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S-e-PRIME ELEMENT PROPERTY IN LATTICES

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ABSTRACT. Let e be a fixed element of a bounded distributive lattice $\mathcal{L}$ and S a join-subset of $\mathcal{L}$. Following the concept of S-F-prime filters [16], we define S-e-prime elements of $\mathcal{L}$ as a new generalization of weakly S-prime elements [17]. Let p be a proper element of $\mathcal{L}$ with $S \wedge p = 0$ (i.e. $s \wedge p = 0$ for all $s \in S$). We say that p is an S-e-prime element of $\mathcal{L}$ if there is an element $s \in S$ such that for all $x, y \in \mathcal{L}$ if $p \leqslant x \vee y$ and $p \vee e \nleqslant x \vee y$, then $p \leqslant s \vee x$ or $p \leqslant s \vee y$. We will make an intensive investigate the basic properties and possible structures of these elements.

1. Introduction

All lattices considered in this paper are assumed to have a least element denoted by 0 and a greatest element denoted by 1, in other words they are bounded. Recently, the study of algebraic structures, using the properties of lattice theory, tends to a useful research topic. Associating a lattice with an algebraic structure allows us to obtain characterizations and representations of special classes of algebraic structures in terms of lattices and vice versa (see for example [7, 9, 11-18]).

Various generalizations of prime ideals of commutative rings have been studied. Recall from [1], a prime ideal P of R is a proper ideal having the property that $ab \in P$ implies either $a \in P$ or $b \in P$ for each $a, b \in R$. In 2003, Anderson and Smith in [3] defined weakly prime ideals which is a generalization of prime ideals (also see [10]). A proper ideal P of a ring R is said to be a weakly prime if $0 \neq xy \in P$ for each $x, y \in R$ implies either $x \in P$ or $y \in P$. Thus every prime ideal is weakly prime. In 2020, Hamed and Malek [19] introduced the notion of an

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S-prime ideal (also see [18, 20]), i.e. let $S \subseteq R$ be a multiplicative set and I an ideal of R disjoint from S. We say that I is S-prime if there exists an $s \in S$ such that for all $a, b \in R$ with $ab \in I$, we have $sa \in I$ or $sb \in I$. Almahdi et. al. [4] introduced the notion of a weakly S-prime ideal as follows (also see [17]): We say that I is a weakly S-prime ideal of R if there is an element $s \in S$ such that for all $x, y \in R$ if $0 \neq xy \in I$, then $sx \in I$ or $sy \in I$. Akray and Hussein generalized the concept of I-prime submodules in [6] (also see [5]). Let R be a commutative ring and I be a fixed ideal of R. Then a proper submodule P of an R-module M is called I-prime submodule of M if $rm \in P - IP$ for all $r \in R$ and $m \in M$ implies that either $m \in P$ or $r \in (P :_R M)$, and a proper ideal P of R is I-prime if for $a, b \in R$ with $ab \in P - IP$ implies either $a \in P$ or $b \in P$. So every weakly prime is I-prime. Let $\mathcal{L}$ be a bounded lattice. We say that a subset $S \subseteq \mathcal{L}$ is *join subset* if $0 \in S$ and $s_1 \vee s_2 \in S$ for all $s_1, s_2 \in S$. Let F be a fixed filter of a bounded distributive lattice $\mathcal{L}$ and S a join subset of $\mathcal{L}$. In [16], the present author, introduced the concepts of F-prime filters and S-F-prime filters as a new generalizations of weakly prime filters and S-prime filters, respectively, i.e. A proper filter P of $\mathcal{L}$ is F-prime if for $a, b \in \mathcal{L}$ with $a \vee b \in P - (F \vee P)$ implies either $a \in P$ or $b \in P$ and we say that a proper filter P of $\mathcal{L}$ with $P \cap S = \emptyset$ is an S-F-prime filter if there is an element $s \in S$ such that for all $a, b \in \mathcal{L}$ if $a \vee b \in P - (P \vee F)$, then $s \vee a \in P$ or $s \vee b \in P$. Also, in [17], the present author, introduced the concepts of S-prime elements (as a new generalization of prime elements) and weakly S-prime elements (as a new generalization of weakly prime elements). A proper element p of $\mathcal{L}$ is called *prime* (resp. *weakly prime*) if $p \leqslant x \vee y$ (resp. $p \leqslant x \vee y \neq 1$), then $p \leqslant x$ or $p \leqslant y$. Let S be a join subset of $\mathcal{L}$. An element p of $\mathcal{L}$ satisfying $S \wedge p = 0$ is said to be *S-prime* (resp. *weakly S-prime*) if there exists an element $s \in S$ such that, whenever $x, y \in \mathcal{L}$, $p \leqslant x \vee y$ (resp. $p \leqslant x \vee y \neq 1$) implies $p \leqslant s \vee x$ or $p \leqslant s \vee y$.

Let e be a fixed element of $\mathcal{L}$. Among many other results in this paper, the first, preliminaries section contains elementary observations needed later on. Section 3 concentrates on some basic properties of e-prime elements of $\mathcal{L}$ as a new generalization of weakly prime elements. A proper element p of $\mathcal{L}$ is e-prime if for $a, b \in \mathcal{L}$ with $p \leqslant a \vee b$ and $p \vee e \nleq a \vee b$ implies either $p \leqslant a$ or $p \leqslant b$. At first, some informations about the structure of such elements are given in Example 3.1, Example 3.2 and Example 3.3. We give an example (Example 3.2 (3)) of a e-prime element of $\mathcal{L}$ that is not a weakly prime element (so it is not prime). Proposition 3.1 shows that if $\mathcal{L}$ is a local complete lattice with unique coatom element c and $1 \neq p \in \mathcal{L}$, then c is an p-prime element. It is shown (Theorem 3.1) that if p is an e-prime element of $\mathcal{L}$ that is not prime, then $e \leqslant p$. In the Corollary 3.2, we give a condition under which an e-prime element of $\mathcal{L}$ is a prime element. In the Theorem 3.2, We give three other characterizations of e-prime elements. In the rest of this section, we investigate the properties of e-prime elements similar to prime elements. In particular, we investigate the behavior of e-prime elements under homomorphism, in factor lattices, $\vee$-lattice and in cartesian products of lattices (see Theorem 3.3, Theorem 3.4, Theorem 3.5 and Theorem 3.6).

Let S be a join subset of $\mathcal{L}$. Section 4 concentrates on some basic properties of S-e-prime elements of $\mathcal{L}$ as a new generalization of S-prime elements. At first, we give the definition of S-e-prime element (Definition 4.1) and provide an example (Example 4.1 (5)) of an S-e-prime element of $\mathcal{L}$ that is not an S-prime element. In the Theorem 4.1, We give a characterization of S-e-prime elements. We provide some conditions under which a join of a family of S-e-prime elements of $\mathcal{L}$ is an S-e-prime element (see Theorem 4.2). It is proved (Theorem 4.3) that if S is a join subset of $\mathcal{L}$ and p is an S-e-prime element of $\mathcal{L}$ that is not S-prime, then $e \leqslant p$. The remaining part of this section is mainly devoted to investigate S-e-prime elements under various contexts of constructions such as homomorphism, in factor lattices, S-$\vee$-lattices, cartesian products of lattices and S-c-Noetherian lattices (see Theorem 4.4, Theorem 4.5, Theorem 4.6, Theorem 4.7, Theorem 4.8, Theorem 4.9 and Theorem 4.10).

2. Preliminaries

A *poset* $(\mathcal{L}, \leqslant)$ is a *lattice* if $\sup\{a, b\} = a \vee b$ and $\inf\{a, b\} = a \wedge b$ exist for all $a, b \in \mathcal{L}$ (and call $\wedge$ the *meet* and $\vee$ the *join*).

Definition 2.1. (1) A lattice $\mathcal{L}$ is *complete* when each of its subsets X has a least upper bound and a greatest lower bound in $\mathcal{L}$. Setting $X = \mathcal{L}$, we see that any nonvoid complete lattice contains a least element 0 and greatest element 1 (in this case, we say that $\mathcal{L}$ is a lattice with 0 and 1).

(2) A lattice $\mathcal{L}$ is called a *distributive lattice* if $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ for all $x, y, z \in \mathcal{L}$ (equivalently, $\mathcal{L}$ is distributive if $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$ for all $x, y, z \in \mathcal{L}$).

(3) An element x of a lattice $\mathcal{L}$ is *nontrivial* (resp. *proper*) if $x \neq 0, 1$ (resp. $x \neq 1$).

(4) We say that an element x in a lattice $\mathcal{L}$ is an *atom* (resp. *coatom*) if there is no $y \in \mathcal{L}$ such that $0 < y < x$ (resp. $x < y < 1$). The set of all coatom (resp. atom) elements of $\mathcal{L}$ is denoted by $\mathcal{CA}(\mathcal{L})$ (resp. $\mathcal{A}(\mathcal{L})$).

(5) A lattice $\mathcal{L}$ is called *local* if it has exactly one coatom element c that $x \leqslant c$ for each proper element x.

(6) If $\mathcal{L}$ and $\mathcal{L}'$ are lattices, then a *lattice homomorphism* $v : \mathcal{L} \to \mathcal{L}'$ is a map from $\mathcal{L}$ to $\mathcal{L}'$ satisfying $v(x \vee y) = v(x) \vee v(y)$ and $v(x \wedge y) = v(x) \wedge v(y)$ for $x, y \in \mathcal{L}$. A lattice homomorphism is said to be a *order lattice homomorphism* if $x \leqslant y$ if and only if $v(x) \leqslant v(y)$ for $x, y \in \mathcal{L}$.

(7) A non-empty subset F of a lattice $\mathcal{L}$ is called a *filter*, if for $a \in F$, $b \in \mathcal{L}$, $a \leqslant b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if $\mathcal{L}$ is a lattice with 1, then $1 \in F$ and $\{1\}$ is a filter of $\mathcal{L}$).

(8) Let D be subset of a lattice $\mathcal{L}$. Then the filter generated by D, denoted by $T(D)$, is the intersection of all filters that is containing D. A filter F is called finitely generated (resp. cyclic) if there is a finite subset A of F (resp. if there is an element $a \in F$) such that $F = T(A)$ (resp. $F = T(\{a\}) = T(a)$).

For undefined notations or terminologies in lattice theory, we refer the reader to [7, 9]. First we need the following easy observations proved in [11, 12, 13, 14].

LEMMA 2.1. *Let $\mathcal{L}$ be a lattice.*

(1) A non-empty subset F of $\mathcal{L}$ is a filter of $\mathcal{L}$ if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F$, $z \in \mathcal{L}$. Moreover, since $x = x \vee (x \wedge y)$, $y = y \vee (x \wedge y)$ and F is a filter, $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in \mathcal{L}$.

(2) Let A be an arbitrary non-empty subset of $\mathcal{L}$. Then

$$T(A) = \{x \in \mathcal{L} : \ a_1 \wedge a_2 \wedge \cdots \wedge a_n \leqslant x \text{ for some } a_i \in A \ (1 \leqslant i \leqslant n)\}.$$

Moreover, if $a, b \in \mathcal{L}$, then $T(\{a\}) = T(a) = \{x \vee a : x \in \mathcal{L}\}$ and $a \leqslant b$ if and only if $T(b) \subseteq T(a)$.

(3) If $\mathcal{L}$ is distributive, then $(G :_{\mathcal{L}} F) = \{x \in \mathcal{L} : x \vee F \subseteq G\}$ and $(F :_{\mathcal{L}} T(\{z\})) = (F :_{\mathcal{L}} z) = \{a \in \mathcal{L} : a \vee z \in F\}$ are filters of $\mathcal{L}$.

3. Characterization of e-prime elements

In this section, we collect some basic properties concerning e-prime elements and remind the reader the following definition.

DEFINITION 3.1. Let e be a fixed element of a lattice $\mathcal{L}$. A proper element p of $\mathcal{L}$ is e-prime if for $a, b \in \mathcal{L}$ with $p \leqslant a \vee b$ and $p \vee e \nleqslant a \vee b$ implies either $p \leqslant a$ or $p \leqslant b$.

EXAMPLE 3.1. (1) If p is a proper element of $\mathcal{L}$, then p is always p-prime (by definition).

(2) Let p and q be elements of $\mathcal{L}$ with $q \leqslant p$. If p is proper, then p is always q-prime (by definition). In particular, p is always 0-prime.

EXAMPLE 3.2. (1) If $e = 1$, then the 1-prime and the weakly prime elements of $\mathcal{L}$ are the same.

(2) Suppose that p is a weakly prime (i.e. 1-prime) element of $\mathcal{L}$ and let e be an element of $\mathcal{L}$. Let $a, b \in \mathcal{L}$ such that $p \leqslant a \vee b$ and $p \vee e \nleqslant a \vee b$; so $a \vee b \neq 1$ since always $p \vee e \leqslant 1$. It follows that $p \leqslant a$ or $p \leqslant b$. Thus, every weakly prime element (prime element) is e-prime.

(3) Let $\mathcal{L} = \{0, a, b, c, 1\}$ be a lattice with the relations $0 \leqslant a \leqslant c \leqslant 1$, $0 \leqslant b \leqslant c \leqslant 1$, $a \vee b = c$ and $a \wedge b = 0$. Then c is an a-prime element of $\mathcal{L}$ by Example 3.1 (2). Also, c is not a weakly prime (prime) element of $\mathcal{L}$ because $c \leqslant a \vee b = c \neq 1$, $c \nleqslant a$ and $c \nleqslant b$. Thus an e-prime element need not be a weakly prime element (prime element).

EXAMPLE 3.3. Let e and f be elements of $\mathcal{L}$ such that $e \leqslant f$. Suppose that p is an f-prime element of $\mathcal{L}$ and let $x, y \in \mathcal{L}$ such that $p \leqslant x \vee y$ and $p \vee e \nleqslant x \vee y$. Then $e \leqslant f$ gives $p \vee f \nleqslant x \vee y$ which implies that $p \leqslant x$ or $p \leqslant y$. Therefor, p is an e-prim element of $\mathcal{L}$. However, the converse is not true in general. Indeed, assume that $\mathcal{L}$ is the lattice as in Example 3.2 (3) and let $a \leqslant 1$. Then $p = c$ is an a-prime element of $\mathcal{L}$ but not an 1-prime element of $\mathcal{L}$.

The proof of the following lemma can be found in [14, Lemma 2.1], but we give the details for convenience.

LEMMA 3.1. *If p is a non-trivial element of a complete lattice $\mathcal{L}$, then $p \leqslant c$ for some coatom element c of $\mathcal{L}$.*

PROOF. Set $\Omega = \{b : b \text{ is an element of } \mathcal{L} \text{ with } p \leqslant b < 1\}$. Then $\Omega \neq \emptyset$ since $p \in \Omega$. Moreover, $(\Omega, \leqslant)$ is a partial order. Clearly, Ω is closed under taking joins of chains and so Ω has at least one maximal element (coatom element) by Zorn's Lemma, say $p \leqslant c$, as needed. $\square$

PROPOSITION 3.1. *Let $\mathcal{L}$ be a local complete lattice with unique coatom element c. If p is a proper element of $\mathcal{L}$, then c is an p-prime element.*

PROOF. Let p be a proper element of $\mathcal{L}$. If $p = 0$, we are done by Example 3.1 (2). So assume that $p \neq 0$. Then $p \leqslant c$ by Lemma 3.1, i.e. the result follows from Example 3.1 (2). $\square$

An element $p \in \mathcal{L}$ is called *irreducible* if $p = x \vee y$, then either $x \in T(p)$ or $y \in T(p)$. Compare the next example with Theorem 2.11 (2) and Theorem 2.11 (3) in [5].

EXAMPLE 3.4. Let $\mathcal{L}$ be the lattice as in Example 3.2 (3). Then $\mathcal{L}$ is a local complete lattice with unique coatom element c. Consider the filter $T(c) = \{1, c\}$.

(1) Since $c \leqslant a \vee b = c \neq 1$ with $c \nleq a$ and $c \nleq b$, we conclude that c is not a weakly prime (1-prime) element. Moreover, c is u-prime for every element $u \leqslant c$ of $\mathcal{L}$ but it is not 1-prime.

(2) Since $c = a \vee b$ with $a, b \notin T(c)$, we infer that c is not an irreducible element, but it is u-prime by (1). Thus an u-prime element need not be irreducible.

Henceforth we will assume that e is a fixed element of $\mathcal{L}$.

THEOREM 3.1. *Let p be an e-prime element of $\mathcal{L}$. If p is not prime, then $e \leqslant p$.*

PROOF. Suppose that $e \nleq p$; we show that p is prime. Let $x, y \in \mathcal{L}$ such that $p \leqslant x \vee y$. If $p \vee e \nleq x \vee y$, then p is an e-prime gives $p \leqslant x$ or $p \leqslant y$. So we may assume that $p \vee e \leqslant x \vee y$. Since $e \nleq p = p \wedge (x \vee y)$, we conclude that $p \vee e \nleq p \wedge (x \vee y)$. Then $p \leqslant p \wedge (x \vee y) = (p \wedge x) \vee (p \wedge y)$ implies that $p \leqslant p \wedge x \leqslant x$ or $p \leqslant p \wedge y \leqslant y$, i.e. p is prime. $\square$

COROLLARY 3.1. *Let p be an 1-prime element of $\mathcal{L}$. If p is not a prime element, then $p = 1$.*

PROOF. By Theorem 3.1, $1 \leqslant p$, i.e. $p = 1$. $\square$

COROLLARY 3.2. *Let p be an e-prime element of $\mathcal{L}$. If $e \nleq p$, then p is a prime element of $\mathcal{L}$.*

PROOF. This is a direct consequence of Theorem 3.1. $\square$

One can easily show that if $p, a \in \mathcal{L}$, then $\mathcal{A}_a(p) = \{x \in \mathcal{L} : p \leqslant x \vee a\}$ is a filter of $\mathcal{L}$. We next give three other characterizations of e-prime elements. Compare the next theorem with Theorem 3.2 in [16].

Theorem 3.2. *If p is a proper element of $\mathcal{L}$, then the following statements are equivalent:*

(1) p is an e-prime element of $\mathcal{L}$;

(2) For $p \nleqslant a$, $\mathcal{A}_a(p) = T(p) \cup \mathcal{A}_a(p \vee e)$;

(3) For $p \nleqslant a$, $\mathcal{A}_a(p) = T(p)$ or $\mathcal{A}_a(p) = \mathcal{A}_a(p \vee e)$;

(4) For elements x and y of $\mathcal{L}$ with $T(x) \vee T(y) \subseteq T(p)$ and $T(x) \vee T(y) \nsubseteq T(p) \vee T(e)$, either $T(x) \subseteq T(p)$ or $T(y) \subseteq T(p)$.

Proof. (1) $\Rightarrow$ (2) Let $p \nleqslant a$. If $z \in T(p)$, then $p \leqslant z = p \vee s \leqslant z \vee a$ for some $s \in \mathcal{L}$ gives $z \in \mathcal{A}_a(p)$; so $T(p) \subseteq \mathcal{A}_a(p)$. If $z \in \mathcal{A}_a(p \vee e)$, then $p \leqslant p \vee e \leqslant z \vee a$ implies that $z \in \mathcal{A}_a(p)$. Therefore, $T(p) \cup \mathcal{A}_a(p \vee e) \subseteq \mathcal{A}_a(p)$. For the reverse inclusion, assume that $z \in \mathcal{A}_a(p)$ (so $p \leqslant z \vee a$). If $p \vee e \nleqslant z \vee a$, then $p \leqslant z$ by (1) which implies that $z = z \vee p \in T(p)$. If $p \vee e \leqslant z \vee a$, then $z \in \mathcal{A}_a(p \vee e)$ and so we have equality.

(2) $\Rightarrow$ (3) Since $\mathcal{A}_a(p) \subseteq T(p) \cup \mathcal{A}_a(p \vee e)$ by (2), we conclude that either $\mathcal{A}_a(p) \subseteq T(p)$ or $\mathcal{A}_a(p) \subseteq \mathcal{A}_a(p \vee e)$ by [11, Remark 2.3 (i)], and so (3) holds.

(3) $\Rightarrow$ (1) Let $a, b \in \mathcal{L}$ such that $p \leqslant a \vee b$ and $p \vee e \nleqslant a \vee b$. On the contrary, assume that $p \nleqslant a$ and $p \nleqslant b$. It suffices to show that $p \vee e \leqslant a \vee b$. Since $p \nleqslant a$ and $b \in \mathcal{A}_a(p)$, we infer that $b \in T(p)$ or $b \in \mathcal{A}_a(p \vee e)$ by (3). If $b \in T(p)$, then $p \leqslant b = p \vee c$ for some $c \in \mathcal{L}$, a contradiction. If $b \in \mathcal{A}_a(p \vee e)$, then $p \vee e \leqslant a \vee b$.

(3) $\Rightarrow$ (4) On the contrary, assume that $T(x) \nsubseteq T(p)$ and $T(y) \nsubseteq T(p)$. It suffices to show that $T(x) \vee T(y) \subseteq T(p) \vee T(e)$. Let $z \in T(x)$. If $z \notin T(p)$, then $z \vee T(y) \subseteq T(p)$ implies that $T(y) \subseteq \mathcal{A}_z(p)$. Now, $T(y) \nsubseteq T(p)$ gives $T(y) \subseteq \mathcal{A}_z(p \vee e)$ by (3); hence $x \vee T(y) \subseteq T(p) \vee T(e)$. So we may assume that $z \in T(P)$. By assumption, there exists $z' \in T(x)$ such that $z' \notin T(p)$; thus $z \wedge z' \notin T(p)$ and $z \wedge z' \in T(x)$ by Lemma 2.1 (1). By an argument like that as above, $(z \wedge z') \vee T(y) \subseteq T(p) \vee T(e)$. Let $u \in T(y)$. Then $(z \wedge z') \vee u = (z \vee u) \wedge (z' \vee u) \in T(p) \vee T(e)$ gives $z \vee u \in T(p) \vee T(e)$ by lemma 2.1 (1); so $z \vee T(y) \subseteq T(p) \vee T(e)$. Hence, $T(x) \vee T(y) \subseteq T(p) \vee T(e)$, as needed.

(4) $\Rightarrow$ (1) Let $a, b \in \mathcal{L}$ such that $p \leqslant a \vee b$ and $p \vee e \nleqslant a \vee b$. Then $T(a) \vee T(b) \subseteq T(p)$ and $T(a) \vee T(b) \nsubseteq T(p) \vee T(e)$ gives $a \in T(a) \subseteq T(p)$ or $b \in T(b) \subseteq T(p)$ by (4); hence $p \leqslant a$ or $p \leqslant b$, i.e. (1) holds. $\square$

We continue this section with the investigation of the stability of e-prime filters in various lattice-theoretic constructions.

Theorem 3.3. *If $v : \mathcal{L} \to \mathcal{L}'$ is an order lattice homomorphism such that $v(1) = 1$ and $v(0) = 0$, then the following hold:*

(1) If v is an epimorphism and p is an e-prime element of $\mathcal{L}$, then $v(p)$ is an $v(e)$-prime element of $\mathcal{L}'$.

(2) If $v(p)$ is an $v(e)$-prime element of $\mathcal{L}'$, then p is an e-prime element of $\mathcal{L}$.

Proof. (1) Let $x, y \in \mathcal{L}'$ such that $v(p) \leqslant x \vee y$ and $v(p \vee e) = v(p) \vee v(e) \nleqslant x \vee y$. Then there exist $a, b \in \mathcal{L}$ such that $x = v(a)$, $y = v(b)$ and $v(p) \leqslant v(a \vee b) = x \vee y$ and $v(p \vee e) \nleqslant v(a \vee b)$ (so $p \vee e \nleqslant a \vee b$). By the hypothesis, $p \leqslant a \vee b$ gives $p \leqslant a$ or $p \leqslant b$ which implies that $v(p) \leqslant v(a) = x$ or $v(p) \leqslant v(b) = y$), as needed.

(2) Let $a, b \in \mathcal{L}$ such that $p \leqslant a \vee b$ and $p \vee e \nleq a \vee b$ (so $v(p \vee e) \nleq v(a \vee b)$). Now, since $v(p) \leqslant v(a \vee b) = v(a) \vee v(b$ and $v(p)$ is an $v(e)$-prime element, we infer that $v(p) \leqslant v(a)$ or $v(p) \leqslant v(b)$. Hence, $p \leqslant a$ or $p \leqslant b$, and so p is an e-prime element. $\square$

Quotient lattices are determined by equivalence relations rather than by ideals as in the ring case. If F is a filter of a lattice $(\mathcal{L}, \leqslant)$, we define a relation on $\mathcal{L}$, given by $x \sim y$ if and only if there exist $a, b \in F$ satisfying $x \wedge a = y \wedge b$. Then $\sim$ is an equivalence relation on $\mathcal{L}$, and we denote the equivalence class of a by $a \wedge F$ and these collection of all equivalence classes by $\mathcal{L}/F$. We set up a partial order $\leqslant_Q$ on $\mathcal{L}/F$ as follows: for each $a \wedge F, b \wedge F \in \mathcal{L}/F$, we write $a \wedge F \leqslant_Q b \wedge F$ if and only if $a \leqslant b$. The following notation below will be used in this paper: It is straightforward to check that $(\mathcal{L}/F, \leqslant_Q)$ is a lattice with $(a \wedge F) \vee_Q (b \wedge F) = (a \vee b) \wedge F$ and $(a \wedge F) \wedge_Q (b \wedge F) = (a \wedge b) \wedge F$ for all elements $a \wedge F, b \wedge F \in \mathcal{L}/F$. Note that $f \wedge F = F$ if and only if $f \in F$ (see [12, Remark 4.2 and Lemma 4.3]).

THEOREM 3.4. *Let F be a filter of a lattice $\mathcal{L}$ and $p \in \mathcal{L}$. Then p is an e-prime element of $\mathcal{L}$ if and only if $p \wedge F$ is an $e \wedge F$-prime element of $\mathcal{L}/F$.*

PROOF. Assume that p is an e-prime element of $\mathcal{L}$ and let $v : \mathcal{L} \to \mathcal{L}/F$ be the order lattice epimorphism defined by $v(x) = x \wedge F$. Then by Theorem 3.3 (1), $v(p) = p \wedge F$ is an $v(e) = e \wedge F$-prime element of $\mathcal{L}/F$. Conversely, assume that $p \wedge F$ is an $e \wedge F$-prime element of $\mathcal{L}/F$ and let $x, y \in \mathcal{L}$ such that $p \leqslant x \vee y$ and $p \vee e \nleq x \vee y$ (so $(p \vee e) \wedge F \nleq_Q (x \vee y) \wedge F$). By the hypothesis, since $p \wedge F \leqslant_Q (x \wedge F) \vee_Q (y \wedge F)$, we conclude that $p \wedge F \leqslant_Q x \wedge F$ or $p \wedge F \leqslant_Q y \wedge F$; hence $p \leqslant x$ or $p \leqslant y$, as required. $\square$

If p is a proper element of $\mathcal{L}$, then by a $\vee$-*factorization* of p we mean an expression of p as a join $\vee_{i=1}^{n} p_i$ of e-prime elements. We call $\mathcal{L}$ a $\vee$-*lattice* if every proper element has a $\vee$-factorization.

THEOREM 3.5. *Let F be a proper filter of a lattice $\mathcal{L}$. If $\mathcal{L}$ is a $\vee$-lattice, then $\mathcal{L}/F$ is a $\vee$-lattice.*

PROOF. Suppose that $\mathcal{L}$ is a $\vee$-lattice and let x be a proper element of $\mathcal{L}/F$. Then $x = p \wedge F$ for some proper element p of $\mathcal{L}$. Let $p = \vee_{i=1}^{n} p_i$ be a $\vee$-factorization of p. Then $x = (\vee_{i=1}^{n} p_i) \wedge F = (p_1 \wedge F) \vee_Q \cdots \vee_Q (p_n \wedge F)$. By Theorem 3.4, we conclude that $p_i \wedge F$ is a $e \wedge F$-prime element of $\mathcal{L}/F$ for each $i \in \{1, 2, \cdots, n\}$. It means that $\mathcal{L}/F$ is a $\vee$-lattice. $\square$

Assume that $(\mathcal{L})_1, \leqslant_1), (\mathcal{L})_2, \leqslant_2)$ are lattices and let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$. We set up a partial order $\leqslant_c$ on $\mathcal{L}$ as follows: for each $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{L}$, we write $x \leqslant_c y$ if and only if $x_i \leqslant_i y_i$ for each $i \in \{1, 2\}$. The following notation below will be used in this paper: It is straightforward to check that $(\mathcal{L}, \leqslant_c)$ is a lattice with $x \vee_c y = (x_1 \vee y_1, x_2 \vee y_2)$ and $x \wedge_c y = (x_1 \wedge y_1, x_2 \wedge y_2)$. In this case, we say that $\mathcal{L}$ is a *decomposable lattice*.

Compare the next theorem with Theorem 3.7 in [16].

THEOREM 3.6. *Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ be a decomposable lattice and $e = (e_1, e_2)$, where e_i is an element of $\mathcal{L}_i$, $i = 1, 2$. Then the e-prime elements of $\mathcal{L}$ have exactly one of the following three types:*

(1) (p_1, p_2), where p_i is a proper element of $\mathcal{L}_i$ with $e_i \leqslant p_i$, $i = 1, 2$;

(2) $(p_1, 0)$, where p_1 is an e_1-prime element of $\mathcal{L}_1$ and $e_2 = 0$;

(3) $(0, p_2)$, where p_2 is an e_2-prime element of $\mathcal{L}_2$ and $e_1 = 0$.

PROOF. First we discuss these elements and show that they are e-prime elements, then we show that there are no more e-prime elements. Since $(p_1, p_2) \vee_c (e_1, e_2) = (p_1 \vee e_1, p_2 \vee e_2) = (p_1, p_2)$, we conclude that (p_1, p_2) is an e-prime (by definition). Suppose that p_1 is an e_1-prime element of $\mathcal{L}_1$ and $e_2 = 0$. If $(p_1, 0) \leqslant_c (a, b) \vee_c (c, d) = (a \vee c, b \vee d)$ and $(p_1, 0) \vee_c (e_1, 0) = (p_1 \vee e_1, 0) \nleqslant_c (a \vee c, b \vee d)$ (so $p_1 \vee e_1 \nleqslant a \vee c$), then $p_1 \leqslant a \vee c$ gives $p_1 \leqslant a$ or $p_1 \leqslant c$ which implies that $(p_1, 0) \leqslant_c (a, b)$ or $(p_1, 0) \leqslant_c (c, d)$; so $(p_1, 0)$ is e-prime. Similarly, $(0, p_2)$ is e-prime. Now, we show that there are no more e-prime elements. Suppose that (q_1, q_2) is an e-prime element of $\mathcal{L}$ and let $x, y \in \mathcal{L}_1$ such that $q_1 \leqslant x \vee y$ and $q_1 \vee e_1 \nleqslant x \vee y$. Then $(q_1, q_2) \leqslant_c (x, 1) \vee_c (y, 1) = (x \vee y, 1)$ and $(q_1 \vee e_1, q_2 \vee e_2) \nleqslant_c (x \vee y, 1)$ implies that $(q_1, q_2) \leqslant_c (x, 1)$ or $(q_1, q_2) \leqslant_c (y, 1)$ and so $q_1 \leqslant x$ or $q_1 \leqslant y$. Therefore, q_1 is e_1-prime. Similarly, q_2 is e_2-prime. If $(q_1, q_2) = (q_1 \vee e_1, q_2 \vee e_2)$, then $e_1 \leqslant q_1$ and $e_2 \leqslant q_2$ and so we are done. So we may assume that $(q_1, q_2) \neq (q_1 \vee e_1, q_2 \vee e_2)$, say $q_1 \neq q_1 \vee e_1$ (so $q_1 < q_1 \vee e_1$). Then $(q_1, q_2) \leqslant_c (q_1, 0) \vee_c (0, q_2)$ and $(q_1 \vee e_1, q_2 \vee e_2) \nleqslant_c (q_1, 0) \vee_c (0, q_2)$ gives $(q_1, q_2) \leqslant_c (q_1, 0)$ or $(q_1, q_2) \leqslant_c (0, q_2)$; hence $q_2 \leqslant 0$ or $q_1 \leqslant 0$ which implies that $q_1 = 0$ or $q_2 = q_2 = 0$. Let $q_1 = 0$. Then $(0, q_2)$ is e-prime, where q_2 is an e_2-prime element of $\mathcal{L}_2$. $\square$

COROLLARY 3.3. *Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ be a decomposable lattice. Then the weakly prime elements of $\mathcal{L}$ have exactly one of the following three types:*

(1) $(1, 1)$;

(2) $(p_1, 0)$, where p_1 is a weakly prime element of $\mathcal{L}_1$;

(3) $(0, p_2)$, where P_2 is a weakly prime element of $\mathcal{L}_2$.

PROOF. Take $e = (1, 1)$ in the Theorem 3.6. $\square$

4. Characterization of S-e-prime elements

We continue to use the notation already established, so e is a fixed element of $\mathcal{L}$. In this section, we collect some basic properties concerning S-e-prime elements and remind the reader the following definition.

DEFINITION 4.1. Let S be a join subset of $\mathcal{L}$. We say that a proper element p of $\mathcal{L}$ with $S \wedge p = 0$ is an S-e-prime element if there is an element $s \in S$ such that for all $a, b \in \mathcal{L}$ if $p \leqslant a \vee b$ and $p \vee e \nleqslant a \vee b$, then $p \leqslant s \vee a$ or $p \leqslant s \vee b$.

EXAMPLE 4.1. (1) If $S = \{0\}$, then the e-prime and the S-e-prime elements of $\mathcal{L}$ are the same.

(2) If p is a e-prime element of $\mathcal{L}$ with $S \wedge p = 0$, then p is an S-e-prime element. Moreover, since every prime element is e-prime, we infer that every prime element p of $\mathcal{L}$ with $S \wedge p = 0$ is S-e-prime.

(3) If p is a proper element of $\mathcal{L}$, then p is always S-p-prime (by definition).

(4) Let p and q be elements of $\mathcal{L}$ with $q \leqslant p$. If p is proper, then p is always S-q-prime (by definition). In particular, p is always S-0-prime.

(5) Clearly, if p is an S-prime element of $\mathcal{L}$, then p is an S-e-prime element. However, the converse is not true in general. Indeed, let $D = \{a, b, c\}$. Then $\mathcal{L} = \{X : X \subseteq D\}$ forms a distributive lattice under set inclusion with greatest element D and least element $\emptyset$ (note that if $x, y \in \mathcal{L}$, then $x \vee y = x \cup y$ and $x \wedge y = x \cap y$). Set $p = \{b, c\}$, $e = \{b\}$ and $S = \{\{a\}, \emptyset\}$. Then S is a join subset of $\mathcal{L}$ with $S \wedge p = 0$ and p is clearly an S-e-prime element of $\mathcal{L}$ by (4). Since $p \leqslant \{b\} \vee \{c\}$, $p \nleqslant \{a\} \vee \{b\}$ and $p \nleqslant \{a\} \vee \{c\}$, it follows that p is not a S-prime element of $\mathcal{L}$. Thus an S-e-prime element need not be an S-prime element.

Proposition 4.1. *Suppose that p is an element of $\mathcal{L}$ and let S be a join subset of $\mathcal{L}$ with $S \wedge p = 0$. The following assertions are equivalent:*

(1) p is an S-e-prime element of $\mathcal{L}$;

(2) There exists $s \in S$ such that for all a, b two elements of $\mathcal{L}$, if $T(a) \vee T(b) \subseteq T(p)$ and $T(a) \vee T(b) \nsubseteq T(p) \vee T(e)$, then $s \vee T(a) \subseteq T(p)$ or $s \vee T(b) \subseteq T(p)$.

Proof. (1) $\Rightarrow$ (2) By assumption, there exists an element $s \in S$ such that for all $a, b \in \mathcal{L}$, if $p \leqslant a \vee b$ and $p \vee e \nleqslant a \vee b$, then $p \leqslant s \vee a$ or $p \leqslant s \vee b$. On the contrary, assume that for all $t \in S$, there are u_t, w_t two elements of $\mathcal{L}$ with $T(u_t) \vee T(w_t) \subseteq T(p)$ and $T(u_t) \vee T(w_t) \nsubseteq T(p) \vee T(e)$, but $t \vee T(u_t) \nsubseteq T(p)$ and $t \vee T(w_t) \nsubseteq T(p)$. Since $s \in S$, we conclude that there exist u_s, w_s two elements of $\mathcal{L}$ with $T(u_s) \vee T(w_s) \subseteq T(p)$ and $T(u_s) \vee T(w_s) \nsubseteq T(p) \vee T(e)$, but $s \vee T(u_s) \nsubseteq T(p)$ and $s \vee T(w_s) \nsubseteq T(p)$. This shows that there exist $a_s, a'_s \in T(u_s)$ and $b_s, b'_s \in T(w_s)$ such that $s \vee a_s \notin T(p)$ (so $p \nleqslant s \vee a_s$), $s \vee b_s \notin T(p)$ (so $p \nleqslant s \vee b_s$) and $a'_s \vee b'_s \notin T(p) \vee T(e)$ (so $p \vee e \nleqslant a'_s \vee b'_s$). It follows that

$$s \vee (a_s \wedge a'_s) = (s \vee a_s) \wedge (s \vee a'_s) \notin T(p), s \vee (b_s \wedge b'_s) = (s \vee b_s) \wedge (s \vee b'_s) \notin T(p)$$

and $(a_s \wedge a'_s) \vee (b_s \wedge b'_s) = (a_s \vee b_s) \wedge (a'_s \vee b_s) \wedge (a_s \vee b'_s) \wedge (a'_s \vee b'_s) \notin T(p) \vee T(e)$ by Lemma 2.1 (1) which implies that $p \nleqslant s \vee (a_s \wedge a'_s)$, $p \nleqslant s \vee (b_s \wedge b'_s)$, $p \vee e \nleqslant (a_s \wedge a'_s) \vee (b_s \wedge b'_s)$ and $p \leqslant (a_s \wedge a'_s) \vee (b_s \wedge b'_s)$ which is a contradiction, as p is an S-e-prime element, i.e. the result holds.

(2) $\Rightarrow$ (1) Let $x, y \in \mathcal{L}$ such that $p \leqslant x \vee y$ and $p \vee e \nleqslant x \vee y$. Clearly, $T(x) \vee T(y) \subseteq T(p)$ and $T(x) \vee T(y) \nsubseteq T(p) \vee T(e)$. Then by (2), there exits $s \in S$ such that $s \vee x \in s \vee T(x) \subseteq T(p)$ or $s \vee y \in s \vee T(y) \subseteq T(p)$ which gives $p \leqslant s \vee x$ or $p \leqslant s \vee y$, i.e.(1) holds. $\square$

We next give a characterization of S-e-prime elements.

Theorem 4.1. *Suppose that p is an element of $\mathcal{L}$ and let S be a join subset of $\mathcal{L}$ with $S \wedge p = 0$. The following assertions are equivalent:*

(1) p is an S-e-prime element of $\mathcal{L}$;

(2) There exists $s \in S$ such that for all elements $a_1, \cdots, a_n$ of $\mathcal{L}$, if $\bigvee_{i=1}^{n} T(a_i) \subseteq T(p)$ and $\bigvee_{i=1}^{n} T(a_i) \nsubseteq T(p) \vee T(e)$, then $s \vee T(a_i) \subseteq T(p)$ for some $i \in \{1, \cdots, n\}$.

Proof. (1) $\Rightarrow$ (2) Let p be an S-e-prime element of $\mathcal{L}$. Then there is an element $s \in S$ such that for all $a, b \in \mathcal{L}$, if $p \leqslant x \vee y$ and $p \vee e \nleqslant a \vee b$, then

$p \leqslant s \vee a$ or $p \leqslant s \vee b$. We use induction on n. We can take $n = 2$ as a base case by Proposition 4.1. Let $n \geqslant 3$, assume that the property holds up to the order $n-1$ and let $a_1, \cdots, a_n$ be elements of $\mathcal{L}$ such that $\bigvee_{i=1}^{n} T(a_i) = (\bigvee_{i=1}^{n-1} T(a_i)) \vee T(a_n) \subseteq T(p)$ and $(\bigvee_{i=1}^{n-1} T(a_i)) \vee T(a_n) \nsubseteq T(p) \vee T(e)$. Then by Proposition 4.1, $s \vee T(a_n) \subseteq T(p)$ or $(s \vee T(a_1)) \vee (\bigvee_{i=2}^{n-1} T(a_i)) \subseteq T(p)$. Since

$$(s \vee T(a_1)) \vee (\bigvee_{i=2}^{n-1} T(a_i)) \nsubseteq T(p) \vee T(e),$$

we infer from the induction hypothesis that $s \vee T(a_n) \subseteq T(p)$ or $(s \vee s \vee T(a_1) = s \vee T(a_1) \subseteq T(p)$ or $s \vee T(a_i) \subseteq T(p)$ for some $i \in \{2, \cdots, n-1\})$. In the same way we prove that $s \vee T(a_i) \subseteq T(p)$ for some $i \in \{1, 2, \cdots, n\}$. The implication $(2) \Rightarrow (1)$ is clear. □

COROLLARY 4.1. *Let p be a proper element of $\mathcal{L}$. Then p is an e-prime element if and only if for all elements $a_1, \cdots, a_n$ of $\mathcal{L}$, if $\bigvee_{i=1}^{n} T(a_i) \subseteq T(p)$ and $\bigvee_{i=1}^{n} T(a_i) \nsubseteq T(p) \vee T(e)$, then $T(a_i) \subseteq T(p)$ for some $i \in \{1, \cdots, n\}$.*

PROOF. Take $S = \{0\}$ in Theorem 4.1. □

COROLLARY 4.2. *Assume that p is an element of $\mathcal{L}$ and let S be a join subset of $\mathcal{L}$ with $S \wedge p = 0$. Then p is an S-e-prime element if and only if there exists $s \in S$ such that for all elements $a_1, a_2, \cdots, a_n$ of $\mathcal{L}$, if $p \leqslant \bigvee_{i=1}^{n} a_i$ and $p \vee e \nleqslant \bigvee_{i=1}^{n} a_i$, then $p \leqslant s \vee a_i$ for some $i \in \{1, \cdots, n\}$.*

PROOF. Assume that p is an S-e-prime element of $\mathcal{L}$ and let $a_1, \cdots, a_n \in \mathcal{L}$ such that $p \leqslant a_1 \vee \cdots \vee a_n$ and $p \vee e \nleqslant a_1 \vee \cdots \vee a_n$. Therefore, $\bigvee_{i=1}^{n} T(a_i) \subseteq T(p)$ and $\bigvee_{i=1}^{n} T(a_i) \nsubseteq T(p) \vee T(e)$. Then by Theorem 4.1, there exists $s \in S$ such that $s \vee a_i \in s \vee T(\{a_i\}) \subseteq T(p)$ for some $i \in \{1, \cdots, n\}$ which implies that $p \leqslant s \vee a_i$ for some $i \in \{1, \cdots, n\}$. For the converse, take $n = 2$. □

COROLLARY 4.3. *Assume that p is an element of $\mathcal{L}$. Then p is an e-prime element if and only if for all elements $a_1, a_2, \cdots, a_n$ of $\mathcal{L}$, if $p \leqslant \bigvee_{i=1}^{n} a_i$ and $p \vee e \nleqslant \bigvee_{i=1}^{n} a_i$, then $p \leqslant a_i$ for some $i \in \{1, \cdots, n\}$.*

PROOF. Take $S = \{0\}$ in Corollary 4.2. □

Let S be a join subset of $\mathcal{L}$. We say that S is a *strongly join subset* if for each family $\{s_i\}_{i \in I}$ of elements of S we have $(\cap_{i \in I} T(s_i)) \cap S \neq \emptyset$ [15, 16].

THEOREM 4.2. *Suppose that S is a strongly join subset of $\mathcal{L}$ and let $\{p_i\}_{i \in I}$ be a chain of S-e-prime elements of $\mathcal{L}$. Then $p = \bigvee_{i \in I} p_i$ is an S-e-prime element.*

PROOF. Clearly, $S \wedge p = 0$. For each $i \in I$, there exists $s_i \in S$ such that for all $x, y \in \mathcal{L}$ with $p_i \leqslant x \vee y$ and $p_i \vee e \nleqslant x \vee y$ we have $p_i \leqslant s_i \vee x$ or $p_i \leqslant s_i \vee y$. Consider $s \in (\cap_{i \in I} T(s_i)) \cap S$. Then for each $i \in I$, $s = s_i \vee a_i$, where $a_i \in \mathcal{L}$. Let $a, b \in \mathcal{L}$ such that $p \leqslant a \vee b$, $p \vee e \nleqslant a \vee b$ (so $e \nleqslant a \vee b$) and suppose that $p \nleqslant s \vee a$. It suffices to show that $p \leqslant s \vee b$. Since $p \nleqslant s \vee a$, we conclude that $p_j \nleqslant s \vee a$ for some $j \in I$. Let $k \in I$. Then $p_k \leqslant p_j$ or $p_j \leqslant p_k$. We split the proof into two cases.

Case 1: $p_j \leqslant p_k$. Since $p_j \nleq s \vee a$, we infer that $p_k \nleq s \vee a = s_k \vee a_k \vee a$; so $p_k \nleq s_k \vee a$. Clearly, $p_j \leqslant a \vee b$ and $p_j \vee e \nleq a \vee b$. This shows that $p_k \leqslant s_k \vee b$; hence $p_k \leqslant s_k \vee a_k \vee b = s \vee b$. Therefore, $p \leqslant s \vee b$.

Case 2: $p_k \leqslant p_j$. Since $p_j \nleq s \vee a = s_j \vee a_j \vee a$, we infer that $p_j \nleq s_j \vee a$; so $p_j \leqslant s_j \vee b$ which gives $p_k \leqslant s \vee b = s_j \vee a_j \vee b$, and so $p \leqslant s \vee b$. □

THEOREM 4.3. *Assume that S is a join subset of $\mathcal{L}$ and let p be an S-e-prime element of $\mathcal{L}$. If p is not S-prime, then $e \leqslant p$.*

PROOF. Suppose that $e \nleq p$; we show that p is an S-prime element. Let $a, b \in \mathcal{L}$ such that $p \leqslant a \vee b$. If $p \vee e \nleq a \vee b$, Then p is an S-e-prime element implies that $p \leqslant t \vee a$ or $p \leqslant t \vee b$ for some $t \in S$. So suppose that $p \vee e \leqslant a \vee b$. By the hypothesis, $p \leqslant p \wedge (a \vee b)$ and $p \vee e \nleq p \wedge (a \vee b)$ (otherwise, $e \leqslant p \vee e \leqslant p$, a contradiction). Since $p \leqslant (p \wedge a) \vee (p \wedge b) = p \wedge (a \vee b)$ and $p \vee e \nleq p \wedge (a \vee b)$, we conclude that there is an element $s \in S$ such that $p \leqslant s \vee (p \wedge a) = (s \vee p) \wedge (s \vee a) \leqslant s \vee a$ or $p \leqslant s \vee (p \wedge b) = (s \vee p) \wedge (s \vee b) \leqslant s \vee b$, i.e. p is S-prime. □

COROLLARY 4.4. *Let p be an S-1-prime element of $\mathcal{L}$. If p is not an S-prime element, then $p = 1$.*

PROOF. This is a direct consequence of Theorem 4.3. □

COROLLARY 4.5. *Let p be an S-e-prime element of $\mathcal{L}$. If $e \nleq p$, then p is an S-prime element of $\mathcal{L}$.*

PROOF. This is a direct consequence of Theorem 4.3. □

We continue this section with the investigation of the stability of S-e-prime elements in various lattice-theoretic constructions.

THEOREM 4.4. *Suppose that $v : \mathcal{L} \to \mathcal{L}'$ is an order lattice homomorphism such that $v(1) = 1$ and $v(0) = 0$ and let S be a join subset of $\mathcal{L}$. Then the following hold:*

(1) If v is an epimorphism and p is an S-e-prime element of $\mathcal{L}$, then $v(p)$ is an $v(S)$-$v(e)$-prime element of $\mathcal{L}'$.

(2) If $v(p)$ is an $v(S)$-$v(e)$-prime element of $\mathcal{L}'$, then p is an S-e-prime element.

PROOF. (1) Clearly, $v(S) \wedge v(p) = 0$. Let $x, y \in \mathcal{L}'$ such that $v(p) \leqslant x \vee y$ and $v(p \vee e) = v(p) \vee v(e) \nleq x \vee y$. Then there exist $a, b \in \mathcal{L}$ such that $x = v(a)$, $y = v(b)$ and $v(p) \leqslant v(a \vee b) = x \vee y$ (so $p \leqslant a \vee b$) and $v(p \vee e) \nleq v(a \vee b)$ (so $p \vee e \nleq a \vee b$). By the hypothesis, there exists $s \in S$ such that $p \leqslant s \vee a$ or $p \leqslant s \vee b$ which implies that $v(p) \leqslant v(s) \vee v(a) = v(s) \vee x$ or $v(p) \leqslant v(s) \vee v(b) = v(s) \vee y$), i.e. the result holds.

(2) Let $a, b \in \mathcal{L}$ such that $p \leqslant a \vee b$ and $p \vee e \nleq a \vee b$. Now, since $v(p) \leqslant v(a \vee b) = v(a) \vee v(b)$, $v(p \vee e) \nleq v(a \vee b)$ and $v(p)$ is an $v(S)$-$v(e)$-prime element, we infer that there exists $s \in S$ such that $v(p) \leqslant v(s) \vee v(a) = v(s \vee a)$ or $v(p) \leqslant v(s) \vee v(b) = v(s \vee b)$. Hence, $p \leqslant s \vee a$ or $p \leqslant s \vee b$, and so p is an S-e-prime. □

An element x of $\mathcal{L}$ is called identity join of a lattice $\mathcal{L}$, if there exists $1 \neq y \in \mathcal{L}$ such that $x \vee y = 1$. The set of all identity joins of a lattice $\mathcal{L}$ is denoted by $\mathrm{I}(\mathcal{L})$.

Suppose that G is a filter of $\mathcal{L}$ and let S be a join subset of $\mathcal{L}$. An easy inspection will show that $S_Q(G) = \{s \wedge G : s \in S\}$ is a join subset of $\mathcal{L}/G$.

PROPOSITION 4.2. *Assume that p is an element of $\mathcal{L}$ and let S be a join subset of $\mathcal{L}$ with $S \wedge p = 0$ such that $S_Q(T(e)) \cap \mathrm{I}(\mathcal{L}/T(e)) = \emptyset$. The following assertions are equivalent:*

(1) p is an S-e-prime element of $\mathcal{L}$;

(2) $\mathcal{A}_s(p)$ is an $T(e)$-prime filter of $\mathcal{L}$ for some $s \in S$.

PROOF. (1) $\Rightarrow$ (2) Since p is an S-e-prime element, we conclude that there is an element $s \in S$ such that for all $x, y \in \mathcal{L}$ with $p \leqslant x \vee y$ and $p \vee e \nleq x \vee y$ we have $p \leqslant s \vee x$ or $p \leqslant s \vee y$. Now, we show that $\mathcal{A}_s(p)$ is an $T(e)$-prime filter of $\mathcal{L}$. Let $x, y \in \mathcal{L}$ such that $x \vee y \in \mathcal{A}_s(p) - \mathcal{A}_s(p) \vee T(e)$. Then $x \vee y \notin T(e)$ gives $(x \vee y) \wedge T(e) \neq 1 \wedge T(e)$. If $x \vee y \vee s \in T(e)$, then $((x \vee y) \wedge T(e)) \vee_Q (s \wedge T(e)) = 1 \wedge T(e)$ by [12, Remark 4.2] implies that $s \wedge T(e) \in S_Q(T(e)) \cap \mathrm{I}(\mathcal{L}/T(e))$ which is impossible. So we may assume that $x \vee y \vee s \notin T(e)$ (so $e \nleq x \vee y \vee s$; hence $p \vee e \nleq x \vee y \vee s$). Therefore, $p \leqslant x \vee y \vee s$ gives $p \leqslant x \vee s \vee s = x \vee s$ or $p \leqslant y \vee s$ which means that $x \in \mathcal{A}_s(p)$ or $y \in \mathcal{A}_s(p)$. Thus $\mathcal{A}_s(p)$ is an $T(e)$-prime filter of $\mathcal{L}$.

(2) $\Rightarrow$ (1) Suppose that $\mathcal{A}_s(p)$ is an $T(e)$-prime filter of $\mathcal{L}$ for some $s \in S$ and let $a, b \in \mathcal{L}$ such that $p \leqslant a \vee b$ and $p \vee e \nleq a \vee b$ (so $e \nleq a \vee b$ and $p \leqslant a \vee b \vee s$). Since $S_Q(T(e)) \cap \mathrm{I}(\mathcal{L}/T(e)) = \emptyset$ and $a \vee b \notin T(e)$, we conclude that $a \vee b \vee s \notin T(e)$; so $a \vee b \in \mathcal{A}_s(p) - \mathcal{A}_s(p) \vee T(e)$. Now, $\mathcal{A}_s(p)$ is an $T(e)$-prime gives $p \leqslant s \vee a$ or $p \leqslant s \vee b$, as required. □

LEMMA 4.1. *Suppose that p is an element of $\mathcal{L}$ and let S be a join subset of $\mathcal{L}$. Then the following hold:*

(1) p is an S-e-prime element if and only if $T(p)$ is an S-$T(e)$-prime filter.

(2) p is an e-prime element if and only if $T(p)$ is an $T(e)$-prime filter;

PROOF. (1) Suppose that p is an S-e-prime element of $\mathcal{L}$ and let $a, b \in \mathcal{L}$ such that $a \vee b \in T(p)$ (so $p \leqslant a \vee b$) and $a \vee b \notin T(p) \vee T(e)$ (so $a \vee b \notin T(e)$; i.e. $e \nleq a \vee b$). Since p is S-e-prime, we conclude that there exists $s \in S$ such that $p \leqslant s \vee a$ or $p \leqslant s \vee b$; hence $s \vee a \in T(p)$ or $s \vee b \in T(p)$. Conversely, assume that $x, y \in \mathcal{L}$ such that $p \leqslant x \vee y$ and $p \vee e \nleq x \vee y$. It follows that $x \vee y \in T(p)$ and $x \vee y \notin T(p) \vee T(e)$ which implies that $t \vee x \in T(p)$ or $t \vee y \in T(p)$ for some $t \in S$, i.e. $p \leqslant t \vee x$ or $p \leqslant t \vee y$.

(2) Take $S = \{0\}$ in (1). □

In the following theorem, we give a condition under which the e-prime and the S-e-prime elements coincide.

THEOREM 4.5. *Assume that p is an element of $\mathcal{L}$ and let S be a join subset of $\mathcal{L}$ with $S \wedge p = 0$ such that $S_Q(T(e)) \cap \mathrm{I}(\mathcal{L}/T(e)) = \emptyset = S_Q(T(p)) \cap \mathrm{I}(\mathcal{L}/T(p))$. Then p is e-prime if and only if p is S-e-prime.*

PROOF. Clearly, if p is a e-prime element of $\mathcal{L}$ with $S \wedge p = 0$, then p is an S-e-prime element. Conversely, assume that p is an S-e-prime element. It suffices to show that $T(p) = \mathcal{A}_s(p)$ for all $s \in S$ by Proposition 4.2 and Lemma

4.1. Let $s \in S$. If $x \in T(p)$, then $p \leqslant x \leqslant x \vee s$ gives $T(p) \subseteq \mathcal{A}_s(p)$. For the reverse inclusion, let $y \in \mathcal{A}_s(p)$. Then $p \leqslant s \vee y$; so $s \vee y \in T(p)$ which implies that $(s \wedge T(p)) \vee_Q (y \wedge T(p)) = (s \vee y) \wedge T(p) = 1 \wedge T(p)$. Since $S_Q(T(p)) \cap \mathrm{I}(\mathcal{L}/T(p)) = \emptyset$, we conclude that $y \wedge T(p) = 1 \wedge T(p)$; so $x \in T(p)$ by [12, Remark 4.2]. $\square$

Theorem 4.6. *Assume that p is an element of $\mathcal{L}$ and let S be a join subset of $\mathcal{L}$ with $S \wedge p = 0$. If F is a filter $\mathcal{L}$, then p is an S-e-prime element of $\mathcal{L}$ if and only if $p \wedge F$ is an $S_Q(F)$-$(e \wedge F)$-prime element of $\mathcal{L}/F$.*

Proof. Suppose that p is an S-e-prime element of $\mathcal{L}$ and let $v : \mathcal{L} \to \mathcal{L}/F$ be the order lattice epimorphism defined by $v(x) = x \wedge F$. Then by Theorem 4.4 (1), $v(p) = p \wedge F$ is an $v(S)$-$v(e) = S_Q(F)$-$(e \wedge F)$-prime element of $\mathcal{L}/F$. Conversely, assume that $p \wedge F$ is an $S_Q(F)$-$(e \wedge F)$-prime element of $\mathcal{L}/F$ and let $x, y \in \mathcal{L}$ such that $p \leqslant x \vee y$ and $p \vee e \nleq x \vee y$ (so $(p \vee e) \wedge F \nleq_Q (x \vee y) \wedge F$). By the hypothesis, since $p \wedge F \leqslant_Q (x \wedge F) \vee_Q (y \wedge F)$, we conclude that there exists $s \wedge F \in S_Q(F)$ such that $p \wedge F \leqslant_Q (s \vee x) \wedge F$ or $p \wedge F \leqslant_Q (s \vee y) \wedge F$; hence $p \leqslant s \vee x$ or $p \leqslant s \vee y$. $\square$

If p is a proper element of $\mathcal{L}$, then by a *S-$\vee$-factorization* of p we mean an expression of p as a join $\vee_{i=1}^{n} p_i$ of S-e-prime elements. We call $\mathcal{L}$ a *S-$\vee$-lattice* if every proper element has a S-$\vee$-factorization.

Theorem 4.7. *Let F be a proper filter of a lattice $\mathcal{L}$. If $\mathcal{L}$ is a S-$\vee$-lattice, then $\mathcal{L}/F$ is a $S_Q(F)$-$\vee$-lattice.*

Proof. Suppose that $\mathcal{L}$ is a S-$\vee$-lattice and let x be a proper element of $\mathcal{L}/F$. Then $x = p \wedge F$ for some proper element p of $\mathcal{L}$. Let $p = \vee_{i=1}^{n} p_i$ be a S-$\vee$-factorization of p. Then $x = (\vee_{i=1}^{n} p_i) \wedge F = (p_1 \wedge F) \vee_Q \cdots \vee_Q (p_n \wedge F)$. By Theorem 4.6, we infer that $p_i \wedge F$ is a $S_Q(F)$-$e \wedge F$-prime element of $\mathcal{L}/F$ for each $i \in \{1, 2, \cdots, n\}$. It means that $\mathcal{L}/F$ is a $S_Q(F)$-$\vee$-lattice. $\square$

Theorem 4.8. *Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ be a decomposable lattice, $p = (p_1, p_2)$, $e = (e_1, e_2)$ and $S = S_1 \times S_2$, where p_i, e_i are elements of $\mathcal{L}_i$ and S_i is a join subset of $\mathcal{L}_i$, $i = 1, 2$. Then the following hold:*

(1) If p_i is a proper element of $\mathcal{L}_i$ with $e_i \leqslant p_i$, $i = 1, 2$, then p is an S-e-element of $\mathcal{L}$;

(2) If p_1 is an S_1-e_1-prime element of $\mathcal{L}_1$ and $e_2 = 0$, then $(p_1, 0)$ is an S-e-element of $\mathcal{L}$;

(3) If p_2 is an S_2-e_2-prime element of $\mathcal{L}_2$ and $e_1 = 0$, then $(0, p_2)$ is an S-e-element of $\mathcal{L}$.

Proof. (1) Since $p \vee_c e = (p_1 \vee e_1, p_2 \vee e_2) = (p_1, p_2) = p$, we infer that p is an S-e-prime (by definition).

(2) Suppose that p_1 is an S_1-e_1-prime element of $\mathcal{L}_1$ and let $e_2 = 0$. Let $(a, b), (c, d) \in \mathcal{L}$ such that $(p_1, 0) \leqslant_c (a, b) \vee_c (c, d) = (a \vee c, b \vee d)$ (so $p_1 \leqslant a \vee c$) and $(p_1, 0) \vee_c (e_1, 0) = (p_1 \vee e_1, 0) \nleq (a \vee c, b \vee d)$ (so $p_1 \vee e_1 \nleq a \vee c$). Then $p_1 \leqslant a \vee c$ and $p_1 \vee e_1 \nleq a \vee c$ gives there exists $s_1 \in S_1$ such that $p_1 \leqslant s_1 \vee a$ or $p_1 \leqslant s_1 \vee c$ which implies that $(p_1, 0) \leqslant_c (s_1, 0) \vee_c (a, b)$ or $(p_1, 0) \leqslant_c (s_1, 0) \vee_c (c, d)$, where $(s_1, 0) \in S$; so $(p_1, 0)$ is an S-e-prime.

(3) The proof is similar to that in case (2) and we omit it. $\square$

Theorem 4.9. *Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ be a decomposable lattice, $p = (p_1, p_2)$, $e = (e_1, e_2)$ and $S = S_1 \times S_2$, where p_i, e_i are elements of $\mathcal{L}_i$ and S_i is a join subset of $\mathcal{L}_i$, $i = 1, 2$. If p is an S-e-prime element of $\mathcal{L}$, then p_1 is an S_1-e_1-prime element of $\mathcal{L}_1$ and $S_2 \wedge p_2 \neq 0$ or p_2 is an S_2-e_2-prime element of $\mathcal{L}_2$ and $S_1 \wedge p_1 \neq 0$ or p_1 is an S_1-e_1-prime element of $\mathcal{L}_1$ and p_2 is an S_2-e_2-prime element of $\mathcal{L}_2$.*

Proof. Let p be an S-e-prime element of $\mathcal{L}$ and assume that $s = (s_1, s_2) \in S$ satisfies the S-e-prime condition. Since $P \wedge S = 0$, we have either $S_1 \wedge p_1 = 0$ or $S_2 \wedge p_2 = 0$. If $S_1 \wedge p_1 \neq 0$, we will show that p_2 is an S_2-e_2-prime element of $\mathcal{L}_2$. Let $p_2 \leqslant x \vee y$ and $p_2 \vee e_2 \nleq x \vee y$ for some $x, y \in \mathcal{L}_2$ (so $e_2 \nleq x \vee y$). Then $(p_1, p_2) \leqslant_c (1, x) \vee_c (1, y) = (1, x \vee y)$ and $p \vee_c e = (p_1 \vee e_1, p_2 \vee e_2) \nleq_c (1, x \vee y)$ gives $p \leqslant_c s \vee_c (1, x) = (1, s_2 \vee x)$ or $p \leqslant_c s \vee_c (1, y) = (1, s_2 \vee y$. This shows that $p_2 \leqslant s_2 \vee x$ or $p_2 \leqslant s_2 \vee y$. Hence, p_2 is an S_2-e_2-prime element of $\mathcal{L}_2$. Similarly, if $S_2 \wedge p_2 \neq 0$, then p_1 is an S_1-e_1-prime element of $\mathcal{L}_1$. Now assume that $S_1 \wedge p_1 = 0 = S_2 \wedge p_2$. We will show that p_1 is an S_1-e_1-prime element of $\mathcal{L}_1$ and p_2 is an S_2-e_2-prime element of $\mathcal{L}_2$. Suppose that p_1 is not an S_1-e_1-prime element of $\mathcal{L}_1$. Then there exist $a, b \in \mathcal{L}_1$ such that $p_1 \leqslant a \vee b$ and $p_1 \vee e_1 \nleq a \vee b$ (so $e_1 \nleq a \vee b$) but $p_1 \nleq s_1 \vee a$ and $p_1 \nleq s_1 \vee b$. Then $p \leqslant_c (a, 0) \vee_c (b, 1) = (a \vee b, 1)$ and $p \vee e \nleq_c (a \vee b, 1)$ gives $p \leqslant_c s \vee_c (a, 0) = (s_1 \vee a, s_2$ or $p \leqslant_c s \vee_c (b, 1) = (s_1 \vee b, 1)$; so $p_1 \leqslant s_1 \vee a$ or $p_1 \leqslant s_1 \vee b$ which is a contradiction. Therefor, p_1 is an S_1-e_1-prime element of $\mathcal{L}_1$. Similarly, p_2 is an S_2-e_2-prime element of $\mathcal{L}_2$. □

A lattice $\mathcal{L}$ is said to be a *cyclic filter lattice* (*c-lattice for short*) precisely when every filter G of $\mathcal{L}$ generated by some $a \in G$ (i.e. $G = T(a)$ for some $a \in G$). Let $\mathbb{F}(\mathcal{L})$ be the set of all filters of $\mathcal{L}$.

Example 4.2. Let $\mathcal{L}$ be the lattice as in Example 3.2 (3). An inspection will show that the nontrivial filters (i.e. different from $\pounds$ and $\{1\}$) of $\mathcal{L}$ are $T(c) = \{1, c\}$, $T(a) = \{1, c, a\}$ and $T(b) = \{1, c, b\}$. Thus $\mathcal{L}$ is a c-lattice.

Definition 4.2. Suppose that $\mathcal{L}$ is a c-lattice and let S be a join subset of $\mathcal{L}$.

(1) We say that a filter G of $\mathcal{L}$ is S-cyclic if $s \vee G \subseteq K \subseteq G$ for some cyclic filter K of $\mathcal{L}$ and some $s \in S$.

(2) We say that an element p of $\mathcal{L}$ is S-cyclic if and only if $T(p)$ is a S-cyclic filter of $\mathcal{L}$.

(3) We say that $\mathcal{L}$ is S-c-Noetherian if each element of $\mathcal{L}$ is S-cyclic.

Proposition 4.3. *Suppose that $\mathcal{L}$ is a c-lattice and let S be a join subset of $\mathcal{L}$. Then the following hold:*

(1) p is a minimal element $(\mathcal{L}, \leqslant)$ if and only if $T(p)$ is a maximal element of $(\mathbb{F}(\mathcal{L}), \subseteq)$.

(2) Let p be an element of $\mathcal{L}$ which is minimal among all non-S-cyclic elements of $\mathcal{L}$. Then p is a prime element of $\mathcal{L}$. In particular, p is an e-prime element of $\mathcal{L}$.

Proof. (1) Suppose that p is a minimal element $(\mathcal{L}, \leqslant)$ and let $T(p) \subseteq T(c)$ for some filter $T(c)$ of $\mathcal{L}$. Then $p \in T(p)$ gives $c \leqslant p = c \vee a$ for some $a \in \mathcal{L}$; so $p = c$. Hence, $T(p) = T(c)$. In the same way, the opposite direction can be proved.

(2) By (1), $T(p)$ is a filter of $\mathcal{L}$ which is maximal among all non-S-cyclic filters of $\mathcal{L}$. On the contrary, assume that p is not prime. Then there are elements $a, b \in \mathcal{L}$ such that $p \leqslant a \vee b$ with $p \nleq a$ and $p \nleq b$ (so $a \vee b \in T(p)$, $a \notin T(p)$ and $b \notin T(p)$). Since $T(p) \subsetneqq T(p \wedge a)$ and $T(p) \subsetneqq (T(p) :_{\mathcal{L}} a)$, we conclude that there exist $s, t \in S$, $u \in\in \mathcal{L}$ and $c \in (T(p) :_{\mathcal{L}} a)$ such that $s \vee T(p \wedge a) \subseteq T((p \wedge a) \vee u) \subseteq T(p \wedge a)$ and $t \vee (T(p) :_{\mathcal{L}} a) \subseteq T(c) \subseteq (T(p) :_{\mathcal{L}} a)$ (so $c \vee a \in T(p)$).

Now, let $x \in T(p)$. Then $s \vee x \in s \vee T(p \wedge a))$ gives $s \vee x = (s \vee x) \vee ((p \wedge a) \vee w) = (s \vee x \vee p \vee w) \wedge (s \vee x \vee w \vee a) \in T(p)$ for some $w \in \mathcal{L}$; so $s \vee x \vee w \vee a \in T(p)$ by Lemma 2.1 (1). It follows that $y = s \vee x \vee w \in (T(p) :_{\mathcal{L}} a)$ which implies that $t \vee y \in (T(p) :_{\mathcal{L}} a) \subseteq T(c)$. Therefor, $t \vee y = c \vee e$ for some $e \in \mathcal{L}$; thus $t \vee y = t \vee y \vee c$. So, $s \vee x \vee t = (s \vee x \vee w \vee p \vee t) \wedge (s \vee x \vee w \vee a \vee t) = (t \vee y \vee p) \wedge (t \vee y \vee a) = (t \vee y \vee p \vee c) \wedge (t \vee y \vee c \vee a) \in T(p) \cap T(p \wedge (c \vee a))$ which gives $(s \vee t) \vee T(p) \subseteq T(p \wedge (c \vee a)) \subseteq T(p)$; hence $T(p)$ is S-cyclic, a contradiction. Thus p is a prime element. The "in particular" statement is clear. $\square$

Lemma 4.2. *Suppose that $\mathcal{L}$ is a complete c-lattice and let S be a join subset of $\mathcal{L}$. If $\{p_i\}_{i\in\Lambda}$ is a chain of elements of $\mathcal{L}$ such that $p = \bigwedge_{i\in\Lambda} p_i$ is S-cyclic, then p_i is S-cyclic for $i \in \Lambda$.*

Proof. By the hypothesis, $s \vee T(p) \subseteq T(c) \subseteq T(p)$ for some $s \in S$ and $c \in \mathcal{L}$. Now, we show that p_i is S-cyclic. Since $s \vee p_i \in s \vee T(p_i) \subseteq s \vee T(p) \subseteq T(c)$, we conclude that $s \vee p_i = c \vee a$ for some $a \in \mathcal{L}$ which implies that $s \vee p_i = s \vee p_i \vee p_i = c \vee p_i \vee a \in T(c \vee p_i)$. Therefore, $s \vee T(p_i) \subseteq T(c \vee p_i)$. We claim that either $p_i \leqslant \bigwedge_{i\neq j\in\Lambda} p_j$ or $\bigwedge_{i\neq j\in\Lambda} p_j \leqslant p_i$. On the contrary, assume that $p_i \nleq \bigwedge_{i\neq j\in\Lambda} p_j$ and $\bigwedge_{i\neq j\in\Lambda} p_j \nleq p_i$. Since $p_i \nleq \bigwedge_{i\neq j\in\Lambda} p_j$, we infer that $p_i \nleq p_k$ for some $i \neq k \in \Lambda$ which gives $\bigwedge_{i\neq j\in\Lambda} p_j \leqslant p_k \leqslant p_i$, a contradiction (i.e. the claim holds). If $p_i \leqslant \bigwedge_{i\neq j\in\Lambda} p_j$, then $c \in T(c) \subseteq T(p)$ gives $c = (p_i \wedge (\bigwedge_{i\neq j\in\Lambda} p_j)) \vee b = p_i \vee b$ for some $b \in \mathcal{L}$ which implies that $c \vee p_i = p_i \vee p_i \vee b = p_i \vee b \in T(p_i)$. If $\bigwedge_{i\neq j\in\Lambda} p_j \leqslant p_i$, then $c \in T(c) \subseteq T(p)$ gives $c = (\bigwedge_{i\neq j\in\Lambda} p_j) \vee e$ for some $e \in \mathcal{L}$; so $c \vee p_i = (\bigwedge_{i\neq j\in\Lambda} p_j) \vee e \vee p_i = e \vee p_i \in T(p_i)$. Therefore, $T(c \vee p_i) \subseteq T(p_i)$. It follows that $s \vee T(p_i) \subseteq T(c \vee p_i) \subseteq T(p_i)$, as needed. $\square$

We obtain the following S-c-version of Cohen's Theorem [8].

Theorem 4.10. *Let S be a join subset of $\mathcal{L}$. Suppose that $\mathcal{L}$ is a complete c-lattice and let S be a join subset of $\mathcal{L}$. The following assertions are equivalent:*

(1) $\mathcal{L}$ is S-c-Noetherian;

(2) Every S-e-prime element of $\mathcal{L}$ is S-cyclic;

(3) Every e-prime element of $\mathcal{L}$ is S-cyclic.

Proof. (1) $\Rightarrow$ (2) This is clear.

(2) $\Rightarrow$ (3) Let p be an e-prime element of $\mathcal{L}$ (so it is an S-e-element of $\mathcal{L}$). If $S \wedge p \neq 0$, then there is an element $s \in S$ such that $s \wedge p \neq 0$; so $s \vee T(p) \subseteq T(s \vee p) \subseteq T(p)$ which implies that $T(p)$ is S-cyclic. If $S \wedge p = 0$, then by (2), p is S-cyclic.

(3) $\Rightarrow$ (1) On the contrary, assume That $\mathcal{L}$ is not S-c-Noetherian. Then the set $\Omega = \{a \in \mathcal{L} : a \text{ is non-S-cyclic}\}$ is not empty. Moreover, $(\Omega, \leqslant')$ is a partial

order and Ω is inductive, where $b \leqslant' a$ if and only if $a \leqslant b$. Indeed, if $\{p_i\}_{i\in\Lambda}$ is a chain of elements of Ω, then by Lemma 4.2, $p = \bigwedge_{i\in\Lambda} p_i$ is not S-cyclic; hence $p \in \Omega$ is an upper bound for the chain. Then by Zorn's Lemma, Ω has a maximal element q for "$\leqslant'$" and so q is an element of $\mathcal{L}$ which is minimal among all non-S-cyclic elements of $\mathcal{L}$. Then Proposition 4.3 shows that q is an e-prime element. If $S \wedge q \neq 0$, then then there exists $s \in S$ such that $s \wedge q \neq 0$ (so $s \vee q =\neq 0$); so $s \vee T(q) \subseteq T(s \vee q) \subseteq T(q)$ which implies that $T(q)$ is S-cyclic, a contradiction. Thus $S \wedge q = 0$. Now, by assumption, $T(q)$ is S-cyclic which is impossible since $q \in \Omega$. Thus $\mathcal{L}$ is S-c-Noetherian. $\square$

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