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INVERSE PROBLEMS FOR DIRAC EQUATIONS WITH A FINITE NUMBER OF TRANSMISSION CONDITIONS DEPENDING HERGLOTZ-NEVANLINNA TYPE FUNCTIONS

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Abstract. In this study, direct and inverse spectral problem is studied for the system of Dirac equations with rational function of Herglotz-Nevanlinna in boundary and transmission conditions. We give some spectral properties of the problem and also show that the coefficients of the problem are uniquely determined by the Weyl function and classical spectral data consisting of the sequence of two different eigenvalues.

1. Introduction

In this work we consider the system of Dirac equation

$$\ell y(x) := By'(x) + \Omega(x)y(x) = \lambda y(x),\ x \in [a,b] \tag{1.1}$$

where $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & r(x) \end{pmatrix}$, $y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$, real valued functions $p(x)$, $q(x)$ and $r(x)$ are considered within the space $L_2(a,b)$, and λ is identified as the spectral parameter.

We denote by L the boundary value problem generated by equation (1.1) with the following boundary and discontinuity conditions

$$U(y) := y_2(a) + f_1(\lambda)y_1(a) = 0 \tag{1.2}$$

$$V(y) := y_2(b) + f_2(\lambda)y_1(b) = 0 \tag{1.3}$$

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$$\left\{\begin{array}{l} y_1(\omega_i+0)-\alpha_i y_1(\omega_i-0)=0 \\ y_2(\omega_i+0)-\alpha_i^{-1}y_2(\omega_i-0)-h_i(\lambda)\,y_1(\omega_i-0)=0 \end{array}\right. \tag{1.4}$$

where $f_1(\lambda)$, $f_2(\lambda)$ and $h_i(\lambda)$ $(i=1,2,\ldots,n)$ are Herglotz-Nevanlinna type functions such that

$$f_j(\lambda)=a_j\lambda+b_j-\sum_{k=1}^{N_j}\frac{f_{jk}}{\lambda-g_{jk}},\ \ (j=1,2) \tag{1.5}$$

$$h_i(\lambda)=m_i\lambda+n_i-\sum_{k=1}^{p_i}\frac{u_{ik}}{\lambda-v_{ik}},\ \ (i=1,2,\ldots,n) \tag{1.6}$$

a_j, b_j, f_{jk}, g_{ik}, m_i, n_i, u_{ik} and v_{ik} are real numbers, $a_1<0$, $f_{1k}<0$, $a_2>0$, $f_{2k}>0$, $g_{j1}<g_{j2}<\ldots<g_{jN_j}$, $m_i>0$, $u_{ik}>0$, , $v_{i1}<v_{i2}<\ldots<v_{ip_i}$, $\alpha_i\in R^+$, $a<w_1<w_2<\cdots<w_n<b$. In special case, when $f_j(\lambda)=\infty$, conditions (1.2) and (1.3) turn to Dirichlet conditions $y_1(a)=y_1(b)=0$ respectively. Moreover, when $h_i(\lambda)=\infty$, conditions (1.4) turn to $y_1(w_i+0)=y_1(w_i-0)=0$, $i=1,2,\ldots,n$.

$R_1(\lambda)y_1(a)+R_2(\lambda)y_2(a)=0$ is a boundary condition depending spectral parameter where $R_1(\lambda)$ and $R_2(\lambda)$ are polynomials(In the specific case where $f_j(\lambda)=\infty$, conditions (1.2) and (1.3) reduce to Dirichlet boundary conditions $y_1(a)=y_1(b)=0$ respectively. Similarly, if $h_i(\lambda)=\infty$, the conditions (1.4) transform into $y_1(w_i+0)=y_1(w_i-0)=0$, $i=1,2,\ldots,n$. The boundary condition $R_1(\lambda)y_1(a)+R_2(\lambda)y_2(a)=0$ depends on the spectral parameter, where $R_1(\lambda)$ and $R_2(\lambda)$ are polynomials. If the degrees of $R_1(\lambda)$ and $R_2(\lambda)$ are both 1, then this condition is linearly dependent on the spectral parameter. Conversely, analyzing higher-degree polynomials $R_1(\lambda)$ and $R_2(\lambda)$ presents more difficulties. When $\frac{R_1(\lambda)}{R_2(\lambda)}$ is a rational function of Herglotz–Nevanlinna type, with $f(\lambda)=a\lambda+b-\sum_{k=1}^{N}\frac{f_k}{\lambda-g_k}$ in boundary conditions, both direct and inverse problems for the Sturm–Liouville operator have been explored. This paper addresses both the direct and inverse spectral problems for systems of Dirac equations, focusing on cases where the boundary and transmission conditions involve a rational function of Herglotz–Nevanlinna.

Eigenvalue-dependent boundary conditions are a common feature in spectral problems encountered across various application areas and mathematical contexts. In 1973, Walter [**22**] initially focused on an expansion theorem related to this type of eigenvalue problem. In 1977, Fulton explored the Sturm–Liouville eigenvalue problem as well. Various papers have discussed inverse problems for certain differential operators that are linearly dependent on eigenvalues (see [**1**, **2**, **8**, **12**, **14**]). A wider variety of boundary conditions is presented in references [**3**–**5**, **9**, **10**, **15**, **18**–**21**]. On another note, when $f(\lambda)$ takes the form of a rational function of Herglotz–Nevanlinna type, the direct and inverse spectral problems for the Sturm–Liouville operator were analyzed in [**6**, **7**, **16**, **17**]. In the work referenced as [**13**], the direct and inverse spectral problems for the Dirac operator have been analyzed in the context of $f(\lambda)$ being a rational function of Herglotz–Nevanlinna type.

This study aims to present uniqueness theorems for the Dirac problem outlined above, specifically when dealing with eigenvalue-dependent rational functions of the Herglotz–Nevanlinna type, relevant to both boundary conditions and a finite number of transmission conditions. We focus on the inverse problem related to the reconstruction of the discussed boundary value problem by utilizing the Weyl function and the spectral data $\{\lambda_n, \mu_n\}_{n\in\mathbb{Z}}$. Despite the fact that the boundary and transmission conditions are not linearly dependent on the spectral parameter, this situation allows for the eigenvalues to be real and normalizing numbers to be defined.

2. Preliminaries

Let's consider the space
$H := L_2(a,b) \oplus L_2(a,b) \oplus \mathbb{C}^{N_1+1} \oplus \mathbb{C}^{N_2+1} \oplus \mathbb{C}^{p_1+1} \oplus \mathbb{C}^{p_2+1} \oplus \ldots \oplus \mathbb{C}^{p_n+1}$ and element Y in H is in the form $Y = (y(x), \upsilon, \tau, w_1, w_2, \ldots, w_n) \in H$, where $y(x) = (y_1(x), y_2(x))$, $\upsilon = (Y_1, Y_2, \ldots, Y_{N_1+1})$, $\tau = (W_1, W_2, \ldots, W_{N_2+1})$, $w_i = \left(R_1^{(i)}, R_2^{(i)}, \ldots, R_{p_i+1}^{(i)}\right)$, $(i = 1, 2, \ldots, n)$. H is a Hilbert space with the inner product defined by

$$\begin{aligned}\langle Y, Z\rangle &:= \int_a^b (y_1\bar{z}_1 + y_2\bar{z}_2)\,dx - \frac{Y_{N_1+1}\bar{Z}_{N_1+1}}{a_1} + \frac{W_{N_2+1}\bar{W}_{N_2+1}}{a_2} \\ &+ \sum_{i=1}^{n} \frac{\alpha_i}{m_i} R_{p_i+1}^{(i)} \bar{R}_{p_i+1}^{(i)} - \sum_{k=1}^{N_1} Y_k \bar{Z}_k \left(\frac{1}{f_{1k}}\right) \\ &+ \sum_{k=1}^{N_2} W_k \bar{W}_k \left(\frac{1}{f_{2k}}\right) + \sum_{i=1}^{n}\sum_{k=1}^{p_i} R_k^{(i)} \bar{R}_k^{(i)} \frac{\alpha_i}{u_{ik}}\end{aligned} \tag{2.1}$$

for $Y = (y(x), \upsilon, \tau, w_1, w_2, \ldots, w_n)$ and $Z = (z(x), \upsilon', \tau', w_1', w_2', \ldots, w_n')$.

Define an operator T with the domain

$D(T) = \{Y \in H : y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} \in AC\,[a,b]\,,\; ly \in L_2(a,b),$
$y_1(\omega_i + 0) - \alpha_i y_1(\omega_i - 0) = 0,\ (i = 1, 2, \ldots, n)\}$
such that

$$TY := (ly, T\upsilon, T\tau, Tw_1, Tw_2, \ldots, Tw_n) \tag{2.2}$$

where

$$T\upsilon = TY_i = \begin{cases} g_{1i}Y_i - f_{1i}y_1(a)\,, & i = 1, 2, \ldots, N_1 \\ y_2(a) + b_1 y_1(a) + \sum\limits_{k=1}^{N_1} Y_k, & i = N_1 + 1 \end{cases} \tag{2.3}$$

$$T\tau = TW_i = \begin{cases} g_{2i}W_i - f_{2i}y_1(b)\,, & i = 1, 2, \ldots, N_2 \\ y_2(b) + b_2 y_1(b) + \sum\limits_{k=1}^{N_2} W_k, & i = N_2 + 1 \end{cases} \tag{2.4}$$

$$Tw_1 = TR_k^{(1)} = \begin{cases} v_{1k}R_k^{(1)} - u_{1k}y_1\left(\omega_1^-\right), & k = 1, 2, \dots, p_1 \\ -y_2(\omega_1^+) + \alpha_1^{-1}y_2(\omega_1^-) + n_1y_1\left(\omega_1^-\right) + \sum\limits_{i=1}^{p_1} R_i^{(1)}, & k = p_1 + 1 \end{cases} \tag{2.5}$$

$$Tw_2 = TR_k^{(2)} = \begin{cases} v_{2k}R_k^{(2)} - u_{2k}y_1\left(\omega_2^-\right), & k = 1, 2, \dots, p_2 \\ -y_2(\omega_2^+) + \alpha_2^{-1}y_2(\omega_2^-) + n_2y_1\left(\omega_2^-\right) + \sum\limits_{i=1}^{p_2} R_i^{(2)}, & k = p_2 + 1 \end{cases} \tag{2.6}$$

$\vdots$

$$Tw_n = TR_k^{(n)} = \begin{cases} v_{nk}R_k^{(n)} - u_{nk}y_1\left(\omega_n^-\right), & k = 1, 2, \dots, p_n \\ -y_2(\omega_n^+) + \alpha_n^{-1}y_2(\omega_n^-) + n_ny_1\left(\omega_n^-\right) + \sum\limits_{i=1}^{p_n} R_i^{(n)}, & k = p_n + 1 \end{cases} \tag{2.7}$$

Accordingly, equality $TY = \lambda Y$ corresponds to problem $(1.1) - (1.4)$ under the domain $D(T) \subset H$. The following theorem could be proved by using definition of T.

Theorem 2.1. *The eigenvalue problem of operator T is adequate problem of* $(1.1) - (1.4)$*, i.e., eigenvalues of operator T and problem* $(1.1) - (1.4)$ *coincide.*

Proof. Assume that λ is an eigenvalue of T and $Y = (y(x), \upsilon, \tau, w_1, w_2, \dots, w_n) \in H$ is the eigenvector corresponding to λ. Since $Y \in D(T)$, it is obvious that the condition $y_1(\omega_i + 0) - \alpha_i y_1(\omega_i - 0) = 0$ and equation (1.1) hold. On the other hand, boundary conditions $(1.2) - (1.3)$ and the second condition of (1.4) are satisfied by the following equalities;

$$T\upsilon = TY_i = g_{1i}Y_i - f_{1i}y_1(a) = \lambda Y_i, \;\; i = 1, 2, \dots, N_1,$$

$$TY_{N_1+1} = y_2(a) + b_1y_1(a) + \sum_{k=1}^{N_1} Y_k = -a_1y_1(a)\lambda,$$

$$T\tau = TW_i = g_{2i}W_i - f_{2i}y_1(b) = \lambda W_i, \;\; i = 1, 2, \dots, N_2,$$

$$TW_{N_2+1} = y_2(b) + b_2y_1(b) + \sum_{k=1}^{N_2} W_k = -a_2y_1(b)\lambda,$$

$$Tw_1 = TR_k^{(1)} = v_{1k}R_k^{(1)} - u_{1k}y_1\left(\omega_1^-\right), \;\; k = 1, 2, \dots, p_1,$$

$$TR_{p_1+1}^{(1)} = -y_2(\omega_1^+) + \alpha_1^{-1}y_2(\omega_1^-) + n_1y_1\left(\omega_1^-\right) + \sum_{k=1}^{p_1} R_k^{(1)} = -m_1y_1\left(\omega_1^-\right)\lambda,$$

$$Tw_2 = TR_k^{(2)} = v_{2k}R_k^{(2)} - u_{2k}y_1\left(\omega_2^-\right), \;\; k = 1, 2, \dots, p_2,$$

$$TR_{p_2+1}^{(2)} = -y_2(\omega_2^+) + \alpha_2^{-1}y_2(\omega_2^-) + n_2y_1\left(\omega_2^-\right) + \sum_{k=1}^{p_2} R_k^{(2)} = -m_2y_1\left(\omega_2^-\right)\lambda,$$

$\vdots$

$$Tw_N = TR_k^{(n)} = v_{nk}R_k^{(n)} - u_{nk}y_1\left(\omega_n^-\right), \ \ k = 1, 2, \dots, p_n,$$

$$TR_{p_n+1}^{(n)} = -y_2(\omega_n^+) + \alpha_n^{-1}y_2(\omega_n^-) + n_n y_1\left(\omega_n^-\right) + \sum_{k=1}^{p_n} R_k^{(n)} = -m_n y_1\left(\omega_n^-\right)\lambda.$$

If $\lambda = g_{jk}$ ($j = 1, 2$ and $k = 1, 2, \dots, N_j$) are eigenvalues of operator T, then, from above equalities and the domain of T, equalities (1.1), $y_1(a, g_{1k}) = 0$, $y_1(b, g_{2k}) = 0$ and (1.4) are satisfied.

Moreover, if $\lambda = v_{ik}$ ($i = 1, 2$ and $k = 1, 2, \dots, p_i$) are eigenvalues of operator T, from above equalities and the domain of T, equations (1.1)$-$(1.3) and $y_1(\omega_i^-, v_{ik}) = 0 = y_1(\omega_i^+, v_{ik})$ are valid. In that case, λ is also an eigenvalue of L.

Conversely, let λ be an eigenvalue of L and $\begin{pmatrix} y_1\left(x,\lambda\right) \\ y_2\left(x,\lambda\right) \end{pmatrix}$ be an eigenfunction corresponding to λ. If $\lambda \neq g_{ik}$ and $\lambda \neq v_{ik}$ then, it is clear that λ is an eigenvalue of T and the vector

$$Y = \Big(y_1\left(x\right), y_2\left(x\right), \tfrac{f_{11}}{g_{11}-\lambda}y_1(a), \tfrac{f_{12}}{g_{12}-\lambda}y_1(a), \dots, \tfrac{f_{1N_1}}{g_{1N_1}-\lambda}y_1(a), -a_1y_1\left(a\right),$$
$$\tfrac{f_{21}}{g_{21}-\lambda}y_1(b), \tfrac{f_{22}}{g_{22}-\lambda}y_1(b), \dots, \tfrac{f_{2N_2}}{g_{2N_2}-\lambda}y_1(b), -a_2y_1\left(b\right),$$
$$\tfrac{u_{11}}{v_{11}-\lambda}y_1(\omega_1^-), \tfrac{u_{12}}{v_{12}-\lambda}y_1(\omega_1^-), \dots, \tfrac{u_{1p_1}}{v_{1p_1}-\lambda}y_1(\omega_1^-), -m_1y_1\left(\omega_1^-\right),$$
$$\tfrac{u_{21}}{v_{21}-\lambda}y_1(\omega_2^-), \tfrac{u_{22}}{v_{22}-\lambda}y_1(\omega_2^-), \dots, \tfrac{u_{2p_2}}{v_{2p_2}-\lambda}y_1(\omega_2^-), -m_2y_1\left(\omega_2^-\right), \dots,$$
$$\tfrac{u_{n1}}{v_{n1}-\lambda}y_1(\omega_n^-), \tfrac{u_{n2}}{v_{n2}-\lambda}y_1(\omega_n^-), \dots, \tfrac{u_{2p_n}}{v_{2p_n}-\lambda}y_1(\omega_n^-), -m_ny_1\left(\omega_n^-\right)\Big)$$

is the eigenvector corresponding to λ.

If $\lambda = g_{1k}$, then

$$Y = (y_1(x), y_2(x), Y_1, Y_2, \dots, Y_{N_1}, 0, W_1, W_2, \dots, W_{N_2}, W_{N_2+1},$$
$$R_1^{(1)}, R_2^{(1)}, \dots, R_{p_1+1}^{(1)}, R_1^{(2)}, R_2^{(2)}, \dots, R_{p_2+1}^{(2)}, \dots, R_1^{(n)}, R_2^{(n)}, \dots, R_{p_n+1}^{(n)}\Big),$$

$$Y_i = \begin{cases} 0, & i \neq k \\ -y_2(a), & i = k \end{cases}, \ i = 1, 2, \dots, N_1$$

is the eigenvector of T corresponding to g_{1k}.

If $\lambda = g_{2k}$, then

$$Y = (y_1(x), y_2(x), Y_1, Y_2, \dots, Y_{N_1}, Y_{N_1+1}, W_1, W_2, \dots, W_{N_2}, 0,$$
$$R_1^{(1)}, R_2^{(1)}, \dots, R_{p_1+1}^{(1)}, R_1^{(2)}, R_2^{(2)}, \dots, R_{p_2+1}^{(2)}, \dots, R_1^{(n)}, R_2^{(n)}, \dots, R_{p_n+1}^{(n)}\Big),$$

$$W_i = \begin{cases} 0, & i \neq k \\ -y_2(b), & i = k \end{cases}, \ i = 1, 2, \dots, N_2$$

is the eigenvector of T corresponding to g_{2k}.

Furthermore, if $\lambda = v_{1k}$, then

$$Y = (y_1(x), y_2(x), Y_1, Y_2, \dots, Y_{N_1}, Y_{N_1+1}, W_1, W_2, \dots, W_{N_2}, W_{N_2+1},$$
$$R_1^{(1)}, R_2^{(1)}, \dots, R_{p_1}^{(1)}, 0, R_1^{(2)}, R_2^{(2)}, \dots, R_{p_2+1}^{(2)}, \dots, R_1^{(n)}, R_2^{(n)}, \dots, R_{p_n+1}^{(n)}\Big),$$

$$R_i^{(1)} = \begin{cases} 0, & i \neq k \\ y_2(\omega_1^+) - \alpha_1^{-1} y_2(\omega_1^-), & i = k \end{cases}, \ i = 1, 2, \ldots, p_1$$

is the eigenvector of T corresponding to v_{1k}.

If $\lambda = v_{2k}$, then

$$Y = (y_1(x), y_2(x), Y_1, Y_2, \ldots, Y_{N_1}, Y_{N_1+1}, W_1, W_2, \ldots, W_{N_2}, W_{N_2+1},$$
$$R_1^{(1)}, R_2^{(1)}, \ldots, R_{p_1}^{(1)}, R_{p_1+1}^{(1)}, R_1^{(1)}, R_2^{(1)}, \ldots, R_{p_1}^{(1)}, R_{p_1+1}^{(1)},$$
$$R_1^{(2)}, R_2^{(2)}, \ldots, R_{p_2}^{(2)}, 0, \ldots, R_1^{(n)}, R_2^{(n)}, \ldots, R_{p_n+1}^{(n)} \Big),$$

$$R_i^{(2)} = \begin{cases} 0, & i \neq k \\ y_2(\omega_2^+) - \alpha_2^{-1} y_2(\omega_2^-), & i = k \end{cases}, \ i = 1, 2, \ldots, p_2$$

is the eigenvector of T corresponding to v_{2k}.

If it continues like this, $\lambda = v_{nk}$, then

$$Y = (y_1(x), y_2(x), Y_1, Y_2, \ldots, Y_{N_1}, Y_{N_1+1}, W_1, W_2, \ldots, W_{N_2}, W_{N_2+1},$$
$$R_1^{(1)}, R_2^{(1)}, \ldots, R_{p_1}^{(1)}, R_{p_1+1}^{(1)}, R_1^{(2)}, R_2^{(2)}, \ldots, R_{p_2+1}^{(2)}, \ldots, R_1^{(n)}, R_2^{(n)}, \ldots, R_{p_n}^{(n)}, 0 \Big),$$

$$R_i^{(n)} = \begin{cases} 0, & i \neq k \\ y_2(\omega_n^+) - \alpha_n^{-1} y_2(\omega_n^-), & i = k \end{cases}, \ i = 1, 2, \ldots, p_n$$

is the eigenvector of T corresponding to v_{nk}. □

It is possible to write $f_j(\lambda)$ $(j = 1, 2)$ as follows:

$f_1(\lambda) = \frac{a_1(\lambda)}{a_2(\lambda)}$, $f_2(\lambda) = \frac{b_1(\lambda)}{b_2(\lambda)}$, where

$$a_1(\lambda) = (a_1\lambda + b_1) \prod_{k=1}^{N_1} (\lambda - g_{1k}) - \sum_{k=1}^{N_1} \prod_{j=1(j\neq k)}^{N_1} f_{1k} (\lambda - g_{1k}),$$

$$a_2(\lambda) = \prod_{k=1}^{N_1} (\lambda - g_{1k}),$$

$$b_1(\lambda) = (a_2\lambda + b_2) \prod_{k=1}^{N_2} (\lambda - g_{2k}) - \sum_{k=1}^{N_2} \prod_{j=1(j\neq k)}^{N_2} f_{2k} (\lambda - g_{2k}),$$

$$b_2(\lambda) = \prod_{k=1}^{N_2} (\lambda - g_{2k}).$$

Assume that $b_1(\lambda)$ and $b_2(\lambda)$ do not have common zeros.

Let the functions $\varphi(x, \lambda) = \varphi_i(x, \lambda) = (\varphi_{i1}(x, \lambda), \varphi_{i2}(x, \lambda))^T$ and $\psi(x, \lambda) = \psi_i(x, \lambda) = (\psi_{i1}(x, \lambda), \psi_{i2}(x, \lambda))^T$, $x \in (\omega_i, \omega_{i+1})$, $(i = 0, 1, \ldots, n)$ be solutions of equation (1.1) satisfying the initial conditions

$$\begin{aligned} \varphi_0(a, \lambda) &= \begin{pmatrix} \varphi_{01}(a, \lambda) \\ \varphi_{02}(a, \lambda) \end{pmatrix} = \begin{pmatrix} -a_2(\lambda) \\ a_1(\lambda) \end{pmatrix}, \\ \psi_n(b, \lambda) &= \begin{pmatrix} \psi_{n1}(b, \lambda) \\ \psi_{n2}(b, \lambda) \end{pmatrix} = \begin{pmatrix} -b_2(\lambda) \\ b_1(\lambda) \end{pmatrix} \end{aligned} \tag{2.8}$$

and the transmission conditions (1.4) such that $\varphi(x,\lambda) = \varphi_i(x,\lambda)$, $w_i < x < w_{i+1}, (i = 0,1,\dots,n)$, $\psi(x,\lambda) = \psi_i(x,\lambda)$, $w_i < x < w_{i+1}, (i = n, n-1,\dots,1)$ where $w_0 = a$, $w_{n+1} = b$.

Then it can be easily proven that $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are the solutions of the following integral equations:

for $i = 1,2,\dots,n$;

$$\varphi_{i1}(x,\lambda) = \alpha_i\varphi_{i-1,1}(\omega_i,\lambda)\cos\lambda\,(x-\omega_i)$$
$$-\left(\alpha_i^{-1}\varphi_{i-1,2}(\omega_i,\lambda) + h_i\,(\lambda)\,\varphi_{i-1,1}(\omega_i,\lambda)\right)\sin\lambda\,(x-\omega_i)$$
$$+\int\limits_{\omega_i}^{x}\left[p(t)\sin\lambda\,(x-t) + q(t)\cos\lambda\,(x-t)\right]\varphi_{i1}(t,\lambda)dt$$
$$+\int\limits_{\omega_i}^{x}\left[q(t)\sin\lambda\,(x-t) + r(t)\cos\lambda\,(x-t)\right]\varphi_{i2}(t,\lambda)dt,$$
$$\varphi_{i2}(x,\lambda) = \alpha_i\varphi_{i-1,1}(\omega_i,\lambda)\sin\lambda\,(x-\omega_i)$$
$$+\left(\alpha_i^{-1}\varphi_{i-1,2}(\omega_i,\lambda) + h_i\,(\lambda)\,\varphi_{i-1,1}(\omega_i,\lambda)\right)\cos\lambda\,(x-\omega_i)$$
$$+\int\limits_{\omega_i}^{x}\left[-p(t)\cos\lambda\,(x-t) + q(t)\sin\lambda\,(x-t)\right]\varphi_{i1}(t,\lambda)dt$$
$$+\int\limits_{\omega_i}^{x}\left[-q(t)\cos\lambda\,(x-t) + r(t)\sin\lambda\,(x-t)\right]\varphi_{i2}(t,\lambda)dt,$$

for $i = n, n-1,\dots,1$;

$$\psi_{i1}(x,\lambda) = \alpha_i^{-1}\psi_{i+1,1}(\omega_i,\lambda)\cos\lambda\,(\omega_i - x)$$
$$+\left(\alpha_i\psi_{i+1,2}(\omega_i,\lambda) - h_i\,(\lambda)\,\psi_{i+1,1}(\omega_i,\lambda)\right)\sin\lambda\rho_{i-1}\,(\omega_i - x)$$
$$-\int\limits_{x}^{\omega_i}\left[p(t)\sin\lambda\,(x-t) + q(t)\cos\lambda\,(x-t)\right]\psi_{i1}(t,\lambda)dt$$
$$-\int\limits_{x}^{\omega_i}\left[q(t)\sin\lambda\,(x-t) + r(t)\cos\lambda\,(x-t)\right]\psi_{i2}(t,\lambda)dt,$$
$$\psi_{i2}(x,\lambda) = -\alpha_i^{-1}\psi_{i+1,1}(\omega_i,\lambda)\sin\lambda\,(\omega_i - x)$$
$$+\left(\alpha_i\psi_{i+1,2}(\omega_i,\lambda) - h_i\,(\lambda)\,\psi_{i+1,1}(\omega_i,\lambda)\right)\cos\lambda\,(\omega_i - x)$$
$$+\int\limits_{x}^{\omega_i}\left[p(t)\cos\lambda\,(x-t) - q(t)\sin\lambda\,(x-t)\right]\psi_{i1}(t,\lambda)dt$$
$$+\int\limits_{x}^{\omega_i}\left[q(t)\cos\lambda\,(x-t) - r(t)\sin\lambda\,(x-t)\right]\psi_{i2}(t,\lambda)dt$$

LEMMA 2.1. $\varphi(x,\lambda) = \left(\varphi_{i1}(x,\lambda), \varphi_{i2}(x,\lambda)\right)^T$, $x \in (\omega_i, \omega_{i+1})$, $(i = 0,1,\dots,n)$ *are entire functions in* λ *and the following asymptotic relations are true for these solutions as* $|\lambda| \to \infty$.

$$\varphi_{01}(x,\lambda) = -a_1\lambda^{N_1+1}\sin\lambda(x-a) + o(\lambda^{N_1+1}\exp|Im\lambda|\,(x-a)),$$
$$\varphi_{02}(x,\lambda) = a_1\lambda^{N_1+1}\cos\lambda(x-a) + o(\lambda^{N_1+1}\exp|Im\lambda|\,(x-a)\,\rho_0),$$

$\varphi_{11}(x,\lambda) = m_1 a_1 \lambda^{N_1+p_1+2} \sin\lambda(\omega_1 - a) \sin\lambda\,(x-\omega_1)$

$\quad +o(\lambda^{N_1+p_1+2} \exp|Im\lambda|\,((\omega_1 - a) + (x-\omega_1)),$

$\varphi_{12}(x,\lambda) = -m_1 a_1 \lambda^{N_1+p_1+2} \sin\lambda(\omega_1 - a) \cos\lambda(x-\omega_1)$

$\quad +o(\lambda^{N_1+p_1+2} \exp|Im\lambda|((\omega_1 - a) + (x-\omega_1)),$

$\vdots$

$\varphi_{n1}(x,\lambda) = (-1)^{n+1} a_1 \lambda^{N_1+p_1+p_2+\ldots+p_n+n+1} \left(\prod_{i=1}^{n} m_i\right) \sin\lambda(\omega_1 - a) \sin\lambda\,(x-\omega_n) \times$

$\prod_{i=2}^{n} \sin\lambda\,(\omega_i - \omega_{i-1}) + o(\lambda^{N_1+p_1+p_2+\ldots+p_n+n+1} \exp|Im\lambda|\,((b-\omega_n) + \sum_{i=1}^{n} (\omega_i - \omega_{i-1})),$

$\varphi_{n2}(x,\lambda) = (-1)^{n} a_1 \lambda^{N_1+p_1+p_2+\ldots+p_n+n+1} \left(\prod_{i=1}^{n} m_i\right) \sin\lambda(\omega_1 - a) \cos\lambda\,(x-\omega_n) \times$

$\prod_{i=2}^{n} \sin\lambda\,(\omega_i - \omega_{i-1}) + o(\lambda^{N_1+p_1+p_2+\ldots+p_n+n+1} \exp|Im\lambda|\,((b-\omega_n) + \sum_{i=1}^{n} (\omega_i - \omega_{i-1})),$

where $A = r_1 + r_2 + \cdots + r_n$.

LEMMA 2.2. $\psi(x,\lambda) = (\psi_{i1}(x,\lambda), \psi_{i2}(x,\lambda))^T$, $x \in (\omega_i, \omega_{i+1})$, $(i = 0, 1, \ldots, n)$ *are entire functions in* λ *and the following asymptotic relations are true for these solutions as* $|\lambda| \to \infty$.

$\psi_{01}(x,\lambda) = (-1)^{n} a_2 \lambda^{N_2+p_1+p_2+\ldots+p_n+n+1} \left(\prod_{i=1}^{n} m_i\right) \sin\lambda(x-\omega_1) \sin\lambda\,(b-\omega_n) \times$

$\prod_{i=2}^{n} \sin\lambda\,(\omega_i - \omega_{i-1}) + o(\lambda^{N_2+p_1+p_2+\ldots+p_n+n+1} \exp|Im\lambda|\,((x-\omega_1) + \sum_{i=1}^{n} (\omega_i - \omega_{i-1})),$

$\psi_{02}(x,\lambda) = (-1)^{n} a_2 \lambda^{N_2+p_1+p_2+\ldots+p_n+n+1} \left(\prod_{i=1}^{n} m_i\right) \sin\lambda(b-\omega_n) \cos\lambda\,(x-\omega_1) \times$

$\prod_{i=2}^{n} \sin\lambda\,(\omega_i - \omega_{i-1}) + o(\lambda^{N_2+p_1+p_2+\ldots+p_n+n+1} \exp|Im\lambda|\,((x-\omega_1) + \sum_{i=1}^{n} (\omega_i - \omega_{i-1})),$

$\vdots$

$\psi_{n-1,1}(x,\lambda) = -m_1 a_2 \lambda^{N_2+p_n+2} \sin\lambda(b-\omega_n) \sin\lambda\,(\omega_n - x)$

$\quad +o(\lambda^{N_2+p_n+2} \exp|Im\lambda|\,((b-\omega_n) + (\omega_n - x)),$

$\psi_{n-1,2}(x,\lambda) = -m_1 a_2 \lambda^{N_2+p_n+2} \sin\lambda(b-\omega_n) \cos\lambda\,(\omega_n - x)$

$\quad +o(\lambda^{N_2+p_n+2} \exp|Im\lambda|\,((b-\omega_n) + (\omega_n - x)),$

$\psi_{n1}(x,\lambda) = a_2 \lambda^{N_2+1} \sin\lambda(b-x) + o(\lambda^{N_2+1} \exp|Im\lambda|\,(b-x)),$

$\psi_{n2}(x,\lambda) = a_2 \lambda^{N_2+1} \cos\lambda(b-x) + o(\lambda^{N_2+1} \exp|Im\lambda|\,(b-x)).$

On the other hand, the Wronskian of the functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ is defined by $W[\psi,\varphi]=\Delta(\lambda)=\psi_1(x,\lambda)\varphi_2(x,\lambda)-\varphi_1(x,\lambda)\psi_2(x,\lambda)$. Since

$$\frac{\partial\Delta(\lambda)}{\partial x}=\psi_1'(x,\lambda)\,\varphi_2(x,\lambda)+\psi_1(x,\lambda)\,\varphi_2'(x,\lambda)$$

$$-\psi_2'(x,\lambda)\,\varphi_1(x,\lambda)-\psi_2(x,\lambda)\,\varphi_1'(x,\lambda)$$

$$=[q(x)\psi_1(x,\lambda)+r(x)\psi_2(x,\lambda)-\lambda\psi_2(x,\lambda)]\,\varphi_2(x,\lambda)$$

$$+[-p(x)\varphi_1(x,\lambda)-q(x)\varphi_2(x,\lambda)+\lambda\varphi_1(x,\lambda)]\,\psi_1(x,\lambda)$$

$$-[-p(x)\psi_1(x,\lambda)-q(x)\psi_2(x,\lambda)+\lambda\psi_1(x,\lambda)]\,\varphi_1(x,\lambda)$$

$$-[q(x)\varphi_1(x,\lambda)+r(x)\varphi_2(x,\lambda)-\lambda\varphi_2(x,\lambda)]\,\psi_2(x,\lambda)=0$$

and

$$\Delta(\omega_i+0)=\psi_1(\omega_i+0,\lambda)\,\varphi_2(\omega+0,\lambda)-\psi_2(\omega_i+0,\lambda)\,\varphi_1(\omega_i+0,\lambda)$$

$$=\alpha_i\psi_1(\omega_i-0,\lambda)\left[\alpha_i^{-1}\varphi_2(\omega_i-0,\lambda)+h_i(\lambda)\,\varphi_1(\omega_i-0,\lambda)\right]$$

$$-\left[\alpha_i^{-1}\psi_2(\omega_i-0,\lambda)+h_i(\lambda)\,\psi_1(\omega_i-0,\lambda)\right]\alpha_i\varphi_1(\omega_i-0,\lambda)$$

$$=\psi_1(\omega_i-0,\lambda)\,\varphi_2(\omega_i-0,\lambda)-\psi_2(\omega_i-0,\lambda)\,\varphi_1(\omega_i-0,\lambda)$$

$$=\Delta(\omega_i-0).$$

$W[\psi,\varphi]$ does not depend on x and $\varphi(x,\lambda)$, $\psi(x,\lambda)$ are linearly independent iff $W[\psi,\varphi]\neq 0$. The characteristic function of the problem (1.1) – (1.4) is defined as following:

$$\begin{aligned}\Delta(\lambda)\quad &=\psi_1(b,\lambda)\varphi_2(b,\lambda)-\psi_2(b,\lambda)\varphi_1(b,\lambda)\\ &=\psi_1(a,\lambda)\varphi_2(a,\lambda)-\psi_2(a,\lambda)\varphi_1(a,\lambda)\\ &=-b_2(\lambda)\,\varphi_2(b,\lambda)-b_1(\lambda)\,\varphi_1(b,\lambda)\\ &=a_1(\lambda)\,\psi_1(a,\lambda)+a_2(\lambda)\,\psi_2(a,\lambda).\end{aligned}$$

$\Delta(\lambda)$ is analytic function in λ and it's zeros are precisely eigenvalues of the problem L. Numbers $\{\alpha_n\}_{n\in\mathbb{Z}}$ are called the normalizing constant of the problem

(1.1) – (1.4) such that

$$
\begin{aligned}
\alpha_n \quad : \quad &= \langle \varphi(x,\lambda_n), \varphi(x,\lambda_n) \rangle = \int\limits_a^b \left(\varphi_1^2+\varphi_2^2\right) dx \\
&- \frac{\left(-a_1\varphi_1(a,\lambda_n)\right)\left(-a_1\varphi_1(a,\lambda_n)\right)}{a_1} \\
&+ \frac{\left(-a_2\varphi_1(b,\lambda_n)\right)\left(-a_2\varphi_1(b,\lambda_n)\right)}{a_2} \\
&+ \sum_{i=1}^{n} \alpha_i \frac{\left(-m_i\varphi_1(\omega_i^-,\lambda_n)\right)\left(\left(-m_i\varphi_1(\omega_i^-,\lambda_n)\right)\right)}{m_i} \\
&- \sum_{k=1}^{N_1} \left(\frac{-f_{1k}\varphi_1(a,\lambda_n)}{\lambda_1-g_{1k}}\right)\left(\frac{-f_{1k}\varphi_1(a,\lambda_n)}{\lambda_2-g_{1k}}\right)\frac{1}{f_{1k}} \\
&+ \sum_{k=1}^{N_2} \left(\frac{-f_{2k}\varphi_1(b,\lambda_n)}{\lambda_1-g_{2k}}\right)\left(\frac{-f_{2k}\varphi_1(b,\lambda_n)}{\lambda_2-g_{2k}}\right)\frac{1}{f_{2k}} \\
&+ \sum_{k=1}^{p_i} \alpha_i \left(\frac{-u_{ik}\varphi_1(\omega_i^-,\lambda_n)}{\lambda_1-v_{ik}}\right)\left(\frac{-u_{ik}\varphi_1(\omega_i^-,\lambda_n)}{\lambda_2-v_{ik}}\right)\frac{1}{u_{ik}} \\
= \quad & \int\limits_a^b \left(\varphi_1^2+\varphi_2^2\right) dx - \varphi_1^2(a,\lambda_n) f_1^{'}(\lambda_n) \\
&+ \varphi_1^2(b,\lambda_n) f_2^{'}(\lambda_n) + \sum_{i=1}^{n} \alpha_i \varphi_1^2(\omega_i^-,\lambda_n) h_i^{'}(\lambda_n) .
\end{aligned} \tag{2.9}
$$

LEMMA 2.3. *The following equality holds for each eigenvalue* λ_n

$$\dot{\Delta}(\lambda_n) = -\beta_n\alpha_n,$$

where $\{\beta_n\}_{n\in\mathbb{Z}}$ *is the sequence that provides the equation* $\psi(x,\lambda_n) = \beta_n\varphi(x,\lambda_n)$.

PROOF. Since

$$
\begin{aligned}
\varphi_2'(x,\lambda_n) + p(x)\varphi_1(x,\lambda_n) + q(x)\varphi_2(x,\lambda_n) &= \lambda_n\varphi_1(x,\lambda_n), \\
\psi_2^{'}(x,\lambda) + p(x)\psi_1(x,\lambda) + q(x)\psi_2(x,\lambda) &= \lambda\psi_1(x,\lambda)
\end{aligned}
$$

and

$$
\begin{aligned}
-\varphi_1'(x,\lambda_n) + q(x)\varphi_1(x,\lambda_n) + r(x)\varphi_2(x,\lambda_n) &= \lambda_n\varphi_2(x,\lambda_n) \\
-\psi_1^{'}(x,\lambda) + q(x)\psi_1(x,\lambda) + r(x)\psi_2(x,\lambda) &= \lambda\psi_2(x,\lambda)
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\varphi_1(x,\lambda_n)\psi_2(x,\lambda) - \varphi_2(x,\lambda_n)\psi_1(x,\lambda)\left(\Big|_a^{\omega_1} + \Big|_{\omega_1}^{\omega_2} + \ldots + \Big|_{\omega_n}^{b}\right) \\
&= (\lambda-\lambda_n)\int\limits_a^b \left[\varphi_1(x,\lambda_n)\psi_1(x,\lambda) - \varphi_2(x,\lambda_n)\psi_2(x,\lambda)\right] dx \, .
\end{aligned}
$$

After adding and subtracting $\Delta(\lambda)$ on the left hand side of last equality and by using (1.2) – (1.4) we can obtain

$$\begin{aligned}
&\int_a^b [\varphi_1(x,\lambda_n)\psi_1(x,\lambda) - \varphi_2(x,\lambda_n)\psi_2(x,\lambda)]\,dx \\
&+\sum_{i=1}^{n} \alpha_i \psi_1(\omega_i^-,\lambda)\varphi_1(\omega_i^-,\lambda_n)\left(h_i(\lambda) - h_i(\lambda_n)\right) \\
&-\varphi_1(a,\lambda_n)\psi_1(a,\lambda)\left(f_1(\lambda) - f_1(\lambda_n)\right) \\
&+\varphi_1(b,\lambda_n)\psi_1(b,\lambda)\left(f_2(\lambda) - f_2(\lambda_n)\right) \\
&+f_2(\lambda_n)\,\psi_1(b,\lambda)\varphi_1(b,\lambda_n) + \psi_1(b,\lambda)\varphi_2(b,\lambda_n) \\
&-a_2(\lambda_n)\psi_2(a,\lambda) - a_2(\lambda_n) f_1(\lambda)\,\psi_1(a,\lambda) \\
&+a_2(\lambda)\psi_2(a,\lambda) + a_2(\lambda) f_1(\lambda)\,\psi_1(a,\lambda) \\
= \quad & -\frac{\Delta(\lambda) - \Delta(\lambda_n)}{\lambda - \lambda_n}.
\end{aligned}$$

For $\lambda \to \lambda_n$, $\dot{\Delta}(\lambda_n) = -\beta_n \alpha_n$ is obtained by using the equalities $\psi(x,\lambda_n) = \beta_n \varphi(x,\lambda_n)$ and (2.1). □

It can easily be seen from the above lemma that the eigenvalues of the problem L are simple.

3. Inverse problem

From the asymptotic formulas of $\varphi(x,\lambda)$ and $\psi(x,\lambda)$, the following asymptotic formulas hold as $|\lambda| \to \infty$:

$$\Delta(\lambda) = (-1)^n a_1 a_2 \lambda^{N_1+N_2+p_1+p_2+\ldots+p_n+n+2} \left(\prod_{i=1}^{n} m_i\right) \sin\lambda(\omega_1 - a) \sin\lambda\,(b-\omega_n)$$

$$\times \prod_{i=2}^{n} \sin\lambda\,(\omega_i - \omega_{i-1})$$

$$+o(\lambda^{N_1+N_2+p_1+p_2+\ldots+p_n+n+2} \exp|\mathrm{Im}\lambda|\,((b-\omega_n) + \sum_{i=1}^{n}(\omega_i - \omega_{i-1})).$$

Let $\delta > 0$ be sufficiently small and fixed. Denote

$$G_\delta := \left\{\lambda : \left|\lambda - \lambda_n^0\right| \geqslant \delta,\ n = 0, \pm1, \pm2, \ldots\right\}.$$

$$|\Delta(\lambda)| \geqslant C_\delta\, |\lambda|^{N_1+N_2+p_1+p_2+\ldots+p_n+n+2} \exp|\tau|\, b,\ \lambda \in G_\delta,\ \ |\lambda| \geqslant \lambda^*$$

for sufficiently large $\lambda^* = \lambda^*(\delta)$.

Let $\Phi(x,\lambda) = \begin{pmatrix} \Phi_1(x,\lambda) \\ \Phi_2(x,\lambda) \end{pmatrix}$ be solution of equation (1.1) under the conditions $l_1(\Phi) = 1,\ l_2(\Phi) = 0$ and transmission conditions (1.4). Denote $W[\varphi,\Phi]$ as the Wronskian of $\varphi(x,\lambda)$ and $\Phi(x,\lambda)$.

The Wronskian $W[\varphi,\Phi]$ does not depend on x. Taking $x=a$ we get

$$\begin{aligned} W[\varphi,\Phi]|_{x=a} &= \varphi_1(a,\lambda)\Phi_2(a,\lambda)-\varphi_2(a,\lambda)\Phi_1(a,\lambda) \\ &= -a_2(\lambda)\Phi_2(a,\lambda)-a_1(\lambda)\Phi_1(a,\lambda) \\ &= -a_2(\lambda)(1+a_1(\lambda)\Phi_1(a,\lambda))\frac{-1}{a_2(\lambda)}-a_1(\lambda)\Phi_1(a,\lambda)=1. \end{aligned}$$

Thus,

$$W[\varphi,\Phi]\equiv 1.$$

Since $l_2(\Phi)=l_2(\Psi)=0$, we may suppose $\Phi(x,\lambda)=k\Psi(x,\lambda)$, where k is a constant independent of x. By the relation $l_1(\Phi)=1$, we obtain

$$k[a_1(\lambda)\Psi_1(a,\lambda)+a_2(\lambda)\Psi_2(a,\lambda)]=1.$$

In view of

$$\Delta(\lambda)=-l_1(\Psi)=-a_1(\lambda)\Psi_1(a,\lambda)-a_2(\lambda)\Psi_2(a,\lambda)=-\frac{1}{k}$$

we get $\Phi(x,\lambda)=-\dfrac{\Psi(x,\lambda)}{\Delta(\lambda)}$ i.e. $\Phi_i(x,\lambda)=-\dfrac{\Psi_i(x,\lambda)}{\Delta(\lambda)}$ $(i=1,2)$ for $\lambda\neq\lambda_n$.

Let $S(x,\lambda)=\begin{pmatrix} S_1(x,\lambda) \\ S_2(x,\lambda)\end{pmatrix}$ and $C(x,\lambda)=\begin{pmatrix} C_1(x,\lambda) \\ C_2(x,\lambda)\end{pmatrix}$ be solutions of (1.1) satisfy the conditions $S(a,\lambda)=\begin{pmatrix} 0 \\ 1\end{pmatrix}$, $C(a,\lambda)=\begin{pmatrix} 1 \\ 0\end{pmatrix}$ and transmission conditions (1.4).

Denote

$$Z(x,\lambda):=\frac{1}{a_2(\lambda)}(S(x,\lambda)+M(\lambda)\varphi(x,\lambda)).$$

Let us show that $Z(x,\lambda)\equiv\Phi(x,\lambda)$. Indeed

$$Z_1(a,\lambda)=\frac{1}{a_2(\lambda)}(S_1(a,\lambda)+M(\lambda)\varphi_1(a,\lambda))=-M(\lambda)$$

$$\begin{aligned} Z_2(a,\lambda) &= \frac{1}{a_2(\lambda)}(S_2(a,\lambda)+M(\lambda)\varphi_2(a,\lambda)) \\ &= \frac{1}{a_2(\lambda)}(1+M(\lambda)a_1(\lambda)) \end{aligned}$$

and consequently

$$\begin{aligned} l_1 Z &= a_2(\lambda)Z_2(a,\lambda)+a_1(\lambda)Z_1(a,\lambda) \\ &= a_2(\lambda)(S_2(a,\lambda)+M(\lambda)\varphi_2(a,\lambda))\frac{1}{a_2(\lambda)}-M(\lambda)a_1(\lambda)=1. \end{aligned}$$

Thus, the functions $Z(x,\lambda)$ and $\Phi(x,\lambda)$ satisfy equation (1.1) and $Z(a,\lambda)=\varphi(a,\lambda)$, hence

$$\Phi(x,\lambda)=\frac{1}{a_2(\lambda)}(S(x,\lambda)+\Phi_1(a,\lambda)\varphi(x,\lambda)).$$

The function $\Phi(x,\lambda)$ is called Weyl solution and the function $M(\lambda)=\Phi_1(a,\lambda)$ is called Weyl function. The functions $Z(x,\lambda)$ and $\Phi(x,\lambda)$ satisfy equation (1.1) and $Z_1(a,\lambda) = \Phi_1(a,\lambda)$, $Z_2(a,\lambda) = \Phi_2(a,\lambda)$ for all $x \in [a,\omega)$. Moreover, for ω_i $(i = 1,2,\ldots,n)$ and for $x \in [a,b]$

$$\begin{aligned} Z_1(\omega_i+0,\lambda) &= \alpha_i Z_1(\omega_i-0,\lambda) = \alpha_i \Phi_1(\omega_i-0,\lambda) = \Phi_1(\omega_i+0,\lambda), \\ Z_2(\omega_i+0,\lambda) &= \alpha_i^{-1} Z_2(\omega_i-0,\lambda) - h_i(\lambda) Z_1(\omega_i-0,\lambda) \\ &= \alpha_i^{-1}\Phi_2(\omega_i-0,\lambda) - h_i(\lambda)\Phi_1(\omega_i-0,\lambda) = \Phi_2(\omega_i+0,\lambda). \end{aligned}$$

We get $Z(x,\lambda) \equiv \Phi(x,\lambda)$. Hence

$$\Phi(x,\lambda) = \frac{1}{a_2(\lambda)}\left(S(x,\lambda) + M(\lambda)\varphi(x,\lambda)\right).$$

In this section, we investigate the inverse problem of the reconstruction of a boundary value problem L from the Weyl function and two different eigenvalues sequences.

Let us consider the boundary value problem

$$\widetilde{\ell}\,[Y(x)] := BY'(x) + \widetilde{\Omega}(x)Y(x) = \lambda Y(x),\ x \in [a,b],$$

$$\begin{aligned} \widetilde{l}_1 y &:= y_2(a) + \widetilde{f}_1(\lambda) y_1(a) = 0, \\ \widetilde{l}_2 y &:= y_2(b) + \widetilde{f}_2(\lambda) y_1(b) = 0, \\ \widetilde{l}_3 y &:= y_1(\omega_i+0) - \widetilde{\alpha}_i y_1(\omega_i-0) = 0, \\ \widetilde{l}_4 y &:= y_2(\omega_i+0) - \widetilde{\alpha}_i^{-1} y_2(\omega_i-0) - \widetilde{h}_i(\lambda) y_1(\omega_i-0) = 0. \end{aligned}$$

$\widetilde{L} := L(\widetilde{\Omega}, \widetilde{f}_i(\lambda), w_i, \widetilde{\alpha}_i, \widetilde{h}_i(\lambda))$, where $\widetilde{\Omega}(x) = \begin{pmatrix} \widetilde{p}(x) & q(x) \\ q(x) & \widetilde{r}(x) \end{pmatrix}$. It is assumed in what follows that if a certain symbol s denotes an object related to the problem L then $\widetilde{s}$ denotes the coresponding object related to the problem $\widetilde{L}$.

THEOREM 3.1. *The boundary value problem L is uniquely detemined by the Weyl function, i.e., if $M(\lambda) = \widetilde{M}(\lambda)$, $a_i(\lambda) = \widetilde{a}_i(\lambda)$ $(i = 1,2)$, then $\Omega(x) = \widetilde{\Omega}(x)$, a.e., $f_2(\lambda) = \widetilde{f}_2(\lambda)$, $\alpha_i = \widetilde{\alpha}_i$, $h_i(\lambda) = \widetilde{h}_i(\lambda)$ for $i = 1,2,\ldots,n$.*

PROOF. Introduce a matrix $P(x,\lambda) = [P_{ij}(x,\lambda)]_{i,j=1,2}$ by the formula

$$P(x,\lambda) = \phi(x,\lambda)\widetilde{\phi}^{-1}(x,\lambda), \tag{3.1}$$

where $\phi(x,\lambda) = \begin{pmatrix} \varphi_2(x,\lambda) & \dfrac{\psi_2(x,\lambda)}{\Delta(\lambda)} \\ \varphi_1(x,\lambda) & \dfrac{\psi_1(x,\lambda)}{\Delta(\lambda)} \end{pmatrix}$ and $\widetilde{\phi}(x,\lambda) = \begin{pmatrix} \widetilde{\varphi}_2(x,\lambda) & \dfrac{\widetilde{\psi}_2(x,\lambda)}{\widetilde{\Delta}(\lambda)} \\ \widetilde{\varphi}_1(x,\lambda) & \dfrac{\widetilde{\psi}_1(x,\lambda)}{\widetilde{\Delta}(\lambda)} \end{pmatrix}$.

Write (3.1) openly

$$\begin{cases} P_{11}(x,\lambda) &= \varphi_2(x,\lambda)\dfrac{\widetilde{\psi}_1(x,\lambda)}{\widetilde{\Delta}\,(\lambda)} - \widetilde{\varphi}_1(x,\lambda)\dfrac{\psi_2(x,\lambda)}{\Delta(\lambda)} \\ P_{12}(x,\lambda) &= \widetilde{\varphi}_2(x,\lambda)\dfrac{\psi_2(x,\lambda)}{\Delta(\lambda)} - \varphi_2(x,\lambda)\dfrac{\widetilde{\psi}_2(x,\lambda)}{\widetilde{\Delta}\,(\lambda)} \\ P_{21}(x,\lambda) &= \varphi_1(x,\lambda)\dfrac{\widetilde{\psi}_1(x,\lambda)}{\widetilde{\Delta}\,(\lambda)} - \widetilde{\varphi}_1(x,\lambda)\dfrac{\psi_1(x,\lambda)}{\Delta(\lambda)} \\ P_{22}(x,\lambda) &= \widetilde{\varphi}_2(x,\lambda)\dfrac{\psi_1(x,\lambda)}{\Delta(\lambda)} - \varphi_1(x,\lambda)\dfrac{\widetilde{\psi}_2(x,\lambda)}{\widetilde{\Delta}\,(\lambda)} \end{cases}. \tag{3.2}$$

Since $\dfrac{\psi_i(x,\lambda)}{\Delta(\lambda)} = -\Phi_i(x,\lambda) = -\left[\dfrac{1}{a_2\,(\lambda)}\left(S_i\,(x,\lambda) + M\,(\lambda)\,\varphi_i\,(x,\lambda)\right)\right]$ and $\varphi_i(x,\lambda) = a_2\,(\lambda)\,C_i\,(x,\lambda) + a_1\,(\lambda)\,S_i\,(x,\lambda)\,,\ i = 1,2,\ M(\lambda) = \widetilde{M}(\lambda)$, the relations

$$\begin{cases} P_{11}(x,\lambda) &= \widetilde{C}_1(x,\lambda)S_2(x,\lambda) - C_2(x,\lambda)\widetilde{S}_1(x,\lambda) \\ P_{12}(x,\lambda) &= C_2(x,\lambda)\widetilde{S}_2(x,\lambda) - \widetilde{C}_2(x,\lambda)S_2(x,\lambda) \\ P_{21}(x,\lambda) &= \widetilde{C}_1(x,\lambda)S_1(x,\lambda) - C_1(x,\lambda)\widetilde{S}_1(x,\lambda) \\ P_{22}(x,\lambda) &= C_1(x,\lambda)\widetilde{S}_2(x,\lambda) - \widetilde{C}_2(x,\lambda)S_1(x,\lambda) \end{cases}$$

are obtained from (3.2). Hence the functions $P_{ij}(x,\lambda)$ are entire on λ as $M(\lambda) = \widetilde{M}(\lambda)$. In addition, $P_{ij}(x,\lambda)$ are bounded with respect to λ. Therefore, it is obvious from Liouville's theorem that, these functions depend only on x.

On the other hands, from (3.2),

$$P_{11}(x,\lambda) - 1 = \varphi_2(x,\lambda)\left(\frac{\widetilde{\psi}_1(x,\lambda)}{\widetilde{\Delta}\,(\lambda)} - \frac{\psi_1(x,\lambda)}{\Delta(\lambda)}\right) - \frac{\psi_2(x,\lambda)\left(\widetilde{\varphi}_1(x,\lambda) - \varphi_1(x,\lambda)\right)}{\Delta\,(\lambda)},$$

$$P_{12}(x,\lambda) = \widetilde{\varphi}_2(x,\lambda)\left(\frac{\psi_2(x,\lambda)}{\Delta\,(\lambda)} - \frac{\widetilde{\psi}_2(x,\lambda)}{\widetilde{\Delta}\,(\lambda)}\right) - \frac{\widetilde{\psi}_2(x,\lambda)\left(\varphi_2(x,\lambda) - \widetilde{\varphi}_2(x,\lambda)\right)}{\widetilde{\Delta}(\lambda)},$$

$$P_{21}(x,\lambda) = \varphi_1(x,\lambda)\left(\frac{\widetilde{\psi}_1(x,\lambda)}{\widetilde{\Delta}\,(\lambda)} - \frac{\psi_1(x,\lambda)}{\Delta(\lambda)}\right) - \frac{\psi_1(x,\lambda)\left(\widetilde{\varphi}_1(x,\lambda) - \varphi_1(x,\lambda)\right)}{\Delta(\lambda)},$$

$$P_{22}(x,\lambda) - 1 = \widetilde{\varphi}_2(x,\lambda)\left(\frac{\psi_1(x,\lambda)}{\Delta(\lambda)} - \frac{\widetilde{\psi}_1(x,\lambda)}{\widetilde{\Delta}\,(\lambda)}\right) - \frac{\widetilde{\psi}_2(x,\lambda)\left(\varphi_1(x,\lambda) - \widetilde{\varphi}_1(x,\lambda)\right)}{\widetilde{\Delta}\,(\lambda)}.$$

Since $-\dfrac{\psi_i(x,\lambda)}{\Delta\,(\lambda)} = \Phi_i\,(x,\lambda) = \dfrac{1}{a_2\,(\lambda)}\,(S_i(x,\lambda) + M\,(\lambda)\,\varphi_i(x,\lambda)),$

$\varphi_i(x,\lambda) = a_2\,(\lambda)\,C_i\,(x,\lambda) + a_1\,(\lambda)\,S_i\,(x,\lambda)$ and $M(\lambda) = \widetilde{M}(\lambda)$

$P_{11}(x,\lambda) = \widetilde{C}_1\,(x,\lambda)\,S_2\,(x,\lambda) - C_2\,(x,\lambda)\,\widetilde{S}_1\,(x,\lambda)$

$P_{12}(x,\lambda) = \widetilde{C}_2\,(x,\lambda)\,\tilde{S}_2\,(x,\lambda) - \tilde{C}_2\,(x,\lambda)\,S_2\,(x,\lambda)$

$$P_{21}(x,\lambda)=\widetilde{C}_1(x,\lambda)\,S_1(x,\lambda)-C_1(x,\lambda)\,\widetilde{S}_1(x,\lambda)$$
$$P_{22}(x,\lambda)=C_1(x,\lambda)\,\widetilde{S}_2(x,\lambda)-\widetilde{C}_2(x,\lambda)\,S_1(x,\lambda)\,.$$

Therefore, due to the fact that,

$$\lim_{\lambda\in\mathbb{R},\lambda\to\infty}\frac{\psi_2(x,\lambda)\left(\widetilde{\varphi}_1(x,\lambda)-\varphi_1(x,\lambda)\right)}{\Delta(\lambda)}=0,$$
$$\lim_{\lambda\in\mathbb{R},\lambda\to\infty}\varphi_2(x,\lambda)\left(\frac{\widetilde{\psi}_1(x,\lambda)}{\widetilde{\Delta}(\lambda)}-\frac{\psi_1(x,\lambda)}{\Delta(\lambda)}\right)=0,$$

for all $x\in[a,b]$

$$\lim_{\lambda\in\mathbb{R},\lambda\to\infty}\left[P_{11}(x,\lambda)-1\right]=0$$

uniformly with respect to x. Thus $P_{11}(x,\lambda)\equiv 1$ and similarly, $P_{22}(x,\lambda)\equiv 1$ $P_{12}(x,\lambda)=P_{21}(x,\lambda)\equiv 0$. Substituate these relations in (3.2), to get

$$\begin{aligned}\varphi_1(x,\lambda)&\equiv\widetilde{\varphi}_1(x,\lambda),\ \varphi_2(x,\lambda)\equiv\widetilde{\varphi}_2(x,\lambda),\\ \frac{\psi_1(x,\lambda)}{\Delta(\lambda)}&\equiv\frac{\widetilde{\psi}_1(x,\lambda)}{\widetilde{\Delta}(\lambda)},\ \frac{\psi_2(x,\lambda)}{\Delta(\lambda)}\equiv\frac{\widetilde{\psi}_2(x,\lambda)}{\widetilde{\Delta}(\lambda)}.\end{aligned}$$

Hence $\Omega(x)=\widetilde{\Omega}(x)$, a.e., since $\dfrac{\psi_i(x,\lambda)}{\Delta(\lambda)}=\dfrac{\widetilde{\psi}_i(x,\lambda)}{\widetilde{\Delta}(\lambda)}$, for $i=1,2$ and

$$\begin{cases}b_2(\lambda)\,\psi_2(b,\lambda)+b_1(\lambda)\,\psi_1(b,\lambda)=0\\ \widetilde{b}_2(\lambda)\,\widetilde{\psi}_2(b,\lambda)+\widetilde{b}_1(\lambda)\,\widetilde{\psi}_1(b,\lambda)=0\end{cases},$$

$b_1(\lambda)\,\widetilde{b}_2(\lambda)-b_2(\lambda)\,\widetilde{b}_1(\lambda)=0$. Since $b_1(\lambda)$, $b_2(\lambda)$ as well as $\widetilde{b}_1(\lambda)$, $\widetilde{b}_2(\lambda)$ do not have common zeros, $b_1(\lambda)=\widetilde{b}_1(\lambda)$, $b_2(\lambda)=\widetilde{b}_2(\lambda)$ i.e., $f_2(\lambda)=\widetilde{f}_2(\lambda)$. From transmission conditions (1.4), since $\varphi_i(x,\lambda)=\widetilde{\varphi}_i(x,\lambda)$, $i=1,2$, we get $\alpha_i=\widetilde{\alpha}_i$, $h_i(\lambda)=\widetilde{h}_i(\lambda)$ for $i=1,2,\ldots,n$. Thus, $L=\widetilde{L}$. □

Consider the following boundary value problem $L_1(\Omega(x),f_1(\lambda),f_2(\lambda))$:

$$\ell\left[y(x)\right]:=BY'(x)+\Omega(x)Y(x)=\lambda\rho(x)Y(x),\ x\in[a,b]\,,$$

$$\begin{aligned}&l_1y:=y_1(a)=0,\\ &l_2y:=y_2(b)+f_2(\lambda)y_1(b)=0,\\ &l_3y:=y_1(\omega_i+0)-\alpha_iy_1(\omega_i-0)=0,\\ &l_4y:=y_2(\omega_i+0)-\alpha_i^{-1}y_2(\omega_i-0)-h_i(\lambda)\,y_1(\omega_i-0)=0.\end{aligned}$$

Let $\{\mu_n\}_{n\geqslant 0}$ be the eigenvalues of the problem L_1. It is clear that,

$$\Delta_1(\lambda)\ =W\left[\psi,S\right]=\psi_1(a,\lambda)\,S_2(a,\lambda)-\psi_2(a,\lambda)\,S_1(a,\lambda)=\psi_1(a,\lambda)$$

is the characteristic function of L_1.

Theorem 3.2. *If $\lambda_n=\widetilde{\lambda}_n$ and $\mu_n=\widetilde{\mu}_n$ for all $n\in\mathbb{Z}$, $a_i(\lambda)=\widetilde{a}_i(\lambda)$, $(i=1,2)$, $a_2=\tilde{a}_2$ and $m_i=\widetilde{m}_i$ $(i=1,2,\ldots,n)$ then $\Omega(x)=\widetilde{\Omega}(x)$, a.e., $f_2(\lambda)=\widetilde{f}_2(\lambda)$, $h_i(\lambda)=\widetilde{h}_i(\lambda)$, $\alpha_i=\widetilde{\alpha}_i$, $(i=1,2,\ldots,n)$.*

Proof. Since $\lambda_n = \widetilde{\lambda}_n$, and $\mu_n = \widetilde{\mu}_n$ then $\dfrac{\Delta(\lambda)}{\widetilde{\Delta}(\lambda)}$ and $\dfrac{\Delta_1(\lambda)}{\widetilde{\Delta}_1(\lambda)}$ are entire function in λ. On the other hand, since $\lim\limits_{\lambda\to-\infty} \dfrac{\Delta(\lambda)}{\widetilde{\Delta}(\lambda)} = \lim\limits_{\lambda\to-\infty} \dfrac{\Delta_1(\lambda)}{\widetilde{\Delta}_1(\lambda)} = 1$, then $\Delta(\lambda) \equiv \widetilde{\Delta}(\lambda)$ and $\Delta_1(\lambda) \equiv \widetilde{\Delta}_1(\lambda)$. Therefore, $\psi_1(a,\lambda) \equiv \widetilde{\psi}_1(a,\lambda)$ and then $M(\lambda) = \widetilde{M}(\lambda)$ from $\Phi_1(x,\lambda) = -\dfrac{\Psi_1(x,\lambda)}{\Delta(\lambda)}$ and $M(\lambda) = \Phi_1(a,\lambda)$. Thus, the proof is completed by Theorem 3.1. □

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