solving activities and pose conceptual questions formulated in terms of the two contexts considered above: cookies on plates and legs on creatures.

[1] The sequence of numbers 1, 1, 2, 3, 5, 8, 13, … , nowadays commonly associated with the name of Fibonacci, was also referred to as the series of Lamé, e.g., "la célèbre série de Lamé ou de *Fibonacci*" (Catalan, 1884, p. 8, italics in the original). Gabriel Lamé (1795–1870) – a French mathematician and engineer. Eugène Charles Catalan (1814–1894) – a French/Belgian mathematician.

Question 1. *Considering the first and the second plates filled with cookies as the plates of the first and the second type, respectively, and given the rank of the last plate, how can one find the number of plates of each type to put cookies on the last plate through the ATLT rule?*

Remark 1. Figure 4 shows how the total of 20 cookies from the fifth plate can be put on two plates of the first type and three plates of the second type. At the same time, one can check to see that although 20 cookies may be put on one plate of the first type and two plates of the second type in different ways, none of the ways enables having 20 cookies on the fifth plate using the ATLT rule. Indeed, $20 = 1 \cdot 2 + 2 \cdot 9$, yet $2 + 9 = 11, 11 + 9 = 20, 20 + 11 = 31.$ This shows there is something special about the numbers two and three and, in general, about the numbers sought in Question 1.

Question 2. *Given the number of consecutive applications of the ATLT rule when counting the total number of legs among drimps and grimps, how can one find the number of creatures of each type?*

Remark 2. Figure 5 shows how the total of 20 legs in the case of three consecutive applications of the ATLT rule can be distributed among two creatures of the first type and three creatures of second type. At the same time, although 20 legs can be distributed among one drimp and two grimps in different ways (e.g., D = 4 and G = 8), three consecutive applications of the ATLT rule yield $4 + 8 = 12, 12 + 8 = 20, 20 + 12 = 32.$

Remark 3. One may note that when D and G are number of cookies on the first and the second plates, respectively, cookies on the fourth plate form the sum $D + 2G$, cookies on the fifth plate form the sum $2D + 3G$, cookies on the sixth plate form the sum $3D + 5G$, and so on. Likewise, two, three, four, and so on applications of the ATLT rule when counting legs, starting from D and G legs, yield, respectively, $D + 2G$, 2D + 3G, $3D + 5G$ and so on legs. The coefficients 3 and 5 are Fibonacci numbers of ranks four and five, respectively.

5 A spreadsheet as a part of an instrumental act

Nowadays, a computer is commonly used as a technical device which enhances problem solving and mathematical machinery serves as a psychological tool that mediates one's thinking about how the use of the tool makes the device applicable to a specific problem type. One of the most popular computer applications is an electronic spreadsheet – a high-tech artifact which through the appropriate use of mathematics may become an effective instrument of problem solving. In terms of the instrumental genesis theory (Rabardel, 1995) with its origin in the seminal ideas of Vygotsky (1930) about mediating problem solving by psychological instruments and technical devices, a spreadsheet may be considered being inserted between a problem solver and a problem to solve. This modern-day epistemically-oriented troika – user-spreadsheet-problem – makes the process of solving a problem an instrumental act. As the middle term of the troika, a spreadsheet acts bi-directionally. On the one hand, it acts primarily towards solving a problem. On the other hand, the use of the instrument requires significant intellectual efforts on the part of the problem solver. That is, the instrument acts towards the cognitive development of its user. An important role of a mathematics teacher educator, when demonstrating possible uses of mathematics in spreadsheet programming as a means of turning an artifact into an instrument, is to support and encourage epistemic advancement of teacher candidates.

In particular, in order to assist one in answering the above questions, a rather sophisticated yet user-friendly spreadsheet-based environment can be employed. Such a spreadsheet (the programming of which is included in Appendix) is shown in Figure 6. It can also be used for posing and solving problems about cookies on plates or legs on creatures. For instance, by setting the number of cookies at 20 (cell A2) and the number of plates at 5 (cell B2), the spreadsheet generated three asterisks: cell D9 is associated with the cells B9 and D2, the third row in Figures 4 and 5, leading to the quintuple (7, 2, 9, 11, 20) of cookies on the given plates; cell F6 is associated with the cells B6 and F2, the second row in Figures 4 and 5, leading to the quintuple (4, 4, 8, 12, 20); cell H3 is associated with the cells B3 and H2, the first row in Figures 4 and 5, leading to the quintuple (1, 6, 7, 13, 20).

	A	B	C	D	E	F	G	H	I
1	Cookies	Plates							
2	**20**	**5**	1	2	3	4	5	6	7
3	1	1						*	
4	1	2							
5	2	3							
6	3	4				*			
7	5	5							
8	8	6							
9	13	7		*					
10	21	8							

Figure 6. Twenty cookies on five plates of two types (cf. Figure 4).

Such a combination of concrete objects like cookies on plates and digital tools like spreadsheets, being powerful in its simplicity, represents an engaging learning environment which motivates further explorations into the problems about cookies/creatures. In particular, the spreadsheet of Figure 6 can be used by a teacher to pose a problem for students to solve using concrete materials of their choice: either cookies on plates or legs on creatures. Using the spreadsheet of Figure 6, one can check to see that setting any term of Fibonacci numbers as the number of cookies on the last plate and the rank of the term selected as the number of plates results in the unique solution. For example, when there are 8 cookies on the 6th plate (the number 8 is the 6th Fibonacci number), the first and the second plates will both have one cookie only. Likewise, when there are 11 cookies on the 6th plate (the number 11 is the 6th Lucas number), the first and the second plates will have, respectively, two and one cookies only. At the same time, there exist two ways to put cookies on the first and the second plates so that the 5th plate will have 11 cookies. One can confirm the last two statements using hands-on and conceptual shortcut approaches; that is, first to put 11 cookies on three plates of one type and five plates of another type (Figure 8) and solve the equation $3D+5G=11$ (by seeing that only $G = 1$ and $D = 2$ yield the solution), followed by putting 11 cookies on two plates of one type and three plates of another type (Figure 9) and solving the equation $2D+3G=11$ (because $G = 1$ yields $D = 4$ and $G = 3$ yields $D = 1$) .

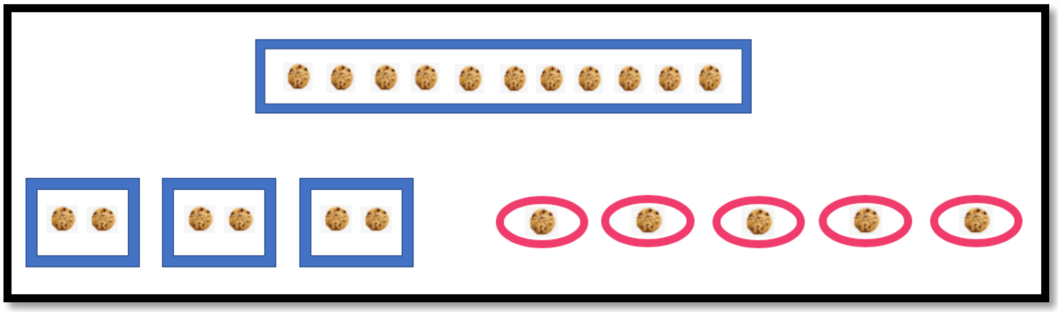

Figure 7. A problem with a single solution: putting 11 cookies on three and five plates.

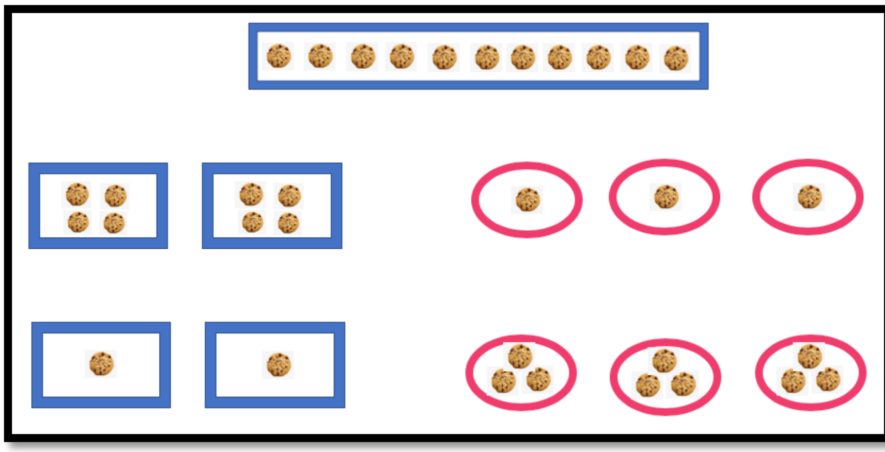

Figure 8. Two ways to put 11 cookies on two and three plates.

If the total number of plates is equal to n (alternatively, n is the rank of the last plate), then, with D and G being the number of cookies on the first and the second plates, respectively, the number of cookies on the last plate is the sum $aD + bG$, where a and b are Fibonacci numbers of the ranks $n - 2$ and $n - 1$. Put another way, these two numbers represent the quantities of the plates of two types. For example, if there are five plates, then one has to find the third and the fourth Fibonacci numbers, which are 2 and 3; therefore, regardless of D and G, there will be two plates of the first type and three plates of the second type. Likewise, in the case of seven plates, one has to consider the equation

$$5D + 8G = 25 \tag{2}$$

decontextualized from the above contexts and use the conceptual shortcut approach to demonstrate that equation (2) does not have integer solutions as G has to be a multiple of 5 and already $G = 5$ makes the left-hand side of (2) greater than its right-hand side. In the case of Question 2, the left-hand side of (2) results from five consecutive applications of the ATLT rule to counting legs among drimps and grimps. The absence of integer solutions to equation (2) means that such count may not result in 25 legs. In general, n consecutive applications of the ATLT rule when counting legs among the creatures with D and G legs would yield the expression $aD + bG$, where a and b are Fibonacci numbers of ranks n and $n + 1$, respectively; these numbers representing the number of creatures of the two types. Without transition from arithmetic to algebra, that is, without replacing numbers by letters, equation (2) and its solution through a conceptual shortcut would never come to light. This answers Questions 1and 2.

6 An instrumental act leads to technology motivated conjecturing

Consider the case when the number of cookies on the plate of rank *n* is the Fibonacci number of rank *n*. For example, let *n* = 34 – the Fibonacci number of rank 9. In order to find the number of cookies on the first two plates, one has to solve the equation 13*D* + 21*G* = 34 the only integer solution of which is *D* = *G* = 1. Here 13 and 21 are Fibonacci numbers of ranks 7 and 8, respectively. At the same time, in the equation 8*D* + 13*G* = 34 (where 8, 13 and 34 are Fibonacci numbers of ranks 6, 7, and 9, respectively) the unknown *G* has to be an even number not greater than 2 (as both 8*D* and 34 are multiples of 2 and already $13 \cdot 4 > 34$) yielding the only solution *D* = 1 and *G* = 2 (where 1 and 2 are Fibonacci numbers of ranks 2 and 3, respectively). In the equation 5*D* + 8*G* = 34 (where 5 and 8 are Fibonacci numbers of ranks 5 and 6, respectively) the unknown *D* has to be an even number not greater than 2 (as both 8*G* and 34 are multiples of 2 and $8 \cdot 4 = 32$) yielding the only solution *D* = 2 and *G* = 3 (where 2 and 3 are Fibonacci numbers of ranks 3 and 4, respectively). Put another way, if the F_n is the Fibonacci number of rank *n*, the above observations suggest that the three identities $F_7 F_1 + F_8 F_2 = F_9, F_6 F_2 + F_7 F_3 = F_9, F_5 F_4 + F_6 F_4 = F_9$ hold true, yielding the following inductively conjectured

Proposition 1.

$$F_{n-(k+1)} F_k + F_{n-k} F_{k+1} = F_n, k \geq 1, n \geq 2. \tag{3}$$

Remark 4. Identity (3) is equivalent to another identity

$$F_{k+n} = F_n F_{k+1} + F_{n-1} F_k, k \geq 0, n \geq 1 \tag{4}$$

among Fibonacci numbers developed in the context of finding the number of ways to tile a $1 \times (k+n-1)$ checkerboard (Grimaldi, 2012, p. 38). Indeed, replacing in (3) *n* by *k* + *n* yields $F_{k+n} = F_{k+n-k-1} F_k + F_{k+n-k} F_{k+1}$ whence $F_{k+n} = F_n F_{k+1} + F_{n-1} F_k$. That is, conceptually different applications of Fibonacci numbers enable almost identical formulations in which they are involved. The recognition of the existence of such formulations requires much more than insight, as putting cookies on plates and tiling one-dimensional checkerboards are quite mathematically and contextually distinct activities. Notwithstanding, one has to keep in mind that, although discoveries in mathematics education rarely lead to new results in purely mathematical sense, such discoveries are very important from a pedagogical perspective and in no way may be considered educationally insignificant, especially when they take place within student-centered activities.

Remark 5. In order to prove identity (3), one has to use the formula

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right]$$

which is a closed formula for Fibonacci numbers, with $F_{n+1} = F_n + F_{n-1}$ being their recursive representation from which it can be derived that $\lim_{m\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$ – the so-called Golden Ratio, the reciprocal of $\frac{\sqrt{5}-1}{2}$. A beauty of the closed formula, named after a French mathematician Jacques Binet (1786-1856), and the importance of making elementary teacher candidates familiar with it is that the formula shows how one class of numbers (in this case, integers) can be represented through another class of numbers (in this case, irrational numbers), the two classes being related in a certain sense. Just as recognizing relation of cookies to creatures required insight, the relation between Fibonacci numbers (introduced by Fibonacci to Western European mathematics in the 13th century) and the Golden Ratio (already known to Pythagoras in the 5th century B.C.) was revealed only in the 16th century. A formal proof Proposition 1 is beyond the level of elementary school mathematics. One may recall that at the beginning of the paper an elementary teacher candidate was cited when appreciating "*a student's ability to think abstractly and apply one idea to many ideas in mathematics*." Indeed, just as the idea of connecting a problem with cookies to the problem with creatures through an equivalent reformulation and explaining the solution of the latter in terms of the former, it (the idea) can be applied to proving identity (3) through the recourse to Binet's formula. Keeping in mind that comprehending an idea is more important than deciphering its technical realization, the proof of Proposition 1 is omitted from the main text of this paper and included in Appendix.

7 Conclusion

This paper was written to reflect on the author's experience as a mathematics instructor of elementary teacher candidates and to share pedagogical ideas behind the courses taught that support current standards for teaching and recommendations for teachers. Several pedagogical ideas of how mathematics for teachers can be structured were highlighted. The paper included excerpts from the teacher candidates' solicited commentaries on the pedagogy of the courses.

The first pedagogical idea dealt with the dual nature of implicit/explicit assumptions which an elementary teacher may encounter when teaching to solve word problems. The paper started with the discussion of how a simple question about putting cookies on two plates in the absence of the assumption of cookies being identical may require a mathematically complicated resolution. This discussion demonstrated an intricate relationship between context from which posing a word problem stems and mathematics which frames the ensuing problem solving. In the real classroom, when the assumption about cookies being identical is implicitly presumed, it may be challenged by a curious student and, likewise, when it is made explicit, another curious student might ask a teacher to explain the need for this, seemingly self-evident declaration. It both cases, to address students' curiosity, an elementary teacher needs to have what is commonly referred to as deep understanding or flexible knowledge of mathematics (e.g., Association of Mathematics Teacher Educators, 2017; Conference Board of the Mathematical Sciences, 2012; Baumert et al., 2010). The discussion around this issue was to give an example of why elementary teachers are expected to possess such knowledge.

Another pedagogical idea dealt with the use of two seemingly unrelated contexts, revealing a hidden relationship between which does require insight (sagacity), just as any new idea in mathematics entails. The need for making this relationship explicit is due to the fact that often, especially in mathematics, questions posed within one context (e.g., cookies on plates) can be answered through the exploration of another context (e.g., legs on creatures) followed by formulating an answer in terms of the relationship between the two contexts. At the same time it was shown that whereas some questions were easier to answer in the context of creatures, other questions were easier to answer in the context of cookies. At the end of the paper, this idea, formulated in pretty authentic terms, was used to explain rather esoteric purpose of representing Fibonacci numbers (integers) through the Golden Ratio (an irrational number), something that enabled proving an identity among Fibonacci numbers through a recourse to Binet's formula involving the Golden Ratio.

With a reference to the Gestalt psychology recommendations for teaching and learning elementary mathematics, the importance of using concrete situations was emphasized as a tool of comprehending the abstractness of the subject matter aimed at successful problem solving. It was shown how the pedagogy of using concrete situations makes sure that instruction "*moves ahead of development*" (Vygotsky, 1987, p. 212, italics in the original) by using pictorial representations of the situations involved as forebear of algebraic equations. In such cognitive milieu, solving those equations does not require one to use developmentally demanding algebra. Rather, one can learn to solve algebraic equations through conceptual shortcuts when numeric properties of coefficients of the equations decide the values of the unknowns. One can say that the pedagogy of using a conceptual shortcut is structured by an early algebra trial and error argument informed by conceptual understanding.

To make a room for the appropriate use of technology, the ATLT rule was eventually formalized by bringing classic Fibonacci numbers into consideration, something that allowed for the development of a spreadsheet capable of exploring conceptual questions posed about cookies and creatures. In turn, the use of the spreadsheet was explained in terms of the concept of the instrumental act (Vygotsky, 1930) when a middle term is inserted between a problem solver and a problem to be solved. Using a spreadsheet as such an element of an instrumental act enabled for finding an answer to conceptual questions in terms of relations among Fibonacci numbers that were generalized to the form of a non-trivial number-theoretical identity stemming from the contexts of cookies and creatures. This identity was shown to be equivalent to an identity developed in a very different context through mathematically very different means. This made it possible to encounter one of the most profound features of mathematics when different problems and methods of solving them lead to either equivalent or identical mathematical outcomes.

Similar to the development of an extended musical composition, the paper represented mathematical ideas in an ascending order, starting from a typical early grades partition problem put in context of putting ten cookies on two plates. By using concreteness of the context as a scaffold for a headway to mathematical formalism needed for the design of a spreadsheet capable of computational experiments leading to technology motivated conjecturing, the paper illustrated how mathematical knowledge for teaching in the modern-day classroom can be developed. We conclude the paper by citing a teacher candidate's understanding of the importance of such knowledge. In her words, "*it is important for K-6 teachers to be able to build the knowledge students need to succeed in the higher grade as they will be learning more formulas and adding to what they were taught in K-6. As a teacher in the primary grades we are really getting them interested in math, and building their foundation that they need for the rest of their life.*"

References

Abramovich, S. (2005). Early algebra with graphics software as a Type II application of technology. *Computers in the Schools*, **22**(3/4): 21-33.

Abramovich, S., Grinshpan, A. Z., and Milligan, D. L. (2019). Teaching mathematics through concept motivation and action learning. *Educational Research International,* volume 2019, Article ID 3745406, 13 pages.

Association of Mathematics Teacher Educators. (2017). *Standards for Preparing Teachers of Mathematics*. [On-line materials]. Available at: https://amte.net/standards.

Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., Klusmann, U., Krauss, S., Neubrand, M., and Tsai, Y.-M. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal*, **47**(1): 133–180.

Becker, J.P., and Selter C. (1996). Elementary school practices. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick and C. Laborde (Eds), *International Handbook of Mathematics Education* (pp. 511-564). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Canobi. K.H. (2005). Children's profiles of addition and subtraction understanding. *Journal of Experimental Child Psychology*, **92**(3): 220–246.

Catalan, E. (1884). Notes sur la théorie des fractions continues et sur certaines séries. *Mémoires de L'Académie Royale des Sciences, des Lettres et des Beaux-Arts de Belgique*, Tome XLV. Bruxelles: F. Hayez, Imprimeur de L'Académie Royale.

Conference Board of the Mathematical Sciences. (2012). *The Mathematical Education of Teachers II*. Washington, DC: The Mathematical Association of America.

Cuoco, A. (2005). *Mathematical Connections: A Companion for Teachers and Others*. Washington, DC: The Mathematical Association of America.

Dilworth, R. L. (1998). Setting the stage. *Performance Improvement Quarterly*, **11**(1): 5-8.

Dunker, K, (1945). On problem solving. *Psychological Monographs*, **58** (whole No. 270).

Grimaldi, R. P. (2012). *Fibonacci and Catalan Numbers: An Introduction*. Hoboken, NJ: John Wiley & Sons.

Hofstadter, D. R. (1985). *Metamagical Themas : Questing for the Essence of Mind and Pattern*. New York, NY: Basic Books.

Koshy, T. (2001). *Fibonacci and Lucas Numbers with Applications*. New York, NY: John Wiley & Sons.

Kuo, E., Hull, M. M., Gupta, A., and Elby, A. (2013). How students blend conceptual and formal mathematical reasoning in solving physics problems. *Science Education*, **97**(1): 32–57.

Pólya, G. (1945). *How to Solve It*. Princeton, NJ: Princeton University Press.

Pólya, G. (1981). *Mathematical Discovery: On Understanding, Learning and Teaching Problem Solving*. Combined edition. New York, NY: Wiley.

Rabardel, P. (1995). *Les Hommes et les Technologies, une Approche Cognitive des Instruments Contemporains* [*Men and Technologies, a Cognitive Approach to Contemporary Instruments*]. Paris: Armand Colin.

Serrazina, L., and Rodrigues, M. (2015). Additive adaptive thinking in 1st and 2nd grades pupils. In K. Krainer and N. Vondrová (Eds), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education* (pp. 368-374). Prague, Czech Republic. Available at: htttp://hal.archives-ouvertes.fr/hal-01286846.

Van de Walle, J. A. (2001). *Elementary and Middle School Mathematics: Teaching Developmentally* (4th edition). New York, NY: Addison Wesley Longman.

Vygotsky, L. S. (1930). The instrumental method in psychology (talk given in 1930 at the Krupskaya Academy of Communist Education). *Lev Vygotsky Archive*. [On-line materials]. Available at: https://www.marxists.org/archive/vygotsky/works/1930/instrumental.htm.

Vygotsky, L.S. (1987). Thinking and speech, In R.W. Rieber & A. S. Carton (Eds.), *The Collected Works of L. S. Vygotsky*, vol. 1 (pp. 39–285). New York, NY: Plenum Press.

Wertheimer, M. (1959). *Productive Thinking*. New York, NY: Harper & Brothers.

Young-Loveridge, J. (2002). Early childhood numeracy: Building an understanding of part-whole relationships. *Australian Journal of Early Childhood*, **27**(4): 36-42.

Appendix

Programming the spreadsheet of Figure 6

Cell A2: given the name *Cookies*. Cell B2: given the name *Plates*. Cell A3: = 1; cell A4: = 1; cell A5: = A3 + A4 – replicated down column A, thereby, generating consecutive Fibonacci numbers. Cell B3: = 1; cell B4: = B3 + 1 – replicated down column B, generating possible numbers of cookies put on the first plate. Cell C2: =1; cell D2: = C2 + 1 – replicated to the right along row 2, generating possible numbers of cookies put on the second plate.

Cell C3: =IF(AND($A$2+Plates-3>0,INDEX($A$2:$A$40,Plates-2,1)*$B2 +INDEX($A$2:$A$40,Plates-1,1)*C$1=Cookies), "*"," ") – replicated to cell I10.

In order to explain how the last (conditional) formula works, note that the spreadsheet of Figure 6 is designed to solve the equation $ax + by = c$ by selecting different pairs of x and y (displayed, respectively, in column B – beginning from cell B3, and row 2 – beginning from cell C2) where a and b, $a \leq b$, are consecutive Fibonacci numbers (displayed in column A starting from cell A3) and c is the number of cookies (displayed in cell A2) on the last plate (which rank is displayed in cell B2). The spreadsheet function INDEX is used to select a certain number from a range of numbers. In our case, one has to select the values of a and b (from consecutive Fibonacci numbers). Therefore, given the number of cookies on the last plate, the pair of Fibonacci numbers is unique. For example, when the number of plates is 8, the pair of ranks of the corresponding Fibonacci numbers is (8 – 1, 8 – 2) = (7, 6) and the relation

INDEX($A$2:$A$40,Plates-2,1)*$B2+INDEX($A$2:$A$40, Plates-1,1)*C$1=Cookies

turns into

INDEX($A$2:$A$40,7,1)*$B2+INDEX($A$2:$A$40, 6,1)*C$1=Cookies

or

8*$B2+13*C$1=Cookies,

as the 7th and the 6th numbers in the range A3:A10 are 13 and 8, respectively.

Proof of Proposition 1

$$F_{n-(k+1)}F_k = \frac{1}{\sqrt{5}}\left[(\frac{1+\sqrt{5}}{2})^{n-k-1} - (\frac{1-\sqrt{5}}{2})^{n-k-1}\right]\frac{1}{\sqrt{5}}\left[(\frac{1+\sqrt{5}}{2})^{k} - (\frac{1-\sqrt{5}}{2})^{k}\right]$$
$$=\frac{1}{5}\left[(\frac{1+\sqrt{5}}{2})^{n-1} + (\frac{1-\sqrt{5}}{2})^{n-1} - (\frac{1-\sqrt{5}}{2})^{n-k-1}(\frac{1+\sqrt{5}}{2})^{k} - (\frac{1+\sqrt{5}}{2})^{n-k-1}(\frac{1-\sqrt{5}}{2})^{k}\right]$$

and

$$F_{n-k}F_{k+1} = \frac{1}{\sqrt{5}}\left[(\frac{1+\sqrt{5}}{2})^{n-k} - (\frac{1-\sqrt{5}}{2})^{n-k}\right]\frac{1}{\sqrt{5}}\left[(\frac{1+\sqrt{5}}{2})^{k+1} - (\frac{1-\sqrt{5}}{2})^{k+1}\right]$$
$$=\frac{1}{5}\left[(\frac{1+\sqrt{5}}{2})^{n+1} + (\frac{1-\sqrt{5}}{2})^{n+1} - (\frac{1-\sqrt{5}}{2})^{n-k}(\frac{1+\sqrt{5}}{2})^{k+1} - (\frac{1+\sqrt{5}}{2})^{n-k}(\frac{1-\sqrt{5}}{2})^{k+1}\right].$$

Thefore,

$$F_{n-(k+1)}F_k + F_{n-k}F_{k+1}$$
$$=\frac{1}{5}\left[(\frac{1+\sqrt{5}}{2})^{n-1} + (\frac{1-\sqrt{5}}{2})^{n-1} - (\frac{1-\sqrt{5}}{2})^{n-k-1}(\frac{1+\sqrt{5}}{2})^{k} - (\frac{1+\sqrt{5}}{2})^{n-k-1}(\frac{1-\sqrt{5}}{2})^{k}\right]$$
$$+\frac{1}{5}\left[(\frac{1+\sqrt{5}}{2})^{n+1} + (\frac{1-\sqrt{5}}{2})^{n+1} - (\frac{1-\sqrt{5}}{2})^{n-k}(\frac{1+\sqrt{5}}{2})^{k+1} - (\frac{1+\sqrt{5}}{2})^{n-k}(\frac{1-\sqrt{5}}{2})^{k+1}\right]$$
$$=\frac{1}{5}\left[(\frac{1+\sqrt{5}}{2})^{n-1} + (\frac{1-\sqrt{5}}{2})^{n-1} + (\frac{1+\sqrt{5}}{2})^{n+1} + (\frac{1-\sqrt{5}}{2})^{n+1}\right]$$
$$+\frac{1}{5}\left[(\frac{1-\sqrt{5}}{2})^{n-k-1}(\frac{1+\sqrt{5}}{2})^{k} + (\frac{1+\sqrt{5}}{2})^{n-k-1}(\frac{1-\sqrt{5}}{2})^{k} + (\frac{1-\sqrt{5}}{2})^{n-k}(\frac{1+\sqrt{5}}{2})^{k+1}\right].$$

Simplifying the sums in the first and the second brackets of the final expression yields, respectively,

$$\frac{1}{5}\left[(\frac{1+\sqrt{5}}{2})^{n-1}+(\frac{1-\sqrt{5}}{2})^{n-1}+(\frac{1+\sqrt{5}}{2})^{n+1}+(\frac{1-\sqrt{5}}{2})^{n+1}\right]$$
$$=\frac{1}{5}\left[(\frac{1+\sqrt{5}}{2})^{n-1}\left[1+(\frac{1+\sqrt{5}}{2})^{2}\right]+(\frac{1-\sqrt{5}}{2})^{n-1}\left[1+(\frac{1-\sqrt{5}}{2})^{2}\right]\right]$$
$$=\frac{1}{5}\left[(\frac{1+\sqrt{5}}{2})^{n-1}\frac{5+\sqrt{5}}{2}+(\frac{1-\sqrt{5}}{2})^{n-1}\frac{5-\sqrt{5}}{2}\right]=\frac{1}{\sqrt{5}}\left[(\frac{1+\sqrt{5}}{2})^{n}-(\frac{1-\sqrt{5}}{2})^{n}\right]=F_n$$

and

$$\frac{1}{5}\left[(\frac{1-\sqrt{5}}{2})^{n-k-1}(\frac{1+\sqrt{5}}{2})^{k}+(\frac{1+\sqrt{5}}{2})^{n-k-1}(\frac{1-\sqrt{5}}{2})^{k}+(\frac{1-\sqrt{5}}{2})^{n-k}(\frac{1+\sqrt{5}}{2})^{k+1}+(\frac{1+\sqrt{5}}{2})^{n-k}(\frac{1-\sqrt{5}}{2})^{k+1}\right]$$
$$=\frac{1}{5}\left[(\frac{1-\sqrt{5}}{2})^{n-k-1}(\frac{1+\sqrt{5}}{2})^{k}(1+\frac{1-\sqrt{5}}{2}\cdot\frac{1+\sqrt{5}}{2})+(\frac{1+\sqrt{5}}{2})^{n-k-1}(\frac{1-\sqrt{5}}{2})^{k}(1+\frac{1-\sqrt{5}}{2}\cdot\frac{1+\sqrt{5}}{2})\right]$$
$$=\frac{1}{5}\left[(\frac{1-\sqrt{5}}{2})^{n-k-1}(\frac{1+\sqrt{5}}{2})^{k}(1-1)+(\frac{1+\sqrt{5}}{2})^{n-k-1}(\frac{1-\sqrt{5}}{2})^{k}(1-1)\right]=0\,.$$

This completes the proof of Proposition 1.