JOURNAL OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4866, ISSN (o) 2303-4947
www.imvibl.org / JOURNALS / JOURNAL
Vol. 9(2)(2019), 361-395
DOI: 10.7251/JIMVI1902361R

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN (p) 0354-5792, ISSN (o) 1986-521X

SOME ALGEBRAIC STRUCTURES WITH APARTNESS, A REVIEW

Daniel A. Romano

Abstract. The logical environment of this text is the Intuitionistic Logic - a logic without the principle of 'Tertium non datur' and the principled-philosophical orientation of the Bishop's constructive algebra. This orientation enables us to construct algebraic structures on the relational structures of the type $(X, =, \neq)$ as basic carriers where $' \neq '$ is a diversity relation / apartness. In the last forty years, many algebraic structures with apartness have been analyzed. In this article, we will expose the recapitulation of some of them and show some basic characteristics of the selected algebraic structures such as semigroups (ordered semigroup under co-quasiorder, semillatice-ordered semigroups, inverse semigroups, implicative semigroups, Γ-semigroups), groups (free Abalian groups, ordered group under co-order) rings (commutative rings, semillatice-ordered semigrings, Γ-semirings, modules over commutative rings).

1. Introduction

1.1. Logical Environment. Our setting is Bishop's constructive mathematics [**Bish**] ([**3**], [**4**], [**19**] and [**59**]), mathematics developed with Constructive logic (or Intuitionistic logic [**IL**] [**59**]) - logic without the Law of Excluded Middle $P \vee \neg P$ [TND]. We have to note that 'the crazy axiom' $\neg P \Longrightarrow (P \Longrightarrow Q)$ is included in the Constructive logic. Precisely, in Constructive logic the 'Double Negation Law' $P \Longleftrightarrow \neg\neg P$ does not hold, but the following implication $P \Longrightarrow \neg\neg P$ holds even in Minimal logic. In Constructive logic 'Weak Law of Excluded Middle' $\neg P \vee \neg\neg P$ does not hold as well. It is interesting, in Constructive logic the following deduction

---

2010 *Mathematics Subject Classification.* 03F65; 06F05, 13A15, 13C05, 20K99, 20M18.
Key words and phrases. Bishop constructive mathematics, algebraic structures with apartness, semigroups, groups, rings.

principle $A \vee B, \neg A \vdash B$ holds, but this is impossible to prove without 'the crazy axiom'.

Bishop's constructive mathematics includes the following two aspects:
(1) The Intuitionistic logic and
(2) The principled-philosophical orientations of constructivism.

1.2. Set with apartness. Intuitionistic logic does not accept the TND principle as an axiom. In addition, Intuitionistic logic does not accept the validity of the 'double negation' principle. This makes it possible to have a difference relation in sets which is not a negation of equality relation. Dual of the equality relations '=' in a set A is diversity relation '$\neq$'. Therefore, we accept that in Bishop's constructive mathematics we consider set A as one relational system $(A, =, \neq)$. Now, we look at the carrier A as a relational system $(A, =, \neq)$, where $' = '$ is the standard equality, and $' \neq '$ is an apartness [**3, 4**]:

$(\forall x, y \in A)(x \neq y \implies \neg(x = y))$ (consistency);
$(\forall x, y \in A)(x \neq y \implies y \neq x)$ (symmetry);
$(\forall x, y, z \in A)(x \neq z \implies (x \neq y \vee y \neq z))$ (co-transitivity).

This last relation is extensive in terms of equality in the following sense:

$= \circ \neq \subseteq \neq$ and $\neq \circ = \subseteq \neq$

where $' \circ '$ is the standard mark for the composition of the relations. It is obvious that the following connection between these relations is valid: $= \subseteq \neg \neq$. In this case for relations $=$ and $\neq$ we say that they are associate. So, it's quite natural to ask the question: Is there the maximal relation '$\neq$' such that it is associated with equality '='?

Generally speaking: Let S be a subset of set $(A, =, \neq)$ determined by a predicate $\mathfrak{P}$. The first task is to construct a dual T of the set S so that the subsets $\neg T = \{a \in A : \neg(a \in T)\}$ and its strong compliment $T^{\lhd} = \{a \in A : a \lhd T\}$ have property $\mathfrak{P}$ where $a \lhd T$ means $(\forall t \in T)(t \neq a)$. In addition, $T^{\lhd} \subseteq \neg T$ holds.

If the relation $' \neq '$ satisfies only the first two conditions, then it is said to be a '*diversity relation*'. Let X and Y be subsets of A. Let us determine $X \neq Y$ if is valid

$(\exists x \in X)\neg(x \in Y) \vee (\exists y \in Y)\neg(y \in X)$.

Obviously, this relation is not an apartness relation in the family $\mathfrak{P}(A)$ of all subsets of A. In this collection, an analogous relation to an apartness relation in a set can be introduced on axiomatic way as it is, for example, done in the [**7**] or, in the way it is shown in article [**16**]. We will write $X \bowtie Y$ if it is

$(\forall x \in X)(x \lhd Y) \wedge (\forall y \in Y)(y \lhd X)$.

This relation is a diversity relation in the family $\mathfrak{P}(A)$ of all subsets of A and it is not an apartness, in the general case.

For example, if $(A, =_A, \neq_A)$ and $(B, =_B, \neq_B)$ are sets with apartnesses, then at the $A \times B$ this relation is determined as follows

$(\forall x, x' \in A)(\forall y, y' \in B)((x, y) \neq (x', y') \iff (x \neq_A x' \vee y \neq_B y'))$.

For function $f : A \longrightarrow B$, it is said to be *strongly extensive* [**54**] if the following

$(\forall x, y \in A)(f(x) \neq_B f(y) \implies x \neq_A y)$

holds. For function $f : A \longrightarrow B$, it is said to be an *embedding* [**54**] if the following

$(\forall x, y \in A)(x \neq y \implies f(x) \neq_B f(y))$

holds.

In the text that follows, in the writing of relations of equality and the relation of apartnesses, we will always omit the indexes whenever it is possible and when it will not allow a different understanding of what the author imagined.

1.3. Concept of co-equality relations. Let ρ be an equivalence relation on the set A. For the relation q we say that it is a *co-equivalence* [**25, 34**] relation to A if and only if the following is valid

$q \subseteq \neq$, (consistency), $q^{-1} = q$, (symmetric) and $q \subseteq q * q$. (co-transitivity).

Here, '$*$' is the filed product between relations defined by the following way: If α and β are relations on set A, then filed product $\beta * \alpha$ of relation α and β is the relation given by $\{(x, z) \in A \times A : (\forall y \in A)((x, y) \in \alpha \vee (y, z) \in \beta)\}$. Of course, the strong compliment $q^{\triangleleft}$ of the relation q is an equivalence in A and the following

$q^{\triangleleft} \subseteq \neg q$, $q \circ q^{\triangleleft} \subseteq q$ and $q^{\triangleleft} \circ q \subseteq q$

are valid. Although the evidence of this claim is known, we will again show it here that the reader can gain an impression of the proof technique that is applied.

Proposition 1.1. *The strong compliment $q^{\triangleleft}$ of the relation q is an equivalence in A and the following $q^{\triangleleft} \subseteq \neg q$, $q \circ q^{\triangleleft} \subseteq q$ and $q^{\triangleleft} \circ q \subseteq q$ holds.*

Proof. Obviously, It is true that $= \subseteq q^{\triangleleft}$ and that $q^{\triangleleft}$ is a symmetric relation. We need to prove that $q^{\triangleleft}$ is a transitive relation. Let $x, y, z, u, v \in A$ arbitrary elements such that $(x.y) \triangleleft q$, $(y, z) \triangleleft q$ and $(u, v) \in q$. Then

$$(u, x) \in q \vee (x, y) \in q \vee (y, z) \in q \vee (z, v) \in q$$

by co-transitivity of q. From here follows $(u, x) \in q \subseteq \neq$ and $(z, v) \in q \subseteq \neq$ by consistency of q and by taking into account the hypothesis of this deduction. So, $u \neq x$ and $z \neq v$ therefore $(x, z) \neq (u, v) \in q$. Finally, we have $(x, z) \triangleleft q$. On this way, the transitivity of the relation q is proven.

For the sake of illustration, we will prove inclusion $q \circ q^{\triangleleft} \subseteq q$. The second inclusion can be proven in an analogous way. Let x, z be arbitrary elements of A such that $(x, z) \in q \circ q^{\triangleleft}$. Then there exists an element $y \in A$ such that $(x, y) \in q^{\triangleleft}$ and $(y, x) \in q$. Thus $(y, x) \in q$ or $(x, z) \in q$ by co-transitivity of q. Since, the first option is impossible because $q^{\triangleleft}$ is a symmetric relation on A and $(y, x) \triangleleft q$, we have $(x, z) \in q$. □

As corollary of above Proposition 1.1 we can construct the quotient-set

$A/(q^{\triangleleft}, q) = \{aq^{\triangleleft} : a \in A\}$

with $aq^{\triangleleft} = bq^{\triangleleft} \iff (a, b) \triangleleft q$ and $aq^{\triangleleft} \neq bq^{\triangleleft} \iff (a, b) \in q$. For the total surjective function $\pi : A \longrightarrow A/(q^{\triangleleft}, q)$, defined by the $\pi(a) = aq(a \in A)$, it is said that *the canonical mapping* from A onto $A/(q^{\triangleleft}, q)$.

For the family $\{qx\}_{x \in A}$, where $qx = \{y \in A : (x, y) \in q\}$, is true:

(i) $x \lhd xq$; (ii) $xq = qx$; (iii) $(x, y) \in q \implies xq \cup yq = A$.

Before proceeding to a further analysis, we recall the term 'strictly extensional subset of the set': for a subset X of the set $(A, =, \neq)$ it is said that a strictly extensional subset of the set A if

$(\forall x, y \in A)(x \in X \implies (x \neq y \lor y \in X))$.

valid. Now, suppose that a family $\{X_t\}_{t\in A}$ of strongly extensional proper subsets of A satisfies the following two conditions:

(a) For any $t \in A$ there exists a strongly extensional subset X_t such that that $t \lhd X_t$;

(b) $X_t \neq X_s \implies X_t \cup X_s = A$ for any $t, s \in A$.

Then the relation R on A defined by

$(x, y) \in R \iff (\exists u \in A)(x \in X_u \land y \lhd X_u)$

is a co-equality in A ([**34, 49**]). For a set $(A, =, \neq)$ and a co-equality relation q on A, the family $\{aq : a \in A\}$ we will indicate with $A : q$. Without major difficulties, it can be verified that there exists a surjective function $\theta : A \longrightarrow A : q$, determined by $\theta(a) = aq$, and the bijection (= strongly extensional surjective, injective and embedding function) $h : A/(q^{\lhd}, q) \longrightarrow A : q$ such that $\theta = h \circ \pi$ and $\pi = h^{-1} \circ \theta$, where $\pi : A \longrightarrow A/(q^{\lhd}, q)$ is the standard canonical surjective function.

Remark 1.1. When this result was first time published in 1996 [**33**], the academic public refused to accept that there were scientific needs for researching of such families of subsets of a set with apartness. The difficulties encountered by researchers of algebraic structures based on sets with apartness relations were nicely described by Bauer in his recently published article [**2**]. Even today, many working mathematicians are extremely reluctant to accept the possibility of existence an academic interest in researching and developing of algebraic structures in a logical environment that is not the Classic logic. Unfortunately, the existence of interest in publishing the results of the research of algebraic structures based on sets with apartness is still being classify in activity with the prefix of exoticism.

For couple ρ and q of a an equality relation and a co-equality relation we say it is associated if holds

$\rho \circ q \subseteq q$ and $q \circ \rho \subseteq q$.

The question naturally arises: For the given equivalence relation ρ is there a maximal co-equality relation q associated with ρ?

1.4. The concept of co-quasiorder relations. On the other hand, some relations can be derived from the co-equality concept by varying the conditions by which this relation is determined. For example, for a relation σ on a set $(A, =, \neq)$ we say that the relation is co-quasiorder if the following holds

$\sigma \subseteq \neq$, and $\sigma \subseteq \sigma * \sigma$.

The complement $\sigma^{\lhd}$ of this relation is a quasi-order relation on A. As the inclusion of $= \subseteq \sigma^{\lhd}$ is obvious, we show the transitivity of $\sigma^{\lhd}$. Let x, y, z, u, v be arbitrary elements of A such that $(x, y) \lhd q$, $(y, z) \lhd q$ and $(u, v) \in \sigma$. Then $(u, x) \in \sigma \subseteq \neq$ or $(x, y) \in \sigma$ or $(y, z) \in \sigma$ or $(z, v) \in \sigma \subseteq \neq$. Thus $u \neq x \lor z \neq v$ and $(x, z) \neq$

$(u, v) \in \sigma$. So, the relation $\sigma^{\triangleleft}$ is a transitive relation. Finally, the relation $\sigma^{\triangleleft}$ is a quasi-order on A.

In addition, for the relation σ on $(A, =, \neq)$ we say that it is a co-order relation on A if the follows holds

$\sigma \subseteq \neq, \quad \sigma \subseteq \sigma * \sigma \quad \text{and} \quad \neq \subseteq \sigma \cup \sigma^{-1}$.

If the relation $\neq$ is a tight apartness, then the complement $\sigma^{\triangleleft}$ of this last relation is an order relation on A. In both cases it is said that the set $(A, =, \neq)$ is ordered under σ.

Analogously as the previous question, the natural question arises: For a given relation (quasi-) order, is there the maximal co(-quasi)-order associated with the first relation?

Let σ be a co-quasiorder on a set S. Then [**49**] (Proposition 3.1) classes $a\sigma$ and σb are strongly extensional subsets of A such that $a \lhd a\sigma$ and $b \lhd \sigma b$, for any $a, b \in A$. Moreover, the following implications hold:

$y \in a\sigma \wedge x \in A \Longrightarrow x \in a\sigma \vee (x, y) \in \sigma$;

$y \in \sigma b \wedge x \in A \Longrightarrow x \in \sigma b \vee (y, x) \in \sigma$;

$(a, b) \in \sigma \Longrightarrow a\sigma \cup \sigma b$.

This idea enables the introduction of the concept of co-ideals and co-filters in ordered algebraic structures.

We complete this subsection with the following terms. Let $f : (A, \sigma) \longrightarrow (B, \tau)$ be a mapping between co-quasiordered sets. For f, it is said that *isotone* if

$(\forall x, y \in A)((x, y) \in \sigma \Longrightarrow (f(x), f(y)) \in \tau)$

holds. Tha mapping f is a *reverse isotone* if

$(\forall x, y \in A)((f(x), f(y)) \in \tau \Longrightarrow (x, y) \in \sigma)$

is valid.

1.5. Concept of groupoids with apartness. For set $(A, =, \neq)$ it is called a *groupoid with apartness* if there exist a strongly extensional total function $w : A \times A \longrightarrow A$. This means that for each pair x, y of elements in A there exists the unique element $w(x, y)$ in A such that the following are valid:

$(\forall x, y, x', y' \in A)((x, y) = (x', y') \Longrightarrow w(x, y) = w(x', y')$ and

$(\forall x, y, x', y' \in A)(w(x, y) \neq w(x', y') \Longrightarrow (x \neq x' \vee y \neq y'))$.

We will write $(A, \cdot)$ for this groupoid instead of $((A, =, \neq), \cdot)$ and xy instead of $w(x, y)$, if it does not lead to misunderstanding.

We use this opportunity to emphasize the complexity of the algebraic concept 'groupoid with apartness'. A subset B of A is subgroupoid of A if holds

$(\forall x, y \in A)(x \in B \wedge y \in B \Longrightarrow xy \in B)$.

The dual of this notion is introduced as follows: For a subset F of groupoid A we say that the *co-subgroupoid* of A if holds

$(\forall x, y \in A)(xy \in F \Longrightarrow (x \in F \vee y \in F))$.

Speaking by classical algebra language, a subset F is a co-subgropoid of A if it is a prime subset of A.

Further, for a subset J of a groupid A it is said that it is:

- a right ideal in A if $(\forall x, y \in A)(x \in J \Longrightarrow xy \in J)$ holds;
- a left ideal in A if $(\forall x, y \in A)(y \in A \Longrightarrow xy \in J)$ holds; and
- an ideal in A if $(\forall x, y \in A)(x \in J \vee y \in J \Longrightarrow xy \in J)$ holds.

The duals of these concepts are introduced in the following way: A subset K of a groupoid A with apartness is

- a *left co-ideal* of A if $(\forall x, y \in A)(xy \in K \Longrightarrow x \in K)$ holds;
- a *right co-ideal* of A if $(\forall x, y \in A)(xy \in K \Longrightarrow y \in K)$ holds; and
- a *co-ideal* of A if $(\forall x, y \in A)(xy \in K \Longrightarrow (x \in K \wedge y \in K))$ holds.

Without major difficulties, it is checked that $\neg K$ and $K^{\lhd}$ are ideals in A such that $K^{\lhd} \subseteq \neg K$ is valid. Speaking by classical algebra language, a subset K is a co-ideal of A if it is a consistent subset of A.

In Bishop's constructive algebra we always encounter the following two problems:

(a) How to choose a predicate (or more predicates) between several classically equivalent ones by which an algebraic concept is determined;

(b) Since every predicate has at least one of its duals, how to construct a dual of the algebraic concept defined with a given predicate(s).

For the illustration, we give the following example. The relation q called *co-congruence* on a groupoid $(A, =, \neq, \cdot)$ if it is strongly extensional (or, it is cancellative with respect to apartness) in the following sense:

$(\forall x, y, u, \in A(((ux, uy) \in q \vee (xu, yu) \in q) \Longrightarrow (x, y) \in q)$.

The following example is inspiring in recognizing, understanding and accepting the duality of an algebraic concept. Let $(A, =, \neq, +, 0, \cdot, 1)$ be a commutative ring with apartness. A subset K of A is a *co-ideal* of the ring A ([**27**], Definition 2) if and only if for any $a, b \in A$ the following is valid

$0 \lhd K$,
$a + b \in K \Longrightarrow a \in K \vee b \in K$,
$-a \in K \Longrightarrow a \in K$,
$ab \in K \Longrightarrow a \in K \wedge b \in K$.

In this case, the relation q, defined with $(a, b) \in q \iff a - b \in K$ is a co-congruence on the ring A and holds $a \in K \iff (a, 0) \in q$ ([**27**], Proposition 2.5).

The concept of co-ideals of commutative rings with apartness was first introduced and analyzed by W. Ruitenberg in 1982 in his dissertation [**54**]. This author also took a considerable part in the development of the ideas of co-ideals in commutative rings with apartness in his dissertation [**25**] and with his first publications available to the entire international academic public (see for example [**27, 29, 36, 37**]). The reader can find some elements of these ideas in the Chapter 'Algebra' in the book [**59**], too.

1.6. Our intention in this article. By choosing the Intuitionistic logic instead of the Classic logic as a logic background, when looking at and developing possible algebraic structures, the first opens much more possibilities than is the case with the Classic algebra. The obligation of mathematicians is to recognize them, describe them and correctly point out their properties and, if possible, prove them in a logically acceptable way. When asked if it might be interesting to most working mathematicians, it must be answer no. The designed material on logically possible ideas and concepts and their relationships with each other should be acceptable to those who affect the perception of the logic of the possible structures that it allows by prior orientation. The vast majority of mathematicians adhere to the orientation that Constructive Mathematics has the meaning in the following sense: The procedures and algorithms used in proving within its aspects must be constructive in the colloquial sense of the word. This attitude greatly narrows the scope of this domain. Perhaps this belief has a temporary (and only partial) justification in the research of applied mathematical theories, such as Analysis. Speaking in a colloquial language, it is rather unusual (it can also be said astoundingly) that the academic community for a long time was reluctant to accept the possibilities that offer the choice of the Intuicionistic logic for background in the perception, understanding and development of algebraic structures.

This article is designed in the following way: Chapter just behind the Introduction is a brief recapitulation of our knowledge of semigroups with apartness relation. The following is the Chapter 3 on ordered semigroups under co-quasiorder relations. Chapter 5 will be a brief introduction to groups with apartness and the last chapter, Chapter 5, is devoted to commutative rings with apartness.

2. Semigroups with apartness

2.1. Concept of semigroups with apartness. Let $((A. =, \neq), \cdot)$ be a groupoid with apartness where the internal operation is strongly extensional in the following sense

$(\forall a, b, x, y \in A)((ay \neq by \implies a \neq b) \wedge (xa \neq xb \implies a \neq b))$.

It is equivalent with the following condition

$(\forall a, b, x, y \in A)(ax \neq by \implies (a \neq b \vee x \neq y))$.

If the internal binary operation $' \cdot '$ in A is associative, that is, if the following formula

$(\forall x, y, z \in A)(x(yz) = (xy)z)$

is valid, then for $(A, \cdot)$ we say that it is *semigroup with apartness*.

Semigroups with apartness are in the focus of interest of one group of researchers. In addition to the texts of this author (Banja Luka) (see, for example: [**38, 39, 40, 42, 49**]) devoted to this topic, S. Crvenković (Novi Sad) (see, for example: [**9, 10, 11, 21**]), M. Mitrović (Nis) (see, for example: [**10, 11, 21**]) and A. Cherubini and A. Frigeri (Milan) ([**8**]) took part in the consideration and development of this class of semigroups and some other authors.

A subset K of a semigroup A with apartness is

- a *left co-ideal* of A if $(\forall x, y \in A)(xy \in K \implies x \in K)$ holds;
- a *right co-ideal* of A if $(\forall x, y \in A)(xy \in K \implies y \in K)$ holds; and
- a *co-ideal* of A if $(\forall x, y \in A)(xy \in K \implies (x \in K \wedge y \in K))$ holds.

The following theorem can be easily proved by direct verification.

THEOREM 2.1. *Let $\{K_i\}_{i\in I}$ be a family of (left, right) co-ideals of a semigroup A. Then the set $\bigcup_{i\in I} K_i$ is a co-ideal in A.*

From this theorem one can easily deduce the following theorem.

THEOREM 2.2. *Let A be a semigroup with apartness. Then the family of all (left, right) co-ideals if A forms a completely lattice.*

PROOF. If $\{K_i\}_{i\in I}$ is the family of all (left, right) co-ideals of A, then $\bigcup_{i\in I} K_i$ is a (left, right) co-ideal in A by previous theorem. Let $\mathfrak{X}$ be the family of all (left, right) co-ideals included in the intersection $\bigcap_{i\in I} K_i$. Then $\bigcup\mathfrak{X}$ is a (left, right) co-ideal in A too. If we put $\sqcup K_i = \bigcup_{i\in I} K_i$ and $\sqcap K_i = \bigcup\mathfrak{X}$, then $(A, \sqcup, \sqcap)$ is completely lattice. □

A coequality relation q on a semigroup A with apartness is called *co-congruence* on A or coequality relation compatible with the semigroup operation on A if the following is valid

(a) $(\forall a,b,x,y \in A)((ax,by) \in q) \implies ((a,b) \in q \vee (x,y) \in q))$.

If we take $a = b$ in the preceding formula, we get it

(b) $(\forall a,x,y \in A)((ax,ay) \in q) \implies (x,y) \in q)$.

Also, if we take $x = y$ in the first formula, we get it

(c) $(\forall a,b,x \in A)((ax,bx) \in q) \implies (a,b) \in q)$.

Of course, vice versa also applies: $(b) \wedge (c) \implies (a)$. Indeed, let for the selected elements $a,v,x,y \in A$ be valid $(ax,by) \in q$. Thus $(ax,bx) \in q \vee (bx,by) \in q$ by co-transitivity of the relation q. From here, it follows (a) due to (c) and (b).

We will start with the following theorem in which we give a very important property of any co-congruence q on a semigroup A:

THEOREM 2.3. *Let q be a co-congruence on a semigroup A with apartness. Then the relation $q^{\lhd}$ is a congruence on A associate with q.*

Although the evidence of this claim is known, we will again show it here that the reader can gain an impression of the proof technique that is applied.

PROOF. Since the first part of this claim is proven in the Proposition 1.1, it remains to show that the relation $q^{\lhd}$ is compatible with the internal operation in the semigroup A.

Let $a,b,u,v \in A$ arbitrary elements such that $(a,b) \lhd q$ and $(u,v) \in q$. Then $(u,ax) \in q$ or $(ax,bx) \in q$ or $(bx,v) \in q$ ny co-transitivity of q. Thus, we have $u \neq ax$ or $bx \neq v$ because $(ax,bx) \in q$ implies $(a,b) \in q$ what is impossible. So, $(ax,bx) \lhd q$. The implication $(a,b) \lhd q \implies (xa,xb) \lhd q$ can be proven analogously to the evidence of the previous implication. □

The preceding theorem enables the construction of factor-semigroup $A/(q^{\lhd}, q)$ with apartness:

THEOREM 2.4 ([**38**], Theorem 2). *If q is an co-congruence on a semigroup A with apartness, then the set $A/(q^{\lhd}, q)$ is a semigroup with*

$aq^{\lhd} = bq^{\lhd} \iff (a,b) \lhd q$, $aq \neq bq \iff (a,b) \in q$ *and* $aq^{\lhd} \cdot bq^{\lhd} = abq^{\rhd}$.

The next assertion is the specificity of this aspect of semigroup studying which have not a counterpart in the classical semigroup theory.

THEOREM 2.5 ([**38**], Theorem 3). *Let q be a co-congruence on a semigroup A with apartness. Then the set $A : q = \{aq : a \in A\}$ is a semigroup with*
$aq = bq \iff (a,b) \lhd q \quad aq \neq bq \iff (a,b) \in q$ *and* $aq \cdot bq = abq$.

This section we will finish with the following results. By the first we will give a construction of co-congruence on semigroup based on given coequality relation.

THEOREM 2.6 ([**38**], Theorem 5). *Let q be a coequality relation on a semigroup A with apartness. Then the relation*
$q^* = \{(x,y) \in A \times A : (\exists a,b \in A)((axb, ayb) \in q)\}$
is a co-congruence on A such that $q \subseteq q^*$. If κ is a co-congruence on A such that $q \subseteq \kappa$, then $q^* \subseteq \kappa$.

Let A and B semigroups with apartness. A strongly extensional mapping $f : A \longrightarrow B$ is a homomorphism if
$(\forall x,y \in A)(f(xy) = f(x)f(y))$.
Let us recall that the following formulas are valid
$(\forall x,y \in A)(x = y \implies f(x) = f(y))$ and $(\forall x, \in A)(f(x) \neq f(y) \implies x \neq y)$.
Without major difficulties, it can be verified ([**38**], Theorem 4) that $Coker(f) = \{(x,y) \in A \times A : f(x) \neq f(y)\}$ is a co-congruence on A compatible with the congruence $Ker(f)$.

Mapping $\pi : A \longrightarrow A/(q^{\lhd}, q)$, $\theta : A \longrightarrow A : q$ and $h : A/(q^{\lhd}, q) \longrightarrow A : q$ appears in Subsection 1.3 are homomorphisms between semigroups with apartness. The following theorem can be seen as the First theorem of isomorphism between semigroups with apartness.

THEOREM 2.7. *Let $f : A \longrightarrow B$ be a homomorphism between semigroups with apartness. Then there exists the strongly extensional injective and embedding homomorphism $g : A/(Ker(f), Coker(f)) \longrightarrow B$ such that $f = g \circ \pi$ where $\pi : A \longrightarrow A/(Ker(f), Coker(f))$ is the canonical epimorphism.*

Keeping in mind the result in Theorem 2.5, the following theorem can also be viewed as the First theorem on isomorphism between semigroups with apartness. This assertion stems from the specificity of the constructive algebra and does not have its counterpart in the classical semigroup theory.

THEOREM 2.8. *Let $f : A \longrightarrow B$ be a homomorphism between semigroups with apartness. Then there exists the strongly extensional injective and embedding homomorphism $\varphi : A : Coker(f) \longrightarrow B$ such that $f = \varphi \circ \theta$ where $\theta : A \longrightarrow A : Coker(f)$ is the strongly extensional standard epimorphism defined by $\theta(a) = aCoker(f)$ $(a \in A)$.*

The next proposition is a standard claim and it can be proved by direct verification.

Proposition 2.1. *Let* $g : ((A, =, \neq), \cdot, 1_A) \longrightarrow ((B, =, \neq), \cdot, 1_B)$ *be a semigroup homomorphism, Then*

- $f(1_A) = 1_B$;
- *If* L *is an ideal of the semigroup* B*, then* $f^{-1}(L)$ *is an ideal in* A*;*
- *If* M *is a co-ideal of the semigroup* B*, then* $f^{-1}(M)$ *is a co-ideal in* A*.*

If we wish to have strongly extensional consistent classes of co-congruence on a semigroup with apartness we need another condition:

Theorem 2.9. *Let* q *be a co-congruence on semigroup* A*. Then classes* aq $(a \in A)$ *are strongly extensional consistent subsets of* A *if and only if*

$(\forall a, b \in A)((ab, a) \lhd q) \wedge (ab, b) \lhd q)$

holds.

Proof. (1) Let the condition $(\forall a, b \in A)((ab, a) \lhd q) \wedge (ab, b) \lhd q)$ holds in semigroup A and let a be an arbitrary element of A. Let xy be element of aq, i.e. $(xy, a) \in q$. Thus from $(xy, x) \in q$ or $(x, a) \in q$ and $(xy, y) \in q$ or $(y, a) \in q$ we conclude that $x \in aq$ and $y \in aq$. So, the subset aq is a consistent subset of A.

(2) Opposite, suppose that the class aq is a consistent subset of A for every a in A. Let (u, v) be an arbitrary element of q. Then from $(u, ab) \in q$ or $(ab, a) \in q$ or $(a, v) \in q$ follows $(u, v) \neq (ab, a)$ or the implication $ab \in aq \Longrightarrow a \in aq \wedge b \in aq$ holds. Last implication is impossible because $a \lhd aq$ holds. Hence, we have $(u, v) \neq (ab, a)$, i.e. the condition $(\forall a, b \in A)((ab, a) \lhd q)$ holds. The second part we get analogously. □

2.2. Inverse semigroups with apartness. In their recently published paper [**8**], A. Cherubini and A. Frigeri introduced a definition of inverse semigroups with apartness, a useful tool to describe partial symmetries in sets with apartness. They proved the constructive analogue of the isomorphism theorem for inverse semigroups, and provide a characterisation of the co-congruences on inverse semigroups.

An *inverse semigroup with apartness* is a tuple $((A, =, \neq), \cdot, ^{-1})$, where

- $(A, =, \neq)$ os an inhabited set with apartness:
- $' \cdot '$ is an internal binary operation on A such that:
  (a) $(\forall x, y, z \in A)(x \cdot (y \cdot z) = (x \cdot y) \cdot z)$ and
  (b) $(\forall x, y, u, v \in A)(x \cdot u \neq y \cdot v \Longrightarrow (x \neq y \vee u \neq v))$
- $^{-1}$ is a unary operation such that:
  (c) $(\forall x \in A)((x^{-1})^{-1} = x)$ and (d) $(\forall x, y \in A)(x^{-1} \neq y^{-1} \Longrightarrow x \neq y)$;
- $(\forall x \in A)(x \cdot x^{-1} \cdot x = x)$ and
- $(\forall x, y \in A)(x \cdot x^{-1} \cdot y \cdot y^{-1} = y \cdot y^{-1} \cdot x \cdot x^{-1})$.

The first result refers to the recognition of inverse semigroups with apartness.

Theorem 2.10 ([**8**], Proposition 2). *A semigroup with apartness* $((A, =, \neq), \cdot)$ *is an inverse semigroup if and only if the relation*

$\{(x, y) \in A \times A : x = xyx \wedge y = yxyg\}$

is a strongly extensional mapping from A *to* A*.*

A partial apartness bijection on A is an apartness bijection from X to Y, where X and Y are subsets of A. As usual, the sets X and Y are called respectively domain and image of f and denoted by $Dom(f)$ and $Im(f)$. We call a partial apartness bijection, whose domain and codomain are detachable subsets of A, a d-partial bijection on A. We denote by I^A the set of all d-partial bijections on A. On I^A we define an equality as usual. We define $\neq$ by
$Dom(f) \neq Dom(g) \vee (\exists x)(f(x) \neq g(x))$

Theorem 2.11 ([**8**], Theorem 2). *Let $(A, =, \neq)$ be a set with apartness. Then $((I^A, =, \neq,), \circ, ^{-1})$ is an inverse semigroup with apartness, where where $' \circ '$ is the usual product of relations and $' .^{-1} '$ denotes the inverse relation.*

There remains an open question as to how to determine the co-quasiorder relation (or co-order relation) in such semigroups compatible with the semigroup operation in a natural and acceptable way.

2.3. Γ-semigroups with apartness. In this subsection we interested in Γ - semigroups with apartness. Also, we will find and analyze some doubles of substructures of these semigroups. Our investigation the concept of Γ-semigroups with appartness consists of the observation of specificities that arise by placing the classically defined algebraic structure of Γ-semigroups ([**55, 56**]) into a different logical environment and using specific Bishop's constructive algebra tools. The concept of Γ-semigroups with apartness was introduced and analyzed in our recently published article [**51**]. In addition, we introduced the concepts of co-ideals in such semigroups and give some properties of the family of such substructures. In addition to introducing the concept of Γ - cocongruences of Γ - semigroup, we also by analyzing the connection between strong extensional homomorphisms of Γ - semigroups and congruences and co-congruences, we proved some assertions in related with co-ideals in such semigroups.

Let $(S, =, \neq)$ and $(\Gamma, =, \neq)$ be two non-empty sets with apartness. Then A is called a Γ - *semigroup with apartness* if there exist a strongly extensional mapping from $A \times \Gamma \times S \ni (x, a, y) \longmapsto xay \in S$ satisfying the condition

$$(\forall x, y, z \in S)(\forall a, b \in \Gamma)((xay)bz = xa(ybz)).$$

We recognize immediately that the following implications

$$(\forall x, y, u, v \in A)(\forall a, b \in \Gamma)(xay \neq ubv \implies (x \neq u \vee a \neq b \vee y \neq v)),$$

$$(\forall x, y \in A)(\forall a, b \in \Gamma)(xay \neq xby \implies a \neq b)$$

are valid, because f is a strongly extensional function.

Let A be a Γ-semigroup with apartness. A subset T of A is said to be a Γ-*cosubsemigroup* of A if the following holds

$$(\forall x, y \in A)(\forall a \in \Gamma)(xay \in T \implies (x \in T \vee y \in T)).$$

We will assume that the empty set $\emptyset$ is a Γ - cosubsemigroup of a Γ - semigroup A by definition.

Now, We will introduce the concept of a (left, right) Γ - coideal in a Γ - semigroup. A strongly extensional subset B of a Γ - semigroup with apartness A is said to be a *left* Γ - *coideal* of A if the following implication holds

$$(\forall x, y \in A)(\forall a \in \Gamma)(xay \in B \implies y \in B).$$

A strongly extensional subset B of a Γ - semigroup A with apartness is said to be a *right* Γ - *coideal* of A if the following implication is valid

$$(\forall x, y \in A)(\forall a \in B)(xay \in B \implies x \in B).$$

A strongly extensional subset B of a Γ - semigroup A with apartness is said to be a (two side) Γ- *coideal* of A if the following implication is valid

$$(\forall x, y \in A)(\forall a \in B)(xay \in B \implies (x \in B \wedge y \in B)).$$

From this definition, it immediately follows that if B is a left Γ - coideal of a Γ - semigroup with apartness A, then B is a Γ - cosubsemigroup of A.

Proposition 2.2 ([**51**], Proposition 2.4, 2.5 and 2.6). *If B is a* (*left, right*) *Γ - coideal of a Γ - semigroup S, then the set $B^{\lhd}$ is a* (*left, right*) *Γ - ideal of S.*

The proof of the following theorem is obtained by direct verification.

Theorem 2.12 ([**51**], Theorem 2.2, 2.3 and 2.4). *The family of all* (*left, right*) *Γ-coideal of Γ-semigroup A forms a completely lattice.*

In our intention to show one form of the first theorem on isomorphism between such semi-groups, we will first determine notions of Γ-homomorphisms and Γ-co-congruences. Let A be a Γ-semigroup with apartness. A coequality relation $q \subseteq A \times A$ is called a Γ - *cocongruence* on A if the following holds

$$(\forall x, y, u, v \in A)(\forall a, b \in \Gamma)((xau, ybv) \in q \implies ((x, y) \in q \vee a \neq b \vee (u, v) \in q)).$$

It is obvious that the following implication

$(\forall x, y \in A)(\forall a, b \in \Gamma)((xay, xby) \in q \implies a \neq b)$

is valid. Without major difficulties, it can be checked

Proposition 2.3 ([**51**], Proposition 2.8). *If q is a Γ - cocongruence on a Γ-semigroup with apartness A, then the relation $q^{\lhd}$ is a Γ - congruence on A.*

Then any class xq, generated by the element $x \in A$, is a strongly extensional subset of A.

Theorem 2.13. *If q is a Γ - cocongruence on a Γ-semigroup with apartness A, then the family $A : q$ of all classes of q is Γ - semigroup with $(xq)a(yq) = (xay)q$ for any $x, y \in A$ and $a \in \Gamma$.*

Let A is a Γ-semigroup and B a Λ - semigroups with apartness. A pair (f, φ) of strongly extensional functions $f : A \longrightarrow B$ and $\varphi : \Gamma \longrightarrow \Lambda$ is called a *homomorphism* from Γ-semigroup A to Λ - semigroup B if the following holds

$$(\forall x, y \in A)(\forall a \in \Gamma)((f, \varphi)(xay) = f(x)\varphi(a)f(y)).$$

It is easily verified that (f, φ) is a correctly determined strongly extensive function. Also, it is easy to see that $Coker(f, \varphi) = \{(x, y) : (f, \varphi)(x) \neq (f, \varphi)(y)\}$ is a Γ - cocongruence on A and $(f, \varphi)(A)$ is a Λ - subsemigroup of Λ - semigroup B.

In our last theorem in this subsection, we show that the family $A : q$ of all classes of Γ - cocongruence q in a Γ-semigroups A plays a significant role in the so-called first isomorphism theorem.

LEMMA 2.1. *If q is a Γ - cocongruence on a Γ-semigroup A, then the pair (θ, i) is a homomorphism between Γ-semigroup A onto Γ - semigroup $A : q$ where $\theta : A \longrightarrow A : q$ is a canonical surjection and i is the identity function on Γ.*

THEOREM 2.14 ([**51**], Theorem 2.12). *Let A be a Γ - semigroup and B be Λ - semigroups and $(f, \varphi) : A \longrightarrow B$ be a homomorphism.*

(1) *Then there is a strongly extensional isomorphism*
$(g, \varphi) : A/(Ker(f, \varphi), Coker(f, \varphi)) \longrightarrow (f, \varphi)(A) \subseteq B$
such that $(f, \varphi) = (g, \varphi) \circ ((Ker(f, \varphi))^\natural, i)$ holds, where $(Ker(f, \varphi))^\natural : A \longrightarrow A/(Ker(f, \varphi), Coker(f, \varphi))$ is the natural epimorphism.

(2) *Then there is a strongly extensional isomorphism*
$(h, \varphi) : A : Coker(f, \varphi) \longrightarrow (f, \varphi)(A) \subseteq B$
such that $(f, \varphi) = (h, \varphi) \circ (\theta, i)$ holds.

3. Ordered semigroups under co-quasiorder

Ordered semigroups with apartness under co-quasiorder has been studied by this author himself or in cooperation with others in several of his articles (see, for example:[**9, 38, 40, 42, 49, 52**]).

Let $A = ((A, =, \neq), \cdot)$ be a semigroup with apartness. A relation σ on A is a co-quasiorder (co-order) on A if it is a co-quasiorder (co-order) on the set A and the following holds
$(\forall x, y, z \in A)(((xz, yz) \in \sigma \vee (zx, zy) \in \sigma) \implies (x, y) \in \sigma)$.
Speaking by the language of the classical algebra, the σ is a left and right cancellative relation. In both cases, for the semigroup A with apartness is said to be *ordered* in under the co-quasiorder (co-order).

EXAMPLE 3.1. ([**9**], Example 2) Let T be a strongly extensional consistent subset of a semigroup A. Then the relation $\sigma \subseteq A \times A$, defined by
$(\forall a, b \in A)((a, b) \in \sigma \iff (a \neq b \wedge a \in T)$,
is a co-quasiorder relation on A but it is not a co-order on A.

EXAMPLE 3.2. ([**9**], Lemma 2.0) Let T be a strongly extensional subset of a semigroup A. Then, the relation $\sigma \subseteq A \times A$, defined by
$(\forall a, b \in A)((a, b) \in \sigma \iff (\exists\, x, y \in A \cup \{1\})(xby \in T \wedge xay \lhd T)$,
is a co-quasiorder relation on A.

Let $A = ((A, =, \neq), \cdot)$ be a semigroup with apartness and σ be a co-quasiorder relation on A. Our first proposition shows the existence of the co-quasiorder Q on $A/(q^\lhd, q)$ and $A : q$, where $q = \sigma \cup \sigma^{-1}$.

LEMMA 3.1. *Let A be a semigroup with apartness ordered under a co-quasiorder σ. Then the relation $q = \sigma \cup \sigma^{-1}$ is a co-congruence on A.*

PROPOSITION 3.1 ([**40**], Lemma 1). *Let A be a semigroup with apartness and σ be a co-quasiorder relation on A. The relation Q on $A/(q^{\lhd}, q)$ and $A : q$, where $q = \sigma \cup \sigma^{-1}$, defined by*

$(\forall a, b \in A)((aq, bq) \in Q \iff (a, b) \in \sigma)$

is a consistent, co-transitive and linear relation on semigroups $S/(q^{\lhd}, q)$ and $A : q$ compatible with the semigroup operation on $A/(q^{\lhd}, q)$ and $A : q$ respectively.

We will start this section with the following statement.

PROPOSITION 3.2 ([**49**], Proposition 2.1). *Let σ be a co-quasiorder on a semigroup A with apartness. Then the left class $L(a)$ and the right class $R(b)$ are strongly extensional subsets of A such that $a \lhd L(a)$ and $b \lhd R(b)$, for any $a, b \in A$. Moreover, the following implications hold:*

- Classes $L(a)$ and $R(b)$ are co-subsemigroups of semigroup A;
- $(\forall x, y \in A)(y \in L(a) \implies (x \in L(a) \vee (x, y) \in \sigma))$;
- $(\forall x, y \in A)(y \in R(b) \implies (x \in R(b) \vee (y, x) \in \sigma))$; and
- $(\forall a, b \in A)((a, b) \in \sigma \implies L(a) \cup R(b) = A)$.

Previous analysis justifies the introduction of the following notions. Hereinafter, we intend to describe some substructures of a ordered semigroup A under a co-quasiorder σ. For a subset K of A , it is said that *co-ideal* [**9**] of A if the following holds

$(\forall x, y \in A)(xy \in K \implies (x \in K \vee y \in K))$ and
$(\forall x, y \in A)((y \in K \implies (y, x) \in \sigma \vee x \in K))$.

So, the subset $R(a)$ is a principal co-ideal of A generated by the element a.

The concept of co-filters in an ordered semigroup A is introduced by the following definition [**49**]. For a subset G of A it is said that it is a *co-filter* in A if

$(\forall x, y \in A)(xy \in G \implies (x \in G \vee y \in G))$ and
$(\forall x, y \in A)((y \in G \implies (x, y) \in \sigma \vee x \in G))$.

So, the subset $L(a)$ is a principal co-filter of A generated by the element a. According to the first property, the co-ideal and the co-filter is a co-subsemigroup in a semigroup A. From another property, immediately follows that any co-ideal and any co-filter in semigroup A is a strongly extensional subset in A.

If K is a co-ideal and a an arbitrary element of A, then the sets $[a : K] = \{x \in A : ax \in K\}$ and $[K : a] = \{y \in A : ya \in K\}$ are co-deals of A.

In the following statement we show that a strong complement $K^{\lhd}$ of a co-ideal K is an ideal in A. To that intention, we will first show one necessary lemma

LEMMA 3.2 ([**42**], Lemma 2.2). *If σ is a co-quasiorder on a semigroup A, then the relation $\sigma^{\lhd}$ is a quasi-order on A.*

PROPOSITION 3.3. *If K is a co-ideal of ordered semigroup A under a quasi-order σ, then $K^{\lhd}$ is an ideal in ordered semigroup A under the quasi-order $\sigma^{\lhd}$.*

In the following statement we show that a strong complement $G^{\lhd}$ of a co-filter G is a filter.

PROPOSITION 3.4 ([**49**], Theorem 2.1). *If G is a co-filter of ordered semigroup A under a quasi-order σ, then $G^{\lhd}$ is a filter in ordered semigroup A under the quasi-order $\sigma^{\lhd}$.*

THEOREM 3.1 ([**49**], Theorem 2.3 and Corollary 2.1). *The family of all co-filters in a ordered semigroup A under a under a co-quasiorder forms a join semi-lattice. The greatest element in this semi-lattice is A.*

.

This specific environment enables [**49, 52**] the introduction of the concept of ordered co-ideals and the concept of ordered co-filters as well as the concepts of normal co-ideals and normal co-filters.

For an element a of A we put $a\alpha = \{x \in A : (a,x) \in \alpha\}$ and $\alpha a = \{x \in A : (x,a) \in \alpha\}$. In the following theorems, we give some fundamental properties of co-quasiorder σ in semigroup A with apartness:

THEOREM 3.2. *The following conditions for a co-quasiorder σ on a semigroup A are equivalent:*

(1) $(\forall a,b \in A)((a,ab) \lhd \sigma \wedge (a,ba) \lhd \sigma)$;

(2) $(\forall a,b \in A)(a\sigma \cup b\sigma \subseteq (ab)\sigma)$;

(3) $(\forall a,b \in A)(\sigma(ab) \subseteq \sigma a \cap \sigma b)$;

(4) *a is a strongly extensional consistent subset of A such that $a \lhd a\sigma$ for each $a \in A$; and*

(5) *σb is a strongly extensional ideal of A such that $b \lhd \sigma b$, for each $b \in A$.*

PROOF. $(1) \Longrightarrow (3)$

$$\begin{aligned} y \in \sigma(ab) &\Longleftrightarrow (y,ab) \in \sigma \\ &\Longrightarrow ((y,a) \in \sigma \vee (a,ab) \in \sigma) \wedge ((y,b) \in \sigma \vee (b,ab) \in \sigma) \\ &\Longrightarrow (y,a) \in \sigma \wedge (y,b) \in \sigma \\ &\Longleftrightarrow y \in \sigma a \cap \sigma b. \end{aligned}$$

$(3) \Longrightarrow (4)$

$$\begin{aligned} xy \in a\sigma &\Longleftrightarrow (a,xy) \in \sigma \\ &\Longleftrightarrow a \in \sigma(xy) \subseteq \sigma x \cap \sigma y \\ &\Longrightarrow a \in \sigma x \wedge a \in \sigma y \\ &\Longleftrightarrow x \in a\sigma \wedge y \in a\sigma. \end{aligned}$$

$(4) \Longrightarrow (1)$. Let (u,v) be an arbitrary element of σ. Then $(u,a) \in \sigma \vee (a,ab) \in \sigma \vee (ab,v) \in \sigma$ and since from $ab \in a\sigma$ follows $(a \in a\sigma \wedge b \in a\sigma)$, which is impossible, we have $u \neq a$ or $ab \neq v$. So, $(a,ab) \neq (u,v) \in \sigma$. For the assertion $(a,ba) \lhd \sigma$ the proof is similar.

$(1) \Longrightarrow (2)$.

$$\begin{aligned} x \in a\sigma \cup b\sigma &\Longleftrightarrow x \in a\sigma \vee x \in b\sigma \\ &\Longleftrightarrow (a,x) \in \sigma \vee (b,x) \in \sigma \\ &\Longrightarrow ((a,ab) \in \sigma \vee (ab,x) \in \sigma) \wedge ((b,ab) \in \sigma \vee (ab,x) \in \sigma) \\ &\Longrightarrow (ab,x) \in \sigma \end{aligned}$$

$\iff x \in (ab)\sigma.$

$(2) \Longrightarrow (5)$

$x \in \sigma a \vee y \in \sigma \iff a \in x\sigma \vee a \in y\sigma$
$\iff a \in x\sigma \cup y\sigma \subseteq (xy)\sigma$
$\Longrightarrow xy \in \sigma a.$

$(5) \Longrightarrow (1)$ Let (u, v) be an arbitrary element of σ and $a, b \in A$. Then $(u, a) \in \sigma \vee (a, ab) \in \sigma \vee (ab, v) \in \sigma$.. Thus $u \neq a \vee a \in \sigma(ab) \vee ab \neq v$. Since, by (5), from $a \in \sigma(ab)$ follows $ab \in \sigma(ab)$ which is impossible, we have that $(a, ab) \neq (u, v) \in \sigma$. The fact $(a, ba) \lhd \sigma$ we got analogously. □

In the following examples we give constructions of a co-quasiorder relation on a semigroup with apartness that satisfies the condition (1) in Theorem 3.2:

Example 3.3. 1. Let J be a proper strongly extensional ideal of A. Then the relation σ on A, defined by $(a, b) \in \sigma \iff a \in J \wedge b \lhd J$, is a co-quasiorder on A satisfies the condition (1) in Theorem 3.2.

2. Let K be a strongly extensional consistent subset of A. Then the relation σ on A, defined by $(a, b) \in \sigma \iff a \lhd K \wedge b \in K$, is a co-quasiprder on A satisfies the condition (1) in the Theorem 3.2.

Theorem 3.3. *Let σ be a co-quasiorder relation on a semigroup A that satisfies one of the equivalent conditions of the previous Theorem 3.2. Then following conditions are equivalent:*

(1) $(\forall a, b \in A)((ab)\sigma = a\sigma \cup b\sigma)$;

(2) *σb is strongly extensional completely prime ideal of A for every b in A;*

(3) $(\forall a, b, c \in A)((ab, c) \in \sigma \Longrightarrow ((a, c) \in \sigma \vee (b, c) \in \sigma))$.

Proof. $(1) \Longrightarrow (2)$ Let $xy \in \sigma b$. Then, $b \in (xy)\sigma = x\sigma \cup y\sigma$. Thus, $b \in x\sigma$ or $b \in y\sigma$. So, $x \in \sigma b$ or $y \in \sigma b$.

$(2) \Longrightarrow (1)$ If x is an arbitrary element of $(ab)\sigma$, then $ab \in x\sigma$ and $a \in x\sigma \vee b \in x\sigma$ because $x\sigma$ is a strongly extensional completely prime ideal of A. Therefore, the following implication $x \in (ab)\sigma \Longrightarrow x \in a\sigma \cup b\sigma$ holds.

$(3) \Longrightarrow (1)$ Let (3) holds. Then, for $x \in (ab)\sigma$ we have $(ab, x) \in \sigma$. Thus, $(a, x) \in \sigma$ or $(b, x) \in \sigma$. Hence, finally, we have $x \in a\sigma \cup b\sigma$.

$(1) \Longrightarrow (3)$ Let the formula (1) is valid. Suppose that a, b and c are elements of A such that $(ab, c) \in \sigma$. Then, $c \in (ab)\sigma = a\sigma \cup b\sigma$. So, we have $c \in a\sigma \vee c \in b\sigma$ and, finally $(a, c) \in \sigma$ or $(b, c) \in \sigma$. □

Corollary 3.1. *If σ is a co-quasiorder relation on semigroup A which satisfies one of conditions (1), (2) or (3) in the above Theorem, then the following implication holds:*

(4) $(\forall a, b, c \in A)((a, c) \lhd \sigma \wedge (b, c) \lhd \sigma \Longrightarrow (ab, c) \lhd \sigma)$.

Proof. Let a, b and c be elements of A such that $(a, c) \lhd \sigma$ and $(b, c) \lhd \sigma$ and let (u, v) be an arbitrary element of σ. Then:

$(u, v) \in \sigma \Longrightarrow ((u, ab) \in \sigma \vee (ab, c) \in \sigma \vee (c, v) \in \sigma)$
$\Longrightarrow u \neq ab \vee (a, c) \in \sigma \vee (b, c) \in \sigma \vee c \neq v$
$\Longrightarrow (ab, c) \neq (u, v) \in \sigma.$ □

THEOREM 3.4. *Let σ be a co-quasiorder relation on a semigroup A that satisfies one of the equivalent conditions of the previous Theorem 3.2. Then following conditions are equivalent:*

(a) $(\forall a, b \in A)((ab, a) \lhd \sigma \vee (ab, b) \lhd \sigma)$;

(b) $(\forall a, b \in A)(\sigma(ab) = \sigma a \cap \sigma b)$;

(c) $a\sigma$ *is a strongly extensional filter of A for every a in A; and*

(d) $(\forall a, b \in A)((a, b) \lhd \sigma \vee (b, a) \lhd \sigma)$.

PROOF. $(a) \Longrightarrow (b)$

$$\begin{aligned} x \in \sigma a \cap \sigma b &\Longleftrightarrow (x, a) \in \sigma \wedge (x, b) \in \sigma \\ &\Longrightarrow ((x, ab) \in \sigma \vee (ab, a) \in \sigma) \wedge ((x, ab) \in \sigma \vee (ab, b) \in \sigma) \\ &\Longrightarrow (x, ab) \in \sigma \\ &\Longleftrightarrow x \in \sigma(ab). \end{aligned}$$

$(b) \Longleftrightarrow (c)$

$$\begin{aligned} x \in a\sigma \wedge y \in a\sigma &\Longleftrightarrow a \in \sigma x \wedge a \in \sigma y \\ &\Longleftrightarrow a \in \sigma x \cap \sigma y = \sigma(xy) \\ &\Longleftrightarrow xy \in a\sigma. \end{aligned}$$

$(c) \Longrightarrow (a)$ Let (u, v) be an arbitrary element of σ and let a, b, c be arbitrary elements of A. Then, $((u, ab) \in \sigma \vee (ab, a) \in \sigma \vee (a, v) \in \sigma)$ and $((u, ab) \in \sigma \vee (ab, b) \in \sigma \vee (b, v) \in \sigma)$, and thus,

$$(u, v) \neq (ab, a) \vee (u, v) \neq (ab, b) \vee (ab \in \sigma a \cap \sigma b = \sigma(ab)).$$

So, we have $(ab, a) \lhd \sigma$ or $(ab, b) \lhd \sigma$, since $ab \lhd \tau(ab)$.

$(a) \Longleftrightarrow (d)$ Out of (d) immediately follows (a). Let (d) holds for elements a, b of semigroup A and let (u, v) be an arbitrary element of σ. Particularly, we have $(a^2, a) \lhd \sigma$ and $(a, a^2) \lhd \sigma$. Thus,

$$((u, ab) \in \sigma \vee (ab, aa) \in \sigma \vee (aa, a) \in \sigma \vee (a, v) \in \sigma)\text{and}$$
$$((u, ab) \in \sigma \vee (ab, bb) \in \sigma \vee (bb, b) \in \sigma \vee (b, v) \in \sigma).$$

Hence,

$$(u \neq ab \vee (b, a) \in \sigma \vee a \neq v) \wedge (u \neq ab \vee (a, b) \in \sigma \vee b \neq v).$$

So, we have $(ab, a) \neq (u, v)$ or $(ab, b) \neq (u, v)$. □

For a given co-ordered semigroup $((A, =, \neq), \cdot, \alpha)$ is essential to know if there exists a co-congruence q on A such that $A : q$ be a co-ordered semigroup. Now, we introduce new important notion: A co-congruence q on A is called *regular* if there is a co-order Θ on $A : q$ satisfying the following conditions:

(1) $((A : q, =, \neq), \cdot, \Theta)$ is a co-ordered semigroup;

(2) The mapping $\theta : A \longrightarrow vA : q$ is a reverse isotone epimorphism.

In order to obtain the relationship between regular co-congruence and co-quasiorder on S, following theorem is essential.

THEOREM 3.5. *Let $(A, \cdot, \alpha)$ be a co-ordered semigroup and q be a co-congruence on A. The following are equivalent:*

(1) *q is regular.*

(2) *there exists a co-quasiorder* σ *on* A, *such that* $q = \sigma \cup \sigma^{-1}$.

Proof. (2) $\Longrightarrow$ (1). By Proposition 3.1, since $q = \sigma \cup \sigma^{-1}$, the semigroup $((A : q, =, \neq), \cdot)$ is a co-ordered semigroup with respect under the co-order Θ, defined by $(\forall x, y \in A)((qx, qy) \in \Theta \iff (x, y) \in \sigma)$. If $x, y \in A$ and $(qx, qy) \in \Theta$, then $(x, y) \in \sigma \subseteq \alpha$. So, q is a regular co-congruence on A by definition.

(1) $\Longrightarrow$ (2). Let q be a regular co-congruence on a semigroup (A, α). Then there exists a co-order relation Θ on the semigroup $A : qS$ such that $(A : q, \Theta)$ is a co-ordered semigroup, and $\theta : A \longrightarrow A : q$ is a strongly extensional reverse isotone homomorphism of co-ordered semigroups. Let $\sigma = \{(x, y) \in S \times S : (qx; qy) \in \Theta\}$. So, σ is a co-antiorder on A and it is easy to check that $q = \sigma \cup \sigma^{-1}$ according to Lemma 1 in the article [**40**]. □

3.1. Semillatice-ordered semigroups. Semilattice-ordered semigroups are important algebraic structure. It studied, for example, by Martin Kuril and Libor Polka [**14**]. In this subsection we describe semilattice-ordered semigroup with apartness. Following to classical definition in [**14**], for algebraic structure $((A, =, \neq, 1), \otimes)$ is called that it is a *semilattice-ordered semigroup with apartness* if ([**20, 41, 43, 46**]):

(i) $((A, =, \neq, 1,), \cdot)$ is a semigroup with apartness:

(ii) $((A, =, \neq), \otimes)$ is a semilattice, i.e. $(A, \otimes)$ is a commutative semigroup with $(\forall x \in A)(x \otimes x = x)$ where the semigroup operation is strongly extensional:

$(\forall a, b, c \in A)((a \otimes c \neq b \otimes c \vee c \otimes a \neq c \otimes b) \Longrightarrow a \neq b)$;

(iii) $(\forall a, b, c \in A)((a \cdot (b \otimes c) = a \cdot b \otimes a \cdot c) \wedge ((a \otimes b) \cdot c = a \cdot c \otimes b \cdot c))$ and

(iv) $(\forall x \in A)(x \otimes 1 = 1)$.

Some examples of these algebraic reader structures can be found in the article [**41**]. In the following lemma we show that semilattice-ordered semigroup with apartness is equipped with the natural defined co-order relation:

Lemma 3.3 ([**41**], Lemma 2.2). *Let* $(A, =, \neq, 1, \cdot, \otimes)$ *be a semilattice-ordered semigroup with apartness. The the relation* α *on* A, *defined by*

$(\forall a, b \in A)((a, b) \in \alpha \iff a \otimes b \neq b)$,

is a co-order relation in A.

The concept of a co-ideal in semilattice-ordered semigroup with apartness is introduced as follows: Let $(A, =, \neq, 1), \cdot, \otimes)$ be a semilattice-ordered semigroup with apartness. A subset K of A is its *co-ideal* if the following jolds

(1) $(\forall a, v \in A)(a \otimes b \in K \Longrightarrow (a \in K \vee b \in K))$, and

(2) $(\forall a, b \in A)(b \in K \Longrightarrow ((a, b) \in \alpha \vee a \in K))$.

The term 'co-filter' as a dual of notion filter we introduce by the following way. Let $(A, =, \neq, \cdot, \otimes)$ be a semilattice-ordered semigroup with apartness. A subset G of A is a *co-filter* of A if the following holds

$(\forall a, b \in A)(a \otimes b \in G \Longrightarrow (a \in G \vee b \in G)$ and

$(\forall a.b \in A)(b \in G \Longrightarrow ((b, a) \in \alpha \vee a \in G))$.

In addition to these concepts, the above-mentioned article introduced and analyzed the concepts of ordered co-filters and normal co-filters.

3.2. Implicative semigroups. The concept of an implicative semigroups with apartness was introduced and analyzed by this author in his articles [**44, 45, 53**]. In this subsection, we assume that the basic apartness is tight, i.e. it satisfies the following

$(\forall x, y \in A)(\neg(x \neq y) \implies x = y)$.

For a *negatively anti-ordered* semigroup (briefly, n.a-o. semigroup) we mean a set $(A, =, \neq)$ with an co-order α and a binary operation '$\cdot$' (we will write xy instead $x \cdot y$) such that for all $x, y, z \in A$, we have to have

(1) $(xy)z = x(yz)$,

(2) $(xz, yz) \in \alpha$ or $(zx, zy) \in \alpha$ implies $(x, y) \in \alpha$, and

(3) $(xy, x) \lhd \alpha$ and $(xy, y) \lhd \alpha$.

In that case for co-order α we will say that it is *negative co-order relation* on semigroup. Let us note that, in that case, we have

(3') $(x, xy) \lhd \alpha^{-1}$ and $(y, xy) \lhd \alpha^{-1}$.

In fact, for $(v, u) \in \alpha^{-1}$ we have

$$(u, v) \in \alpha \Longrightarrow ((u, xy) \in \alpha \ \vee \ (xy, x) \in \alpha \ \vee \ (x, v) \in \alpha).$$

Thus, by (3), we have $u \neq xy$ or $x \neq v$. So, we proved $(x, xy) \neq (v, u) \in \alpha^{-1}$. The second part of (3') we prove analogously.

Let α be a relation on A. For an element a of A we put $a\alpha = \{x \in A : (a, x) \in \alpha\}$ and $\alpha a = \{x \in A : (x, a) \in \alpha\}$. In the following proposition we give some fundamental properties of negative co-order relation on semigroup.

THEOREM 3.6 ([**44**], Theorem 3.1). *If $\alpha \subseteq A \times A$ is a co-order relation on a semigroup A, then the following statements are equivalent:*

(i) *α is a negative co-order relation;*

(ii) *αb is a consistent subset of A for any b in A;*

(iii) *$(\forall a, b \in A)(\alpha a \cup \alpha b \subseteq \alpha(ab))$;*

(iv) *$a\alpha$ is an ideal of A for any a in A;*

(v) *$(\forall a, b \in A)((ab)\alpha \subseteq a\alpha \cap b\alpha)$.*

An n.a-o. semigroup $((A, =, \neq), \cdot, \alpha)$ is said to be *implicative semigroup with apartness* if there is an additional binary operation $\otimes : A \times A \longrightarrow A$ such that for any elements x, y, z of A, the following is true

(4) $(z, x \otimes y) \in \alpha \Longleftrightarrow (zx, y) \in \alpha$.

Let us point out, as in the classical case, that in definition of implicative semigroup we can take the following lower demand

(4') $(z, x \otimes y) \lhd \alpha \Longleftrightarrow (zx, y) \lhd \alpha$

instead demand (4).

In addition, let us recall that the internal binary operation must satisfy the following implications:

$(a, b) = (u, v) \Longrightarrow a \otimes b = u \otimes v$ and $a \otimes b \neq u \otimes v \Longrightarrow (a, b) \neq (u, v)$.

The operation $\otimes$ is called *implication.* From now on, an implicative n.a-o. semigroup is simply called an *implicative semigroup with apartness.*

In any implicative semigroup A there exist a special element of A, the biggest element in $(A, \alpha^{\lhd})$, which is an almost neutral element in $(A, \cdot)$. In the following proposition we describe role of this special element.

THEOREM 3.7 ([**44**], Theorem 3.4). *If A is an implicative semigroup, then for every $x, y \in A$ holds*
$(x, y) \lhd \alpha \Longleftrightarrow 1 = x \otimes y$ *and* $(x, y) \in \alpha \Longleftrightarrow 1 \neq x \otimes y$.

A subset G of A we called *co-filter* if it satisfies the following conditions:
$xy \in G \Longrightarrow x \in G \vee y \in G$, that is, G is a co-subsemigroup of A and
$y \in G \Longrightarrow (x, y) \in \alpha \vee x \in G$.
It is easy to check that a co-filter is a strongly extensional subset of A. Moreover, strong compliment $G^{\lhd}$ of a co-filter G is a filter in A.

The following two theorems give equivalent conditions for G to be a co-filter in implicative semigroup A.

THEOREM 3.8 ([**44**], Theorem 3.7). *An inhabited proper subset G of an implicative semigroup A is a co-filter of A if and only if it satisfies the following conditions:*
(i) $1 \lhd G$, (ii) $(\forall x, y \in A)(y \in G \implies (x \otimes y \in G \vee x \in G))$.

THEOREM 3.9 ([**44**], Theorem 3.8). *An inhabited proper subset G of an implicative semigroup A is a co-filter of A if and only if it satisfies the following condition:*
$(\forall x, y, z \in A)(z \in G \implies ((x, y \otimes z) \in \alpha \vee x \in G \vee y \in G))$.

3.3. Some results of contemporary research. In the recently published paper [**48**] the author was interested in semigroup action on groupoid ordered under co-order. He first introduced the concept of acting semigroup with apartness $(S, =, \neq, \cdot)$ on ordered grupoid with apartness $(G, =, \neq, +)$ under co-order $\nleqslant$. Using this concepts he showb that if a commutative semigroup $(S, =, \neq, \cdot)$ acts on an ordered grupoid $(G, =, \neq, +)$ under co-quasirder $\nleqslant_G$, then there exists a commutative semigroup $\mathfrak{S}$, constructed by S, acting on the groupoid $(G \times S)/q$ ordered under $\nleqslant_q$, where $(x, a)q \nleqslant_q (y, b)q \Longleftrightarrow bx \nleqslant_G ay$ and $q = \rho \cup \rho^{-1}$, $((x, a), (y, b)) \in \rho \Longleftrightarrow bx \nleqslant_G ay$ for any $(x, a), (y, b) \in G \times S$.

4. Groups with apartness

Groups with apartness relations are very rarely researched. Some elements of the theory of these groups can be found in the Ruitenberge's dissertation [**54**] and in our dissertation [**25**] and in our articles [**26, 31**]. The constructions of free Abelian groups with apartness can be found in the text [**13**] written bt D. van Dalen, F.-J. de Vries. and in our article [**28**]. In this section, we will present some of the elements of the theory of commutative groups with apartnes.

A groupoid $(A, =, \neq), \cdot)$ with apartness and with the unity $'1'$ is a *group with apartness* if the following holds
$(\forall x \in A)(x \cdot 1 = 1 \cdot x = x)$,
$(\forall x \in A)(\exists x^{-1} \in A)(x \cdot x^{-1} = x^{-1} \cdot x = 1)$.

Group A is said to be a *commutative group* (or an Abelian group) if valid

$(\forall x, y \in A)(x \cdot y = y \cdot x)$

In our further exposition, we assume that all the groups that appear are commutative. It is clear that the following implication is valid

$(\forall x, y \in A)(x \neq y \implies x^{-1} \neq y^{-1})$.

Indeed, from $x \neq y$ follows $1 \cdot x \neq y \cdot 1$ and $y \cdot y^{-1} \cdot x = y \cdot x^{-1} \cdot x$. From here follows $y^{-1} = x^{-1}$ due to the cancellativity of the internal operation with respect to the apartness.

From the previous implication immediately follows

$(\forall x, y \in A)(x^{-1} \neq y^{-1} \implies x \neq y)$.

Let $x^{-1} \neq y^{-1}$. Since $(z^{-1})^{-1} = z$ for any $z \in A$, we have $(x^{-1})^{-1} \neq (y^{-1})^{-1}$ by previous implication. So, we have $x \neq y$.

For a subset S of a group A, it is said that it is a *subgroup* of the group A if the following is valid

$1 \in S$,

$(\forall x, y \in A)(x \in S \wedge y \in S \implies xy \in S)$,

$(\forall x \in A)(x \in S \implies x^{-1} \in S)$.

Let A be a group with apartness and let H be a subset of A. For H is said to be a *co-subgroup* of A if holds

$1 \lhd H$,

$(\forall x, y \in A)(xy \in H \implies (x \in H \vee y \in H))$ and

$(\forall x \in A)(x^{-1} \in H \implies x \in H)$.

Let q be a co-equality on a group A with apartness. q it is called a *co-congruence* relation on A if holds

$(\forall x, y, u, v \in A)((xu, yv) \in q \implies ((x, y) \in q \vee (u, v) \in q))$.

The following proposition describes the relationship between the concept of co-subgroups and the concept of co-congruences.

Proposition 4.1 ([**26**], Proposition 4). *Let H be a co-subgroup of a group A. Then the relation q on A, defined by*

$(\forall x, y \in A)((x, y) \in q \iff xy^{-1} \in H)$,

is a co-congruence on A and $H = \{a \in A : (a, 1) \in q\}$ holds.

Oppositely, if q is a co-congruence on a group A, then the set $H = \{a \in A : (a, 1) \in q\}$ is a co-subgroup of A.

Without major difficulties, it is checked that if q is a co-congruence on the group A, then $q^{\lhd}$ is a congruence on A ([**26**], Proposition 3). Analogously, if H is a co-subgroup of a group A, then $H^{\lhd}$ is a subgroup of A ([**26**], Proposition 4).

Let H be a subgroup of group A and let $a \in A$ be an arbitrary element. Define $aH^{\lhd} = \{ab : b \in H^{\lhd}\}$ and the family $A/(H^{\lhd}, H) = \{aH^{\lhd} : a \in A\}$ with

$(\forall a, b \in A)(aH^{\lhd} = bH^{\lhd} \iff ab^{-1} \lhd H)$ and

$(\forall a, b \in A)(aH^{\lhd} \neq bH^{\lhd} \iff ab^{-1} \in H)$.

Then this family is ([**25**], Theorem 7) a group with the internal operation defined by $aH^{\lhd} \cdot bH^{\lhd} = abH^{\lhd}$ and the unity $H^{\lhd}$.

Let A and B be groups with apartness. A total strongly extensional function $f; A \longrightarrow B$ is a homomorphism between groups with apartness if

$(\forall x, y \in A)(f(x \cdot y_{=} f(x) \cdot f(y))$.

A homomorphism is a *bijection* if it is injective (monomorphism) and surjective (epimorphism) mapping. A homomorphism is an *isomorphism* if it is bijective and embedding. The following theorems are important.

THEOREM 4.1 ([**25**], Theorem 10). *Let $f : A \longrightarrow B$ a homomorphism between groups with apartness. Then there exist the unique isomorphism*

$g : A/(Ker(f), Coker(f) \longrightarrow Im(f)$

such that $f = g \circ \pi$, where $\pi : A \longrightarrow A/(Ker(f), Coker(f))$, is the canonical epimorphism.

On the previous theorem we can look at the first theorem on isomorphism between Abalian groups with apartnes.

THEOREM 4.2 ([**25**], Theorem 11). *Let $f : A \longrightarrow B$ be a homomorphism between groups with apartness. Let H and K be a pair of a subgroup and a co-subgroup of $Im(f)$. Then:*

- The set $f^{-1}(H)$ is a subgroup of A such that $Ker(f) \subseteq f^{-1}(G)$;
- The set $f^{-1}(K)$ is a co-subgroup of A such that $f^{-1}(K) \subseteq A_f$;
- $f^{-1}(H) \subseteq \neg f^{-1}(K)$;
- There exist the unique isomorphism
$h : A/(f^{-1}(H), f^{-1}(K)) \longrightarrow Im(f)/(H, K)$.

4.1. A construction of free Abelian group with apartness. In what follows it will be shown how a free Abel group with apartness can be constructed (see for example [**13**] and [**28**]).

Let $(X, =, \neq)$ be a set with apartness. We form the following class X^+ of all strictly finite sequences of elements of X

$$x^+ \in X^+ \Longleftrightarrow (n_x \in \mathbf{N})(\exists f_x)(f_x : \{1, 2, ..., n_x\} \longrightarrow x^+)$$

with

$$(\forall i \in \{1, 2, ..., n_x\})(f_x(i) \in X).$$

As usual the concatenation of x^+ and y^+ is denoted by $x^+ \circ y^+$. If

$$x^+ = (f_x(1), ..., f_x(n_x)) \text{ and } y^+ = (f_y(1), ..., f_y(n_y)),$$

then

$$\begin{gathered} n_{xy} = n_x + n_y \text{ and} \\ i \in \{1, 2, ..., n_x\} \Longrightarrow f_{xy}(i) = f_x(i), \\ i = n_x + j\,(j \in \{1, 2, ..., n_y\}) \Longrightarrow f_{xy}(i) = f_y(j), \end{gathered}$$

i.e.

$$x^+ \circ y^+ = (f_x(1), ..., f_x(n_x), f_y(1), ..., f_y(n_y)).$$

On the class X^+ we define

$$\begin{gathered} x^+ =_1 y^+ \Longleftrightarrow (n_x = n_y \wedge f_x = f_y) \text{ and} \\ x^+ \neq_1 y^+ \Longleftrightarrow (\neg(n_x = n_y) \vee f_x \neq f_y). \end{gathered}$$

It is obvious that the relation $=_1$ is an equality relation on the class X^+. It is clear that the relation $\neq_1$ is consistent $\neg(x^+ \neq_1 x^+)$ and symmetric $(x^+ \neq_1 y^+ \Longrightarrow y^+ \neq_1 x^+)$. We have to prove that the relation $\neq_1$ is compatible with the equality $=_1$ and co-transitive.
$x^+ =_1 y^+ \wedge y^+ \neq_1 z^+ \Longleftrightarrow$
$(x^+ =_1 y^+(\Longleftrightarrow (n_x = n_y \wedge f_x = f_y))) \wedge (y^+ \neq_1 z^+(\Longleftrightarrow \neg(n_y = n_z) \wedge f_y \neq f_z))$
$\Longrightarrow \neg(n_x = n_z) \vee f_x \neq f_z$
$\Longrightarrow x^+ \neq_1 z^+$.
Let x^+, z^+ be arbitrary elements of X^+ such that $x^+ \neq_1 z^+$, and let y^+ be an arbitrary element of X^+. Then there exist natural numbers n_x , n_y, n_z and functions f_x, f_y, f_z such that $\neg(n_x = n_z) \vee f_x \neq f_z$. Thus,

$$\neg(n_x = n_y) \vee \neg(n_y = n_z) \vee f_x \neq f_y \vee f_y \neq f_z.$$

Therefore, $x^+ \neq_1 y^+ \vee y^+ \neq_1 z^+$ holds.
The mapping $\circ : X^+ \times X^+ \ni (x^+, z^+) \longmapsto x^+ \circ z^+ \in X^+ \times X^+$ is an internal binary operation on the set $(X^+, =_1, \neq_1)$. Indeed: Let (x^+, z^+) and (a^+, b^+) be two pairs of elements of X^+ and let $x^+ =_1 a^+$ and $y^+ =_1 b^+$. Then $x^+ =_1 a^+ \Longleftrightarrow (n_x = n_a \wedge f_x = f_a)$ and $y^+ =_1 b^+ \Longleftrightarrow (n_y = n_b \wedge f^y = f_b)$. We have $n_x + n_y = n_a + n_b$ and
$x^+ \circ y^+ =$
$(f_x(1), ..., f_x(n_x), f_y(1), ..., f_y(n_y)) =_1 (f_a(1), ..., f_a(n_a), f_b(1), ..., f_b(n_b))$
$= a^+ \circ b^+$
Let $x^+ \circ y^+ \neq_1 a^+ \circ b^+$, i.e. let $\neg(n_{xy} = n_{ab}) \vee f_{xy} \neq f_{ab}$. Thus:
(i) If $\neg(n_{xy} = n_x + n_y = n_a + n_b = n_{ab})$, then $\neg(n_x = n_a) \vee \neg(n_y = n_b)$.
(ii) If

$$(f_x(1), ..., f_x(n_x), f_y(1), ..., f_y(n_y)) \neq_1 (f_a(1), ..., f_a(n_x), f_b(1), ..., f_b(n_y)),$$

then there exists the natural number $i \in \{1, ..., n_{xy}\}$ such that $f_{xy}(i) \neq f_{ab}(i)$. If $i \in \{1, ..., n_x\}$, then $f_x(i) = f_{xy}(i) \neq f_{ab}(i) = f_a(i)$; if $i \in \{n_{x+1}, ..., n_{xy}\}$, then $i = j + n_x$ and $f_y(j) = f_{xy}(i) \neq f_{ab}(i) = f_b(j)$. So, from the both cases, we conclude that $x^+ \neq_1 a^+$ or $y^+ \neq_1 b^+$. Therefore, the operation "$\circ$" is strongly extensional.
Since $x^+ \circ y^+ = y^+ \circ x^+$, the operation $\circ$ is commutative and the unity element in X^+ is the empty sequence v.

THEOREM 4.3 ([**28**], Theorem). *The structure* $((X^+ \times X^+, =_2, \neq_2), v, +)$ *is a free Abelian group with apartness over a set* $(X, =, \neq)$, *where*
$(x^+, y^+) =_2 (a^+, b^+) \Longleftrightarrow x^+ \circ b^+ =_1 a^+ \circ y^+$,
$(x^+, y^+) \neq_2 (a^+, b^+) \Longleftrightarrow x^+ \circ b^+ \neq_1 a^+ \circ y^+$, *and*
$(x^+, y^+) + (a^+, b^+) =_2 (x^+ \circ a^+, y^+ \circ b^+)$.

4.2. Co-ordered Abelian groups. Let $(A, =, \neq), 0, +)$ be an additive Abelian group with apartness. Am co-order α on a set A is a co-order on group A if

$$(\forall a, b, c, d \in A)((a + c, b + c) \in \alpha \Longrightarrow ((a, b) \in \alpha \vee (c, d) \in \alpha)).$$

In this case, the group A is called ordered group under co-order α.

PROPOSITION 4.2. *Let* α *be a co-order on an additive Abelian group* A. *Then:*

- *If the apartness is tight, the relation $\alpha^{\triangleleft}$ is a partial order relation on A compatible with the group operation.*

- $(\forall a, b \in A)((a, b) \in \alpha, \Longrightarrow (a + c, b + c) \in \alpha)$.

- $(\forall a, b \in A)((a, b) \in \alpha \Longrightarrow (-a, -b) \in \alpha)$

PROPOSITION 4.3. *Let a subset P of an additive abelian group A satisfies the following conditions:*

(1) $0 \triangleleft P$,

(2) $P \cup (-P) = P$ *and* $P \cap (-P) = \emptyset$ *and*

(3) $(\forall a, b \in A)(a + b \in P \Longrightarrow a \in P \vee b \in P)$.

Then the relation α on defined by $(\forall z, b \in A)((a, b) \in \alpha \iff a-b \in P)$ is a co-order relation on A compatible with the group operation and $P = \{a \in A : (a, 0) \in \alpha\}$ holds.

5. Rings with apartness

5.1. Concept of rings with apartness. Some elements of ring theory with apartness can be found in the early seventies in the papers [**57, 58**] written by John Staples. Rings with apartness was studied by W. Ruitenburg in his dissertation [**54**]. Rings from the aspect of constructive mathematics have been studied by R. Mines and F. Richman in several of their texts (See, for example: [**17, 18**]), D. S. Bridges ([**5**]) and D. S. Bridges and R. S. Havea ([**6**]). This author independently, or in cooperation with some of his colleagues, took part in the development of the theory of the ring with apartnes (see, for example: [**25, 27, 29, 37**]).

We begin by recall the definition of ring with apartness. A *ring with apartness* R is a nonempty set $(R, =, \neq)$ with apartness together with two strongly extensional internal operations $+ : E \times R \ni (a, b) \longmapsto a+b \in R$ and $\cdot : R \times R \ni (a, b) \longmapsto ab \in R$ for which the following conditions are valid

$(\forall a, b, c \in R)(a + (b + c) = (a + b) + c)$,

$(\forall a, b \in R)(a + b = b + a)$,

$(\exists 0 \in T)(\forall a \in R)(a + 0 = a0 + a)$,

$(\forall a \in R)(\exists(-a) \in R)(a + (-a) = 0 = (-a) + a)$,

$(\forall a, b, c \in R)(a(bc) = (ab)c)$,

$(\forall a, b, c \in R)(a(b + c) = ab + ac = (b + c)a)$.

A ring R is said to be commutative if $(\forall a, b \in R)(ab = ba)$ holds. An element of R is called a unity, and it is denoted by $1 (\neq 0)$, of $1a = a1 = a$ for all $a \in R$. The inverse $(-a)$ of an element $a \in R$ with be denoted by $-a$ and $a + (-b)$ will be written as $a - b$. We shall assume throughout that all rings are commutative rings with the unity $'1'$. Since the additio and the multiplication in R are total strongly extensional functions, we have

$(\forall a, b, x, y \in R)((a = x \wedge b = y) \Longrightarrow (a + b = c + y \wedge ab = xy)$,

$(\forall a, b, x, y \in R)(a + b \neq x + y \Longrightarrow (a \neq x \vee b \neq y))$ and

$(\forall a, b, x, y \in R)(ab \neq xy \Longrightarrow (a \neq x \vee b \neq y))$.

Specially, we have

$(\forall a, b \in R)(ab \neq 0 \Longrightarrow (a \neq 0 \wedge b \neq 0))$.

A ring R is a *Richman field* ([**27**]) if R is a commutative ring with
$(\forall a \in R)(a \neq 0 \implies (\exists a^{-1} \in R)(aa^{-1} = 1))$.
A *Heyting field* ([**19**]) is a Richman field with tight apartness.

By *subring* of a ring R we mean a ring S such that the set $(S, =, \neq)$ is a subset of the set $(R, =, \neq)$ and such that the binay operaions of R yield the binary operations in S when restrictd to $S \times S$.

Let $((R, =, \neq), +, 0, \cdot, 1)$ be a commutative ring with apartness and let $(S, =, \neq)$ be a subset of R. S is a *co-ideal* of R if the following holds
$0 \lhd S$,
$(\forall a \in R)(-a \in S \implies a \in S)$,
$(\forall a, b \in R)(a + b \in S \implies (a \in S \vee b \in S))$,
$(\forall a, b \in R)(ab \in S \implies (a \in S \wedge b \in S))$.

Any inhabited co-ideal S of a ring R is strongly extensional subset of R:
$$\begin{aligned} a \in S &\iff a - b + b \in S \\ &\implies (a - b \in S \vee v \in S) \\ &\implies a \neq b \vee b \in S. \end{aligned}$$

For an ingabited co-ideal S of a ring R with the unit 1, we have
$$\begin{aligned} S \neq \emptyset &\iff (\exists a \in R)(a \in S) \\ &\iff (\exists a \in R)(a \cdot 1 \in S) \\ &\iff (\exists a \in R)(a \in S \wedge 1 \in S) \\ &\iff 1 \in S. \end{aligned}$$
Now, we can define the following relation q in R by
$(\forall a, b \in R)((a, b) \in q \iff a - b \in S)$.
For it, the author is proved the following result:

THEOREM 5.1 ([**27**], Proposition 2.5). *The relation q is a co-equality relation on R and it satisfies the following properties:*
$(\forall a, b, u, v \in R)((a + u, b + v) \in q \implies ((a, b) \in q \vee (u, v) \in q))$ *and*
$(\forall a, b, u, v \in R)((au, bv) \in q \implies ((a, b) \in q \vee (u, v) \in q))$.

Tee relation q on R is called a *co-congruence* on R. Inversely, we have

THEOREM 5.2 ([**27**], Proposition 2.5). *If q is a co-congruence on a ring R, then the set $S = \{a \in R : (a, 0) \in q\}$ is a co-ideal of R.*

Let J be an ideal of a ring R and let S be a co-ideal of R. Ruitenburg, in his dissertation ([**54**], page 33), first stated the demand that
$J \subseteq \neg S$.
This condition is equivalent with the following condition
$(\forall a, b \in R)((a \in J \wedge b \in S), \implies a + b \in S)$.
In this case, we say that they are compatible.

Let $((R, =, \neq), +, 0, \cdot, 1)$ be a commutative ring with apartness. The following theorem gives a construction of a co-congruence on R on the basis of the given coequality relation q on R.

THEOREM 5.3 ([**35**], Theorem 3). *Let q be a coequality relation on a ring R. Then the relation*
$q^* = \{(x, y \in R \times R : (\exists s, t \in R)((xt + s, yt + s) \in q)\}$
is a co-congruence on R. If r is a co-congruence on R such that $q \subseteq r$, then $q^* \subseteq r$.

THEOREM 5.4. *Let q be a co-congruence on a ring R. Then the relation $q^{\lhd}$ is a congruence on R.*

As a corollary of above theorem, we can construct the ideal $\{a \in R : (a, 0) \in q^{\lhd}\}$. The following theorem gives a connection between ideal $\{a \in R : (a, 0) \in q^{\lhd}\}$ and the ideal $\{b \in R : (b, 0) \in q\}^{\lhd}$.

THEOREM 5.5 ([**30**]). *Let q be a co-congruence on a ring R. Then*
$\{a \in R : (a, 0) \in q^{\lhd}\} = \{b \in R : (b, 0) \in q\}^{\lhd}$.

Further, we have

THEOREM 5.6. *Let $\{S_i\}_{i \in I}$ be a family of co-ideals of a ring R. Then the union $\bigcup_{i \in I} S_i$ is a co-ideal of R.*

Let J and S be compatible an ideal and a co-ideal of a ring R. In the following theorem we construct the quotient-ring $R/(J, S)$.

THEOREM 5.7 ([**27**], Theorem 2.7). *Let J and S be compatible an ideal and a co-ideal of a ring R. Then the set $R/(J, S) = \{aJ : a \in R\}$ is a ring with*
$(\forall a, b \in R)(aJ = bJ \iff a - b \in J)$,
$(\forall a, b \in R)(aJ \neq bJ \iff a - b \in S)$,
$(\forall a, b \in R)(aJ + bJ = (a + b)J)$ *and*
$(\forall a, b \in R)(aJ \cdot bJ = (ab)J)$.

We say that co-ideal S of a ring R is a *prime* co-ideal ([**25**], [**54**], [**29**]) if the following is valid
$(\forall a, b \in R)((a \in S \wedge b \in S) \implies ab \in S)$.
A co-ideal S of R is a *minimal* co-ideal ([**54, 27**]) if holds
$(\forall a \in R)(a \in S \implies (\exists b \in R)(ab - 1 \lhd S))$.

COROLLARY 5.1 ([**27**], Proposition 3.7). *Let R be a ring and S be a co-ideal of R. Then the quotient-ring $R/(S^{\lhd}, S)$ is an integral domain if and only if S is a prime co-ideal of R.*

COROLLARY 5.2 ([**27**], Theorem 3.4). *Let R be a ring and S be a co-ideal of R. Then the quotient -ring $R/(S^{\lhd}, S)$ is a field if and only if S is a minimal co-ideal of R.*

A function $f : R \longrightarrow T$ from a ring R to a ring T is a homomorphism of rings if holds
$(\forall a, b \in R)(f(a + b) = f(a) + f(b) \wedge f(ab) = f(a)f(b))$
In the collection $Hom(R, T)$ of all homomorphisms from R to T we define
$(\forall f \in Hom(R, T))(f = 0 \iff (\forall a \in R)(f(a) = 0))$ and

$(\forall f \in Hom(R,T))(f \neq 0 \iff (\exists a \in R)(f(a) \neq 0))$.

We have

LEMMA 5.1 ([**19**], Theorem I, 2.2). *Let S be a set with a diversity relation and let X be a set. If the diversity on S is consistent, symmetric, co-transitive, or tight, then so, respectively, is the diversity on S^X*

COROLLARY 5.3. *Let R be a ring with apartness. Then the set $End(R) = Hom(R,R)$ of all homomorphism from R to R is a noncommutative ring with apartness.*

THEOREM 5.8 ([**27**]). *Let R and T be wings with apartess and lt $f : R \longrightarrow T$ be a strongly extensional homomorphism. Then the following holds:*

(1) *The set $Ker(f) = \{a \in R : f(a) = 0\}$ is an ideal of R.*

(20 *The set $R_f = \{b \in R : f(b) \neq 0\}$ is a co-ideal of R and $R_f \subseteq R_0]\{a \in R : a \neq 0\}$ is valid.*

(3) *$Ker(f)$ and R_f are compatible.*

(4) *$Om(f)]\{f(a) \in T : a \in R\}$ is a subring of T.*

(5) *The homomorphism f is injective if and only if $Ker(f) = (0)$.*

(6) *The homomorphism f is an embedding if and only if $R_f = R_0$.*

(7) *There exists a strongly extensional embedding isomorphism*

$h : R/(Ker(f), R_f) \longrightarrow Im(f) \subseteq T$

such that $f = h \circ \pi$, where $\pi : R \longrightarrow R/(Ker(f), R_f)$ is the canonical epimorphism.

EXAMPLE 5.1. (1) Let m and $i \in \{1, 2, ..., m-1\}$ be integers. We set $m\mathbf{Z} + i = \{mz + i : z \in \mathbf{Z}\}$. Then the set $C_m = \bigcup\{m\mathbf{Z} + i : i \in \{1, 2, ..., m-1\}\}$ is a co-ideal of the ring $\mathbf{Z}$. The co-ideal C_m is a prime co-ideal of $\mathbf{Z}$ if and only if m is a prime integer.

(2) Let K be a field and let x be an unknown variable under K. Then the set $C = \{f \in k[x] : f(0) \neq 0\}$ is a co-ideal of the ring $K[x]$. This co-ideal is a minimal co-ideal.

(3) ([**32**], Theorem 1) Let $\{R_t\}_{t \in T}$ be a family of rings and let $\mathfrak{H}$ be a nonempty subfamily of $\mathfrak{P}(T)$. Then the set

$S(\mathfrak{H}) = \{r \in \prod_{t \in T} R_t : (\exists A \in \mathfrak{H})(A \cap \{t \in T : r(t) \neq 0\} \neq \emptyset\}$

is a co-ideal of the ring $\prod_{t \in T} R_t$.

5.2. Concept of Γ-semirings with apartness. In this subsection we introduce the concept of Γ-semirings with apartness. We first consider the ideals and co-ideals of a Γ-semiring with apartness. Also, by using the congruences and co-congruences induced by strongly extensional homomorphisms between such Γ-semirings, we establish an isomorphism theorem.

The concept of Γ-semirings were first introduced and studied by M. K. Rao [**22, 23**] as a generalization of notion of Γ-rings.

We call R a Γ*-semiring with apartness* [**50**] if there exists a map $R \times \Gamma \times R \longrightarrow R$, written image of (x, a, y) by xay, such that it satisfies the following axioms:

$(\forall x, y, z \in R)(\forall a \in \Gamma)(xa(y + z) = xay + xaz$ and $(x + y)az = xaz + yaz)$,

$(\forall x, y \in R)(\forall a, b \in \Gamma)(x(a + b)y = xay + xby)$,

$(\forall x,y,z\in R)(\forall a,b\in\Gamma)((xay)bz=xa(ybz))$.

Remark 5.1. As can be seen, the definition of Γ-semirings with apartness is completely identical to the definition of Γ-semiring in the classical case. However, they do not determine the same algebraic structure. The reader should always keep in mind that the logical setting are different and that the manipulation with them takes place with the previously acceptance of the various principles-philosophical orientations. In this environment, the following implication is valid

$$(\forall x,y,u,v\in R)(\forall a,b\in\Gamma)(xay\neq ubv\implies(x\neq u\vee a\neq b\,y\neq v)).$$

A Γ-semiring with apartness R is said to have a *zero element* if there exists an element $0\in R$ such that the following

$$(\forall x\in R)(\forall a\in\Gamma)(0+x=x=x+0\ \text{ and }\ 0ax=0=xa0)$$

is valid. Of course, we also have

$$(\forall x,y\in R)(x+y\neq 0\implies(x\neq 0\vee y\neq 0))$$

and

$$(\forall x,y\in R)(\forall a\in\Gamma)(xay\neq 0\implies(x\neq 0\wedge y\neq 0)).$$

Also, a Γ-semiring with apartness R is said to be *commutative* if the following holds

$$(\forall x,y\in R)(\forall a\in\Gamma)(xay=yax).$$

Let R be a Γ-semiring and T a Λ-semiring. Then $(f,\varphi):(R,\Gamma)\longrightarrow(T,\Lambda)$ is called a *homomorphism* if $f:R\longrightarrow T$ and $\varphi:\Gamma\longrightarrow\Lambda$ are strongly extensional homomorphisms of semigroups such that

$$(\forall x,y\in R)(\forall a\in\Gamma)((f,\varphi)(xay)=f(x)\varphi(a)f(y))$$

holds. The mapping (f,φ) is called an *epimorphism* if (f,φ) is a homomorphism and f and φ are epimorphisms of semigroups. Similarly, we can define a monomorphism. A homomorphism (f,φ) is an isomorphism if (f,φ) is an epimorphism and a monomorphism and f and φ are embeddings.

Let R be a Γ-semiring with apartness

- A non-empty subset A of R is a *sub-*Γ*-semiring* of R if A is an additive sub-semigroup of R and the following holds

$$(\forall x,y\in R)(\forall a\in\Gamma)((x\in A\wedge y\in A)\implies xay\in A).$$

- A subset B of R is a *cosub-*Γ*-semiring* of R if B is an additive cosub-semigroup of R and the following holds

$$(\forall x,y\in R)(\forall a\in\Gamma)(xay\in B\implies(x\in A\vee y\in A)).$$

Let R be a Γ-semiring with apartness.

- A subset B of R is a *right* Γ*-coideal* of R if B is a additive cosub-semigroup of R and the following holds

$$(\forall x,y\in R)(\forall a\in\Gamma)(xay\in B\implies y\in B).$$

- A subset B of R is a *left* Γ-*coideal* of R if B is a additive cosub-semigroup of R and the following holds

$$(\forall x, y \in R)(\forall a \in \Gamma)(xay \in B \implies x \in B).$$

- A subset B of R is a Γ-*coideal* of R if B is a additive cosub-semigroup of R and the following holds

$$(\forall x, y \in R)(\forall a \in \Gamma)(xay \in B \implies (x \in B \wedge y \in B)).$$

If R is a Γ-semiring with zero element 0, then it is mandatory to assume that $0 \lhd B$.

Proposition 5.1. *If B is (left, right) coideal of a Γ-semiring R, then the set $B^{\lhd}$ is a (letf, ringt) ideal of R.*

Theorem 5.9. *The union of any family $\{B_i\}_{i \in I}$ of (right, left) Γ-coideals of a Γ-semigroup (R, Γ) is a (right, left) Γ-coideal of R.*

Corollary 5.4. *Let X be a subset of Γ-semigroup (R, Γ). Then there exists the maximal (left, rignt) Γ-coideal of Γ-semiring (R, Γ) included in X.*

Corollary 5.5. *Let $\mathfrak{L}(R, \Gamma)$ be the family of all (left, right) Γ-cideals of (R, Γ). Then $(\mathfrak{L}(R, \Gamma), \sqcup, \sqcap)$ is a completely lattice, where $B_1 \sqcup B_2 = B_1 \cup B_2$ and $B_1 \sqcap B_2$ is the maximal coideal included in $B_1 \cap B_2$.*

A co-equality relation q on Γ-semiring (R, Γ) is said to be a *co-congruence* if the following conditions
$(\forall x, y, z \in R)((x + z, y + z) \in q \implies (x, y) \in q)$ and
$(\forall x, y, z \in R)(\forall a \in \Gamma)(((xaz, yaz) \in q \vee (zax, zay) \in q) \implies (x, y) \in q)$
are satisfied.

It is known ([**22**]: pp. 51, [**12**]: Theorem 4.5) that if ρ is a congruence relation on a Γ-semiring (R, Γ), then $R/\rho = \{[x]_\rho : x \in R\}$ is also Γ-semiring where it is

$$(\forall x, y \in R)([x]_\rho + [y]_\rho = [x + y]_\rho),$$

$$(\forall x, y \in R)(\forall a \in \Gamma)([x]_\rho a [y]_\rho = [xay]_\rho).$$

Proposition 5.2. *If q is a Γ-cocongruence on a Γ-semiring (R, Γ), then the relation $q^{\lhd}$ is a Γ-congruence on (R, Γ).*

Theorem 5.10. *Let q be a Γ-cocongruence on a Γ-semigrong (R, Γ). Then the family $R : q = \{[x]_q : x \in R\}$ is a Γ-semiting also with*

$$(\forall x, y \in R)([x]_q =_1 [y]_1 \iff (x, y) \lhd q),$$

$$(\forall x, y \in R)([x]_q \neq_1 [y]_q \iff (x, y) \in q),$$

$$(\forall x, y \in R)([x]_q + [y]_q =_1 [x + y]_q),$$

$$(\forall x, y \in R)(\forall a \in \Gamma)([x]_q a [y]_q =_1 [xay]_q).$$

Lemma 5.2. *Let q be a Γ-cocongruence on a Γ-semigrong (R, Γ). Then the mapping $(\pi, i) : R \longrightarrow R : q$, defined by $\pi(x) = [x]_q$ and $i(a) = a$, is a strongly extensional epimorphism.*

LEMMA 5.3. *If the mapping* $(f, \varphi) : (R, \Gamma) \longrightarrow (T, \Lambda)$ *is a strongly extensional homomoephism, then the relation* $q(f)$ *on* R*, defined by*

$$(\forall x, y \in R)((x, y) \in q(f) \iff f(x) \neq f(y)),$$

is a Γ*-cocongruence on* R*.*

Without major difficulties, the following theorem can be proved. We can be viewed on this theorem as on the First Theorem on Isomorphisms using co-congruences in Γ-semirigns with apartness.

THEOREM 5.11. *Let* $(f, \varphi) : (R, \Gamma) \longrightarrow (T, \Lambda)$ *be a strongly extensional homomorphism, then there exists the strongly extensional injective and embedding homomorphism* $(g, \varphi) : (R : q(f), \Gamma) \longrightarrow (T, \Lambda)$ *such that*

$$(f, \varphi) = (g, \varphi) \circ (\pi, i).$$

5.3. Semivaluation on Heytin fields. The first investigation of valuation theory from the constructive point of view is in te article [**57**] by J. Staples. However, Staples was not interested in valuation per se. There are also developments [**17**] and [**18**]. They dea rank one valuation to b a function v from a field K to the nonnegative real numbers satisfying

(1) $v(x) \neq 0 \iff x \neq 0$, (2) $v(xy) = v(x)v(y)$ and (3) $v(x + y) \leqslant v(x) + v(y)$. In our papaer [**36**] we given a theory of semivaluation Heyting filed different from above.

Let $((R, =, \neq), +, 0, \cdot, 1)$ be a ring with apartness. A subset D of R is a *co-subring* of R if the following holds

$0 \lhd D$ and $1 \lhd D$,

$(\forall a \in R) - a \in D \implies a \in D)$,

$(\forall a, b \in R)(a + b \in D \implies (a \in D \vee b \in D))$ and

$(\forall a, b \in R)(ab \in D \implies (a \in D \vee b \in D))$.

Some examples of co-subrings can be found in Examples in the article [**36**].

PROPOSITION 5.3 ([**36**], Theorem 5.1). *Let* R *be a commutative ring with apartness and let* D *be a co-subring of* R*. Then the set* $D^{\lhd}$ *is a subring of* R *compatible with* S *in the following sense that* $a \in D \wedge b \in D^{\lhd} \implies a + b \in D$*.*

THEOREM 5.12 ([**36**], Theorem 8.2). *Let* R *be a fields with apartness and let* D *be a co-subring of* R*. Then:*

(1) *The set* $S = \{a \in R : a \in D \vee a^{-1} \in D\}$ *is a strongly extensional co-subgroup of the multiplicative group* $R^* = \{a \in R : a \neq 0\}$ *compatible with the subgroup* $S^{\lhd}$ *such that that we can the factor-group* $G = R^*/(S, S^{\lhd})$*. (We write the group operation on* G *as additive.)*

(2) *The relation* α *on* G*, defined by* $(aS^{\lhd}, bS^{\lhd}) \in \alpha \iff a^{-1}b \in D$*, is a co-order relation on* G *compatible with the group operation in* G*.*

(3) *If* w *is the canonical homomorphism from* R^* *onto* G*, then* w *has the following properties:*

(0) *w is a strongly extensional homomorphism;*

(i) $w(ab) = w(a) + w(b)$*;*

(ii) $w(-1) = S^{\lhd}$;
(iii) *Let* $a, b, t \in R^*$ *such that* $a + b \in R^*$. *If* $(w(t), w(a + b)) \in \alpha$, *then* $w(t, w(a)) \in \alpha$ *or* $(w(t), w(b)) \in \alpha$.

This observation motivates a definition of the concept of semivaluations on Heyting field ([**36**]). Let H be an ordered additive Abelian group under a co-order α, and let let K be a Heyting fields. A strong semivaluation on K is a strongly extensional mapping v from K^* onto H such that
(1) $(\forall a, b \in K^*)(v(a + b) = v(a) + v(b))$,
(2) $v(-1_{=}0$ and
(3) $(c(t), v(a + b)) \in \alpha \implies ((v(t), v(a)) \in \alpha \vee (v(t), v(b)) \in \alpha)$.
for any $r, a, b \in K^*$ such that $a + b \in K^*$.

Theorem 5.13 ([**36**], Theorem 6.1). *Let* $v : K^* \longrightarrow H$ *be a strong semivaluation. Then*
(1) v *is a strongly extensional homomorphism of groups.*
(2) *The set* $Ker(v)$ *and* $Coker(v)$ *are compatible a subgroup and a co-subgroup of the multiplicative group* K^*.
(3) *The set* $D_v = \{c \in K^* : (0, v(c)) \in \alpha\}$ *is a co-subring of* K *and holds* $Coker(v) = D_v \cup D_v^{-1}$ *and* $D_v \cap D_v^{-1} = \emptyset$.

5.4. Modules over commutative ring. The definition of the concept of modules with apartness over commutative rings and a description of some of its fundamental properties is taken from article [**25, 29**]. Some elements of a constructive aspect in discovering linear spaces over rings can be found in the texts [**1, 15**].

Let R be a ring and let M be an Abelian group. It is said that M is an A-module if there exists is a strongly extensional function $A \times M \longrightarrow M$ on M over A such that
$(\forall a, b \in A)(\forall x \in M)((ab)x = ac + bx)$,
$(\forall x \in M)(1 \cdot x = x)$,
$(\forall a, b \in A)(\forall x \in M)((a + b)x = ax + bx)$, and
$(\forall a \in A)(\forall x, y \in M)(a(x + y) = ax + ay)$.

A linear space is a module over a field.

Let M be a module over a ring R. Then
$(\forall a, b \in A)(\forall x, y \in M)(ax \neq by \implies a \neq b \vee x \neq y)$
by strongly extensionality of the function $A \times M \longrightarrow M$.

Let M be a module over a commutative ring R. A coequality relation q on M is a *co-congruence* on R-module M ([**29**], Definition 3.1) if the following holds
$(\forall a \in R)(\forall x \in M)((ax, 0) \in q \implies (a \neq 0 \wedge (x, 0) \in q))$and
$(\forall x, y, u, v \in M)((x = u, y + v) \in q \implies ((x, y) \in q \vee (u, v) \in q))$.

A subset H of M is a *co-submodule* ([**29**], Definition 3.2) of M if holds
- $0 \lhd H$,
- $(\forall x, y \in M)(x \in H \vee y \in H)$ and
- $(\forall a \in R)(\forall \in M)(ax \in H \implies (a \neq 0 \wedge x \in H))$.

Proposition 5.4 ([**29**], Proposition 3.2). *Let M be a R-module and let q be a relation on M. Then q is a co-congruence on M if and only if the set $P = \{x \in M : (x,0) \in q\}$ is a co-submodule of M and $(x,y) \in q \iff x - y \in P$ holds.*

Theorem 5.14 ([**29**], Theorem 3.4). *Let M be a module over a commutative ring A and let e and q be compatible a congruence and a co-congruence on M. Then the family $M/(e,q) = \{ae : a \in M\}$ be a module over A with the equality and apartness determined by*

$(\forall x, y \in M)(xe = ye \iff (x,y) \in e)$ *and*
$(\forall x, y \in M)(xe \neq ye \iff (x,y) \in q)$

and the internal operation determined by

$(\forall x, y \in M)(xe + ye = (x+y)e)$ *and*
$(\forall a \in A)(\forall x \in M)(a \cdot xe = (ax)e)$.

Let N and K be compatible a submodule and a co-submodule of a A-module M. Then the relation $e \subseteq M \times M$, defined by $(x,y) \in e \iff x - y \in N$, is a congruence on M and the relation $q \subseteq M \times M$, defined by $(x,y) \in q \iff x - y \in K$ is a co-congruence on M compatible with each other. Let we define a family $M/(M,K) = \{x + N = \{x + t : t \in N\} : x \in M\}$ with

$(\forall x, y \in M)(x + N = y + N \iff x - y \in N)$,
$(\forall x, y \in M)(x + N \neq y + N \iff x - y \in K)$
$(\forall x, y \in M)((x + N) + (y + N) = (x + y) + N)$ and
$(\forall a \in A)(\forall x \in M)(a(x + N) = ax + N)$.

In this case we write $M/(N,K)$ instead of $M/(e,q)$.

Analogously, we can define the family $M : q = \{xq : x \in M\}$ with

$(\forall x, y \in M)(xq = yq \iff (x,y) \lhd q)$,
$(\forall x, y \in M)(xq \neq yq \iff (x,y) \in q)$,
$(\forall x, y \in M)(xq + yq = (x+y)q)$ and
$(\forall a \in A)(\forall x \in M)(a(xq) = (ax)q)$.

Theorem 5.15. *Let M be a module over a commutative ring A and let q be a co-congruence on M. Then the family $M : q = \{aq : a \in M\}$ be a module over A.*

If we define $N : K = \{x + K = x + t : t \in K\} : x \in M\}$ with

$(\forall x, y \in M)(x + K = y + K \iff x - y \lhd K)$,
$(\forall x, y \in M)(x + K \neq y + K \iff x - y \in K)$,
$(\forall x, y \in M)((x + K) + (y + K) = (x + y) + K)$ and
$(\forall a \in A)(\forall x \in M)(a(x + K) = ax + K)$

we can write $M : K$ instead of $M : q$.

Let M and N be A-modules. A total strongly extensional function $f : M \longrightarrow N$ is a homomorphism of modules if holds

$(\forall a \in A)(\forall x, y \in M)(f(x+y) = f(x) + f(y) \wedge f(ax) = af(x))$.

Lemma 5.4 ([**29**], Proposition 4.1). *If $f : M \longrightarrow N$ is a homomorphism between A modules, then the sets $Ker(f)$ and $Coker(f)$ are compatible a submodule and a co-submodule of M in the sense*

$(\forall x, y \in M)(x \in Ker(f) \wedge y \in Coker(f) \implies x + y \in Coker(f))$.

Theorem 5.16 ([**29**], Theorem 4.4). *Let* $f : M \longrightarrow N$ *be a homomorphism between A-modules. Then:*

- there exists the unique embedding monomorphism

$g : M/(Ker(f), Coker(f)) \longrightarrow Im(f)$

such that $f = g \circ \pi$, *where* $\pi : M \longrightarrow /M/(Ker(f), Coker(f))$ *is the canonical epimorphism.*

- there exists the unique embedding monomorphism

$h : M : Coker(f) \ni x + Coker(f) \longmapsto f(x) \in Im(f) \subseteq M$

such that $f = g \circ \theta$, *where* $\theta : M \longrightarrow M : Coker(f)$ *is standard monomorphism.*

The previous theorem can be seen as the First isomorphism theorem between the modules with apartness under a commutative ring. Of course, the statement of the theorem in this principled-philosophical orientation is differs significantly from the analogous theorem in the classical theory. In this case, the theorem has an important part that does not have its own dual in the classical module theory over commutative rings.

References

[1] M. A. Baroni. Suprema in ordered vector spaces: a constructive approach. Supplement Proceedings of ICMI 45, Bacau, Sept.18-20, 2006. *Stud. Cercet. Stiint., Ser.Mat.*, **16**(2006), 39–56

[2] A. Bauer. Five stages of accepting Constructive mathematics. *Bull. Amer. Math. Soc.*, **54**(3)(2017), 481-498.

[3] E. Bishop. *Foundations of Constructive Analysis.* New York: McGraw-Hill, 1967.

[4] D. S. Bridges and F. Richman. *Varieties of Constructive Mathematics.* Cambridge: London Mathematical Society Lecture Notes, No. 97, Cambridge University Press, 1987.

[5] D. S. Bridges. Prime and maximal ideals in constructive ring theory. *Comm. Algebra.*, **29**(7)(2001), 2787–2803.

[6] D. S. Bridges and R. S. Havea. Constructive notions of maximality for ideals. J. Uni. Comput. Sci., **14**(22)(2008), 3648–3657.

[7] D. S. Bridges and L.S. Viţă. *Apartness and Uniformity: A Constructive Development.* in: CiE series "Theory and Applications of Computability", Springer Verlag, Heidelberg, 2011.

[8] A. Cherubini and A. Frigeri. Inverse semigroups with apartness. *Semigroup Forum*, (2019). https://doi.org/10.1007/s00233-019-10019-y

[9] S. Crvenković and D. A. Romano. A theorem on anti-ordered factor-semigroups. *Publications de l'Institut Mathmatique*, **82**(96)(2007), 119–128.

[10] S. Crvenković, M. Mitrović and D. A. Romano. Semigroups with apartness. *Math. Logic Quarterly*, **59**(6)(2013), 407–414.

[11] S. Crvenković, M. Mitrović and D. A. Romano. Basic notions of (Constructive) semigroups with apartness. *Semigroup Forum*, **92**(3)(2016), 659–674.

[12] H. Hedayati and K. P. Shum. *An introduction to* Γ*-semirings.* Int. J. Algebra, **5**(15)(2011), 709–726.

[13] D. van Dalen, F.-J. de Vries. Intuitionistic free Abelian groups. *Math. Logic Quaterly (Formerly: Z. Math. Logik Grundl. Math.)*, **34**(1)(1988), 3–12.

[14] M. Kuril and L. Polak. On varieties of semilattice-ordered semigroups. *Semigroup Forum*, **71**(1) (2005), 27–48.

[15] H. Lombardi and C. Quitté. Commutative algebra: Constructive methods. Springer Netherlands, 2015.

[16] R. Milošević, D. A. Romano and M. Vinčić. A basic separation on sets with apartnesses. *Univ. Beograd , Publ. ETF , Math. Ser.*, **8**(1997), 36–43.

[17] R. Mines and F. Richman. Valuation theory: a constructive view. *J. Number Theory*, **19**(1984), 40–62.
[18] R. Mines amd F. Richman. Archimedean valuations. *J. London Math. Soc.*, **34**(1986), 403–410.
[19] R. Mines, F. Richman and W. Ruitenburg. *A course of constructive algebra*. New York: Springer-Verlag, 1988.
[20] M. Mitrović, D. A. Romano and M. Vinčić. A theorem on semilattice-ordered semigroup. *Int. Math. Forum*, **4**(5)(2009), 227–232.
[21] M. Mitrović, S. Crvenković and D. A. Romano. Semigroups with apartness: constructive versions of some classical theorems. *The 46th Annual Iranian Mathematics Conference 25-28 August 2015* (pp. 64-67), Yazd University, Yazd, Iran, 2016.
[22] M. K. Rao. Γ*-semirings, 1*. Southeast Asian Bull. Math., **19**(1995), 49–54.
[23] M. K. Rao. Γ*-semirings, II*. Southeast Asian Bull. Math., **21**(1997), 281–287.
[24] F. Richman. Constructive aspects of Noetherian rings. *Proc. Amer. Math. Soc.*, **44**(2)(1974), 436–441.
[25] D. A. Romano. *Constructive algebra - algebraic structures*. Ph.D. Thesis, Faculty of Mathematics, University of Belgrade, Belgrade 1985. (In Serbian)
[26] D. A. Romano. Aspect of constructive Abelian groups. In: Z. Stojaković (Ed.). *Proceedings of the 5th conference Algebra and Logic, Cetinje 1986*, (pp. 167-174). Novi Sad: University of Novi Sad, Institute of Mathematics, 1987.
[27] D. A. Romano. Rings and fields, a constructive view. *Math. Logic Quarterly (Formerly: Z. Math. Logik Grundl. Math.)*, **34**(1)(1988), 25–40.
[28] D. A. Romano. Construction of free Abelian groups. *Sarajevo J. Math. (Formerly: Radovi mat).*, **4**(1)(1988), 151–158.
[29] D. A. Romano. Rings and modules, a constructive view. In: S. Crvenković (Ed.). *Proceedings of the 6th conference Algebra and Logic, Sarajevo 1987* (pp. 113–140). Novi Sad: University of Novi Sad, Institute of Mathematics, 1989.
[30] D. A. Romano. A theorem on co-congruence of rings. *Math. Logic Quarterly (Formerly: Z.Math.Logik Grundl. Math.)*, **36**(1)(1990), 87–88.
[31] D. A. Romano. Some generalizations of inverse limits of Abelian groups with apartnesses. *Kragujevac J. of Math. (Formely Coll. Sci. Papers of Fac. Sci. Kragujevac)*, **15**(1994), 27–43.
[32] D. A. Romano. On coideals of product of commutative rings with apartness. *Bull. Soc. Math. Banja Luka*, **2**(1995), 1–7.
[33] D. A. Romano. Some notes on coequality relations. *Mathematica Montisnigri*, **6**(1996), 93–98.
[34] D. A. Romano. Coequality relations, a survey. *Bull. Soc. Math. Banja Luka*, **3**(1996), 1–36.
[35] D. A. Romano. A note on two problems in constructive commutative coideal theory. *Sci. Rev. Ser. Math.*, **19-20**(1996), 91–96.
[36] D. A. Romano. Semivaluation on Heyting field. *Kragujevac J. Mat. (Formely Coll. Sci. Papers of Fac. Sci. Kragujevac)*, **20**(1998), 24–40.
[37] D. A. Romano. The coideal theory of commutative ring with apartness, a Survey. *Bull. Soc. Math. Banja Luka*, **8**(2001), 1–19.
[38] D. A. Romano. Some relations and subsets of semigroup with apartness generated by the principal consistent subset. *Univ. Beograd, Publ. Elektroteh. Fak. Ser. Math*, **13**(2002), 7–25.
[39] D. A. Romano. A note on a family of quasi-antiorder on semigroup. *Kragujevac J. Math.*, **27**(2005), 11–18.
[40] D. A. Romano. A note on quasi-antiorder in semigroup. *Novi Sad J. Math.*, **37**(1)(2007), 3–8.
[41] D. A. Romano. On semilattice-ordered semigroups. A constructive point of view. *Sci. Stud. Research. Ser. Math. Inf.*, **21**(2)(2011), 117–134.
[42] D. A. Romano. On quasi-antiorder relation on semigroups. *Matematiki vesnik*, **64**(3)(2012), 190–199.

[43] D. A. Romano. Semilattice-ordered semigroup with apartness representation problem. *J. Adv. Math. Stud.*, **5**(2)(2012), 13–19.
[44] D. A. Romano. An introduction to implicative semigroups with apartness. *Sarajevo J. Math.*, **12**(2)(2016), 155–165.
[45] D. A. Romano. Strongly extensional homomorphism of implicative semigroups with apartness. *Sarajevo J. Math.*, **13**(2)(2017), 155-162
[46] D A. Romano. Co-filters in semilattice-ordered semigroup with apartness. *J. Advan. Math. Stud.*, **11**(1)(2018), 124–131.
[47] D. A. Romano. Semilattice-ordered semiring with apartness. *J. Adv. Math. Stud.*, **11**(3) (2018), 496–502.
[48] D. A. Romano. Semigroup action on grupoid ordered under co-order. *Quasigroups and Related Systems*, **26**(2)(2018), 299-308.
[49] D. A. Romano. Co-ideals and co-filters in ordered set under co-quasiorder. *Bull. Int. Math. Virtual Inst.*, **8**(1)(2018), 177–188.
[50] D. A. Romano. Γ-semirings with apartness. *Rom. J. Math. Comp. Sci.*, **9**(2019) (To appear)
[51] D. A. Romano. Γ-semigroups with apartness. (To appear).
[52] D. A. Romano. On co-filters in semigroup with apartness. *Kragujevac J. Math.*, **45**(2021) (To appear)
[53] D. A. Romano. On co-filters in implicative semiroups with apartness. (To appear)
[54] W. Ruitenburg. *Intuitionistic algebra.* Ph.D. Thesis, University of Utrecht, Utrecht 1982.
[55] M. K Sen. On Γ-semigroups. In *Proceeding of International Conference on 'Algebra and its Applications, (New Delhi, 1981)'* (pp. 301–308). Lecture Notes in Pure and Appl. Math. 9, New York: Decker Publication, 1984.
[56] M. K. Sen and N. K. Saha. On Γ-semigroup, I. *Bull. Calcutta Math. Soc.*, **78**(1986), 181–186.
[57] J. Staples, On constructive fields. *Proc. London Math. Soc.*, **23**(3)(1971), 753–768.
[58] J. Staples. Axioms for constructive fields. *Bull. Austral. Math. Soc.*, **8**(1973), 221–232.
[59] A. S. Troelstra and D. van Dalen. *Constructivism in mathematics, An introduction.* North-Holland, Amsterdam 1988.

Received by editors 07.03.2019; Available online 01.07.2019.

International Mathematical Virtual Institute,
6, Kordunaška Street, 78000 Banja LUka, Bosnia and Herzegovina
E-mail address: bato49@hotmail.com