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# BI-QUASI-INTERIOR IDEALS OF SEMIRING

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ABSTRACT. In this paper, as a further generalization of ideals, we introduce the notion of bi-quasi interior ideals as a generalization of bi-ideal, quasi ideal, interior ideal, bi-interior ideal, bi-quasi ideal of semiring and study their properties.

## 1. Introduction

The notion of a semiring was introduced by Vandiver [**30**] in 1934, but semirings had appeared in earlier studies on the theory of ideals of rings. A universal algebra $(S,+,\cdot)$ is called a semiring if and only if $(S,+),(S,\cdot)$ are semigroups which are connected by distributive laws, *i.e.*, $a(b+c)=ab+ac,\ (a+b)c=ac+bc,$ for all $a,b,c\in S$. The theory of rings and theory of semigroups have considerable impact on the development of theory of semirings. Semirings play an important role in studying matrices and determinants. Semirings are useful in the areas of theoretical computer science as well as in the solution of graph theory, optimization theory, in particular for studying automata, coding theory and formal languages. Ideals play an important role in advance studies and uses of algebraic structures.

Generalization of ideals of algebraic structures and algebraic structure plays a very remarkable role and also necessary for further advance studies and applications of various algebraic structures. Many mathematicians proved important results and characterization of algebraic structures by using the concept and the properties of generalization of ideals in algebraic structures.

Between 1980 and 2016 there have been no new generalization of ideals.Then the author [**19, 20**] introduced and studied weak interior ideals, bi-interior ideals,

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bi quasi ideals, quasi interior ideals and bi quasi interior ideals of $\Gamma-$semirings, semirings, $\Gamma-$semigroups, semigroups as a generalization of bi-ideal, quasi ideal and interior ideal of algebraic structures and charecterized regular algebraic structures as well as simple algebraic structures using these ideals. Murali Krishna Rao et al. [**21**, **23**, **24**] introduced the notion of $\Gamma-$ incline and studied generalized right derivation of $\Gamma-$ incline and right derivation of ordered $\Gamma-$semiring.

We know that the notion of a one sided ideal of any algebraic structure is a generalization of an ideal. The quasi ideals are generalization of left ideal and right ideal whereas the bi-ideals are generalization of quasi ideals. In 1952, the concept of bi-ideals was introduced by Good and Hughes [**2**] for semigroups. The notion of bi-ideals in rings and semigroups were introduced by Lajos and Szasz [**9**, **10**]. Bi-ideal is a special case of ( m-n) ideal. In 1976, the concept of interior ideals was introduced by Lajos [**11**] for semigroups. Steinfeld [**29**] first introduced the notion of quasi ideals for semigroups and then for rings. Iseki and Izuka [**4**–**6**] introduced the concept of quasi ideal for a semiring. Quasi ideals, bi-ideals in $\Gamma-$semirings studied by Jagtap and Pawar [**8**]. Henriksen [**3**] and Shabir et al. [**28**] studied ideals in semirings. During 2016-2018, Murali Krishna Rao et al. [**13**–**15**] studied ideals in $\Gamma-$semirings, semirings and semigroups. In this paper, as a further generalization of ideals, we introduce the notion of bi- quasi interior ideal as a generalization of bi-ideal, quasi ideal, interior ideal, bi-interior ideal and bi-quasi ideal of semiring and study the properties of semiring. Some charecterization of bi-quasi-interior ideals of semiring, regular semiring and simple semiring. A necessary and sufficient condition for a semiring to be regular and simple is proved using bi- quasi interior ideals semiring.

## 2. Preliminaries

In this section, we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1. [1] A set $S$ together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) will be called semiring provided

(i) addition is a commutative operation.
(ii) multiplication distributes over addition both from the left and from the right.
(iii) there exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$.

Example 2.1. Let $M$ be the set of all natural numbers. Then ($M$, $max$, $min$) is a semiring.

Definition 2.2. Let $M$ be a semiring. If there exists $1 \in M$ such that $a \cdot 1 = 1 \cdot a = a$, for all $a \in M$, is called an unity element of $M$ then $M$ is said to be semiring with unity.

Definition 2.3. An element $a$ of a semiring $S$ is called a regular element if there exists an element $b$ of $S$ such that $a = aba$.

Definition 2.4. A semiring $S$ is called a regular semiring if every element of $S$ is a regular element.

Definition 2.5. An element $a$ of a semiring $S$ is called a multiplicatively idempotent (an additively idempotent) element if $aa = a(a + a = a)$.

Definition 2.6. An element $b$ of a semiring $M$ is called an inverse element of $a$ of $M$ if $ab = ba = 1$.

Definition 2.7. A non-empty subset $A$ of semiring $M$ is called

(i) a subsemiring of $M$ if $A$ is an additive subsemigroup of $M$ and $AA \subseteq A$.
(ii) a left(right) ideal of $M$ if $A$ is an additive subsemigroup of $M$ and $MA \subseteq A(AM \subseteq A)$.
(iii) an ideal if $A$ is an additive subsemigroup of $M$, $MA \subseteq A$ and $AM \subseteq A$.
(iv) a $k-$ideal if $A$ is a subsemiring of $M, AM \subseteq A, MA \subseteq A$ and $x \in M,\ x+y \in A, y \in A$ then $x \in A$.

Definition 2.8. A semiring $M$ is called a division semiring if for each non-zero element of $M$ has multiplication inverse.

Definition 2.9. A non-empty subset $A$ of semiring $M$ is called

(i) an interior ideal of $M$ if $A$ is a subsemiring of $M$ and $MAM \subseteq A$.
(ii) a quasi ideal of $M$ if $A$ is a subsemiring of $M$ and $AM \cap MA \subseteq A$.
(iii) a bi-ideal of $M$ if $A$ is a subsemiring of $M$ and $AMA \subseteq A$.

Definition 2.10. A non-empty subset $B$ of semiring $M$ is said to be bi-interior ideal of $M$ if $B$ is a subsemiring of $M$ and $MBM \cap BMB \subseteq B$.

Definition 2.11. Let $M$ be a semiring. A non-empty subset $L$ of $M$ is said to be left bi-quasi ideal (right bi-quasi ideal) of $M$ if $L$ is a subsemigroup of $(M, +)$ and $ML \cap LML \subseteq L$ $(LM \cap LML \subseteq L)$.

Definition 2.12. Let $M$ be a semiring. $L$ is said to be bi-quasi ideal of $M$ if it is both a left bi-quasi and a right bi-quasi ideal of $M$.

Example 2.2.

(i) Let $Q$ be the set of all rational numbers, $M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in Q \right\}$ be the additive semigroup of $M$ matrices.With respect to usual matrix multiplication, M is a semiring

(a) If $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid 0 \neq a, 0 \neq b \in Q \right\}$ then $R$ is a quasi ideal of semiring $M$ and $R$ is neither a left ideal nor a right ideal.
(b) If $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid 0 \neq a \in Q \right\}$ then $S$ is a bi-ideal of semiring $M$.

(ii) If $M = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in Q \right\}$ and $\Gamma = M$ then $M$ is a semiring with respect to usual addition of matrices and ternary operation is defined

as usual matrix multiplication and $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid 0 \neq a, 0 \neq b \in Q \right\}$.
Then $A$ is not a bi-ideal of semiring $M$.

DEFINITION 2.13. A semiring $M$ is called a left bi-quasi simple semiring if $M$ has no left bi-quasi ideal other than $M$ itself.

## 3. Bi-quasi-interior ideals of semirings

In this section we introduce the notion of bi-quasi-interior ideal as a generalization of bi-ideal, quasi-ideal and interior ideal of semiring and study the properties of bi-quasi interior ideal of semiring.

DEFINITION 3.1. A non-empty subset $B$ of semiring $M$ is said to be bi-quasi interior ideal of $M$ if $B$ is a subsemiring of $M$ and $BMBMB \subseteq B$

Every bi-quasi-interior ideal of semiring $M$ need not be bi-ideal, quasi-ideal, interior ideal bi-interior ideal. and bi-quasi ideals of semiring $M$.

EXAMPLE 3.1. Let $N$ be a set of all natural numbers. $N$ is additive abelian semigroup with respect to usual addition of integers.Multiplication operation is also defined as the usual addition of integers. Then $N$ is a semiring. A subset $I = 2N$ of $N$ is a bi-quasi interior ideal of $N$ but not bi-ideal, quasi-ideal, interior ideal, bi-interior ideal and bi-quasi ideal of semiring $N$.

THEOREM 3.1. *Every bi-ideal of semiring $M$ is a bi-quasi interior ideal of semiring $M$.*

PROOF. Let $B$ be a bi-ideal of semiring $M$. Then $BMB \subseteq B$.
Therefore $BMBMB \subseteq BMB \subseteq B$.
Hence every bi-ideal of semiring $M$ is a bi-quasi-interior ideal of $M$. □

THEOREM 3.2. *Every interior ideal of semiring $M$ is a bi-quasi interior ideal of $M$.*

PROOF. Let $I$ be an interior ideal of semiring $M$.
Then $IMIMI \subseteq MIM \subseteq I$.
Hence $I$ is a bi-quasi interior ideal of semiring $M$. □

In the following theorems, we mention some important properties and we omit the proofs since proofs are straightforward.

THEOREM 3.3. *Let $M$ be a semiring. Every left ideal is a bi-quasi interior of $M$.*

THEOREM 3.4. *Let $M$ be a semiring. Every right ideal is a bi-quasi interior of $M$.*

THEOREM 3.5. *Let $M$ be a semiring.Every quasi ideal is bi-quasi interior of $M$.*

THEOREM 3.6. *Let $M$ be a semiring. Every ideal is a bi-quasi interior ideal of $M$.*

Theorem 3.7. *Let $M$ be a semiring. Intersection of a right ideal and a left ideal of $M$ is a bi-quasi interior ideal of $M$.*

Theorem 3.8. *Let $M$ be a semiring. If $L$ is a left ideal and $R$ is a right ideal of semiring $M$ then $B = RL$ is a bi-quasi interior ideal of $M$.*

Theorem 3.9. *Let $M$ be a semiring. If $B$ is a bi-quasi-interior ideal and $T$ is a subsemiring of $M$ then $B \cap T$ is a bi-quasi interior ideals of semiring $M$.*

Theorem 3.10. *Let $M$ be a semiring and $B$ be a subsemiring of $M$.If $BMMMB \subseteq B$ then $B$ is a bi-quasi interior ideal of $M$.*

Theorem 3.11. *Let $M$ be a semiring and $B$ be a subsemiring of $M$.If $MMMMB \cap BMMMM \subseteq B$ then $B$ is a bi-quasi interior ideal of $M$.*

Theorem 3.12. *Let $M$ be a semiring and $B$ be a subsemiring of $M$. $B$ is a bi-quasi interior ideal of $M$ if and only if there exist a left ideal $L$ and a right ideal $R$ of $M$ such that $RL \subseteq B \subseteq R \cap L$ .*

Proof. Suppose $B$ is a bi-quasi interior ideal of semiring$M$.
Then $BMBMB \subseteq B$. Let $R = BMBM$ and $L = MB$.
Then $L$ and $R$ are left and right ideals of $M$ respectively.
Therefore $RL \subseteq B \subseteq R \cap L$ .
Conversely suppose that there exist $L$ and $R$ are left and right ideals of $M$ respectively such that $RL \subseteq B \subseteq R \cap L$ .Then
$BMBMB \subseteq (R \cap L)M(R \cap L)M(R \cap L)$
$\subseteq RMRML$
$\subseteq RL \subseteq B$ .
Hence $B$ is a bi-quasi interior ideal of semiring $M$. □

Theorem 3.13. *The intersection of a bi-quasi interior ideal $B$ of semiring $M$ and a right ideal $A$ of $M$ is always bi-quasi interior ideal of $M$.*

Proof. Suppose $C = B \cap A$.

$$CMCMC \subseteq BMBMB \subseteq B$$
$$CMCMC \subseteq AMAMA \subseteq A\ , \quad \text{since } A \text{ is a left ideal of } M$$

Therefore, $\quad CACMCM \subseteq B \cap A = C.$

Hence the intersection of a bi-quasi interior ideal $B$ of semiring $M$ and a subsemiring $A$ of $M$ is always bi-quasi interior ideal of $M$. □

Theorem 3.14. *Let $A$ and $C$ be bi-quasi interior ideals of semiring $M$ and $B = AC$. If $CC = C$ then $B$ is a bi-quasi-interior ideal of $M$.*

Proof. Let $A$ and $C$ be bi-quasi interior ideals of semiring $M$ and $B = AC$. $BB = ACAC = ACCCMCAC \subseteq ACMCMC \subseteq AC = B$. Obviously $B = AC$ is a subsemiring of $M$

$$BMBMB = ACMACMAC$$
$$\subseteq AMAMAC \subseteq AC = B.$$

Hence $B$ is a bi-quasi interior ideal of $M$. □

Corollary 3.1. *Let $A$ and $C$ be bi-quasi interior ideals of semiring $M$ and $B = CA$. If $CC = C$ then $B$ is a bi-quasi-interior ideal of $M$.*

Theorem 3.15. *Let $A$ and $C$ be subsemirings of $M$ and $B = AC$. If $A$ is the left ideal then $B$ is a bi-quasi-interior ideal of $M$.*

Proof. Let $A$ and $C$ be subsemirings of $M$ and $B = AC$ Suppose $A$ is the left ideal of $M$. $BB = ACAC \subseteq AC = B$.

$$\begin{aligned} BMBMB &= ACMACMAC \\ &\subseteq AC = B. \end{aligned}$$

Hence $B$ is a bi-quasi interior ideal of $M$. □

Corollary 3.2. *Let $A$ and $C$ be subsemirings of semiring $M$ and $B = AC$. If $C$ is a right ideal then $B$ is a bi-quasi-interior ideal of $M$.*

Theorem 3.16. *Let $M$ be a semiring and $T$ be a non-empty subset of $M$. Then every subsemiring of $T$ containing $TMTMT$ is a bi-quasi interior ideal of semiring $M$.*

Proof. Let $B$ be a subsemiring of $T$ containing $TMTMT$. Then

$$\begin{aligned} BMB\Gamma MB &\subseteq TMTMT \\ &\subseteq B. \end{aligned}$$

Therefore, $BMBMB \subseteq B$.
Hence $B$ is a bi-quasi interior ideal of $M$. □

Theorem 3.17. *$B$ is a bi-quasi interior ideal of semiring $M$ if and only if $B$ is a left ideal of some right ideal of semiringM.*

Proof. Suppose $B$ is a left ideal of some right ideal $R$ of semiring $M$. Then

$$MB \subseteq B, RM \subseteq B.$$

Hence $BMBMB \subseteq BMB \subseteq RMB \subseteq RB \subseteq B$. Therefore $B$ is a bi-quasi-interior ideal of semiring $M$.

Conversely, suppose that $B$ is a bi-quasi interior ideal of semiring $M$. Then

$$BMBMB \subseteq B.$$

Therefore $B$ is a left ideal of right ideal $BMBM$ of semiring $M$. □

Corollary 3.3. *$B$ is a bi-quasi interior ideal of semiring $M$ if and only if $B$ is a right ideal of some left ideal of semiringM.*

Theorem 3.18. *If $B$ is a bi-quasi interior ideal of semiring $M$, $T$ is a subsemiring of $M$ and $T \subseteq B$ then $BT$ is a bi-quasi interior ideal of $M$.*

PROOF. Obviously, $BT$ is a $\Gamma-$subsemigroup of $(M,+)$. $BTBT \subseteq BT$. Hence $BT$ is a subsemiring of $M$.

$$\begin{aligned} \text{We have } & MBTM \subseteq MBM \\ \text{and } & BTMBT \subseteq BMB \\ \Rightarrow & BTMBTMBT \subseteq BMBMB\Gamma T \subseteq B\Gamma T. \end{aligned}$$

Hence $BT$ is a bi-quasi interior ideal of semiring $M$. □

THEOREM 3.19. *Let $B$ be bi-ideal of semiring $M$ and $I$ be interior ideal of $M$. Then $B \cap I$ is bi-quasi-interior ideal of $M$.*

PROOF. Obviously $B \cap I$ is subsemiring of $M$. Suppose $B$ is a bi-ideal of $M$ and $I$ is an interior ideal of $M$. Then

$$(B \cap I)M(B \cap I)M(B \cap I) \subseteq BMBMB \subseteq B,$$
$$(B \cap I)M(B \cap I)M(B \cap I) \subseteq IMIMI \subseteq I$$

Therefore, $(B \cap I)M(B \cap I)M(BI)M \subseteq B \cap I$.
Hence $B \cap I$ is a bi-quasi interior ideal of $M$. □

THEOREM 3.20. *Let $M$ be a semiring and $T$ be an additive subsemigroup of $M$. Then every additive subsemigroup of $T$ containing $TMTMT$ is a bi-quasi-interior ideal of $M$.*

PROOF. Let $C$ be an additive subsemigroup of $T$ containing $TMTMT$. Then

$$\begin{aligned} CMCMC &\subseteq TMTMT \\ &\subseteq C. \end{aligned}$$

Hence $C$ is a bi-quasi-interior ideal of semiring $M$. □

THEOREM 3.21. *Let $M$ be a semiring. If $M = MA$, for all $a \in M$. Then every bi-quasi-interior ideal of $M$ is a quasi ideal of $M$.*

PROOF. Let $B$ be a bi-quasi interior ideal of a semiring $M$ and $a \in B$. Then

$$\begin{aligned} & BMBMB \subseteq B \\ \Rightarrow & MA \subseteq MB, \ (BM = M) \\ \Rightarrow & M \subseteq MB \subseteq M \\ \Rightarrow & MB = M \\ \Rightarrow & BM = BMB \subseteq BMBMB \subseteq B \\ \Rightarrow & MB \cap BM \subseteq MM \cap BM \subseteq B. \end{aligned}$$

Therefore $B$ is a quasi ideal of $M$. □

THEOREM 3.22. *The intersection of $\{B_\lambda \mid \lambda \in A\}$ bi-quasi-interior ideals of a semiring $M$ is a bi-quasi interior ideal of $M$.*

Proof. Let $B = \bigcap_{\lambda \in A} B_\lambda$. Then $B$ is a subsemiring of $M$.
Since $B_\lambda$ is a bi-quasi interior ideal of $M$, we have

$$B_\lambda M B_\lambda M B_\lambda \subseteq B_\lambda, \text{ for all } \lambda \in A$$
$$\Rightarrow \cap B_\lambda M \cap B_\lambda M \cap B_\lambda) \subseteq \cap B_\lambda$$
$$\Rightarrow BMBMB \subseteq B.$$

Hence $B$ is a bi-quasi interior ideal of $M$. □

Theorem 3.23. *Let $B$ be a bi-quasi interior ideal of semiring $M$,$e \in B$ and $e$ be an idempotent. Then $eB$ is a bi-quasi interior ideal of $M$.*

Proof. Let $B$ be a bi-quasi interior ideal of semiring $M$. Suppose $x \in B \cap eM$. Then $x \in B$ and $x = ey, y \in M$.

$$\begin{aligned} x &= ey \\ &= eey \\ &= e(ey) \\ &= ex \in eB. \end{aligned}$$
$$\text{Therefore } B \cap eM \subseteq eB$$
$$eB \subseteq B \text{ and } eB \subseteq eM$$
$$\Rightarrow eB \subseteq B \cap eM$$
$$\Rightarrow eB = B \cap eM.$$

Hence $eB$ is a bi-quasi interior ideal of $M$. □

Corollary 3.4. *Let $M$ be a semiring $M$ and $e$ be an idempotent. Then $eM$ and $Me$ are bi-quasi interior ideals of $M$.*

Theorem 3.24. *If $B$ be a left bi-quasi ideal of semiring $M$, then $B$ is a bi-quasi interior ideal of $M$.*

Proof. Suppose $B$ is a left bi-quasi ideal of semiring $M$.Then $BMBMB \subseteq MB$ and $BMBMB \subseteq BMB$.
Therefore,

$$BMBMB \subseteq MB \cap MBM \subseteq B.$$

Hence, $B$ is a bi-quasi interior ideal of $M$. □

Corollary 3.5. *If $B$ be a right bi-quasi ideal of semiring $M$, then $B$ is a bi-quasi interior ideal of $M$.*

Corollary 3.6. *If $B$ be a bi-quasi ideal of semiring $M$, then $B$ is a bi-quasi interior ideal of $M$.*

Theorem 3.25. *If $B$ be a bi-interior ideal of semiring of $M$, then $B$ is a bi-quasi interior ideal of $M$.*

PROOF. Suppose $B$ is a bi-interior ideal of semiring $M$.Then

$$BMBMB \subseteq MBM$$

and

$$BMBMB \subseteq MBM.$$

Therefore,

$$BMBMB \subseteq MBM \cap MBM \subseteq B.$$

Hence $B$ is a bi-quasi interior ideal of $M$. □

We introduce the notion of bi-quasi-interior simple semiring and characterize the bi-quasi-interior simple semiring using bi-quasi-interior ideals of semiring and study the properties of minimal bi-quasi interior ideals of semiring..

DEFINITION 3.2. A semiring $M$ is said to be bi-quasi-interior simple semiring if $M$ has no bi-quasi interior ideals other than $M$ itself.

THEOREM 3.26. *If $M$ is a division semiring then $M$ is bi-quasi interior simple semiring.*

PROOF. Let $B$ be a proper bi-quasi interior ideal of division semiring and $0 \neq a \in B$. Since $M$ is a division semiring, there exists $b \in M$ such that $ab = 1$. Then $abx = x = xab$,for all $x \in M$. Then $x \in BM$. Therefore $M \subseteq BM$. We have $BM \subseteq M$. Hence $M = BM$. Similarly we can prove $MB = M$.

$$\begin{aligned} M &= MB \\ &= BMBMB \subseteq B \\ M &\subseteq B \\ \text{Therefore,} \quad M &= B. \end{aligned}$$

Hence division semiring $M$ has no proper bi-quasi interior ideals. □

THEOREM 3.27. *Let $M$ be a simple semiring. Every bi-quasi-interior ideal is bi-ideal of $M$.*

PROOF. Let $M$ be a simple semiring and $B$ be a bi-quasi interior ideal of $M$. Then $BMBMB \subseteq B$ and $MBM$ is an ideal of $M$.
Since $M$ is a simple semiring, we have $MBM = M$. Hence

$$\begin{aligned} &BMBMB \subseteq B \\ \Rightarrow &BMB \subseteq B. \end{aligned}$$

□

THEOREM 3.28. *Let $M$ be a semiring. Then $M$ is a bi-quasi interior simple semiring if and only if $(a)_{bqi} = M$, for all $a \in M$, where $(a)_{bqi}$ is the bi-quasi interior ideal generated by $a$.*

Proof. Let $M$ be a semiring. Suppose that $(a)_{bqi}$ is a bi-quasi interior ideal generated by $a$ and $M$ is a bi-quasi interior simple semiring. Then $(a)_{bqi} = M$, for all $a \in M$.

Conversely suppose that $B$ is a bi-quasi-interior ideal of semiring $M$ and $(a)_{bqi} = M$, for all $a \in M$. Let $b \in B$.
Then $(b)_{bqi} \subseteq B \Rightarrow M = (b)_{bqi} \subseteq B \subseteq M$.
Therefore $M$ is a bi-quasi interior simple semiring. □

Theorem 3.29. *Let $M$ be a semiring. $M$ is a bi-quasi interior simple semiring if and only if $< a >= M$, for all $a \in M$ and where $< a >$ is the smallest bi-quasi interior ideal generated by $a$.*

Proof. Let $M$ be a semiring. Suppose $M$ is a bi-quasi interior simple semiring, $a \in M$ and $B = MA$.
Then $B$ is a left ideal of $M$.
Therefore, by Theorem 3.3, $B$ is a bi-quasi interior ideal of $M$.
Therefore $B = M$. Hence $Ma = M$, for all $a \in M$.

$$\begin{aligned} Ma &\subseteq < a > \subseteq M \\ \Rightarrow M &\subseteq < a > \subseteq M. \\ \text{Therefore } M &= < a > . \end{aligned}$$

Suppose $< a >$ is the smallest bi-quasi interior ideal of $M$ generated by $a$ and $< a >= M$ and $A$ is the bi-quasi interior ideal and $a \in A$. Then

$$\begin{aligned} < a > &\subseteq A \subseteq M \\ \Rightarrow M &\subseteq A \subseteq M. \end{aligned}$$

Therefore $A = M$. Hence $M$ is a bi-quasi-interior simple semiring. □

Theorem 3.30. *Let $M$ be a semiring. Then $M$ is a bi-quasi-interior simple semriring if and only if $aMaMa = M$, for all $a \in M$.*

Proof. Suppose $M$ is left bi-quasi simple semiring and $a \in M$.
Therefore $aMaMa$ is a bi-quasi interior ideal of $M$.
Hence $aMaMa = M$, for all $a \in M$.

Conversely suppose that $aMaMa = M$, for all $a \in M$.
Let $B$ be a bi-quasi interior ideal of semiring $M$ and $a \in B$.

$$\begin{aligned} M &= aMaMa \\ &\subseteq BMBMB \subseteq B \\ \text{Therefore,} \quad M &= B. \end{aligned}$$

Hence $M$ is a bi-quasi-interior simple semiring. □

Definition 3.3. A semiring $M$ is a left (right) simple semiring if $M$ has no proper left (right) ideal of $M$.

A semiring $M$ is said to be simple semiring if $M$ has no proper ideals.

Theorem 3.31. *If semiring $M$ is left simple semiring then every bi-quasi-interior ideal of $M$ is a right ideal of $M$.*

Proof. Let $B$ be a bi-quasi interior of left simple semiring. Then $MB$ is a left ideal of $M$ and $MB \subseteq M$. Therefore $MB = M$. Then

$$\begin{aligned} &MBM = MM = M\\ \Rightarrow &BM = BMB\\ \Rightarrow &\ BM = BMBMB \subseteq B\\ \Rightarrow &BM \subseteq B. \end{aligned}$$

Hence every bi-quasi interior ideal is a right ideal of $M$.

□

Corollary 3.7. *If semiring $M$ is right simple semiring then every bi-quasi-interior ideal of $M$ is a left ideal of $M$.*

Corollary 3.8. *Every bi-quasi-interior ideal of left and right simple semiring $M$ is an ideal of $M$.*

Theorem 3.32. *Let $M$ be a semiring and $B$ be bi-quasi-interior ideal of $M$. Then $B$ is minimal bi-quasi-interior ideal of $M$ if and only if $B$ is a bi-quasi-interior simple subsemiring.*

Proof. Let $B$ be a minimal bi-quasi-interior ideal of semiring $M$ and $C$ be a bi-quasi interior ideal of $B$. Then $CBCBC \subseteq C$.
Therefore $CBCBCB$ is a bi-quasi interior ideal of $M$.
Since $C$ is a bi-quasi interior ideal of $B$,

$$\begin{aligned} &CBCBC = B\\ \Rightarrow &B = CBCBC \subseteq C\\ \Rightarrow &B = C. \end{aligned}$$

Conversely suppose that $B$ is a bi-quasi-interior simple subsemiring of $M$. Let $C$ be a bi-quasi-interior ideal of $M$ and $C \subseteq B$.

$$\begin{aligned} &CBCBC = C\\ \Rightarrow &CBCBC \subseteq CMCMC \subseteq BMBMB \subseteq B,\\ \Rightarrow &B = C, \text{since } B \text{ is a bi-quasi interior simple semiring.} \end{aligned}$$

Hence $B$ is a minimal bi-quasi interior ideal of $M$. □

Theorem 3.33. *Let $M$ be a semiring and $B = RL$, where $L$ and $R$ are minimal left ideal and right ideal of $M$ respectively. Then $B$ is a minimal bi-quasi interior ideal of $M$.*

Proof. Obviously $B = RL$ is bi-quasi-interior ideal of $M$. Let $A$ be bi-quasi-interior ideal of $M$ such that $A \subseteq B$.

We have $MA$ is a right ideal. Then

$$\begin{aligned} MA &\subseteq MB \\ &= MRL \\ &\subseteq L, \text{ since } L \text{ is a left ideal of } M. \\ \text{Similarly, we can prove } AM &\subseteq R \\ \text{Therefore } MA &= L,\ AM = R \\ \text{Hence } \ B &= AMMA \\ &\subseteq AMAMA. \\ &\subseteq A \end{aligned}$$

Therefore $A = B$. Hence $B$ is a minimal bi-quasi interior ideal of $M$.

□

We characterize regular semiring using bi-quasi interior ideals of semiring.

THEOREM 3.34. *Let $M$ be a regular semiring. Then every bi-quasi interior ideal of $M$ is an ideal of $M$.*

PROOF. Let $B$ be a bi-quasi interior ideal of $M$. Then

$$\begin{aligned} &BMBMB \subseteq B \\ \Rightarrow &BM \subseteq BMB, \text{ since } M \text{ is regular} \\ \Rightarrow &BM \subseteq BMBMB \subseteq B. \end{aligned}$$

Similarly, we can show that $MB \subseteq BMBMB \subseteq B$. □

THEOREM 3.35. *Let $M$ be a regular semiring. Then $B$ is a bi-quasi-interior ideal of $M$ if and only if $BMBMB = B$, for all bi-quasi interior ideals $B$ of $M$.*

PROOF. Suppose $M$ is a regular semiring, $B$ is a bi-interior ideal of $M$ and $x \in B$. Then $BMBMB \subseteq B$ and there exist $y \in M$, such that $x = xyxyx \in BMBMB$.
Therefore $x \in BMBMB$.
Hence $BMBMB = B$.

Conversely, suppose that $BMBMB = B$, for all bi-quasi interior ideals $B$ of $M$.
Let $B = R \cap L$, where $R$ is a right ideal and $L$ is a left ideal of $M$. Then $B$ is a bi-interior ideal of $M$.
Therefore $(R \cap L)M(R \cap L)M(R \cap L) = R \cap L$

$$\begin{aligned} R \cap L &= (R \cap L)M(R \cap L)M(R \cap L) \\ &\subseteq RMLML \\ &\subseteq RL \\ &\subseteq R \cap L \text{ (since } RL \subseteq L \text{ and } RL \subseteq R). \end{aligned}$$

Therefore $R \cap L = RL$. Hence $M$ is a regular semiring. □

Theorem 3.36. *Let $B$ be the bi-quasi interior ideal of regular semiring $M$. If $B$ is a bi-quasi interior ideal of $M$ and $B$ is regular subsemiring of $M$ then any bi-quasi interior ideal of $B$ is a bi-quasi interior ideal of $M$.*

Proof. Let $A$ be a bi-quasi interior ideal of regular subsemiring $B$ of $M$. Then by Theorem[3.35], $ABABA = A$. We have $BMBMB = B$ and $A \subseteq AB$.

$$\begin{aligned}&\Rightarrow AMAMA \subseteq BMBMB = B\\&\Rightarrow ABABA = A \subseteq B\\&\Rightarrow A = ABABA \subseteq AMAMA\\&\Rightarrow AMAMA = A..\end{aligned}$$

Hence $A$ is a bi-quasi interior ideal of $M$. □

Theorem 3.37. *$M$ is regular semiring if and only if $AB = A \cap B$ for any right ideal $A$ and left ideal $B$ of semiring $M$.*

Theorem 3.38. *Let $B$ besubsemiring of regular semiring $M$. If $B$ can be represented as $B = RL$, where $R$ is a right ideal and $L$ is a left ideal of $M$ then $B$ is a bi-quasi interior ideal of $M$.*

Proof. Suppose $B = RL$, where $R$ is right ideal of $M$ and $L$ is a left ideal of $M$.

$$\begin{aligned}BMBMB &= RLMRLMRL\\&\subseteq RL = B.\end{aligned}$$

Hence $B$ is a bi-quasi interior ideal of semiring $M$.

Conversely suppose that $B$ is a bi-quasi interior ideal of regular semiring $M$. We have $BMBMB = B$. Let $R = BM$ and $L = MB$.
Then $R = BM$ is a right ideal of $M$ and $L = MB$ is a left ideal of $M$.

$$\begin{aligned}&BM \cap MB \subseteq BMBMB = B\\&\Rightarrow BM \cap MB \subseteq B\\&\Rightarrow R \cap L \subseteq B.\\\text{We have } &B \subseteq BM = R \text{ and } B \subseteq MB = L\\&\Rightarrow B \subseteq R \cap L\\&\Rightarrow B = R \cap L = RL, \text{ since } M \text{ is a regular semiring.}\end{aligned}$$

Hence $B$ can be represented as $RL$, where $R$ is the right ideal and $L$ is the left ideal of $M$. □

The following theorem is a necessary and sufficient condition for semiring $M$ to be regular using bi-quasi-interior ideal.

Theorem 3.39. *$M$ is a regular semriring if and only if $B \cap I \cap L \subseteq BIL$, for any bi-quasi interior ideal $B$, ideal $I$ and left ideal $L$ of $M$.*

PROOF. Suppose $M$ be a regular semiring, $B, I$ and $L$ are bi-quasi interior ideal, ideal and left ideal of $M$ respectively.
Let $a \in B \cap I \cap L$. Then $a \in aMa$, since $M$ is regular.

$$\begin{aligned} a &\in aMa \subseteq aMaMa \\ &\subseteq BIB \end{aligned}$$

Hence $B \cap I \cap L \subseteq BIL$.
Conversely suppose that $B \cap I \cap L \subseteq BIL$, for any bi-quasi interior ideal $B$, ideal $I$ and left ideal $L$ of $M$. Let $R$ be a right ideal and $L$ be left ideal of $M$. Then by assumption,

$$R \cap L = R \cap M \cap L \subseteq RML \subseteq RL$$
$$\text{We have } RL \subseteq R,\ RL \subseteq L.$$

Therefore $RL \subseteq R \cap L$. Hence $R \cap L = RL$.
Thus $M$ is a regular semiring. □

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