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EXISTENCE OF POSITIVE SOLUTIONS FOR SINGULAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we investigate the existence of positive solutions for a singular fractional differential equation. Our analysis is based upon the Avery Peterson fixed point theorem and the Leray-Schauder nonlinear alternative theorem. As applications, we present examples for the demonstration of our main results

1. Introduction

Fractional differential equations describe many phenomena in various fields of engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing and so on see [**7**, **14**, **16**, **17**].

Singular fractional differential equations have high signifiance due to the variety of applications in areas of mathematical and natural sciences. Many author study on singular boundary value problems using diverse methods, such as the Krasnoselskii fixed point theorem on cones, the Legett-Williams fixed point theorem, the fixed point index theory in cones, the Avery-Peterson fixed point theorem. For more details, see [**2**, **5**, **8**, **9**, **12**, **15**, **18**–**25**] and references there in.

In [**12**], the authors investigated positive solutions for the singular fractional boundary value problem:

$$
\begin{aligned}
&{}^cD^{\alpha}_{0^+}u(t) + f(t, u(t), u'(t)) = 0, \quad 0 < t < 1\\
&u(0) = u'(1) = u''(0) = 0,
\end{aligned}
$$

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where $2 < \alpha \leqslant 3$, ${}^cD^{\alpha}_{0+}$ is the Caputo derivative of order α and $f(t,x,y)$ may be singular at $t = 0$.

In [**6**], the authors established to existence and uniqueness of positive solution for the singular fractional boundary value problem:

$$\begin{aligned} D^{\alpha}_{0+}u(t) + f(t,u(t)) = 0, \quad 0 < t < 1 \\ u(0) = u(1) = 0, \end{aligned}$$

where $1 < \alpha \leqslant 2$, D^{α}_{0+} is the Riemann-Liouville derivative of order α, $f(t,x)$ is may be singular at $t = 0$.

In [**9**], the authors investigated positive solutions for the singular fractional boundary value problem:

$$\begin{aligned} D^{\alpha}_{0+}u(t) + \lambda f(t,u(t)) = 0, \quad 0 < t < 1 \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0 \\ D^{p}_{0+}u(t)|_{t=1} = \sum_{i=1}^{m} a_i D^{q}_{0+}u(t)|_{t=\xi_i}, \end{aligned}$$

where λ is a positive parameter, $n-1 < \alpha \leqslant n$, $n \in \mathbb{N}$, $n \geqslant 3$, $\xi_i \in \mathbb{R}$ for all $i = 1,2,\ldots m$, $0 < \xi_1 < \cdots < \xi_m < 1$, $p,q \in \mathbb{R}$, $p \in [1, n-2]$, $q \in [0,p]$, D^{α}_{0+} is the Riemann-Liouville derivative of order α, f may change sign and may be singular at $t = 0$ or $t = 1$.

The aim of this paper is to establish multiple positive solutions for the fractional differential equation with Caputo derivative of order $\alpha \in (3,4]$.

$$(1.1) \qquad {}^cD^{\alpha}_{0+}u(t) + f(t,u(t)) = 0, \quad t \in (0,1)$$

$$(1.2) \qquad u^{''}(0) = u^{'''}(0) = 0, \quad u^{'}(0) = u(1) = \sum_{j=1}^{\infty} \eta_j \, u(\xi_j)$$

in which $0 < \xi_1 < \xi_2 < \ldots < \xi_{j-1} < \xi_j < \ldots < 1$, $j = 1,2,3,\ldots$, $\eta_j > 0$, $f(t,u)$ may be singular at t=0. In this paper we will suppose that the following conditions hold.

(H_1) $1 - \sum_{j=1}^{\infty} \eta_j \xi_j > 0$

(H_2) $f(t,x) : (0,1] \times [0,\infty) \to [0,\infty)$ and there exists a constant $0 < \sigma < 1$ such that $t^{\sigma} f(t,x)$ is continuous in $[0,1] \times [0,\infty)$.

The organization of this paper is as follows. In section 2, we provide some definitions and preliminary lemmas which are key tools for our main result. In section 3, we give and prove our main results. Finally, we give examples to illustrate how the main results can be used in practice.

In order to assert our main results, we will give Avery Peterson fixed point theorem and Leray-Schauder nonlinear alternative theorem.

Let φ and θ be nonnegative continuous convex functionals on P, ϕ be a nonnegative continuous concave functional on P and ψ be a nonnegative continuous functional on P. Then, for positive numbers a,b,c,d, we define the following convex sets:

$$P(\varphi,d)=\{x\in P:\varphi(x)<d\}$$
$$P(\varphi,\phi,b,d)=\{x\in P:\phi(x)\geqslant b,\varphi(x)\leqslant d\}$$
$$P(\varphi,\theta,\phi,b,c,d)=\{x\in P:\phi(x)\geqslant b,\theta(x)\leqslant c,\varphi(x)\leqslant d\}$$
$$R(\varphi,\psi,a,d)=\{x\in P:\psi(x)\geqslant a,\varphi(x)\leqslant d\}$$

Theorem 1.1. *[4] Let P be a cone of E, φ and θ be nonnegative continuous convex functionals on P, ϕ be a nonnegative continuous concave functional on P and ψ be a nonnegative continuous functional on P. Provides $\psi(lx)\leqslant l\psi(x)$ for $l\in[0,1]$ such that for some positive numbers d and K,*

$$\phi(x)\leqslant\psi(x)\quad and\quad \|x\|\leqslant K\varphi(x)$$

for all $x\in\overline{P(\varphi,d)}$. In that case

$$T:\overline{P(\varphi,d)}\rightarrow\overline{P(\varphi,d)}$$

is completely continuous, there are positive numbers a,b,c such that and $a<b$ such that it satisfies the following conditions:

- $(S1)$ *$\{x\in P(\varphi,\theta,\phi,b,c,d):\phi(x)>b\}\neq\emptyset$ and $\phi(Tx)>b$ for $x\in P(\varphi,\theta,\phi,b,c,d)$;*
- $(S2)$ *$\phi(Tx)>b$ for $x\in P(\varphi,\phi,b,d)$ with $\theta(Tx)>c$;*
- $(S3)$ *$0\notin R(\varphi,\psi,a,d)$ and $\psi(Tx)<a$ for $x\in R(\varphi,\psi,a,d)$ with $\psi(x)=a$.*

Then T has at least three fixed points $u_1,u_2,u_3\in\overline{P(\varphi,d)}$, such that

$$\varphi(u_j)\leqslant d,\quad for\ j=1,2,3$$

and

$$b<\phi(u_1),\quad a<\psi(u_2),\quad \phi(u_2)<b,\quad a>\psi(u_3).$$

Theorem 1.2. *[**1, 7**] Let E be a Banach space, C is a closed, convex subset of E, U an open subset of C and 0 ∈ U. Suppose that $A:\bar{U}\rightarrow C$ is a continuous, compact map.Then either*

- (A_1) *A has a fixed point in $\bar{U}$; or*
- (A_2) *There is a $x\in\partial U$ and $\lambda\in(0,1)$ with $x=\lambda A(x)$.*

2. Preliminaries

In this section, some lemmas and definitions required for fractional calculations that will be used later are given in [**3, 10, 11, 13, 14, 17**].

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \to$ R is given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds.$$

provided the right-hand is pointwise defined on $(0, \infty)$ and where Γ is the gamma function.

Definition 2.2. The Caputo fractional derivative of order $\alpha > 0$ of a function $y : (0, \infty) \to R$ is given by

$${}^{c}D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} y^{(n)}(s) ds$$

where α is fractional number $n = [\alpha] + 1$, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 2.1. *Let $n - 1 < \alpha < n$, $u \in C^n[0, 1]$, then*

$$I_{0+}^{\alpha}\, {}^{c}D_{0+}^{\alpha} u(t) = u(t) - c_1 - c_2 t - ... - c_n t^{n-1}$$

where $n = [\alpha] + 1$, $[\alpha]$ is the smallest integer greater than or equal to α, $c_i \in R$ $(i = 1, 2, .., n)$

Lemma 2.2. *If $y \in C[0, 1]$, then the fractional differential equation*

$$ {}^{c}D_{0+}^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1 \tag{2.1}$$

with the boundary condition (1.2) has a unique solution

$$u(t) = \int_0^1 G(t, s)\, y(s) ds, \quad t \in [0, 1]$$

where,

$$G(t, s) = g(t, s) + \frac{t}{\triangle} \sum_{j=1}^{\infty} \eta_j g(\xi_j, s) \tag{2.2}$$

$$g(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leqslant s \leqslant t \leqslant 1 \\ (1-s)^{\alpha-1}, & 0 \leqslant t \leqslant s \leqslant 1 \end{cases} \tag{2.3}$$

and

$$\triangle = 1 - \sum_{j=1}^{\infty} \eta_j \xi_j.$$

PROOF. The equation (2.1) can be translated into the following equations:

$$u(t) = c_1 - c_2 t - c_3 t^2 - c_4 t^3 - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds.$$

By using the condition $u''(0) = u'''(0) = 0$, we obtain $c_3 = c_4 = 0$. Then we can get that,

$$u(t) = c_1 - c_2 t - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds. \tag{2.4}$$

For the resulting function (2.4), then the condition $u'(0) = \sum_{j=1}^{\infty} \eta_j u(\xi_j)$, we can get that $c_2 = \sum_{j=1}^{\infty} \eta_j u(\xi_j)$.

By other condition $u(1) = \sum_{j=1}^{\infty} \eta_j u(\xi_j)$, we obtain $c_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds$. So, the unique solution of the problem (2.1) is given by

$$\begin{aligned}
u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds + \frac{t}{\Gamma(\alpha)\triangle} \sum_{j=1}^{\infty} \eta_j \int_0^1 (1-s)^{\alpha-1} y(s) ds \\
&\quad - \frac{t}{\Gamma(\alpha)\triangle} \sum_{j=1}^{\infty} \eta_j \int_0^{\xi_j} (\xi_j - s)^{\alpha-1} y(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds \\
&= \frac{1}{\Gamma(\alpha)} \left[\int_0^t \left[(1-s)^{\alpha-1} - (t-s)^{\alpha-1} \right] y(s) ds + \int_t^1 (1-s)^{\alpha-1} y(s) ds \right] \\
&\quad + \frac{t}{\Gamma(\alpha)\triangle} \sum_{j=1}^{\infty} \eta_j \left[\int_0^{\xi_j} \left[(1-s)^{\alpha-1} - (\xi_j - s)^{\alpha-1} \right] y(s) ds \right. \\
&\quad \left. + \int_{\xi_j}^1 (1-s)^{\alpha-1} y(s) ds \right].
\end{aligned}$$

So,

$$u(t) = \int_0^1 \left[g(t,s) + \frac{t}{\triangle} \sum_{j=1}^{\infty} \eta_j g(\xi_j, s) \right] y(s) ds = \int_0^1 G(t,s) y(s) ds.$$

□

Lemma 2.3. *The function $g(t,s)$ expressed by (2.3) involve the following properties:*

a) *$g(t,s)$ is continuous and $g(t,s) \geqslant 0$, $t,s \in [0,1]$.*
b) *$g(t,s) \leqslant \gamma(s)$, $t,s \in [0,1]$.*
c) *$g(t,s) \geqslant \rho_1 \gamma(s)$, $t \in [k,\zeta]$, $s \in [0,1]$ where $k,\zeta \in (0,1)$ with $k < \zeta$,*

and

$$\gamma(s) = \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}, \quad \rho_1 = 1 - \zeta^{\alpha-1}.$$

Proof. a) By definition of the function g we deduce that g is a continuous function and $g(t,s) \geqslant 0$, $t,s \in [0,1]$.

b) if $s \leqslant t$, we obtain,

$$\begin{aligned} g(t,s) &= \frac{(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &\leqslant \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &= \gamma(s) \end{aligned}$$

if $t \leqslant s$, we deduce,

$$g(t,s) = \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} = \gamma(s).$$

Hence, we conclude

$$g(t,s) \leqslant \gamma(s), \quad t,s \in [0,1].$$

c) if $s \leqslant t$, we obtain,

$$\begin{aligned} g(t,s) &= \frac{(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &\geqslant \frac{1}{\Gamma(\alpha)}\Big[(1-s)^{\alpha-1} - (t-ts)^{\alpha-1}\Big] \\ &= \frac{1}{\Gamma(\alpha)}\Big[(1-s)^{\alpha-1} - t^{\alpha-1}(1-s)^{\alpha-1}\Big] \\ &= \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1}(1-t^{\alpha-1}) \\ &\geqslant \gamma(s)(1-\zeta^{\alpha-1}) \\ &= \rho_1\gamma(s) \end{aligned}$$

if $t \leqslant s$, we deduce

$$\begin{aligned} g(t,s) &= \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &= \gamma(s) \\ &\geqslant \gamma(s)(1-\zeta^{\alpha-1}) \\ &= \rho_1\gamma(s). \end{aligned}$$

Hence, we conclude

$$g(t,s) \geqslant \rho_1\gamma(s), \quad t \in [k,\zeta] \text{ and } s \in [0,1].$$

□

LEMMA 2.4. *The function G given by (2.2) is a continuous function on $[0,1]\times[0,1]$ and satisfies the inequalities:*

a) $G(t,s) \leqslant \rho_2\gamma(s), \quad t,s \in [0,1]$ *where* $\rho_2 = 1 + \frac{1}{\triangle}\sum_{j=1}^{\infty}\eta_j$

b) $G(t,s) \geqslant \rho_1\gamma(s), \quad t \in [k,\zeta]$ *and* $s \in [0,1]$ *where* $\rho_1 = 1-\zeta^{\alpha-1}$.

PROOF. *a)* This property follows from the definition of function g and Lemma 2.3,

$$\begin{aligned} G(t,s) &= g(t,s) + \frac{t}{\triangle}\sum_{j=1}^{\infty}\eta_j g(\xi_j,s) \\ &\leqslant \gamma(s) + \frac{1}{\triangle}\sum_{j=1}^{\infty}\eta_j\gamma(s) \\ &= \gamma(s)\left[1 + \frac{1}{\triangle}\sum_{j=1}^{\infty}\eta_j\right] \end{aligned}$$

So,

$$G(t,s) \leqslant \rho_2\gamma(s).$$

b) for t $\in [k,\zeta]$ and $s \in [0,1]$, we get

$$\begin{aligned} G(t,s) &= g(t,s) + \frac{t}{\triangle}\sum_{j=1}^{\infty}\eta_j g(\xi_j,s) \\ &\geqslant g(t,s) \\ &\geqslant \rho_1\gamma(s). \end{aligned}$$

So,

$$G(t,s) \geqslant \rho_1\gamma(s).$$

Thus, the proof is completed.

□

Let B= C [0,1] with the norm $\|u\| \ = \max_{t\in[0,1]} \left|u(t)\right|$ for u $\in B$. Define a cone,

$$P = \{u \in E, u(t) \geqslant 0, \ \ t \in [0,1], \ \min_{t\in[k,\zeta]} u(t) \geqslant \frac{\rho_1}{\rho_2}\|u\|\}$$

an operator $T : P \to B$ given by

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))ds.$$

Lemma 2.5. *$T : P \to P$ is completely continuous operator.*

Proof. Firstly, let's prove in that $T(P) \subset P$. $G(t,s)$ and $s^\sigma f(s,u(s))$ are continuous functions for all t,s $\in [0,1]$ and $s^{-\sigma}$ is integrable on [0,1]. We know $Tu(t) \geqslant 0$ from Lemma 2.4. Also, for all $t \in [k,\zeta]$ and for all $s \in [0,1]$, we obtain that

$$\|Tu\| \leqslant \rho_2 \int_0^1 \gamma(s)f(s,u(s))ds$$

So,

$$\begin{aligned}
\min_{t\in[k,\zeta]} Tu(t) &= \min_{t\in[k,\zeta]} \int_0^1 G(t,s)f(s,u(s))ds \\
&\geqslant \rho_1 \int_0^1 \gamma(s)f(s,u(s))ds \\
&\geqslant \frac{\rho_1}{\rho_2}\|Tu\|
\end{aligned}$$

Thus, T(P) $\subset$ P.

According to Lemma 2.3 and for $u \in P$, it is clear that Tu(t) is nonnegative. Also we can be saying that the $T : P \to P$ is continuous due to (H2).

Next, we will show that for bounded $V \subset P$, T(V) is relatively compact.There is a $M > 0$ such that $t^\sigma f(t,u(t)) \leqslant M$ for $t \in [0,1]$ and for any $u \in V$.

$$\begin{aligned}
Tu(t) = \int_0^1 G(t,s)f(s,u(s))ds &= \int_0^1 G(t,s)s^{-\sigma}s^\sigma f(s,u(s))ds \\
&\leqslant \rho_2 M \int_0^1 \gamma(s)s^{-\sigma}ds \\
&= \frac{\rho_2 M\Gamma(1-\sigma)}{\Gamma(\alpha+1-\sigma)}.
\end{aligned}$$

Therefore,

$$\|Tu(t)\| \leqslant \frac{\rho_2 M\Gamma(1-\sigma)}{\Gamma(\alpha+1-\sigma)} < \infty, u \in V.$$

It show that T(V) is uniformly bounded. Finally, we show that T(V) is equicontinuous. For u ∈ V and t_1, $t_2 \in [0,1]$ such that $t_1 < t_2$, we have

$$\begin{aligned}
\mid Tu(t_2) - Tu(t_1) \mid &= \left| \int_0^1 G(t_2,s)f(s,u(s)) - \int_0^1 G(t_1,s)f(s,u(s)) \right| \\
&= \left| \int_0^1 G(t_2,s)s^{-\sigma}s^{\sigma}f(s,u(s))ds - \int_0^1 G(t_1,s)s^{-\sigma}s^{\sigma}f(s,u(s))ds \right| \\
&= \left| \int_0^1 \Big(G(t_2,s) - G(t_1,s)\Big)s^{-\sigma}s^{\sigma}f(s,u(s))ds \right| \\
&\leqslant M \left| \int_0^1 G(t_2,s)s^{-\sigma}ds - \int_0^1 G(t_1,s)s^{-\sigma}ds \right| \\
&= M \Bigg| \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma}ds - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma}ds \\
&- \frac{t_1}{\triangle}\sum_{j=1}^{\infty}\eta_j \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma}ds + \frac{t_2}{\triangle}\sum_{j=1}^{\infty}\eta_j \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma}ds \\
&- \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma}ds - \frac{t_2}{\triangle}\sum_{j=1}^{\infty}\eta_j \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma}ds \\
&+ \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma}ds + \frac{t_1}{\triangle}\sum_{j=1}^{\infty}\eta_j \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma}ds \Bigg| \\
&= M\Bigg[- \frac{t_1 B_1}{\triangle\Gamma(\alpha)}\sum_{j=1}^{\infty}\eta_j + \frac{t_2 B_1}{\triangle\Gamma(\alpha)}\sum_{j=1}^{\infty}\eta_j + t_2^{\alpha-\sigma}B_2 \\
&- \frac{t_2 B_2}{\triangle\Gamma(\alpha)}\sum_{j=1}^{\infty}\eta_j - t_1^{\alpha-\sigma}B_2 + \frac{t_1 B_2}{\triangle\Gamma(\alpha)}\sum_{j=1}^{\infty}\eta_j \Bigg]
\end{aligned}$$

So we get the result, $\mid Tu(t_2) - Tu(t_1) \mid \to 0$ when $(t_1 \to t_2)$.

Consequently, applying the Arzela-Ascoli theorem, we conclude that $T : P \to P$ is a completely continuous operator.

□

3. Main results

The prove that (1.1), (1.2) has three positive solutions the following three functionals are defined by convex functional $\theta(u) = \varphi(u) = \psi(u) = \|u\|$ and the concave functional $\phi(u) = \min\limits_{t\in[k,\zeta]} |u(t)|$.

THEOREM 3.1. *Assume that there exist positive constants a,b,c,d with* $a < b$, $c > max\{e^k, \frac{\rho_2}{\rho_1}\}b$, $d > \frac{br}{\rho_1 N}$ *and* $d \geqslant c$ *and f holds the following conditions:*

(H3) $t^\sigma f(t,u) \leqslant \frac{d}{r}, \quad (t,u) \in [0,1] \times [0,d]$

(H4) $f(t,u) > \frac{b}{\rho_1 N}, \quad (t,u) \in [k,\zeta] \times [b,c]$

(H5) $t^\sigma f(t,u) < \frac{a}{r}, \quad (t,u) \in [0,1] \times [0,a]$

where $r = \rho_2 \int\limits_0^1 \gamma(s)s^{-\sigma}ds$ *and* $N = \int\limits_k^\zeta \gamma(s)ds$.

Then the problem (1.1), (1.2) *has at least three positive solutions* u_1, u_2 *and* u_3 *such that*

$$\varphi(u_j) \leqslant d, \quad for\ j = 1,2,3$$

and

$$b < \phi(u_1), \quad a < \psi(u_2), \quad \phi(u_2) < b, \quad a > \psi(u_3).$$

PROOF. First, we indicate that T: $\overline{P(\varphi,d)} \to \overline{P(\varphi,d)}$. If u $\in \overline{P(\varphi,d)}$, then $\varphi(u) \leqslant d$, $\|u\| \leqslant d$. In view of (H3), we can get,

$$\begin{aligned}
\varphi(Tu) &= \|Tu\| \\
&= \max_{t\in[0,1]} \left| \int_0^1 G(t,s)f(s,u(s))ds \right| \\
&\leqslant \int_0^1 \gamma(s)\rho_2 s^{-\sigma} s^\sigma f(s,u(s))ds \\
&\leqslant \frac{d}{r}\rho_2 \int_0^1 \gamma(s)s^{-\sigma}ds \\
&= d
\end{aligned}$$

So, we obtain $T : \overline{P(\varphi,d)} \to \overline{P(\varphi,d)}$.

Next, we indicate that condition $(S1)$ of Theorem 1.1 is fulfilled. Let's choose a function $u(t) = be^t$ for t $\in [0,1]$. We have $\phi(u) > b$ for $u \in P(\varphi,\theta,\phi,b,c,d)$,

$$\begin{aligned}\phi(u) &= \phi\big(be^{t}\big)\\ &= \min_{t\in[k,\zeta]}\big|be^{t}\big|\\ &\geqslant be^{k}\\ &> b.\end{aligned}$$

Hence, $\{u \in P(\varphi,\theta,\phi,b,c,d) : \phi(u) > b\} \neq \emptyset$. Choose $u \in P(\varphi,\theta,\phi,b,c,d)$, then this means $u(t) \in [b,c]$ for any $t \in [0,1]$. By (H4) we get,

$$\begin{aligned}\phi(Tu) &= \min_{t\in[k,\zeta]}\big|Tu(t)\big|\\ &= \min_{t\in[k,\zeta]}\left|\int_0^1 G(t,s)f(s,u(s))ds\right|\\ &\geqslant \rho_1\int_0^1 \gamma(s)f(s,u(s))ds\\ &\geqslant \rho_1\int_k^\zeta \gamma(s)f(s,u(s))ds\\ &> \rho_1\,\frac{b}{\rho_1 N}\int_k^\zeta \gamma(s)ds\\ &= b.\end{aligned}$$

Thus, condition (S1) of Theorem 3.1 holds.

Let's get $u \in P(\varphi,\phi,b,d)$. So we have, $\theta(Tu) = \|Tu\| > c$. Since $Tu \in P$, from the definition of cone and from $\frac{\rho_2}{\rho_1}b < c$, we get

$$\begin{aligned}\phi(Tu) &= \min_{t\in[k,\zeta]}\big|Tu(t)\big|\\ &\geqslant \frac{\rho_1}{\rho_2}\|Tu\|\\ &> \frac{\rho_1}{\rho_2}\,c\\ &> b\end{aligned}$$

So, (S2) holds.

Since $a > 0$, 0 is not member of $R(\varphi,\psi,a,d)$ with $\psi(u) = a$, then using (H5), we get

$$\begin{aligned}\psi(Tu) &= \max_{t\in[0,1]}\left|\int_0^1 G(t,s)f(s,u(s))ds\right|\\ &\leqslant \int_0^1 \rho_2\ \gamma(s)s^{-\sigma}s^{\sigma}f(s,u(s))ds\\ &< \rho_2\ \frac{a}{r}\int_0^1 \gamma(s)s^{-\sigma}ds\\ &= a\end{aligned}$$

So, the condition (S3) holds. By Theorem 3.1, we can get that (1.1) , (1.2) has at the least three positive solutions u_1, u_2, u_3 satisfying:

$$\varphi(u_j) \leqslant d, \quad for\ j = 1, 2, 3$$

and

$$b < \phi(u_1), \quad a < \psi(u_2), \quad b > \phi(u_2), \quad a > \psi(u_3).$$

□

THEOREM 3.2. *Let $f(t,x) : (0,1] \times [0,\infty) \to [0,\infty)$ and $0 < \sigma < 1$ such that $t^{\sigma} f(t,x)$ is continuous in $[0,1] \times [0,\infty)$*

(H6) *There exist a function $p \in C([0,1],(0,\infty))$ and a nondecreasing function $q : (0,\infty) \to (0,\infty)$ such that*

$$t^{\sigma} f(t,u) \leqslant p(t)q(\|u\|)\ for\ all\ (t,u) \in [0,1] \times [0,\infty)$$

(H7) *There exists a constant $r > 0$ such that*

$$\frac{r\Gamma(\alpha+1-\sigma)}{\rho_2\Gamma(1-\sigma)\|p\|q(r)} > 1$$

Then, the problem (1.1), (1.2) has at least one positive solution on [0,1].

PROOF. The first step is to show that the operator T maps bounded sets into bounded set in E. For a positive number v, let $B_v = \{u \in E : \|u\| \leqslant v\}$ be a bounded set in E.

Then

$$\begin{aligned} |Tu(t)| &= \left| \int_0^1 G(t,s) f(s,u(s)) ds \right| \\ &= \left| \int_0^1 G(t,s) s^{\sigma} s^{-\sigma} f(s,u(s)) ds \right| \\ &\leqslant \int_0^1 \rho_2 \gamma(s) s^{-\sigma} p(s) q(\|u\|) ds \\ &\leqslant \rho_2 \frac{\Gamma(1-\sigma)}{\Gamma(\alpha+1-\sigma)} \|p\| q(v) \end{aligned}$$

and consequently,

$$\|Tu\| \leqslant \rho_2 \frac{\Gamma(1-\sigma)}{\Gamma(\alpha+1-\sigma)} \|p\| q(v)$$

Hence, $T(B_v)$ is uniformly bounded.

The next step is to verify that the operator T maps bounded sets into equicontinuous sets of E. Let $t_1 < t_2$ and $t_1, t_2 \in [0,1]$ for $u \in B_v$. So we have,

$$\begin{aligned} \big|Tu(t_2) - Tu(t_1)\big| &\leqslant \int_0^1 \big|G(t_2,s) - G(t_1,s)\big| s^{\sigma} s^{-\sigma} f(s,u(s)) ds \\ &\leqslant \|p\| q(v) \Bigg[-\frac{t_1 B_1}{\triangle\Gamma(\alpha)} \sum_{j=1}^{\infty} \eta_j + \frac{t_2 B_1}{\triangle\Gamma(\alpha)} \sum_{j=1}^{\infty} \eta_j + t_2^{\alpha-\sigma} B_2 \\ &\quad - \frac{t_2 B_2}{\triangle\Gamma(\alpha)} \sum_{j=1}^{\infty} \eta_j - t_1^{\alpha-\sigma} B_2 + \frac{t_1 B_2}{\triangle\Gamma(\alpha)} \sum_{j=1}^{\infty} \eta_j \Bigg] \end{aligned}$$

Hence, by Arzela-Ascoli theorem, the operator T is completely continuous since the right hand side tends to zero independent of $u \in B_v$ as $t_2 \to t_1$.

Let $U = \{u \in E : \|u\| < r\}$. We claim that there is no $u \in \partial U$, such that $u = \lambda(Tu)$ for $\lambda \in (0,1)$.

Let's admit that $u = \lambda(Tu)$ for all $\lambda \in (0,1)$ and for $t \in [0,1]$.

$$\begin{aligned}
\|u\| &= \|\lambda(Tu)\| \\
&= \max_{t\in[0,1]} \left| \lambda \int_0^1 G(t,s) f(s,u(s))ds \right| \\
&= \max_{t\in[0,1]} \left| \lambda \int_0^1 G(t,s) s^{\sigma} s^{-\sigma} f(s,u(s))ds \right| \\
&\leqslant \rho_2 \int_0^1 \gamma(s) s^{-\sigma} p(s) q(\|u\|)ds \\
&\leqslant \rho_2 \left[\frac{\Gamma(1-\sigma)}{\Gamma(\alpha+1-\sigma)} \|p\| q(r) \right]
\end{aligned}$$

This gives

$$\frac{r\Gamma(\alpha+1-\sigma)}{\rho_2 \Gamma(1-\sigma)\|p\|q(r)} \leqslant 1$$

which is contradiction with (H7). There is no $u \in \partial U$ that can satisfy $u = \lambda(Tu)$ for $\lambda \in (0,1)$. By the nonlinear alternative theorem of Leray Schauder type (Theorem 1.2) we deduce that T has a fixed point $u \in \overline{U}$ which is a solution of the problem (1.1), (1.2).

This completes the proof.

□

Example 3.1. Consider the following boundary value problem:

$$\begin{cases} {}^{c}D_{0^+}^{\frac{7}{2}} u(t) + f(t,u(t)) = 0, \quad 0 < t < 1 \\ u''(0) = u'''(0) = 0, \\ u'(0) = u(1) = \sum\limits_{j=1}^{\infty} \frac{1}{2^{j+4}} u\Big(1 - \frac{1}{2^j}\Big), \end{cases} \tag{3.1}$$

where

$$f(t,u) = \begin{cases} \frac{19}{12\sqrt{t}}\big(1+\sqrt[3]{u}\big), & (t,u) \in \text{ (0,1] x [0,2]} \\ \frac{300\sqrt[6]{u}}{\sqrt{\pi t}}, & (t,u) \in \text{ (0,1] x [4,9]} \\ \frac{299}{\sqrt{t}}, & (t,u) \in \text{ (0,1] x [100,}\infty) \end{cases}$$

$\alpha = \frac{7}{2}$, $\sigma = \frac{1}{2}$, $k = \frac{1}{4}$, $\zeta = \frac{2}{4}$, $\eta_j = \frac{1}{2^{j+4}}$, $\xi_j = 1 - \frac{1}{2^j}$, $\triangle \approx 0,95833$, $\gamma(s) = \frac{8(1-s)^{\frac{5}{2}}}{15\sqrt{\pi}}$, $r \approx 0,3147359$, $N \approx 0,0238184$, $\rho_1 \approx 0,82322$, $\rho_2 \approx 1,06521$.

Let we choose a= 2, b=4, c=9, d=300.

We see that all the conditions of Theorem 3.1 are satisfied. Namely, the problem (3.1) has at least three positive solutions satisfying u_1,u_2 and u_3

$$\varphi(u_j) \leqslant 300, \quad for\ j = 1, 2, 3$$

and

$$4 < \phi(u_1), \quad 2 < \psi(u_2), \quad \phi(u_2) < 4, \quad 2 > \psi(u_3).$$

Example 3.2. Consider the following boundary value problem:

$$\begin{cases} {}^cD_{0^+}^{\frac{15}{4}}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1 \\ u^{''}(0) = u^{'''}(0) = 0, \\ u^{'}(0) = u(1) = \sum\limits_{j=1}^{\infty} \frac{1}{2^{j+4}}\, u\big(1 - \frac{1}{2^j}\big) \end{cases} \tag{3.2}$$

where $f(t, u) = \frac{e^{-t}+2}{\sqrt[3]{t}}(u + 5)$, $\alpha = \frac{15}{4}$, $\sigma = \frac{1}{3}$, $\eta_j = \frac{1}{2^{j+4}}$, $\xi_j = 1 - \frac{1}{2^j}$. So, we get $\Delta \approx 0,95833$, $\rho_2 \approx 1,06522$. Clearly $t^{1/3}f(t, u) \leqslant p(t)q(r)$ with $p(t) = e^{-t} + 2$, $q(r) = r + 5$. Consequently,

$$\frac{r\Gamma(\alpha + 1 - \sigma)}{\rho_2\Gamma(1 - \sigma)\|p\|q(r)} = \frac{r\Gamma(\frac{53}{12})}{(1,06522)\Gamma(\frac{2}{3})3(r + 5)} > 1$$

is holds when $r > 3,58092$. So, Theorem 3.2 all conditions of satisifed. Thus, this problem has at least one positive solution.

References

1. R. P. Agarwal, M. Meehan, and D. O'regan, *Fixed point theory and applications*, Cambridge university press, **141** (2001).
2. R. P. Agarwal, D. O'Regan, and S. Stanek, *Positive solutions for drichlet problems of singular nonlinear fractional differential equations*, J. Math. Anal. Appl., *371*, (2010), 57–68.
3. B. Ahmad, J. Henderson, and R. Luca, *Boundary value problems for fractional differential equations and systems*, Trends in abstract and applied analysis, Vol. 9, USA, (2021), 451.
4. R. I. Avery and A. C. Peterson, *Three positive fixed points of nonlinear operators on order Banach spaces*, Comput. Math. Appl. **42** (2001), 313–322.
5. Z. Bai and W. Sun, *Existence and multiplicity of positive solutions for singular fractional boundary value problems*, Comput. Math. Apll., **63** (2012), 1369–1381.
6. J. Caballero, J. Harjani, and K. Sadarangani, *Positive solutions for a class of singular fractional boundary value problems*, Comput. Math. Appl., Vol. 62, No. 3, (2011), 1325-1332.
7. A. Granas and J. Dugundji, *Fixed point theory*, Springer **14**(2003).
8. L. Guo, L. Liu, and Y. Wu, *Existence of positive solutions for singular fractional differential equations with infinite-point boundary conditions*, Nonlinear Analysis: Modelling and Control, Vol. 21, No. 5, (2015), 635–650.
9. J.Henderson and R. Luca, *Existence of positive solutions for a singular fractional boundary value problem*, Nonlinear Analysis: Modelling and Control, Vol. 22, No. 1, (2016) 99–114.
10. J. Henderson and R. Luca, *Boundary value problems for sytems of differential, difference and fractioan equations*, Amsterdam, (2016), 307.

11. W. Jiang, X. Huang, W. Guo, and Q. Ziang, *The existence of positive solutions for the singular fractional differential equation*, J. Appl. Math. Comput., **41** (2013), 171–182.
12. A. A. Kilbas, H.M. Srivastava, and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Math. Stud., Vol. 204, Elsevier, Amsterdam, (2006).
13. K. Miller and B. Ross, *An introduction to the fractional calculus and fractional differntial equations*, Canada, (1993), 366.
14. I. Podlubny, *Fractional differential equations, mathematics in science and engineering* Academic Press, New York, (1999), 340.
15. T. Qui and Z. Bai, *Existence of positive solutions for singular fractional differential equations*, Electronic Journal of Differential Equations, Vol. (2008), No. 146, 1–9.
16. J. Sabatier, O.P. Agrawal, and J.A.T. Machado (Eds.), *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, (2007).
17. G. Samko, A. A. Kilbas, and O. Marichev, *Fractional integrals and derivatives: Theory and applications*, Gordon and Breach, Switzerland, (1993), 976.
18. S. Stanek, *The existence of positive solutions of singular fractional boundary value problems*, Comput. Math. Appl., **62** (2011), 1379–1388.
19. X. Su, *Positive solutions to singular boundary value problems for fractional functional differential equations with changing sign nonlinearity*, Comput. Math. App., **64** (2012), 3425–3435.
20. X. Xu, D. Jiang, and C. Yuan, *Multiple positive solutions to singular positone and semi-positone Dirichlet-type boundary value problems of nonlinear fractional differential equations*, Nonlinear Arial, **74** (2011), 5685–5696.
21. X. Zhang, *Positive solutions for a class of singular fractional differential equation with infinite-point boundary value conditions*, Appl. Math. Lett., (2014).
22. X. Zhang and Q. Zhong, *Triple positive solutions for nonlocal fractional differential equations with singularities both on time and space variables*, Appl. Math. Lett.,(2018).
23. X. Zhang, L.Liu, and Y. Wu, *The eigenvalue problem for a singular higher order fractional differential equation involving fractional derivatives*, Apll. Math. Comput., **218** (2012), 8526–8536.
24. S. Zhang, *Positive solutions to boundary value problem for nonlinear fractional differential equation*, Comput Math Apll. **59** (2010), 1300–1309.
25. X. Zhang, L. Liu, and Y, Wu *The uniquennes of positive solution for a singular fractional differential system involving derivatives*, Commun nonliear science and numerical simulation, **18** (2013), 1400–1409.

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