JOURNAL OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4866, ISSN (o) 2303-4947
www.imvibl.org /JOURNALS/JOURNAL
J. Int. Math. Virtual Inst., **15**(1)(2025), 65-85
DOI: 10.7251/JIMVI2501065T

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

SOME PROPERTIES OF EXTENDED b-METRIC PRESERVING FUNCTIONS AND GLUING LEMMA

Nihal Taş and Ayşenur Şen

Abstract. In the literature, metric preserving and b-metric preserving functions are given with some basic properties. In this paper, we investigate some properties of extended b-metric preserving functions and give some relations between the known metric preserving functions and extended b-metric preserving functions. Also, we prove two gluing lemmas for this preserving function family.

1. Introduction

What does a metric function look like?

A definition for the metric function is that a topological space function that provides a value representing the distance between any two points in the space. The most general environment in which to examine many of the ideas of geometry and mathematical analysis is a metric space. Three-dimensional Euclidean space, with its typical concept of distance, is the most well-known example of a metric space. Metric spaces are a technique utilized in many different areas of mathematics because of their generality.

The concept of generalized metric spaces appears in the literature as a generalization of the concept of metric spaces. There are many examples of generalized metric spaces in the literature. Some of these examples are the concept of b-metric spaces and various generalizations of b-metric spaces. Fixed-point results are being

2020 *Mathematics Subject Classification.* Primary 54C50; Secondary 54E35, 54E40, 26A21, 26A99.

Key words and phrases. Extended b-metric space, extended b-metric preserving function, Gluing lemma, fixed point.

Communicated by Özgür Ege.

studied on these spaces (For example, see [**7**], [**9**], [**16**], [**19**], [**21**] and the references therein)

A new area of research for metric functions is the study of metric preserving functions. The idea of metric preserving functions appears to have been initially mentioned in the literature by [**22**]. Metric preserving functions have been the topic of a substantial body of literature. Afterwards, research on the idea of a metric preserving function for various generalized metric spaces started. The notion of the b-metric preserving function was proposed and several relationships between the concepts of the metric preserving function and the concept of b-metric space were investigated [**13**]. The weak-ultrametric notion was used to present a new metric preserving function concept to the literature, while certain aspects of the b-metric preserving function concept were still being investigated [**14**]. Two distinct Pasting Lemmas (or Gluing Lemmas) have been formulated and proven for b-metric preserving functions using a simple topological technique [**15**].

In this paper, we use the extended b-metric concept to offer a new notion of metric preserving function based on all the previously described motivations. We first describe the extended b-metric preserving function and look at the fundamental connections between it and certain concepts of metric maintaining functions that are known from the literature. Using a topological approach, we develop and prove two distinct Gluing Lemmas for extended b-metric preserving functions.

2. Preliminaries

In this section, we give some basic concepts related to metric and generalized metric preserving functions.

Let X be a nonempty set and $d : X \times X \to [0, \infty)$ a function. If the following conditions are satisfied for all $x, y, z \in X$, then d is called a metric:

$(d1)$ $d(x, y) = 0$ if and only if $x = y$,
$(d2)$ $d(x, y) = d(y, x)$,
$(d3)$ $d(x, y) \leqslant d(x, z) + d(z, y)$.

Then the pair (X, d) is said to be a metric space.

This metric space was generalized using different approaches as seen in the following notions:

Definition 2.1. [**1**] Let X be a nonempty set and $d_s : X \times X \to [0, \infty)$ a function. If the following conditions are satisfied for all $x, y, z \in X$, then d_s is called a b-metric:

(d_s1) $d_s(x, y) = 0$ if and only if $x = y$,
(d_s2) $d_s(x, y) = d_s(y, x)$,
(d_s3) There is $s \geqslant 1$ such that

$$d_s(x, y) \leqslant s\left[d_s(x, z) + d_s(z, y)\right].$$

Then the pair (X, d_s) is said to be a b-metric space.

Remark 2.1. The notion of a b-metric is a generalization of a metric. Indeed, we take $s = 1$ in Definition 2.1, then the concepts coincide. In the literature, there exist some examples of b-metric which is not a metric. For example, let $X = l_p(\mathbb{R})$ with $p \in (0,1)$ where

$$l_p(\mathbb{R}) = \left\{ \{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}.$$

Assume that for $x = \{x_n\}$ and $y = \{y_n\}$, the function $d_s : X \times X \to [0,\infty)$ is defined by

$$d_s(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}}.$$

Then d_s is a b-metric with $s = 2^{\frac{1}{p}}$ (see, [**2**], [**6**] and [**10**] for more details).

The notions of a metric and a b-metric were generalized to an extended b-metric as follows:

DEFINITION 2.2. [**12**] Let X be a nonempty set, $\theta : X \times X \to [1,\infty)$ and $d_s : X \times X \to [0,\infty)$ be two functions. If the following conditions are satisfied for all $x, y, z \in X$, then d_θ is called an extended b-metric:
($d_\theta 1$) $d_\theta(x,y) = 0$ if and only if $x = y$,
($d_\theta 2$) $d_\theta(x,y) = d_\theta(y,x)$,
($d_\theta 3$) $d_\theta(x,y) \leqslant \theta(x,y)[d_\theta(x,z) + d_\theta(z,y)]$.
Then the pair (X, d_θ) is said to be an extended b-metric space.

EXAMPLE 2.1. [**11**] Let $X = [0,\infty)$. Let us define the functions $\theta : X \times X \to [1,\infty)$ and $d_s : X \times X \to [0,\infty)$ for $x, y \in X$ as follows:

$$\theta(x,y) = x + y + 1$$

and

$$d_\theta(x,y) = \begin{cases} x+y & ; \quad x \neq y \\ 0 & ; \quad x = y \end{cases}.$$

Then (X, d_θ) is an extended b-metric space.

REMARK 2.2. If we take $\theta(x,y) = s$ for $s \geqslant 1$, then the concepts of a b-metric and an extended b-metric coincide.

The relationships among the notions of a metric space, a b-metric space and an extended b-metric space are as follows:

metric space
$\downarrow$
b-metric space
$\downarrow$
extended b-metric space

Using the concepts of a metric and a b-metric, the following definitions were introduced:

DEFINITION 2.3. [4] [**18**] [**20**] Let (X, d) be a metric space and $\psi : [0,\infty) \to [0,\infty)$ be a function. Then ψ is called metric preserving function if $\psi \circ d$ is a metric on X.

Definition 2.4. [**13**] Let (X,d) be a metric space, (X,d_S) be a b-metric space and $\psi:[0,\infty)\to[0,\infty)$ be a function. Then

(i) ψ is called a b-metric preserving function if $\psi\circ d_s$ is a b-metric on X.

(ii) ψ is called a metric-b-metric preserving function if $\psi\circ d$ is a b-metric on X.

(iii) ψ is called a b-metric-metric preserving function if $\psi\circ d_s$ is a metric on X.

Let $\mathcal{M}$ be the set of all metric preserving function, $\mathcal{B}$ be the set of all b-metric preserving functions, $\mathcal{MB}$ be the set of all metric-b-metric preserving functions and $\mathcal{BM}$ be the set of all b-metric-metric preserving functions [**13**].

Now we recall the following basic notions:

Let $\psi:[0,\infty)\to[0,\infty)$ and $I\subseteq[0,\infty)$. Then ψ is called

- increasing on I if $\psi(x)\leqslant\psi(y)$ for all $x,y\in I$ such that $x<y$,
- strictly increasing on I if $\psi(x)<\psi(y)$ for all $x,y\in I$ such that $x<y$,
- decreasing on I if $\psi(x)\geqslant\psi(y)$ for all $x,y\in I$ such that $x<y$,
- strictly decreasing on I if $\psi(x)>\psi(y)$ for all $x,y\in I$ such that $x<y$,
- amenable if $\psi^{-1}(\{0\})=\{0\}$,
- tightly bounded on $(0,\infty)$ if there is $v>0$ such that $\psi(x)\in[v,2v]$ for all $x\in(0,\infty)$,
- subadditive if $\psi(a+b)\leqslant\psi(a)+\psi(b)$ for all $a,b\in[0,\infty)$,
- quasi-subadditive if there exists $s\geqslant 1$ such that $\psi(a+b)\leqslant s[\psi(a)+\psi(b)]$ for all $a,b\in[0,\infty)$,
- convex if $\psi((1-t)x+ty)\leqslant(1-t)\psi(x)+t\psi(y)$ for all $x,y\in[0,\infty)$ and $t\in[0,1]$,
- concave if $\psi((1-t)x+ty)\geqslant(1-t)\psi(x)+t\psi(y)$ for all $x,y\in[0,\infty)$ and $t\in[0,1]$,
- linear if $\psi(a+b)=\psi(a)+\psi(b)$ and $\psi(ka)=k\psi(a)$ for all $a,b\in[0,\infty)$ with a constant k.

Definition 2.5. (i) [**13**] Let $a,b,c\geqslant 0$. A triple (a,b,c) is called a triangle triplet if

$$a\leqslant b+c,\ b\leqslant a+c \text{ and } c\leqslant a+b.$$

(ii) [**13**] Let $s\geqslant 1$ and $a,b,c\geqslant 0$. A triple (a,b,c) is called an s-triangle triplet if

$$a\leqslant s(b+c),\ b\leqslant s(a+c) \text{ and } c\leqslant s(a+b).$$

(iii) [**17**] Let $\theta:X\times X\to[1,\infty)$ and $a,b,c\geqslant 0$. A triple (a,b,c) is called θ-triangle triplet if

$$a\leqslant\theta(x,y)(b+c),\ b\leqslant\theta(x,z)(a+c) \text{ and } c\leqslant\theta(z,y)(a+b),$$

for all $x,y,z\in X$.

Let Δ, Δ_s and Δ_θ be the set of all triangle triplets, s-triangle triplets and θ-triangle triplets, respectively.

Lemma 2.1. *[**5**] [**8**] If $\psi\in\mathcal{M}$, then ψ is amenable and subadditive.*

Lemma 2.2. *[**4**] [**5**] [**8**] Assume that $\psi:[0,\infty)\to[0,\infty)$ is subadditive. Then for all positive integers n and for all $x\in[0,\infty)$, we have*

$$\psi(nx)\leqslant n\psi(x).$$

LEMMA 2.3. *[**3**] [**5**] [**8**] Suppose that* $\psi : [0,\infty) \to [0,\infty)$ *is amenable. Then the followings are equivalent*:

(a) $\psi \in \mathcal{M}$.

(b) *For each* $(\alpha, \beta, \gamma) \in \Delta$, *then* $(\psi(\alpha), \psi(\beta), \psi(\gamma)) \in \Delta$.

LEMMA 2.4. *[**8**]Let* $\psi : [0,\infty) \to [0,\infty)$ *be amenable. Then* ψ *is concave if and only if for all* $t \geqslant 0$ *and* $x, y, z \in [0, t]$,

$$x + t = y + z \Longrightarrow \psi(x) + \psi(t) = \psi(y) + \psi(z).$$

3. Main results

In this section, we investigate some properties of extended b-metric preserving functions.

DEFINITION 3.1. Let (X, d) be a metric space, (X, d_S) be a b-metric space, (X, d_θ) be a extended b-metric space and $\psi : [0,\infty) \to [0,\infty)$ be a function. Then

(i) ψ is called an extended b-metric preserving function if $\psi \circ d_\theta$ is an extended b-metric on X [**17**].

(ii) ψ is called a metric-extended b-metric preserving function if $\psi \circ d$ is an extended b-metric on X.

(iii) ψ is called an extended b-metric-metric preserving function if $\psi \circ d_\theta$ is a metric on X.

(iv) ψ is called a b-metric-extended b-metric preserving function if $\psi \circ d_S$ is an extended b-metric on X.

(v) ψ is called an extended b-metric-b-metric preserving function if $\psi \circ d_\theta$ is a b-metric on X.

Let $\mathcal{E}_b$ be the set of all extended b-metric preserving function, $\mathcal{M}\mathcal{E}_b$ be the set of all metric-extended b-metric preserving functions, $\mathcal{E}_b\mathcal{M}$ be the set of all extended b-metric-metric preserving functions, $\mathcal{B}\mathcal{E}_b$ be the set of all b-metric-extended b-metric preserving functions and $\mathcal{E}_b\mathcal{B}$ be the set of all extended b-metric-b-metric preserving functions.

THEOREM 3.1. *Let n be a positive real number. Let us define* $\psi : [0,\infty) \to [0,\infty)$ *by*

$$\psi(x) = x^n.$$

Then the followings are satisfied:

(a) *If* $n \in (0, 1]$, *then* $\psi \in \mathcal{M}$.

(b) *If* $n > 1$, *then* $\psi \in \mathcal{E}_b$, *but* $\psi \notin \mathcal{M}$.

PROOF. (a) It can be easily seen in the proof of Theorem 12 given in [**13**].

(b) Let $n > 1$. Let us define the function $\phi : [0,\infty) \to \mathbb{R}$ as

$$\phi(x) = \frac{(1+x)^n}{1+x^n}.$$

Then we get

$$\phi'(x) = \frac{n(1+x)^{n-1}\left(1 - x^{n-1}\right)}{(1+x^n)^2}$$

and

$$\phi'(x) \geqslant 0 \Leftrightarrow x \leqslant 1.$$

Hence, we say that ϕ is increasing on $[0,1]$ and decreasing $[1,\infty)$. For all $x \in [0,\infty)$, we have

$$\phi(x) \leqslant \phi(1) = 2^{n-1}. \tag{3.1}$$

To show $\psi \in \mathcal{E}_b$, let d_θ be an extended b-metric on X and $\theta : X \times X \to [1,\infty)$ be a function such that

$$d_\theta(x,y) \leqslant \theta(x,y)\left[d_\theta(x,z) + d_\theta(z,y)\right],$$

for all $x,y,z \in X$.

Now, we check that $\psi \circ d_\theta$ satisfies the conditions $(d_\theta 1)$, $(d_\theta 2)$ and $(d_\theta 3)$ as follows:

$(d_\theta 1)$ For all $x,y \in X$, we get

$$\begin{aligned} \psi \circ d_\theta(x,y) &= 0 \Leftrightarrow \psi(d_\theta(x,y)) = 0 \Leftrightarrow [d_\theta(x,y)]^n = 0 \\ &\Leftrightarrow d_\theta(x,y) = 0 \Leftrightarrow x = y. \end{aligned}$$

$(d_\theta 2)$ For all $x,y \in X$, we have

$$\psi \circ d_\theta(x,y) = \psi(d_\theta(x,y)) = \psi(d_\theta(y,x)) = \psi \circ d_\theta(y,x).$$

$(d_\theta 3)$ Let $x,y,z \in X$. If $x = z$, then it is clear that

$$\psi \circ d_\theta(x,y) \leqslant [\theta(x,y)]^n 2^{n-1}\left[\psi \circ d_\theta(x,z) + \psi \circ d_\theta(z,y)\right].$$

Assume that $x \neq z$. By (3.1), we obtain

$$\psi\left(\frac{d_\theta(z,y)}{d_\theta(x,z)}\right) \leqslant 2^{n-1}$$

and

$$\begin{aligned} [d_\theta(x,z) + d_\theta(z,y)]^n &\leqslant 2^{n-1}\left[d_\theta(x,z)^n + d_\theta(z,y)^n\right] \\ &= 2^{n-1}\left[\psi \circ d_\theta(x,z) + \psi \circ d_\theta(z,y)\right]. \end{aligned} \tag{3.2}$$

Using (3.2) and the properties of extended b-metric, we get

$$\begin{aligned} \psi \circ d_\theta(x,y) &= [d_\theta(x,y)]^n \leqslant [\theta(x,y)(d_\theta(x,z) + d_\theta(z,y))]^n \\ &\leqslant [\theta(x,y)]^n 2^{n-1}\left[\psi \circ d_\theta(x,z) + \psi \circ d_\theta(z,y)\right]. \end{aligned}$$

Consequently, $\psi \in \mathcal{E}_b$. It is easily seen in the proof of Theorem 12 given in [**13**] that $\psi \notin \mathcal{M}$. □

Example 3.1. Let us consider the usual metric space on $\mathbb{R}$ with the metric function

$$d(x,y) = |x-y|,$$

for all $x,y \in \mathbb{R}$. Then (X,d) is also an extended b-metric space $\theta(x,y) = 1$. Assume that the function $\psi : [0,\infty) \to [0,\infty)$ is defined by

$$\psi(x) = x^2.$$

Then $\psi \in \mathcal{E}_b$, but $\psi \notin \mathcal{M}$.

LEMMA 3.1. *Let (X,d) be a metric space, (X,d_θ) be an extended b-metric space and $\psi:[0,\infty)\to[0,\infty)$ be a function. Then we have*
(a) $\mathcal{E}_b\mathcal{M}\subseteq\mathcal{M}$.
(b) $\mathcal{E}_b\subseteq\mathcal{M}\mathcal{E}_b$.

PROOF. (a) Let $\psi\in\mathcal{E}_b\mathcal{M}$ and d be a metric on X. Since every metric is an extended b-metric, d is an extended b-metric on X. As $\psi\in\mathcal{E}_b\mathcal{M}$, then $\psi\circ d$ is a metric on X. Hence, we get $\psi\in\mathcal{M}$.

(b) Let $\psi\in\mathcal{E}_b$ and d be a metric on X. Since every metric is an extended b-metric, d is an extended b-metric on X. As $\psi\in\mathcal{E}_b$, then $\psi\circ d$ is an extended b-metric on X. Hence, we get $\psi\in\mathcal{M}\mathcal{E}_b$. □

LEMMA 3.2. *Let (X,d) be a metric space, (X,d_θ) be an extended b-metric space with the bounded function θ and $\psi:[0,\infty)\to[0,\infty)$ be a function. Then we have*

$$\mathcal{M}\subseteq\mathcal{E}_b.$$

PROOF. Let $\psi\in\mathcal{M}$ and d_θ be an extended b-metric on X. We show that $\psi\circ d_\theta$ is an extended b-metric on X as follows:

$(d_\theta 1)$ Since $\psi\in\mathcal{M}$, then by Lemma 2.1, ψ is amenable. For all $x,y\in X$, we get

$$\begin{aligned}\psi\circ d_\theta(x,y) &= 0\Leftrightarrow\psi(d_\theta(x,y))=0\\ &\Leftrightarrow d_\theta(x,y)=0\Leftrightarrow x=y.\end{aligned}$$

$(d_\theta 2)$ For all $x,y\in X$, we find

$$\psi\circ d_\theta(x,y)=\psi(d_\theta(x,y))=\psi(d_\theta(y,x))=\psi\circ d_\theta(y,x).$$

$(d_\theta 3)$ For all $x,y,z\in X$, assume that

$$\alpha=d_\theta(x,y),\ \beta=d_\theta(x,z)\ \text{and}\ \gamma=d_\theta(z,y).$$

Since d_θ is an extended b-metric on X, we have

$$d_\theta(x,y)\leqslant\theta(x,y)[d_\theta(x,z)+d_\theta(z,y)]\Longrightarrow\alpha\leqslant\theta(x,y)[\beta+c\gamma],$$

$$d_\theta(x,z)\leqslant\theta(x,z)[d_\theta(x,y)+d_\theta(y,z)]\Longrightarrow\beta\leqslant\theta(x,z)[\alpha+\gamma]$$

and

$$d_\theta(y,z)\leqslant\theta(y,z)[d_\theta(y,x)+d_\theta(x,z)]\Longrightarrow\gamma\leqslant\theta(y,z)[\alpha+\beta].$$

Let

$$N=\sup\{\theta(x,y):x,y\in X\}$$

and n be a positive integer larger than N. Then, we have

$$\alpha\leqslant\theta(x,y)[\beta+\gamma]\leqslant n(\beta+\gamma)=n\beta+n\gamma$$

and so $(\alpha,n\beta+n\gamma,n\beta+n\gamma)$ is a triangle triplet. Since $\psi\in\mathcal{M}$, by Lemmas 2.1, 2.2 and 2.3, we get

$$\begin{aligned}\psi(\alpha) &\leqslant \psi(n\beta+n\gamma)+\psi(n\beta+n\gamma)=2\psi(n\beta+n\gamma)\\ &\leqslant 2[\psi(n\beta)+\psi(n\gamma)]\leqslant 2[n\psi(\beta)+n\psi(\gamma)]\\ &= 2n[\psi(\beta)+\psi(\gamma)]\end{aligned}$$

and so

$$\psi \circ d_\theta(x,y) \leqslant 2n\left[\psi \circ d_\theta(x,z) + \psi \circ d_\theta(z,y)\right].$$

Consequently, $\psi \circ d_\theta$ is an extended b-metric with the function $\theta : X \times X \to [1,\infty)$ defined as

$$\theta(x,y) = 2n,$$

for all $x, y \in X$, that is, $\psi \in \mathcal{E}_b$. □

From Lemma 3.1 and Lemma 3.2, we get the following corollary:

Corollary 3.1. *Let (X,d) be a metric space, (X,d_θ) be an extended b-metric space with the bounded function θ and $\psi : [0,\infty) \to [0,\infty)$ be a function. Then we have*

$$\mathcal{E}_b\mathcal{M} \subseteq \mathcal{M} \subseteq \mathcal{E}_b \subseteq \mathcal{M}\mathcal{E}_b.$$

Lemma 3.3. *[**17**] Let (X,d_s) be a b-metric space, (X,d_θ) be an extended b-metric space and $\psi : [0,\infty) \to [0,\infty)$ be a function. Then we have*

$$\mathcal{B} \subseteq \mathcal{E}_b.$$

Lemma 3.4. *Let (X,d_s) be a b-metric space, (X,d_θ) be an extended b-metric space and $\psi : [0,\infty) \to [0,\infty)$ be a function. Then we have*
(a) $\mathcal{E}_b\mathcal{B} \subseteq \mathcal{B}$.
(b) $\mathcal{E}_b \subseteq \mathcal{B}\mathcal{E}_b$.

Proof. (a) Let $\psi \in \mathcal{E}_b\mathcal{B}$ and d_s be a b-metric on X. Since every b-metric is an extended b-metric, d_s is an extended b-metric on X. As $\psi \in \mathcal{E}_b\mathcal{B}$, then $\psi \circ d_s$ is a b-metric on X. Hence, we get $\psi \in \mathcal{B}$.

(b) Let $\psi \in \mathcal{E}_b$ and d_s be a b-metric on X. Since every b-metric is an extended b-metric, d_s is an extended b-metric on X. As $\psi \in \mathcal{E}_b$, then $\psi \circ d_s$ is an extended b-metric on X. Hence, we get $\psi \in \mathcal{B}\mathcal{E}_b$. □

From Lemma 3.3 and Lemma 3.4, we get the following corollary:

Corollary 3.2. *Let (X,d_s) be a b-metric space, (X,d_θ) be an extended b-metric space and $\psi : [0,\infty) \to [0,\infty)$ be a function. Then we have*

$$\mathcal{E}_b\mathcal{B} \subseteq \mathcal{B} \subseteq \mathcal{E}_b \subseteq \mathcal{B}\mathcal{E}_b.$$

Lemma 3.5. *Let (X,d) be a metric space, (X,d_s) be a b-metric space, (X,d_θ) be an extended b-metric space and $\psi : [0,\infty) \to [0,\infty)$ be a function. Then we have*
(a) $\mathcal{E}_b\mathcal{M} \subseteq \mathcal{E}_b\mathcal{B}$.
(b) $\mathcal{E}_b\mathcal{M} \subseteq \mathcal{B}\mathcal{E}_b$.

Proof. (a) Let $\psi \in \mathcal{E}_b\mathcal{M}$ and d_θ be an extended b-metric on X. As $\psi \in \mathcal{E}_b\mathcal{M}$, then $\psi \circ d_\theta$ is a metric on X. Since every metric is a b-metric, $\psi \circ d_\theta$ is a b-metric on X. Hence, we get $\psi \in \mathcal{E}_b\mathcal{B}$.

(b) Let $\psi \in \mathcal{E}_b\mathcal{M}$ and d_s be a b-metric on X. Since every b-metric is an extended b-metric, d_s is an extended b-metric on X. As $\psi \in \mathcal{E}_b\mathcal{M}$, then $\psi \circ d_s$ is a metric on X. Also, since every metric is an extended b-metric,then we get $\psi \in \mathcal{B}\mathcal{E}_b$. □

From Corollary 3.1, Corollary 3.2 and Lemma 3.5, we get the following corollary:

COROLLARY 3.3. *Let* (X, d) *be a metric space,* (X, d_s) *be a b-metric space,* (X, d_θ) *be an extended b-metric space with the bounded function* θ *and* $\psi : [0, \infty) \to [0, \infty)$ *be a function. Then we have*

$$\mathcal{E}_b\mathcal{M} \subseteq \mathcal{E}_b\mathcal{B} \subseteq \mathcal{B} \subseteq \mathcal{E}_b \subseteq \mathcal{B}\mathcal{E}_b$$

and

$$\mathcal{E}_b\mathcal{M} \subseteq \mathcal{M} \subseteq \mathcal{B} \subseteq \mathcal{E}_b \subseteq \mathcal{M}\mathcal{E}_b.$$

THEOREM 3.2. *Let* (X, d) *be a metric space and* (X, d_θ) *be an extended b-metric space. Assume that* $\psi : [0, \infty) \to [0, \infty)$ *is amenable. Then the followings are equivalent*:

(*a*) $\psi \in \mathcal{M}\mathcal{E}_b$.

(*b*) *There exists a function* $\theta : X \times X \to [1, \infty)$ *such that* $(\psi(x), \psi(y), \psi(z)) \in \Delta_\theta$ *for all* $(x, y, z) \in \Delta$.

PROOF. Suppose that $\psi \in \mathcal{M}\mathcal{E}_b$. Let d be a usual metric on $\mathbb{R}^2$. Then $\psi \circ d$ is a extended b-metric and so there exists a function $\theta : X \times X \to [1, \infty)$ such that

$$\psi \circ d(u, w) \leqslant \theta(u, w)\left[\psi \circ d(u, v) + \psi \circ d(v, w)\right],$$

for all $u, v, w \in \mathbb{R}^2$. Let $(x, y, z) \in \Delta$. According to the Euclidean geometry, there are $u, v, w \in \mathbb{R}^2$ such that

$$x = d(u, w),\ y = d(u, v) \text{ and } z = d(v, w).$$

Then, we have

$$\begin{aligned}
\psi(x) &= \psi \circ d(u, w) \leqslant \theta(u, w)\left[\psi \circ d(u, v) + \psi \circ d(v, w)\right] \\
&= \theta(u, w)\left[\psi(y) + \psi(z)\right],
\end{aligned}$$

$$\begin{aligned}
\psi(y) &= \psi \circ d(u, v) \leqslant \theta(u, v)\left[\psi \circ d(u, w) + \psi \circ d(w, v)\right] \\
&= \theta(u, v)\left[\psi(x) + \psi(z)\right],
\end{aligned}$$

$$\begin{aligned}
\psi(z) &= \psi \circ d(v, w) \leqslant \theta(v, w)\left[\psi \circ d(v, u) + \psi \circ d(u, w)\right] \\
&= \theta(v, w)\left[\psi(x) + \psi(y)\right]
\end{aligned}$$

and so

$$(\psi(x), \psi(y), \psi(z)) \in \Delta_\theta.$$

Assume that there exists a function $\theta : X \times X \to [1, \infty)$ such that

$$(\psi(x), \psi(y), \psi(z)) \in \Delta_\theta$$

for all $(x, y, z) \in \Delta$. Let (X, d) be a metric space. Now we show that $\psi \circ d$ is an extended b-metric as follows:

$(d_\theta 1)$ Since ψ is amenable, for all $x, y \in X$, we have

$$\psi \circ d(x, y) = 0 \Longleftrightarrow \psi(d(x, y)) = 0 \Longleftrightarrow d(x, y) = 0 \Longleftrightarrow x = y.$$

($d_\theta 2$) For all $x, y \in X$, we get

$$\psi \circ d(x,y) = \psi(d(x,y)) = \psi(d(y,x)) = \psi \circ d(y,x).$$

($d_\theta 3$) Using hypothesis, since $(d(x,y), d(x,z), d(z,y)) \in \Delta$ for all $x, y, z \in X$, then

$$(\psi \circ d(x,y), \psi \circ d(x,z), \psi \circ d(z,y)) \in \Delta_\theta,$$

that is,

$$\psi \circ d(x,y) \leqslant \theta(x,y)[\psi \circ d(x,z) + \psi \circ d(z,y)].$$

Consequently, $\psi \circ d$ is an extended b-metric and $\psi \in \mathcal{ME}_b$. □

THEOREM 3.3. *Let (X, d_s) be a b-metric space and (X, d_θ) be an extended b-metric space. Assume that $\psi : [0,\infty) \to [0,\infty)$ is amenable. Then the followings are equivalent*:

(a) $\psi \in \mathcal{BE}_b$.

(b) There exists a function $\theta : X \times X \to [1,\infty)$ such that $(\psi(x), \psi(y), \psi(z)) \in \Delta_\theta$ for all $(x,y,z) \in \Delta_s$.

PROOF. Suppose that $\psi \in \mathcal{BE}_b$. Let d_s be a usual metric on $\mathbb{R}^2$. Since every metric is a b-metric, hence d_s is a b-metric. Then $\psi \circ d_s$ is a extended b-metric and so there exists a function $\theta : X \times X \to [1,\infty)$ such that

$$\psi \circ d_s(u,w) \leqslant \theta(u,w)[\psi \circ d_s(u,v) + \psi \circ d_s(v,w)],$$

for all $u, v, w \in \mathbb{R}^2$. Let $(x,y,z) \in \Delta_s$. Then we have

$$x \leqslant s(y+z),\ y \leqslant s(x+z) \text{ and } z \leqslant s(x+y).$$

Since d_s is a b-metric, there are $u, v, w \in \mathbb{R}^2$ such that

$$x = d_s(u,w),\ y = d_s(u,v) \text{ and } z = d_s(v,w).$$

Then, we have

$$\begin{aligned} \psi(x) &= \psi \circ d_s(u,w) \leqslant \theta(u,w)[\psi \circ d_s(u,v) + \psi \circ d_s(v,w)] \\ &= \theta(u,w)[\psi(y) + \psi(z)], \end{aligned}$$

$$\begin{aligned} \psi(y) &= \psi \circ d_s(u,v) \leqslant \theta(u,v)[\psi \circ d_s(u,w) + \psi \circ d_s(w,v)] \\ &= \theta(u,v)[\psi(x) + \psi(z)], \end{aligned}$$

$$\begin{aligned} \psi(z) &= \psi \circ d_s(v,w) \leqslant \theta(v,w)[\psi \circ d_s(v,u) + \psi \circ d_s(u,w)] \\ &= \theta(v,w)[\psi(x) + \psi(y)] \end{aligned}$$

and so

$$(\psi(x), \psi(y), \psi(z)) \in \Delta_\theta.$$

Assume that there exists a function $\theta : X \times X \to [1,\infty)$ such that

$$(\psi(x), \psi(y), \psi(z)) \in \Delta_\theta$$

for all $(x,y,z) \in \Delta_s$. Let (X, d_s) be a b-metric space. Now we show that $\psi \circ d_s$ is an extended b-metric as follows:

($d_\theta 1$) Since ψ is amenable, for all $x, y \in X$, we have

$$\psi \circ d_s(x,y) = 0 \Longleftrightarrow \psi(d_s(x,y)) = 0 \Longleftrightarrow d_s(x,y) = 0 \Longleftrightarrow x = y.$$

($d_\theta 2$) For all $x, y \in X$, we get

$$\psi \circ d_s(x,y) = \psi(d_s(x,y)) = \psi(d_s(y,x)) = \psi \circ d_s(y,x).$$

($d_\theta 3$) Using hypothesis, since $(d(x,y), d(x,z), d(z,y)) \in \Delta_s$ for all $x, y, z \in X$, then

$$(\psi \circ d_s(x,y), \psi \circ d_s(x,z), \psi \circ d_s(z,y)) \in \Delta_\theta,$$

with the function $\theta : X \times X \to [1, \infty)$ defined as $\theta(x,y) = s$ for all $x, y \in X$. Hence, we obtain

$$\psi \circ d_s(x,y) \leqslant \theta(x,y)[\psi \circ d_s(x,z) + \psi \circ d_s(z,y)].$$

Consequently, $\psi \circ d_s$ is an extended b-metric and $\psi \in \mathcal{BE}_b$. □

THEOREM 3.4. *Let (X,d) be a metric space and (X,d_θ) be an extended b-metric space. If $\psi : [0,\infty) \to [0,\infty)$ is increasing, quasi-subadditive and amenable on $[0,\infty)$, then $\psi \in \mathcal{ME}_b$.*

PROOF. Assume that ψ is increasing, quasi-subadditive and amenable. Let (X,d) be a metric space. Now, we show that $\psi \circ d$ is an extended b-metric as follows:

($d_\theta 1$) Since d is a metric and ψ is amenable, for all $x, y \in X$, we have

$$\psi \circ d(x,y) = 0 \Longleftrightarrow \psi(d(x,y)) = 0 \Longleftrightarrow d(x,y) = 0 \Longleftrightarrow x = y.$$

($d_\theta 2$) Since d is a metric, using the symmetry property of d, for all $x, y \in X$, we get

$$\psi \circ d(x,y) = \psi(d(x,y)) = \psi(d(y,x)) = \psi \circ d(y,x).$$

($d_\theta 3$) Since ψ is quasi-subadditive, there exists $s \geqslant 1$ such that

$$\psi(u+v) \leqslant s[\psi(u) + \psi(v)], \tag{3.3}$$

for all $u, v \in [0,\infty)$. Let the function $\theta : X \times X \to [1,\infty)$ be defined as

$$\theta(x,y) = s,$$

for all $x, y \in X$. Using the increasing property and the inequality (3.3), we obtain

$$\begin{aligned} \psi \circ d(x,y) &= \psi(d(x,y)) \leqslant \psi(d(x,z) + d(z,y)) \\ &\leqslant s[\psi(d(x,z)) + \psi(d(z,y))] \\ &= \theta(x,y)[\psi \circ d(x,z) + \psi \circ d(z,y)], \end{aligned}$$

for all $x, y, z \in X$.

Consequently, $\psi \in \mathcal{ME}_b$. □

THEOREM 3.5. *Let (X,d_s) be a b-metric space and (X,d_θ) be an extended b-metric space. If $\psi : [0,\infty) \to [0,\infty)$ is increasing, linear and amenable on $[0,\infty)$, then $\psi \in \mathcal{BE}_b$.*

PROOF. Assume that ψ is increasing, linear and amenable. Let (X, d_s) be a b-metric space. Now, we show that $\psi \circ d_s$ is an extended b-metric as follows:

$(d_\theta 1)$ Since d_s is a b-metric and ψ is amenable, for all $x, y \in X$, we have

$$\psi \circ d_s(x,y) = 0 \Longleftrightarrow \psi(d_s(x,y)) = 0 \Longleftrightarrow d_s(x,y) = 0 \Longleftrightarrow x = y.$$

$(d_\theta 2)$ Since d_s is a b-metric, using the symmetry property of d_s, for all $x, y \in X$, we get

$$\psi \circ d_s(x,y) = \psi(d_s(x,y)) = \psi(d_s(y,x)) = \psi \circ d_s(y,x).$$

$(d_\theta 3)$ Let the function $\theta : X \times X \to [1, \infty)$ be defined as

$$\theta(x,y) = s,$$

for all $x, y \in X$. Using the increasing and linear property, we obtain

$$\begin{aligned}
\psi \circ d_s(x,y) &= \psi(d_s(x,y)) \leqslant \psi(s(d_s(x,z) + d_s(z,y))) \\
&= s\psi(d_s(x,z)) + s\psi(d_s(z,y)) \\
&= s[\psi \circ d_s(x,z) + \psi \circ d_s(z,y)] \\
&= \theta(x,y)[\psi \circ d_s(x,z) + \psi \circ d_s(z,y)],
\end{aligned}$$

for all $x, y, z \in X$.

Consequently, $\psi \in \mathcal{BE}_b$. □

THEOREM 3.6. *Let (X, d) be a metric space, (X, d_θ) be an extended b-metric space with the bounded function θ and $\psi : [0, \infty) \to [0, \infty)$. If $\psi \in \mathcal{ME}_b$, then ψ is quasi-subadditive and amenable.*

PROOF. Let $\psi \in \mathcal{ME}_b$ and d be a usual metric on $\mathbb{R}$. So, $\psi \circ d$ is an extended b-metric on $\mathbb{R}$. Then we get

$$\psi(0) = \psi(d(0,0)) = \psi \circ d(0,0) = 0.$$

Also, suppose that $x \in [0, \infty)$ and $\psi(x) = 0$. Hence, we obtain

$$0 = \psi(x) = \psi(d(x,0)) = \psi \circ d(x,0)$$

and

$$\psi \circ d(x,0) = 0.$$

Since $\psi \circ d$ is an extended b-metric, we have

$$x = 0$$

and

$$\psi^{-1}(\{0\}) = \{0\},$$

that is, ψ is amenable.

Now, we show that ψ is quasi-subadditive. Since $\psi \circ d$ is an extended b-metric, there exists a function $\theta : X \times X \to [1, \infty)$ such that

$$\psi \circ d(x,y) \leqslant \theta(x,y)[\psi \circ d(x,z) + \psi \circ d(z,y)],$$

for all $x, y, z \in \mathbb{R}$. Let

$$S = \sup\{\theta(x,y) : x, y \in X\}.$$

To show that ψ is quasi-subadditive, let $u, v \in [0, \infty)$. Thereby, we obtain

$$\begin{aligned} \psi(u+v) &\leqslant \psi \circ d(0, u+v) \\ &\leqslant \theta(0, u+v)[\psi \circ d(0, u) + \psi \circ d(u, u+v)] \\ &= \theta(0, u+v)[\psi(u) + \psi(v)] \\ &\leqslant S(\psi(u) + \psi(v)). \end{aligned}$$

Consequently, ψ is quasi-subadditive. □

THEOREM 3.7. *Let (X, d_s) be a b-metric space, (X, d_θ) be an extended b-metric space with the bounded function θ and $\psi : [0, \infty) \to [0, \infty)$. If $\psi \in \mathcal{BE}_b$, then ψ is quasi-subadditive and amenable.*

PROOF. We can see this readily by using the same reasoning as in the proof of Theorem 3.6. □

THEOREM 3.8. *Let (X, d) be a metric space, (X, d_θ) be an extended b-metric space and $\psi : [0, \infty) \to [0, \infty)$. Then $\psi \in \mathcal{E}_b\mathcal{M}$ if and only if ψ is tightly bounded and amenable.*

PROOF. Suppose that ψ is tightly bounded and amenable. Let $a > 0$ be a constant such that

$$\psi(x) \in [a, 2a], \tag{3.4}$$

for all $x > 0$ and (X, d_θ) be an extended b-metric space. Now we show that $\psi \circ d_\theta$ is a metric as follows:

$(d1)$ Since d_θ is an extended b-metric space and ψ is amenable, we have

$$\psi \circ d_\theta(x, y) = 0 \Longleftrightarrow \psi(d_\theta(x, y)) = 0 \Longleftrightarrow d_\theta(x, y) = 0 \Longleftrightarrow x = y,$$

for all $x, y \in X$.

$(d2)$ Since d_θ is an extended b-metric space, we get

$$\psi \circ d_\theta(x, y) = \psi(d_\theta(x, y)) = \psi(d_\theta(y, x)) = \psi \circ d_\theta(y, x),$$

for all $x, y \in X$.

$(d3)$ Let $x, y, z \in X$. If $x = y$, $x = z$ or $y = z$, then the following inequality is satisfied:

$$\psi \circ d_\theta(x, y) \leqslant \psi \circ d_\theta(x, z) + \psi \circ d_\theta(z, y).$$

Assume that $x \neq y$, $x \neq z$ and $y \neq z$. Then $d_\theta(x, y)$, $d_\theta(x, z)$ and $d_\theta(y, z)$ are positive. From (3.4), we obtain

$$\psi \circ d_\theta(x, y), \psi \circ d_\theta(x, z), \psi \circ d_\theta(z, y) \in [a, 2a].$$

Then, we get

$$\psi \circ d_\theta(x, y) \leqslant 2a = a + a \leqslant \psi \circ d_\theta(x, z) + \psi \circ d_\theta(z, y)$$

and so $\psi \circ d_\theta$ is a metric, that is, $\psi \in \mathcal{E}_b\mathcal{M}$.

Let $\psi \in \mathcal{E}_b\mathcal{M}$. From Corollary 3.1, we have $\psi \in \mathcal{M}$ and so since ψ is a metric preserving function, by Lemma 2.1, ψ is amenable. To show that ψ is tightly bounded, assume that d is a usual metric on $\mathbb{R}$ and

$$d_\theta = \varsigma \circ d,$$

where

$$\varsigma(x) = x^n \ \ (n > 1),$$

for all $x \in [0, \infty)$. From Theorem 3.1, we say that

$$d_\theta(x, y) = |x - y|^n$$

is an extended b-metric. Since $\psi \in \mathcal{E}_b\mathcal{M}$, then $\psi \circ d_\theta$ is a metric. Using the similar approaches given in Theorem 24 in [**13**], it can be easily proved that ψ is tightly bounded. □

THEOREM 3.9. *Let (X, d) be a metric space, (X, d_θ) be an extended b-metric space with the bounded function θ and $\psi : [0, \infty) \to [0, \infty)$. Then ψ is a metric-extended b-metric preserving function if and only if ψ is an extended b-metric preserving function, that is,*

$$\mathcal{M}\mathcal{E}_b = \mathcal{E}_b.$$

PROOF. From Lemma 3.1, we have

$$\mathcal{E}_b \subseteq \mathcal{M}\mathcal{E}_b.$$

Now, we show

$$\mathcal{M}\mathcal{E}_b \subseteq \mathcal{E}_b.$$

Let $\psi \in \mathcal{M}\mathcal{E}_b$. We prove that $\psi \in \mathcal{E}_b$, that is, $\psi \circ d_\theta$ is an extended b-metric. By Theorem 3.6, ψ is amenable and quasi-subadditive. Then, we can easily see that $\psi \circ d_\theta$ satisfies the conditions of an extended b-metric as follows:

($d_\theta 1$) Since ψ is amenable, for all $x, y \in X$, we have

$$\psi \circ d_\theta(x, y) = 0 \Longleftrightarrow \psi(d_\theta(x, y)) = 0 \Longleftrightarrow d_\theta(x, y) = 0 \Longleftrightarrow x = y.$$

($d_\theta 2$) For all $x, y \in X$, we get

$$\psi \circ d_\theta(x, y) = \psi(d_\theta(x, y)) = \psi(d_\theta(y, x)) = \psi \circ d_\theta(y, x).$$

($d_\theta 3$) Since ψ is quasi-subadditive, there exists $s \geqslant 1$ such that

$$\psi(u + v) \leqslant s[\psi(u) + \psi(v)], \tag{3.5}$$

for all $u, v \in [0, \infty)$. Since d_θ is an extended b-metric, by hypothesis, there exists a bounded function θ_1 such that

$$d_\theta(x, y) \leqslant \theta_1(x, y)[d_\theta(x, z) + d_\theta(z, y)],$$

for all $x, y, z \in X$. Let

$$n_1 = \sup\{\theta_1(x, y) : x, y \in X\}.$$

Then, there exists $m \in \mathbb{N}$ such that $m > n_1$ and

$$d_\theta(x, y) \leqslant m[d_\theta(x, z) + d_\theta(z, y)]. \tag{3.6}$$

Since $\psi \in \mathcal{ME}_b$, by Theorem 3.2 and hypothesis, there exists a bounded function θ_2 such that

$$(\psi(u), \psi(v), \psi(w)) \in \Delta_{\theta_2}$$

for each $(u, v, w) \in \Delta$. Let

$$n_2 = \sup\{\theta_2(x, y) : x, y \in X\}$$

and

$$n = 2n_2 m s^m.$$

Assume that

$$u = d_\theta(x, y), \ v = d_\theta(x, z) \text{ and } w = d_\theta(z, y),$$

for all $u, v, w \in X$. From the inequality (3.6), we have

$$u \leqslant mv + mw$$

and so we get

$$(u, mv + mw, mv + mw) \in \Delta.$$

By Theorem 3.2, we find

$$(\psi(u), \psi(mv + mw), \psi(mv + mw)) \in \Delta_{\theta_2}$$

and we obtain

$$\begin{aligned} \psi(u) &= \psi \circ d_\theta(x, y) \leqslant \theta_2(x, y)[\psi(mv + mw) + \psi(mv + mw)] \\ &= 2\theta_2(x, y)\psi(m(v + w)) \\ &\leqslant 2n_2\psi(m(v + w)). \end{aligned} \tag{3.7}$$

Now, we show that

$$\psi(\alpha x) = \alpha s^{\alpha-1}\psi(x), \tag{3.8}$$

for all $x \in [0, \infty)$ and $\alpha \in \mathbb{N}$. To do this, we apply mathematical induction on α. For $\alpha = 1$, it is clear that the equality (3.8) is satisfied. Let $\alpha \geqslant 1$ and assume that the equality (3.8) is satisfied for α. Since $\alpha \geqslant 1$, we have

$$\alpha s^{\alpha-1} + 1 \leqslant (\alpha + 1) s^{\alpha-1}.$$

By the inequality (3.5) and the induction hypothesis, we obtain

$$\begin{aligned} \psi((\alpha + 1) x) &\leqslant s[\psi(\alpha x) + \psi(x)] \leqslant s\left[\alpha s^{\alpha-1}\psi(x) + \psi(x)\right] \\ &= s\left(\alpha s^{\alpha-1} + 1\right)\psi(x) \leqslant s(\alpha + 1) s^{\alpha-1}\psi(x) \\ &= (\alpha + 1) s^\alpha \psi(x). \end{aligned}$$

Hence, the inequality (3.8) is satisfied. By (3.5), (3.7) and (3.8), we get

$$\begin{aligned} \psi \circ d_\theta(x, y) &\leqslant 2n_2\psi(m(v + w)) \leqslant 2n_2 m s^{m-1}\psi(v + w) \\ &\leqslant 2n_2 m s^{m-1} s[\psi(v) + \psi(w)] \\ &= 2n_2 m s^m[\psi(v) + \psi(w)] \\ &= n[\psi \circ d_\theta(x, z) + \psi \circ d_\theta(z, y)]. \end{aligned}$$

If we take $\theta(x, y) = n$ for all $x, y \in X$, then the condition $(d_\theta 3)$ is satisfied.

Consequently, $\psi \circ d_\theta$ is an extended b-metric on X. □

THEOREM 3.10. *Let (X, d_s) be a b-metric space, (X, d_θ) be an extended b-metric space with the bounded function θ and $\psi : [0, \infty) \to [0, \infty)$. Then ψ is a b-metric-extended b-metric preserving function if and only if ψ is an extended b-metric preserving function, that is,*

$$\mathcal{BE}_b = \mathcal{E}_b.$$

PROOF. We can demonstrate this readily by applying the same reasoning that were used in the Theorem 3.9 proof. □

4. Gluing lemmas for extended b-metric preserving functions

In this section, we prove two gluing lemmas for extended b-metric preserving functions.

THEOREM 4.1. (*A gluing lemma for functions in $\mathcal{E}_b$ and $\mathcal{ME}_b$*) *Let (X, d) be a metric space, (X, d_θ) be an extended b-metric space with the bounded function θ and $\psi : [0, \infty) \to [0, \infty)$. Suppose that $\psi_1, \psi_2 \in \mathcal{E}_b$, $\varepsilon > 0$ and $\psi_1(\varepsilon) = \psi_2(\varepsilon)$. Let us define the function $\psi : [0, \infty) \to [0, \infty)$ as*

$$\psi(x) = \begin{cases} \psi_1(x) & ; \quad x \in [0, \varepsilon) \\ \psi_2(x) & ; \quad x \in [\varepsilon, \infty) \end{cases},$$

for all $x \in [0, \infty)$. Assume that ψ_1 is a concave and increasing function such that

$$|x - y| \leqslant \varepsilon \Longrightarrow |\psi_2(x) - \psi_2(y)| \leqslant \psi_1(|x - y|),$$

for all $x, y \in [\varepsilon, \infty)$. Then

$$\psi \in \mathcal{E}_b.$$

PROOF. Since $\psi_1, \psi_2 \in \mathcal{E}_b$, by Theorems 3.2 and 3.9, there exist the bounded functions $\theta_1, \theta_2 : X \times X \to [1, \infty)$ such that

$$(\psi_1(a), \psi_1(b), \psi_1(c)) \in \Delta_{\theta_1}$$

and

$$(\psi_2(a), \psi_2(b), \psi_2(c)) \in \Delta_{\theta_2},$$

for all $(a, b, c) \in \Delta$. Let $(a, b, c) \in \Delta$ and the function $\theta : X \times X \to [1, \infty)$ be defined as

$$\theta(x, y) = \max\{\theta_1(x, y), \theta_2(x, y)\},$$

for all $x, y \in X$. Keeping generality intact, suppose

$$0 \leqslant a \leqslant b \leqslant c \leqslant a + b. \tag{4.1}$$

If $a, b, c \in [0, \varepsilon)$, then we get

$$(\psi(a), \psi(b), \psi(c)) = (\psi_1(a), \psi_1(b), \psi_1(c)) \in \Delta_{\theta_1} \subseteq \Delta_\theta.$$

If $a, b, c \in [\varepsilon, \infty)$, then we obtain

$$(\psi(a), \psi(b), \psi(c)) = (\psi_2(a), \psi_2(b), \psi_2(c)) \in \Delta_{\theta_2} \subseteq \Delta_\theta.$$

Consider the scenarios in which a, b and c do not fall inside the same interval.

Since if $c \in [0, \varepsilon)$ then by (4.1), we have $a, b \in [0, \varepsilon)$, we only investigate the following cases:

Case 1: Let $a, b \in [0, \varepsilon)$ and $c \in [\varepsilon, \infty)$. Then we have

$$\psi(a) = \psi_1(a) \leqslant \psi_1(b) = \psi(b) \leqslant \psi(b) + \psi(c) \leqslant \theta(x, y)(\psi(b) + \psi(c)). \tag{4.2}$$

Since

$$|\varepsilon - c| = c - \varepsilon \leqslant a + b - \varepsilon < \varepsilon + \varepsilon - \varepsilon,$$

we get

$$|\psi_1(\varepsilon) - \psi_2(c)| = |\psi_2(\varepsilon) - \psi_2(c)| \leqslant \psi_1(|\varepsilon - c|) = \psi_1(c - \varepsilon).$$

Then we find

$$-\psi_1(c - \varepsilon) \leqslant \psi_1(\varepsilon) - \psi_2(c) \leqslant \psi_1(c - \varepsilon) \tag{4.3}$$

and

$$\psi_1(\varepsilon) - \psi_1(c - \varepsilon) \leqslant \psi_2(c).$$

Since

$$c \leqslant a + b,$$

we have

$$c - \varepsilon \leqslant a + b - \varepsilon \leqslant a.$$

Using the increasing property of ψ_1, we get

$$\psi_1(c - \varepsilon) \leqslant \psi_1(a)$$

and so

$$\begin{aligned} \psi(b) &= \psi_1(b) \leqslant \psi_1(\varepsilon) \leqslant \psi_1(\varepsilon) + \psi_1(a) - \psi_1(c - \varepsilon) \\ &= [\psi_1(\varepsilon) - \psi_1(c - \varepsilon)] + \psi_1(a) \\ &\leqslant \psi_2(c) + \psi_1(a) = \psi(c) + \psi(a) \\ &\leqslant \theta(x, y)[\psi(c) + \psi(a)]. \end{aligned} \tag{4.4}$$

For $t = \varepsilon$, $x = a + b - \varepsilon$, $y = a$ and $z = b$, since ψ_1 is concave, we get

$$\psi_1(a + b - \varepsilon) + \psi_1(\varepsilon) \leqslant \psi_1(a) + \psi_1(b).$$

By (4.3), we obtain

$$\psi_2(c) \leqslant \psi_1(\varepsilon) + \psi_1(c - \varepsilon)$$

and

$$\begin{aligned} \psi(c) &= \psi_2(c) \leqslant \psi_1(\varepsilon) + \psi_1(c - \varepsilon) \leqslant \psi_1(\varepsilon) + \psi_1(a + b - \varepsilon) \\ &\leqslant \psi_1(a) + \psi_1(b) = \psi(a) + \psi(b) \\ &\leqslant \theta(x, y)[\psi(a) + \psi(b)]. \end{aligned} \tag{4.5}$$

Using (4.2), (4.4) and (4.5), we have

$$(\psi(a), \psi(b), \psi(c)) \in \Delta_\theta.$$

Case 2: Let $a \in [0, \varepsilon)$ and $b, c \in [\varepsilon, \infty)$. Since

$$(\psi_2(\varepsilon), \psi_2(b), \psi_2(c)) \in \Delta_{\theta_2},$$

we get

$$\varepsilon \leqslant b + c,\ b \leqslant c \leqslant c + \varepsilon,\ c \leqslant a + b \leqslant \varepsilon + b$$

and so

$$(\varepsilon, b, c) \in \Delta.$$

Then we have

$$\begin{aligned} \psi(a) &= \psi_1(a) \leqslant \psi_1(\varepsilon) = \psi_2(\varepsilon) \leqslant \theta_2(x,y)[\psi_2(b) + \psi_2(c)] \\ &\leqslant \theta(x,y)[\psi_2(b) + \psi_2(c)] = \theta(x,y)[\psi(b) + \psi(c)]. \end{aligned} \tag{4.6}$$

Since

$$|b - c| = c - b \leqslant \varepsilon,$$

we find

$$|\psi_2(b) - \psi_2(c)| \leqslant \psi_1(|b - c|) = \psi_1(c - b),$$
$$-\psi_1(c - b) \leqslant \psi_2(b) - \psi_2(c) \leqslant \psi_1(c - b),$$

$$\begin{aligned} \psi(b) &= \psi_2(b) \leqslant \psi_1(c - b) + \psi_2(c) \leqslant \psi_1(a) + \psi_2(c) \\ &= \psi(a) + \psi(c) \leqslant \theta(x,y)[\psi(a) + \psi(c)] \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} \psi(c) &= \psi_2(c) \leqslant \psi_1(c - b) + \psi_2(b) \leqslant \psi_1(a) + \psi_2(b) \\ &= \psi(a) + \psi(b) \leqslant \theta(x,y)[\psi(a) + \psi(b)]. \end{aligned} \tag{4.8}$$

By (4.6), (4.7) and (4.8), we get

$$(\psi(a), \psi(b), \psi(c)) \in \Delta_\theta.$$

Under the all cases, we say

$$(\psi(a), \psi(b), \psi(c)) \in \Delta_\theta.$$

Consequently, by Theorems 3.2 and 3.9, we see that

$$\psi \in \mathcal{E}_b.$$

□

THEOREM 4.2. *(A gluing lemma for functions in $\mathcal{E}_b\mathcal{M}$) Let (X, d) be a metric space, (X, d_θ) be an extended b-metric space with the bounded function θ and $\psi : [0, \infty) \to [0, \infty)$. Suppose that $\psi_1, \psi_2 \in \mathcal{E}_b\mathcal{M}$, $\varepsilon > 0$ and $\psi_1(\varepsilon) = \psi_2(\varepsilon)$. Let us define the function $\psi : [0, \infty) \to [0, \infty)$ as*

$$\psi(x) = \begin{cases} \psi_1(x) & ; \quad x \in [0, \varepsilon) \\ \psi_2(x) & ; \quad x \in [\varepsilon, \infty) \end{cases},$$

for all $x \in [0, \infty)$. Let

$$A = \sup_{x \in (0,\infty)} \psi(x)$$

and

$$B = \inf_{x \in (0,\infty)} \psi(x).$$

Then we have

$$A = \max\left\{ \sup_{x \in (0,r)} \psi_1(x), \sup_{x \in [r,\infty)} \psi_2(x) \right\}, \tag{4.9}$$

$$B=\min\left\{\inf_{x\in(0,r)}\psi_1(x),\inf_{x\in[r,\infty)}\psi_2(x)\right\} \tag{4.10}$$

and the followings are equivalent:

(*i*) $\psi\in\mathcal{E}_b\mathcal{M}$,

(*ii*) $A\leqslant 2B$,

(*iii*) $\sup_{x\in(0,r)}\psi_1(x)\leqslant 2\inf_{x\in[r,\infty)}\psi_2(x)$ *and* $\sup_{x\in[r,\infty)}\psi_2(x)\leqslant 2\inf_{x\in(0,r)}\psi_1(x)$.

PROOF. Using Theorem 3.8, $\sup_{x\in(0,r)}\psi_1(x)$, $\sup_{x\in[r,\infty)}\psi_2(x)$, $\inf_{x\in(0,r)}\psi_1(x)$ and $\inf_{x\in[r,\infty)}\psi_2(x)$ exist. The satisfaction of conditions (4.9) and (4.10) is evident.

By Theorem 3.8, we say that ψ is tightly bounded and amenable. So, there exists a $\upsilon>0$ such that

$$\upsilon\leqslant\psi(x)\leqslant 2\upsilon.$$

Then we get

$$\upsilon\leqslant B\leqslant A\leqslant 2\upsilon.$$

Hence

$$2B\geqslant 2\upsilon\geqslant A$$

and the condition (*ii*) is satisfied. On the contrary, we suppose that the second requirement is satisfied. For all $x\in(0,\infty)$, we obtain

$$B=\inf_{x\in(0,\infty)}\psi(x)\leqslant\psi(x)\leqslant\sup_{x\in(0,\infty)}\psi(x)=A+2B.$$

Then we see that ψ is tightly bounded. By Theorem 3.8, since $\psi_1,\psi_2\in\mathcal{E}_b\mathcal{M}$, the functions ψ_1 and ψ_2 are amenable. So ψ is amenable. Again using Theorem 3.8, we say $\psi\in\mathcal{E}_b\mathcal{M}$. Then the conditions (*i*) and (*ii*) are equivalent.

We now assume that the second condition (*ii*) holds true. We obtain

$$\begin{aligned}\sup_{x\in(0,r)}\psi_1(x) &\leqslant \max\left\{\sup_{x\in(0,r)}\psi_1(x),\sup_{x\in[r,\infty)}\psi_2(x)\right\}\\ &= A\leqslant 2B\\ &= 2\min\left\{\inf_{x\in(0,r)}\psi_1(x),\inf_{x\in[r,\infty)}\psi_2(x)\right\}\\ &\leqslant 2\inf_{x\in[r,\infty)}\psi_2(x)\end{aligned}$$

and

$$\sup_{x\in[r,\infty)}\psi_2(x)\leqslant A\leqslant 2B\leqslant 2\inf_{x\in(0,r)}\psi_1(x).$$

Thus, the requirement (*iii*) is met. Now, we consider the following cases for the converse statement:

Case 1: We have

$$\sup_{x\in(0,r)}\psi_1(x)\geqslant\sup_{x\in[r,\infty)}\psi_2(x)$$

and so

$$A=\sup_{x\in(0,r)}\psi_1(x).$$

Since $\psi_1 \in \mathcal{E}_b\mathcal{M}$, utilizing the comparable justifications provided in the proof of $(i) \Longrightarrow (ii)$, we get

$$\sup_{x\in(0,r)} \psi_1(x) \leqslant 2 \inf_{x\in(0,r)} \psi_1(x)$$

and by (iii), we have

$$\sup_{x\in(0,r)} \psi_1(x) \leqslant 2 \inf_{x\in[r,\infty)} \psi_2(x).$$

Therefore, we acquire

$$\begin{aligned} A &\leqslant \min\left\{2\inf_{x\in(0,r)} \psi_1(x), 2\inf_{x\in[r,\infty)} \psi_2(x)\right\} \\ &= 2\min\left\{\inf_{x\in(0,r)} \psi_1(x), \inf_{x\in[r,\infty)} \psi_2(x)\right\} = 2B. \end{aligned}$$

Case 2: Here, we have

$$\sup_{x\in(0,r)} \psi_1(x) < \sup_{x\in[r,\infty)} \psi_2(x)$$

and

$$A = \sup_{x\in[r,\infty)} \psi_2(x).$$

Since $\psi_2 \in \mathcal{E}_b\mathcal{M}$, we obtain

$$\sup_{x\in[r,\infty)} \psi_2(x) \leqslant 2 \inf_{x\in[r,\infty)} \psi_2(x).$$

From (iii), we get

$$\sup_{x\in[r,\infty)} \psi_2(x) \leqslant 2 \inf_{x\in(0,r)} \psi_1(x).$$

Under the all cases, we show $A \leqslant 2B$. □

5. Acknowledgment

This work is supported by the Scientific Research Projects Unit of Balıkesir University under the project number 2023/031.

References

[1] I. A. Bakhtin, The contraction mapping principle in almost metric spaces. *Functional Analysis*, **30** (1989), 26-37.

[2] V. Berinde, Generalized contractions in quasi metric spaces. *In Seminar on Fixed Point Theory; Babes-Bolyai University: Cluj-Napoca, Romania*, (1993), 3-9.

[3] J. Borsik and J. Dobos, Functions whose composition with every metric is a metric. *Mathematica Slovaca*, **31** (1981), 3-12.

[4] J. Borsik and J. Dobos, On metric preserving functions. *Real Analysis Exchange*, **13** (1987-88), 285-293.

[5] P. Corazza, Introduction to metric-preserving functions. *The American Mathematical Monthly*, **106** (4) (1999), 309-323.

[6] S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces. *Atti del Seminario Matematico e Fisico dell'Universita di Modena e Reggio Emilia*, **46** (1998), 263-276.

[7] M. Dhanraj, A. J. Gnanaprakasam, G. Mani, Ö. Ege, and M. De la Sen, Solution to integral equation in an O-Complete branciari b-metric spaces. *Axioms,* **11** (2022) 728.

[8] J. Dobos, Metric preserving functions. *Online Lecture Notes* available at http://web.science.upjs.sk/jozefdobos/wp-content/uploads/2012/03/mpf1.pdf

[9] A. J. Gnanaprakasam, S. K. Prakasam, G. Mani, and Ö. Ege, Fixed point results via O-F_{ϱ} contraction and applications to Fredholm and integro-differential equations. *Filomat*, **38** (29) (2024), 10323-10344.

[10] J. Heinonen, Lectures on analysis on metric spaces. *Springer: Berlin, Germany*, (2001).

[11] H. Huang, Y. M. Singh, M. S. Khan, and S. Radenovic, Rational type contractions in extended b-metric spaces. *Symmetry* **13** (2021), 614.

[12] T. Kamran, M. Samreen, and Q. U. Ain, A generalization of b-metric space and some fixed point theorems. *Mathematics* **5** (2017), 19.

[13] T. Khemaratchatakumthorn and P. Pongsriiam, Remarks on b-metric and metric-preserving functions. *Mathematica Slovaca,* **68** (5) (2018), 1009-1016.

[14] T. Khemaratchatakumthorn, P. Pongsriiam, and S. Samphavat, Further remarks on b-metrics, metric-preserving functions, and other related metrics. *International Journal of Mathematics and Computer Science*, **14** (2) (2019), 473-480.

[15] T. Khemaratchatakumthorn and D. Siriwan, Pasting lemmas for b-metric preserving and related functions. *International Journal of Mathematics and Computer Science*, **16** (4) (2021), 1591-1598.

[16] G. Mani, A. J. Gnanaprakasam, Ö. Ege, N. Fatima, and N. Mlaiki, Solution of Fredholm integral equation via common fixed point theorem on bicomplex valued b-metric space. *Symmetry*, **15** (2) (2023), 297.

[17] R. Martinez-Cruz and E. Hernandez-Pina, Extended b-metric-preserving functions and other related metrics. *Publicacion Semestral Padi,* **9** (18) (2022), 47-55.

[18] A. Petruşel, I. A. Rus, and M. A. Şerban, The role of equivalent metrics in fixed point theory. *Topological Methods in Nonlinear Analysis*, **41** (2013), 85-112.

[19] S. K. Prakasam, A. J. Gnanaprakasam, Ö. Ege, G. Mani, S. Haque, and N. Mlaiki, Fixed point for an $\mathbb{O}g\mathfrak{F}$-**c** in $\mathbb{O}$-complete $\mathfrak{b}$-metric-like spaces. *AIMS Mathematics*, **8** (1) (2023), 1022-1039.

[20] P. Pongsriiam and I. Termwuttipong, On metric-preserving functions and fixed point theorems. *Fixed Point Theory and Applications*, **179** (2014), 14.

[21] W. Shatanawi, K. Abodayeh, and A. Mukheimer, Some fixed point theorems in extended b-metric spaces. *UPB Scientific Bulletin, Series A: Applied Mathematics and Physics*, **80** (4) (2018), 71-78.

[22] W. A. Wilson, On certain types of continuous transformations of metric spaces. *American Journal of Mathematics*, **57** (1935), 62-68.

Received by editors 7.2.2025; Revised version 5.3.2025; Available online 31.5.2025.

Nihal TAŞ, Balikesir University, Department of Mathematics, 10145 Balikesir, Turkey

Email address: `nihaltas@balikesir.edu.tr`

Ayşenur ŞEN, Balikesir University, Department of Mathematics, 10145 Balikesir, Turkey

Email address: `ayseenursenn@gmail.com`