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COMMON FIXED POINTS OF RATIONAL TYPE AND GERAGHTY-SUZUKI TYPE CONTRACTION MAPS IN PARTIAL METRIC SPACES

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Abstract. In this paper, we prove the existence of common fixed points for a pair of maps using F-class function in partial metric spaces. Further, we introduce Geraghty-Suzuki type contraction for two pairs of selfmaps and prove the existence of common fixed points of these maps in a complete subspace of a partial metric space under the assumption that these maps are weakly compatible. Two examples are given to verify the given results.

1. Introduction

The development of fixed point theory is based on the generalization of contraction conditions in one direction or/and generalization of ambient spaces of the operator under consideration on the other. Banach contraction principle plays an important role in solving nonlinear equations, and it is one of the most useful result in fixed point theory. In 1994, Matthews [**16**] introduced the notion of partial metric in which the concept of self distance need not be equal to zero.

Throughout this paper we denote

$$\mathfrak{F} = \{\beta : [0,\infty) \to [0,1) : \beta(t_n) \to 1 \text{ implies } t_n \to 0 \text{ as } n \to \infty\},\ \mathbb{R}^+ = [0,\infty)$$

and $\mathbb{N}$ is the set of all natural numbers.

In 1973, Geraghty [**8**] proved the following theorem which generalizes the Banach contraction principle.

Theorem 1.1 ([**8**]). *Let (X, d) be a complete metric space and let $T : X \to X$ be a selfmap. Suppose that there exists $\beta \in \mathfrak{F}$ such that $d(Tx, Ty) \leqslant \beta(d(x, y))d(x, y)$*

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holds for all $x, y \in X$. Then T has a unique fixed point $u \in X$ and for each $x \in X$ the Picard sequence $\{T^n x\}$ converges to u as $n \to \infty$.

In 1975, Dass and Gupta [**5**] established a fixed point result using contraction condition involving rational expression as follows:

THEOREM 1.2 ([**5**]). *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that there exist $\alpha, \beta > 0$ with $\alpha + \beta < 1$ satisfying*

$$d(Tx, Ty) \leqslant \alpha d(x, y) + \beta \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}$$

for all $x, y \in X$. Then T has a unique fixed point.

In 2008, Suzuki [**18**] proved two fixed point theorems, one of which is a new type of generalization of the Banach contraction principle and does characterize the metric completeness.

The following theorem is due to Suzuki [**18**].

THEOREM 1.3 ([**18**]). *Let (X, d) be a complete metric space and let T be a mapping on X. Define a non-increasing function $\theta : [0, 1) \to (\frac{1}{2}, 1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leqslant r \leqslant \frac{\sqrt{5}-1}{2} \\ (1-r)r^{-2} & \text{if } \frac{\sqrt{5}-1}{2} \leqslant r \leqslant 2^{-\frac{1}{2}} \\ (1+r)^{-1} & \text{if } 2^{-\frac{1}{2}} \leqslant r < 1. \end{cases}$$

Assume that there exists $r \in [0, 1)$, such that

$$\theta(r) d(x, Tx) \leqslant d(x, y) \text{ implies } d(Tx, Ty) \leqslant r d(x, y)$$

for all $x, y \in X$. Then, there exists a unique fixed point z of T. Moreover,

$$\lim_{n \to \infty} T^n x = z, \; x \in X.$$

DEFINITION 1.1. ([**16**]) Let X be a nonempty set. A mapping $p : X \times X \to \mathbb{R}^+$ is said to be a partial metric on X, if it satisfies the following conditions:

(P_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

(P_2) $p(x, x) \leqslant p(x, y)$,

(P_3) $p(x, y) = p(y, x)$,

(P_4) $p(x, y) \leqslant p(x, z) + p(z, y) - p(z, z)$

for all $x, y, z \in X$. Then the pair (X, p) is called a partial metric space.

If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ defined by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a (usual) metric on X.

EXAMPLE 1.1. ([**1, 14, 16**]) Consider $X = \mathbb{R}^+$ with $p(x, y) = \max\{x, y\}$. Then (X, p) is a partial metric space. It is clear that p is not a (usual) metric.

Note that in this case, $p^s(x, y) = |x - y|$.

EXAMPLE 1.2. ([**11**]) Let $X = \{[a, b] : a, b \in \mathbb{R}, a \leqslant b\}$ and define

$$p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}.$$

Then (X, p) is a partial metric space.

Each partial metric p on X generates a τ_0 topology τ_p on X, which has a base, the family of open p-balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$, where

$$B_p(x, \epsilon) = \{y \in X \mid p(x, y) < p(x, x) + \epsilon\}$$

for all $x \in X$ and $\epsilon > 0$.

Clearly, limit of a sequence in a partial metric space need not be unique. Moreover, the function p need not be a continuous.

EXAMPLE 1.3. ([**6**]) Consider $X = \mathbb{R}^+$ with $p(x, y) = \max\{x, y\}$. Set $x_n = 1$, for all $n \in \mathbb{N}$. Then for each $x \geqslant 1$, we have $p(x_n, x) = p(x, x)$.

DEFINITION 1.2. ([**16**]) Let (X, p) be a partial metric space. A sequence $\{x_n\}$ converges to x if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.

DEFINITION 1.3. ([**16**]) Let (X, p) be a partial metric space. A sequence $\{x_n\}$ is said to be a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.

DEFINITION 1.4. ([**16**]) A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p, to a point $x \in X$, such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

The following lemmas in a partial metric space are useful in proving our main results.

LEMMA 1.1 ([**16**]). *Let (X, p) be a partial metric space. Then the sequence $\{x_n\}$ is a Cauchy sequence in X if and only if it is a Cauchy sequence in the metric space (X, p^s).*

LEMMA 1.2 ([**16**]). *A partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete. Moreover,*

$$\lim_{n\to\infty} p^s(x, x_n) = 0 \Leftrightarrow p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m).$$

LEMMA 1.3 ([**16**]). *Let (X, p) be a partial metric space. Assume $x_n \to z$ as $n \to \infty$ such that $p(z, z) = 0$. Then $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ for every $y \in X$.*

LEMMA 1.4 ([**16**]). *Let (X, p) be a partial metric space. Then*
(i) $p(x, y) = 0 \Rightarrow x = y$,
(ii) $x \neq y \Rightarrow p(x, y) > 0$.

The Banach fixed point theorem in the context of partial metric spaces due to Matthews [**16**] is the following:

THEOREM 1.4 ([**16**]). *Let (X, p) be a complete partial metric space, and let $T : X \to X$ be a mapping such that there exists $k \in [0, 1)$, satisfying $p(Tx, Ty) \leqslant kp(x, y)$ for all $x, y \in X$. Then T has a unique fixed point in X.*

Recently, Dukić et al. [**7**] proved a fixed point theorem using Geraghty-type contraction in partial metric spaces as follows:

THEOREM 1.5 ([**7**]). *Let (X,p) be a complete partial metric space and let $T : X \to X$ be a selfmap. Suppose that there exists $\beta \in \mathfrak{F}$ such that $p(Tx,Ty) \leqslant \beta(p(x,y))p(x,y)$ holds for all $x,y \in X$. Then T has a unique fixed point $u \in X$ and for each $x \in X$ the Picard sequence $\{T^n x\}$ converges to u as $n \to \infty$.*

For more works on fixed point results and common fixed point results in partial metric spaces, we refer [**1, 4, 6, 9, 10, 11, 13, 14, 19**].

DEFINITION 1.5. ([**12**]) Let X be a nonempty set. Let $A : X \to X$ and $B : X \to X$ be two selfmaps. If $Ax = Bx$ implies that $ABx = BAx$ for x in X, then we say that the pair (A,B) is weakly compatible.

DEFINITION 1.6. ([**2**]) A mapping $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is called C-class function if it is continuous and satisfies the following conditions:
(i) $F(s,t) \leqslant s$;
(ii) $F(s,t) = s$ implies that either $s = 0$ or $t = 0$ for all $s,t \in \mathbb{R}^+$.

We denote the set of all C-class functions by $\mathcal{C}$.

EXAMPLE 1.4. ([**2**]) The following functions $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ are elements of $\mathcal{C}$, for all $s,t \in \mathbb{R}^+$:
(i) $F(s,t) = s - t,\ F(s,t) = s \Rightarrow t = 0$;
(ii) $F(s,t) = ms,\ 0 < m < 1, F(s,t) = s \Rightarrow s = 0$;
(iii) $F(s,t) = s\beta(s),\ \beta : \mathbb{R}^+ \to [0,1),$ and is continuous, $F(s,t) = s \Rightarrow s = 0$;
(iv) $F(s,t) = s - \varphi(s),\ F(s,t) = s \Rightarrow s = 0,$ where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$;
(v) $F(s,t) = \phi(s),\ F(s,t) = s \Rightarrow s = 0,$ where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that $\phi(0) = 0,$ and $\phi(t) > 0$ for $t > 0$.

Babu and Sudheer [**3**] introduced F-class functions as follows:

DEFINITION 1.7. ([**3**]) A continuous map $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is said to be F-class function if $F(s,t) < s$ for all $s,t > 0$.

We denote F-class functions as $\mathcal{F}$.
Babu and Sudheer [**3**] proved that $\mathcal{C} = \mathcal{F}$ and $F(0,0)$ may not be zero.

DEFINITION 1.8. ([**15**]) (Altering Distance Function) A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an altering distance function if the following properties are satisfied:
(i) ψ is nondecreasing and continuous, and
(ii) $\psi(t) = 0$ if and only if $t = 0$.

DEFINITION 1.9. ([**2**]) (Ultra Altering Distance Function) An ultra altering distance function is a continuous, nondecreasing mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(t) > 0, t > 0$ and $\varphi(0) \geqslant 0$.

We use the following two notations in our discussion.
$\Psi = \{\psi | \psi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is an altering distance function}\}$ and

$\Phi = \{\varphi | \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is an ultra altering distance function$\}$.

Recently, Yildirim et al. [**20**] proved the following theorem in partial metric spaces.

THEOREM 1.6 ([**20**]). *Let (X, p) be a complete partial metric space and $T : X \to X$ ba a selfmap such that there exist a pair of functions $\varphi \in \Psi$, $\phi \in \Phi$, and $F \in \mathcal{C}$ such that*

$$\varphi(p(Tx, Ty)) \leqslant \max\{F(\varphi(p(x, y)), \phi(p(x, y))), F(\varphi(p(y, Ty)\tfrac{1+p(x,Tx)}{1+p(x,y)}), \phi(p(y, Ty)\tfrac{1+p(x,Tx)}{1+p(x,y)}))\}$$

for all $x, y \in X$. Then T has a unique fixed point in X.

In 2017, Hima Bindu et al. [**10**] proved the following theorem in partial metric spaces as follows:

THEOREM 1.7 ([**10**]). *Let (X, p) be a partial metric space and let $S, T, f, g : X \to X$ be mappings satisfying*

$$\tfrac{1}{2}\min\{p(fx, Sx), p(gy, Ty)\} \leqslant p(fx, gy)$$

$$\text{implies } \psi(p(Sx, Ty)) \leqslant \alpha(M(x, y)) - \beta(M(x, y)), \tag{1.1}$$

for all $x, y \in X$, where $\psi, \alpha, \beta : \mathbb{R}^+ \to \mathbb{R}^+$ are such that ψ is an altering distance function, α is continuous, and β is lower semi continuous, $\alpha(0) = \beta(0) = 0$ and $\psi(t) - \alpha(t) + \beta(t) > 0$, for all $t > 0$ and

$$M(x, y) = \max\{p(fx, gy), p(fx, Sx), p(gy, Ty), \tfrac{1}{2}[p(fx, Ty) + p(gy, Sx)]\}.$$

Assume that

(i) $S(X) \subseteq g(X), T(X) \subseteq f(X)$,

(ii) *either $f(X)$ or $g(X)$ is a complete subspace of X,*

(iii) *the pairs (f, S) and (g, T) are weakly compatible.*

Then S, T, f and g have a unique common fixed point in X.

In the following we introduce Geraghty-Suzuki type contraction for two pairs of maps.

DEFINITION 1.10. Let (X, p) be a partial metric space, and let A, B, S and T be selfmaps of X. If there exists $\beta \in \mathfrak{F}$ such that

$$\tfrac{1}{2}\min\{p(Sx, Ax), p(Ty, By)\} \leqslant p(Sx, Ty)$$

$$\text{implies that } p(Ax, By) \leqslant \beta(M(x, y))M(x, y) \tag{1.2}$$

for all $x, y \in X$, where

$$M(x, y) = \max\{p(Sx, Ty), p(Sx, Ax), p(Ty, By), \frac{1}{2}[p(Sx, By) + p(Ty, Ax)]\},$$

then we say that the pairs (A, S) and (B, T) are Geraghty-Suzuki type contraction maps.

EXAMPLE 1.5. Let $X = [0, 1]$. We define $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Then (X, p) is a partial metric space. We define selfmaps A, B, S and T on X by $A(X) = \frac{2x^2}{5}$, $B(X) = \frac{x^2}{5}, S(X) = x^2, T(X) = \frac{3x^2}{5}$ and $\beta(t) = \frac{1}{1+t}$.

Then clearly the pairs (A, S) and (B, T) are Geraghty-Suzuki type contraction maps.

In Section 2, we extend Theorem 1.6 to a pair of maps by using F-class function (Theorem 2.1). Also, we prove the existence of common fixed points by replacing the inequality (1.1) of Theorem 1.7 with Geraghty-Suzuki type contraction for two pairs of maps (Theorem 2.2). In Section 3, we draw some corollaries and provide examples in support of our results.

2. Main results

THEOREM 2.1. *Let (X, p) be a partial metric space and let f and g be selfmaps on X. Assume that there exist $\varphi \in \Psi$, $\phi \in \Phi$ and $F \in \mathcal{F}$ such that*

$$\begin{aligned}\varphi(p(fx, fy)) \leqslant \max\{&F(\varphi(p(gx, gy)), \phi(p(gx, gy))),\\ &F(\varphi(p(gy, fy)\tfrac{1+p(gx,fx)}{1+p(gx,gy)}), \phi(p(gy, fy)\tfrac{1+p(gx,fx)}{1+p(gx,gy)}))\}\end{aligned} \tag{2.1}$$

for all $x, y \in X$. If $f(X) \subseteq g(X)$, the pair (f, g) is weakly compatible and $g(X)$ is a complete subspace of X then f and g have a unique common fixed point in X.

PROOF. Let x_0 be an arbitrary point in X. Since $f(X) \subseteq g(X)$ there exists $x_1 \in X$ such that $fx_0 = gx_1 = y_0$. By induction, a sequence $\{x_n\}$ can be chosen such that $fx_n = gx_{n+1} = y_n$, for all $n \in \mathbb{N} \cup \{0\}$

Case (i): Assume that $p(y_n, y_{n+1}) > 0$ for some n. We show that

$$p(y_n, y_{n+1}) \leqslant p(y_{n-1}, y_n), n \in \mathbb{N}.$$

Then by (2.1) for all $n \in \mathbb{N}$, we have

$$\begin{aligned}\varphi(p(y_n, y_{n+1})) &= \varphi(p(fx_n, fx_{n+1}))\\ &\leqslant \max\{F(\varphi(p(gx_n, gx_{n+1})), \phi(p(gx_n, gx_{n+1}))),\\ &\qquad F(\varphi(p(gx_{n+1}, fx_{n+1})\tfrac{1+p(gx_n,fx_n)}{1+p(gx_n,gx_{n+1})}),\\ &\qquad\qquad \phi(p(gx_{n+1}, fx_{n+1})\tfrac{1+p(gx_n,fx_n)}{1+p(gx_n,gx_{n+1})}))\}\\ &= \max\{F(\varphi(p(y_{n-1}, y_n)), \phi(p(y_{n-1}, y_n))),\\ &\qquad F(\varphi(p(y_n, y_{n+1})\tfrac{1+p(y_{n-1},y_n)}{1+p(y_{n-1},y_n)}), \phi(p(y_n, y_{n+1})\tfrac{1+p(y_{n-1},y_n)}{1+p(y_{n-1},y_n)}))\}\\ &= \max\{F(\varphi(p(y_{n-1}, y_n)), \phi(p(y_{n-1}, y_n))),\\ &\qquad F(\varphi(p(y_n, y_{n+1})), \phi(p(y_n, y_{n+1})))\}.\end{aligned} \tag{2.2}$$

If

$$\begin{aligned}\max\{F(\varphi(p(y_{n-1}, y_n)), \phi(p(y_{n-1}, y_n))), F(\varphi(p(y_n, y_{n+1})), \phi(p(y_n, y_{n+1})))\}\\ = F(\varphi(p(y_n, y_{n+1})), \phi(p(y_n, y_{n+1}))),\end{aligned}$$

then from (2.2), we have

$$\varphi(p(y_n, y_{n+1})) \leqslant F(\varphi(p(y_n, y_{n+1})), \phi(p(y_n, y_{n+1}))) < \varphi(p(y_n, y_{n+1})),$$

which is a contradiction. Therefore

$$\max\{F(\varphi(p(y_{n-1}, y_n)), \phi(p(y_{n-1}, y_n))), F(\varphi(p(y_n, y_{n+1})), \phi(p(y_n, y_{n+1})))\}$$
$$= F(\varphi(p(y_{n-1}, y_n)), \phi(p(y_{n-1}, y_n))).$$

Hence

$$\varphi(p(y_n, y_{n+1})) \leqslant F(\varphi(p(y_{n-1}, y_n)), \phi(p(y_{n-1}, y_n))) < \varphi(p(y_{n-1}, y_n))$$

and by the property of φ we have $p(y_n, y_{n+1}) \leqslant p(y_{n-1}, y_n)$. Then, the sequence $\{p(y_n, y_{n+1})\}$ is a decreasing sequence. Then there exists $r \geqslant 0$ such that

$$\lim_{n\to\infty} p(y_n, y_{n+1}) = r. \tag{2.3}$$

We claim that $r = 0$. On the contrary suppose $r > 0$. On letting $n \to \infty$ in (2.2) and using (2.3), we get

$$\varphi(r) \leqslant F(\varphi(r), \phi(r)) < \varphi(r),$$

it is a contradiction. Hence

$$\lim_{n\to\infty} p(y_n, y_{n+1}) = 0. \tag{2.4}$$

Thus from (P_2), we get that

$$\lim_{n\to\infty} p(y_n, y_n) = 0. \tag{2.5}$$

By the definition of p^s, (2.4) and (2.5), we get

$$\lim_{n\to\infty} p^s(y_n, y_{n+1}) = 0. \tag{2.6}$$

Next, we prove that $\{y_n\}$ is Cauchy in (X, p^s).

On the contrary suppose that $\{y_n\}$ is not a Cauchy sequence. There exist $\epsilon > 0$ and monotone increasing sequence of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k$ with

$$p^s(y_{m_k}, y_{n_k}) \geqslant \epsilon \text{ and } p^s(y_{m_k}, y_{n_k-1}) < \epsilon. \tag{2.7}$$

Now we prove that (i) $\lim_{k\to\infty} p(y_{m_k}, y_{n_k}) = \frac{\epsilon}{2}$.

Since $\epsilon \leqslant p^s(y_{m_k}, y_{n_k})$ for all k, we have

$$\epsilon \leqslant \liminf_{k\to\infty} p^s(y_{m_k}, y_{n_k}). \tag{2.8}$$

Now for each positive integer k, by the triangular inequality, we get

$$p^s(y_{m_k}, y_{n_k}) \leqslant p^s(y_{m_k}, y_{n_k-1}) + p^s(y_{n_k-1}, y_{n_k})$$

On taking limit superior as $k \to \infty$, from (2.6) and (2.7), we have

$$\limsup_{k\to\infty} p^s(y_{m_k}, y_{n_k}) \leqslant \epsilon. \tag{2.9}$$

Hence from (2.8) and (2.9), we get

$$\lim_{k\to\infty} p^s(y_{m_k}, y_{n_k}) \text{ exists and } \lim_{k\to\infty} p^s(y_{m_k}, y_{n_k}) = \epsilon.$$

Hence, from the definition of p^s and (2.5), we have $\lim_{k\to\infty} p(y_{m_k}, y_{n_k}) = \frac{\epsilon}{2}$.

In similar way, it is easy to see that

(ii) $\lim_{k\to\infty} p(y_{n_k+1}, y_{m_k}) = \frac{\epsilon}{2}$;

(iii) $\lim_{k\to\infty} p(y_{n_k}, y_{m_k-1}) = \frac{\epsilon}{2}$.

We now consider

$$\begin{aligned}\varphi(p(y_{n_k+1}, y_{m_k})) &= \varphi(p(fx_{n_k+1}, fx_{m_k})) \\ &\leqslant \max\{F(\varphi(p(gx_{n_k+1}, gx_{m_k})), \phi(p(gx_{n_k+1}, gx_{m_k}))),\end{aligned}$$

$$F(\varphi(p(gx_{m_k}, fx_{m_k})\frac{1+p(gx_{n_k+1}, fx_{n_k+1})}{1+p(gx_{n_k+1}, gx_{m_k})}),$$
$$\phi(p(gx_{m_k}, fx_{m_k})\frac{1+p(gx_{n_k+1}, fx_{n_k+1})}{1+p(gx_{n_k+1}, gx_{m_k})}))\}$$
$$= \max\{F(\varphi(p(y_{n_k}, y_{m_k-1})), \phi(p(y_{n_k}, y_{m_k-1}))),$$
$$F(\varphi(p(y_{m_k-1}, y_{m_k})\frac{1+p(y_{n_k}, y_{n_k+1})}{1+p(y_{n_k}, y_{m_k-1})}),$$
$$\phi(p(y_{m_k-1}, y_{m_k})\frac{1+p(y_{n_k}, y_{n_k+1})}{1+p(y_{n_k}, y_{m_k-1})}))\}$$

On letting $k \to \infty$ and using (2.4), (ii) and (iii), we get

$$\varphi(\frac{\epsilon}{2}) \leqslant \max\{F(\varphi(\frac{\epsilon}{2}), \phi(\frac{\epsilon}{2})), F(\varphi(0), \phi(0))\}.$$

If $F(\varphi(\frac{\epsilon}{2}), \phi(\frac{\epsilon}{2}))$ is maximum then, $\varphi(\frac{\epsilon}{2}) \leqslant F(\varphi(\frac{\epsilon}{2}), \phi(\frac{\epsilon}{2})) < \varphi(\frac{\epsilon}{2})$, which is a contradiction.

Suppose $F(\varphi(0), \phi(0))$ is maximum. Then $\varphi(\frac{\epsilon}{2}) \leqslant F(\varphi(0), \phi(0)) < \varphi(0)$. By the property of φ we have $\frac{\epsilon}{2} \leqslant 0$, a contradiction. Hence $\{y_n\}$ is a Cauchy sequence in (X, p^s).

Case (ii): Assume that $y_n = y_{n+1}$ for some n.

If $p(y_{n+1}, y_{n+2}) > 0$. We have

$$\begin{aligned}
\varphi(p(y_{n+1}, y_{n+2})) &= \varphi(p(fx_{n+1}, fx_{n+2})) \\
&\leqslant \max\{F(\varphi(p(gx_{n+1}, gx_{n+2})), \phi(p(gx_{n+1}, gx_{n+2}))), \\
&\qquad F(\varphi(p(gx_{n+2}, fx_{n+2})\tfrac{1+p(gx_{n+1}, fx_{n+1})}{1+p(gx_{n+1}, gx_{n+2})})), \\
&\qquad\quad \phi(p(gx_{n+2}, fx_{n+2})\tfrac{1+p(gx_{n+1}, fx_{n+1})}{1+p(gx_{n+1}, gx_{n+2})})\} \\
&= \max\{F(\varphi(p(y_n, y_{n+1})), \phi(p(y_n, y_{n+1}))), \\
&\qquad F(\varphi(p(y_{n+1}, y_{n+2})\tfrac{1+p(y_n, y_{n+1})}{1+p(y_n, y_{n+1})}), \\
&\qquad\quad \phi(p(y_{n+1}, y_{n+2})\tfrac{1+p(y_n, y_{n+1})}{1+p(y_n, y_{n+1})}))\} \\
&= \max\{F(\varphi(p(y_n, y_{n+1})), \phi(p(y_n, y_{n+1}))), \\
&\qquad F(\varphi(p(y_{n+1}, y_{n+2})), \phi(p(y_{n+1}, y_{n+2})))\}.
\end{aligned}$$

If

$$\begin{aligned}
&\max\{F(\varphi(p(y_n, y_{n+1})), \phi(p(y_n, y_{n+1}))), F(\varphi(p(y_{n+1}, y_{n+2})), \phi(p(y_{n+1}, y_{n+2})))\} \\
&\qquad = F(\varphi(p(y_{n+1}, y_{n+2})), \phi(p(y_{n+1}, y_{n+2}))),
\end{aligned}$$

then we have

$$\varphi(p(y_{n+1}, y_{n+2})) \leqslant F(\varphi(p(y_{n+1}, y_{n+2})), \phi(p(y_{n+1}, y_{n+2}))) < \varphi(p(y_{n+1}, y_{n+2})),$$

which is a contradiction. Therefore

$$\begin{aligned}
&\max\{F(\varphi(p(y_n, y_{n+1})), \phi(p(y_n, y_{n+1}))), F(\varphi(p(y_{n+1}, y_{n+2})), \phi(p(y_{n+1}, y_{n+2})))\} \\
&\qquad = F(\varphi(p(y_n, y_{n+1})), \phi(p(y_n, y_{n+1}))).
\end{aligned}$$

Hence

$$\varphi(p(y_{n+1}, y_{n+2})) \leqslant F(\varphi(p(y_n, y_{n+1})), \phi(p(y_n, y_{n+1}))) < \varphi(p(y_n, y_{n+1})).$$

Since φ is monotonically increasing, we have

$$p(y_{n+1}, y_{n+2}) \leqslant p(y_n, y_{n+1}) = p(y_{n+1}, y_{n+1}).$$

Hence $y_{n+1} = y_{n+2}$. Continuing in this way, we can conclude that $y_n = y_{n+k}$ for all $k \geqslant 0$. Thus, $\{y_n\}$ is a Cauchy sequence in (X, p^s). From Lemma 1.1, it follows that $\{y_n\}$ is a Cauchy sequence in (X, p). Therefore

$$\lim_{n,m\to\infty} p(y_n, y_m) = 0. \tag{2.10}$$

Suppose $g(X)$ is complete. Since $y_n = fx_n = gx_{n+1}$, it follows that $\{y_n\} \subseteq g(X)$ is a Cauchy sequence in the complete metric space $(g(X), p^s)$, it follows that $\{y_n\}$ converges in $(g(X), p^s)$. Thus, $\lim_{n\to\infty} p^s(y_n, u) = 0$ for some $u \in g(X)$. i.e., $\lim_{n\to\infty} y_n = u = gt \in g(X)$ for some $t \in X$. From Lemma 1.2 and (2.10), we have

$$p(u, u) = \lim_{n\to\infty} p(y_{n+1}, u) = \lim_{n\to\infty} p(y_n, u) = \lim_{n,m\to\infty} p(y_n, y_m).$$

From (2.10), we have

$$p(u, u) = \lim_{n\to\infty} p(y_n, u) = \lim_{n,m\to\infty} p(y_n, y_m) = 0.$$

We now show that $ft = u$. Suppose $p(ft, u) > 0$. From (2.1), we have

$$\begin{aligned}
\varphi(p(ft, y_{n+1})) &= \varphi(p(ft, fx_{n+1})) \\
&\leqslant \max\{F(\varphi(p(gt, gx_{n+1})), \phi(p(gt, gx_{n+1}))), \\
&\qquad F(\varphi(p(gx_{n+1}, fx_{n+1})\tfrac{1+p(gt,ft)}{1+p(gt,gx_{n+1})})), \\
&\qquad\qquad \phi(p(gx_{n+1}, fx_{n+1})\tfrac{1+p(gt,ft)}{1+p(gt,gx_{n+1})})\} \\
&= \max\{F(\varphi(p(u, y_n)), \phi(p(u, y_n))), \\
&\qquad F(\varphi(p(y_n, y_{n+1})\tfrac{1+p(u,ft)}{1+p(u,y_n)})), \phi(p(y_n, y_{n+1})\tfrac{1+p(u,ft)}{1+p(u,y_n)}))\}.
\end{aligned}$$

On letting $n \to \infty$, we get

$$\varphi(p(ft, u)) \leqslant \max\{F(\varphi(0), \phi(0)), F(\varphi(0), \phi(0))\} = F(\varphi(0), \phi(0)) < \varphi(0).$$

Since φ is monotonically increasing, we have $p(ft, u) \leqslant 0$, which is a contradiction. Hence $ft = u$. Therefore $ft = gt = u$. Since the pair (f, g) is weakly compatible and $ft = gt = u$, we have $fu = gu$. We now prove that $fu = u$.

On the contrary, suppose that $p(fu, u) > 0$. From the inequality (2.1), we have

$$\begin{aligned}
\varphi(p(fu, y_{n+1})) &= \varphi(p(fu, fx_{n+1})) \\
&\leqslant \max\{F(\varphi(p(gu, gx_{n+1})), \phi(p(gu, gx_{n+1}))), \\
&\qquad F(\varphi(p(gx_{n+1}, fx_{n+1})\tfrac{1+p(gu,fu)}{1+p(gu,gx_{n+1})})), \\
&\qquad\qquad \phi(p(gx_{n+1}, fx_{n+1})\tfrac{1+p(gu,fu)}{1+p(gu,gx_{n+1})})\} \\
&= \max\{F(\varphi(p(fu, y_n)), \phi(p(fu, y_n))), \\
&\qquad F(\varphi(p(y_n, y_{n+1})\tfrac{1+p(fu,fu)}{1+p(fu,y_n)})), \phi(p(y_n, y_{n+1})\tfrac{1+p(fu,fu)}{1+p(fu,y_n)}))\}.
\end{aligned}$$

On letting $n \to \infty$, we get

$$\begin{aligned}
\varphi(p(fu, u)) &\leqslant \max\{F(\varphi(p(fu, u)), \phi(p(fu, u))), F(\varphi(0), \phi(0))\} \\
&= F(\varphi(p(fu, u)), \phi(p(fu, u))) < \varphi(p(fu, u)),
\end{aligned}$$

which is a contradiction. Hence $fu = gu = u$.

Therefore u is a common fixed point of f and g.

Uniqueness of a common fixed point follows from the inequality (2.1). □

PROPOSITION 2.1. *Let (X, p^s) be a metric space with $\lim_{n\to\infty} p^s(y_n, y_{n+1}) = 0$. If $\{y_{2n}\}$ is a Cauchy sequence in (X, p^s) then $\{y_n\}$ is also Cauchy in (X, p^s).*

PROOF. Suppose that $\{y_{2n}\}$ is Cauchy in (X, p^s). We have

$$\begin{aligned}
p^s(y_{2n+1}, y_{2m+1}) - p^s(y_{2n}, y_{2m}) &= 2p(y_{2n+1}, y_{2m+1}) - p(y_{2n+1}, y_{2n+1}) \\
&\quad - p(y_{2m+1}, y_{2m+1}) - 2p(y_{2n}, y_{2m}) + p(y_{2n}, y_{2n}) \\
&\qquad + p(y_{2m}, y_{2m}) \\
&\leqslant 2[p(y_{2n+1}, y_{2n}) + p(y_{2n}, y_{2m+1}) - p(y_{2n}, y_{2n})] \\
&\quad - p(y_{2n+1}, y_{2n+1}) - p(y_{2m+1}, y_{2m+1}) \\
&\quad - 2p(y_{2n}, y_{2m}) + p(y_{2n}, y_{2n}) + p(y_{2m}, y_{2m}) \\
&= 2p(y_{2n+1}, y_{2n}) + 2p(y_{2n}, y_{2m+1}) - p(y_{2n}, y_{2n}) \\
&\quad - p(y_{2n+1}, y_{2n+1}) - p(y_{2m+1}, y_{2m+1}) \\
&\quad - 2p(y_{2n}, y_{2m}) + p(y_{2m}, y_{2m}) \\
&= p^s(y_{2n+1}, y_{2n}) + 2p(y_{2n}, y_{2m+1}) - 2p(y_{2n}, y_{2m}) \\
&\quad + p(y_{2m}, y_{2m}) - p(y_{2m+1}, y_{2m+1}) \\
&\leqslant p^s(y_{2n+1}, y_{2n}) + 2[p(y_{2n}, y_{2m}) + p(y_{2m}, y_{2m+1}) \\
&\quad - p(y_{2m}, y_{2m})] - 2p(y_{2n}, y_{2m}) \\
&\quad + p(y_{2m}, y_{2m}) - p(y_{2m+1}, y_{2m+1}) \\
&= p^s(y_{2n+1}, y_{2n}) + 2p(y_{2m}, y_{2m+1}) - p(y_{2m}, y_{2m}) \\
&\quad - p(y_{2m+1}, y_{2m+1}) \\
&= p^s(y_{2n+1}, y_{2n}) + p^s(y_{2m+1}, y_{2m})
\end{aligned}$$

so that

$$p^s(y_{2n+1}, y_{2m+1}) - p^s(y_{2n}, y_{2m}) \leqslant p^s(y_{2n+1}, y_{2n}) + p^s(y_{2m+1}, y_{2m}). \tag{2.11}$$

Now, we have

$$\begin{aligned}
p^s(y_{2n}, y_{2m}) - p^s(y_{2n+1}, y_{2m+1}) &= 2p(y_{2n}, y_{2m}) - p(y_{2n}, y_{2n}) - p(y_{2m}, y_{2m}) \\
&\quad - 2p(y_{2n+1}, y_{2m+1}) + p(y_{2n+1}, y_{2n+1}) \\
&\quad + p(y_{2m+1}, y_{2m+1}) \\
&\leqslant 2[p(y_{2n}, y_{2n+1}) + p(y_{2n+1}, y_{2m}) - p(y_{2n+1}, y_{2n+1})] \\
&\quad - p(y_{2n}, y_{2n}) - p(y_{2m}, y_{2m}) - 2p(y_{2n+1}, y_{2m+1}) \\
&\quad + p(y_{2n+1}, y_{2n+1}) + p(y_{2m+1}, y_{2m+1}) \\
&= 2p(y_{2n}, y_{2n+1}) + 2p(y_{2n+1}, y_{2m}) - p(y_{2n+1}, y_{2n+1}) \\
&\quad - p(y_{2n}, y_{2n}) - p(y_{2m}, y_{2m}) \\
&\quad - 2p(y_{2n+1}, y_{2m+1}) + p(y_{2m+1}, y_{2m+1}) \\
&= p^s(y_{2n+1}, y_{2n}) + 2p(y_{2n+1}, y_{2m}) \\
&\quad - 2p(y_{2n+1}, y_{2m+1}) \\
&\quad - p(y_{2m}, y_{2m}) + p(y_{2m+1}, y_{2m+1}) \\
&\leqslant p^s(y_{2n+1}, y_{2n}) + 2[p(y_{2n+1}, y_{2m+1}) \\
&\quad + p(y_{2m+1}, y_{2m}) - p(y_{2m+1}, y_{2m+1})] \\
&\quad - 2p(y_{2n+1}, y_{2m+1}) - p(y_{2m}, y_{2m}) + p(y_{2m+1}, y_{2m+1}) \\
&= p^s(y_{2n+1}, y_{2n}) + 2p(y_{2m+1}, y_{2m}) - p(y_{2m}, y_{2m}) \\
&\quad - p(y_{2m+1}, y_{2m+1}) \\
&= p^s(y_{2n+1}, y_{2n}) + p^s(y_{2m+1}, y_{2m})
\end{aligned}$$

so that

$$p^s(y_{2n}, y_{2m}) - p^s(y_{2n+1}, y_{2m+1}) \leqslant p^s(y_{2n+1}, y_{2n}) + p^s(y_{2m+1}, y_{2m}). \tag{2.12}$$

Thus, from (2.11) and (2.12), we have

$$|p^s(y_{2n+1}, y_{2m+1}) - p^s(y_{2n}, y_{2m})| \leqslant p^s(y_{2n+1}, y_{2n}) + p^s(y_{2m+1}, y_{2m}). \tag{2.13}$$

On letting $n, m \to \infty$ in (2.13), we have $\lim\limits_{n,m\to\infty} p^s(y_{2n+1}, y_{2m+1}) = 0$.

Hence $\{y_{2n+1}\}$ is a Cauchy sequence in (X, p^s). Thus $\{y_n\}$ is Cauchy in (X, p^s). □

PROPOSITION 2.2. *Let (X, p) be a partial metric space, and let A, B, S and T be selfmaps of X. Assume that the pairs (A, S) and (B, T) are Geraghty-Suzuki type contraction maps. Then the following hold:*

(i) *If $A(X) \subseteq T(X)$ and the pair (B, T) is weakly compatible, and if z is a common fixed point of A and S then z is a common fixed point of A, B, S and T and it is unique.*

(ii) *If $B(X) \subseteq S(X)$ and the pair (A, S) is weakly compatible, and if z is a common fixed point of B and T then z is a common fixed point of A, B, S and T and it is unique.*

PROOF. First, we assume that (i) holds. Let z be a common fixed point of A and S. Then $Az = Sz = z$. Since $A(X) \subseteq T(X)$, there exists $u \in X$ such that $Tu = z$. Therefore $Az = Sz = Tu = z$.

We now prove that $Tu = Bu$. Suppose that $Tu \neq Bu$. Since

$$\frac{1}{2}\min\{p(Sz, Az), p(Tu, Bu)\} \leqslant p(Sz, Tu),$$

it follows from the inequality (1.2),

$$\begin{aligned}
p(Tu, Bu) &= p(Az, Bu)\\
&\leqslant \beta(M(z, u))M(z, u)\\
&= \beta(\max\{p(Sz, Tu), p(Sz, Az), p(Tu, Bu), \tfrac{1}{2}[p(Sz, Bu) + p(Tu, Az)]\})\\
&\qquad \max\{p(Sz, Tu), p(Sz, Az), p(Tu, Bu), \tfrac{1}{2}[p(Sz, Bu) + p(Tu, Az)]\}\\
&= \beta(p(Tu, Bu))p(Tu, Bu) < p(Tu, Bu),
\end{aligned}$$

it is a contradiction. Hence $Bu = Tu = z$. Since the pair (B, T) is weakly compatible, it follows that $BTu = TBu$. i.e, $Bz = Tz$.

Suppose $Bz \neq z$. Since $\frac{1}{2}\min\{p(Sz, Az), p(Tz, Bz)\} \leqslant p(Bz, Tz) \leqslant p(Sz, Tz)$, from the inequality (1.2), we have

$$\begin{aligned}
p(z, Bz) &= p(Az, Bz)\\
&\leqslant \beta(M(z, z))M(z, z)\\
&= \beta(\max\{p(Sz, Tz), p(Sz, Az), p(Tz, Bz), \tfrac{1}{2}[p(Sz, Bz) + p(Tz, Az)]\})\\
&\qquad \max\{p(Sz, Tz), p(Sz, Az), p(Tz, Bz), \tfrac{1}{2}[p(Sz, Bz) + p(Tz, Az)]\}\\
&= \beta(p(z, Bz))p(z, Bz) < p(z, Bz),
\end{aligned}$$

which is a contradiction. Thus, $Bz = Tz = z$. Hence $Az = Bz = Sz = Tz = z$. Therefore, z is a common fixed point of A, B, S and T.

Let z' be another common fixed point of A, B, S and T. Since

$$\frac{1}{2}\min\{p(Sz, Az), p(Tz', Bz')\} \leqslant p(z, z) \leqslant p(z, z') = p(Sz, Tz'),$$

from the inequality (1.2), we have

$$\begin{aligned} p(z,z') &= p(Az,Bz') \\ &\leqslant \beta(M(z,z'))M(z,z') \\ &= \beta(\max\{p(Sz,Tz'),p(Sz,Az),p(Tz',Bz'),\tfrac{1}{2}[p(Sz,Bz')+p(Tz',Az)]\}) \\ &\qquad \max\{p(Sz,Tz'),p(Sz,Az),p(Tz',Bz'),\tfrac{1}{2}[p(Sz,Bz')+p(Tz',Az)]\} \\ &= \beta(p(z,z'))p(z,z') < p(z,z'), \end{aligned}$$

which is a contradiction. Hence $z = z'$. Thus z is a unique common fixed point of A, B, S and T.

The proof of (ii) is similar to (i) and hence is omitted. □

Theorem 2.2. *Let (X,p) be a partial metric space, and let A, B, S and T be selfmaps of X. Assume that the pairs (A,S) and (B,T) are Geraghty-Suzuki type contraction maps. If*

(i) *$A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$,*
(ii) *either $S(X)$ or $T(X)$ is a complete subspace of X, and*
(iii) *the pairs (A,S) and (B,T) are weakly compatible,*

then A, B, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point in X. Since $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$, there exist sequences of $\{x_n\}$and $\{y_n\} \in X$, such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \text{ for } n = 0,1,2, \ \dots .$$

Case (i): Assume that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. We now show that $p(y_n, y_{n+1}) \leqslant p(y_{n-1}, y_n), n = 1,2,3, \dots$. Now,

$$\frac{1}{2}\min\{p(Sx_{2n},Ax_{2n}),p(Tx_{2n+1},Bx_{2n+1})\} \leqslant p(Sx_{2n},Ax_{2n}) = p(Sx_{2n},Tx_{2n+1}),$$

it follows from the inequality (1.2), we have

$$p(Ax_{2n},Bx_{2n+1}) \leqslant \beta(M(x_{2n},x_{2n+1}))M(x_{2n},x_{2n+1}),$$

where

$$\begin{aligned} M(x_{2n},x_{2n+1}) &= \max\{p(Sx_{2n},Tx_{2n+1}),p(Sx_{2n},Ax_{2n}),p(Tx_{2n+1},Bx_{2n+1}), \\ &\qquad \tfrac{1}{2}[p(Sx_{2n},Bx_{2n+1})+p(Tx_{2n+1},Ax_{2n})]\} \\ &= \max\{p(y_{2n-1},y_{2n}),p(y_{2n-1},y_{2n}),p(y_{2n},y_{2n+1}), \\ &\qquad \tfrac{1}{2}[p(y_{2n-1},y_{2n+1})+p(y_{2n},y_{2n})]\} \\ &= \max\{p(y_{2n-1},y_{2n}),p(y_{2n},y_{2n+1})\}. \end{aligned}$$

If $\max\{p(y_{2n-1},y_{2n}),p(y_{2n},y_{2n+1})\} = p(y_{2n},y_{2n+1})$, then we have

$$p(y_{2n},y_{2n+1}) \leqslant \beta(p(y_{2n},y_{2n+1}))p(y_{2n},y_{2n+1}) < p(y_{2n},y_{2n+1}),$$

which is a contradiction. Hence $\max\{p(y_{2n-1},y_{2n}),p(y_{2n},y_{2n+1})\} = p(y_{2n-1},y_{2n})$. Thus

$$p(y_{2n},y_{2n+1}) \leqslant \beta(p(y_{2n-1},y_{2n}))p(y_{2n-1},y_{2n}) < p(y_{2n-1},y_{2n}). \tag{2.14}$$

Therefore, $p(y_{2n},y_{2n+1}) < p(y_{2n-1},y_{2n})$. Similarly, we can show that

$$p(y_{2n-1},y_{2n}) < p(y_{2n-2},y_{2n-1}).$$

Thus, $p(y_n, y_{n+1}) < p(y_{n-1}, y_n)$, for all $n = 1, 2, 3, ...$. Therefore $\{p(y_n, y_{n+1})\}$ is a decreasing sequence of nonnegative real numbers and converges to a real number $r \geqslant 0$.

Suppose $r > 0$. On letting $n \to \infty$ in (2.14), we have $r \leqslant \lim_{n\to\infty} \beta(p(y_n, y_{n+1}))r$. Then $1 \leqslant \lim_{n\to\infty} \beta(p(y_n, y_{n+1})) \leqslant 1$ so that we have $\lim_{n\to\infty} \beta(p(y_n, y_{n+1})) = 1$. Since $\beta \in \mathfrak{F}$, we have $\lim_{n\to\infty} p(y_n, y_{n+1}) = 0$, which is a contradiction. Hence $r = 0$. Thus

$$\lim_{n\to\infty} p(y_n, y_{n+1}) = 0. \tag{2.15}$$

Therefore from (P_2), we get that

$$\lim_{n\to\infty} p(y_n, y_n) = 0 \text{ and } \lim_{n\to\infty} p(y_{n+1}, y_{n+1}) = 0. \tag{2.16}$$

By the definition of p^s, using (2.15) and (2.16), we get that

$$\lim_{n\to\infty} p^s(y_n, y_{n+1}) = 0. \tag{2.17}$$

Now, we prove that $\{y_{2n}\}$ is a Cauchy sequence in (X, p^s).
On the contrary, suppose that $\{y_{2n}\}$ is not Cauchy. Then there exist an $\epsilon > 0$ and monotone sequences of natural numbers $\{2m_k\}$ and $\{2n_k\}$ such that $n_k > m_k$,

$$p^s(y_{2m_k}, y_{2n_k}) \geqslant \epsilon \text{ and } p^s(y_{2m_k}, y_{2n_k-2}) < \epsilon. \tag{2.18}$$

Now we prove that (i) $\lim_{k\to\infty} p(y_{2m_k}, y_{2n_k}) = \frac{\epsilon}{2}$.
Since $\epsilon \leqslant p^s(y_{2m_k}, y_{2n_k})$ for all k, we have

$$\epsilon \leqslant \liminf_{k\to\infty} p^s(y_{2m_k}, y_{2n_k}). \tag{2.19}$$

Now for each positive integer k, by the triangular inequality, we get
$p^s(y_{2m_k}, y_{2n_k}) \leqslant p^s(y_{2m_k}, y_{2n_k-2}) + p^s(y_{2n_k-2}, y_{2n_k-1}) + p^s(y_{2n_k-1}, y_{2n_k})$
On taking limit superior as $k \to \infty$, from (2.17) and (2.18), we have

$$\limsup_{k\to\infty} p^s(y_{2m_k}, y_{2n_k}) \leqslant \epsilon. \tag{2.20}$$

Hence from (2.19) and (2.20), we get that $\lim_{k\to\infty} p^s(y_{2m_k}, y_{2n_k})$ exists and that holds $\lim_{k\to\infty} p^s(y_{2m_k}, y_{2n_k}) = \epsilon$. Hence from the definition of p^s and (2.16), we have $\lim_{k\to\infty} p(y_{2m_k}, y_{2n_k}) = \frac{\epsilon}{2}$. In similar way, it is easy to see that
(ii) $\lim_{k\to\infty} p(y_{2n_k+1}, y_{2m_k}) = \frac{\epsilon}{2}$; (iii) $\lim_{k\to\infty} p(y_{2n_k}, y_{2m_k-1}) = \frac{\epsilon}{2}$ and
(iv) $\lim_{k\to\infty} p(y_{2m_k-1}, y_{2n_k+1}) = \frac{\epsilon}{2}$.
If $\frac{1}{2} \min\{p(y_{2m_k-1}, y_{2m_k}), p(y_{2n_k}, y_{2n_k+1})\} > p(y_{2m_k-1}, y_{2n_k})$, then from (2.15) and (iii), on letting $k \to \infty$, we get $0 \geqslant \frac{\epsilon}{2}$, which is a contradiction. Hence

$$\frac{1}{2} \min\{p(y_{2m_k-1}, y_{2m_k}), p(y_{2n_k}, y_{2n_k+1})\} \leqslant p(y_{2m_k-1}, y_{2n_k}) = p(Sx_{2m_k}, Tx_{2n_k+1}).$$

From the inequality (1.2), we have

$$\begin{aligned} p(y_{2m_k}, y_{2n_k+1}) &= p(Ax_{2m_k}, Bx_{2n_k+1}) \\ &\leqslant \beta(M(x_{2m_k}, x_{2n_k+1}))M(x_{2m_k}, x_{2n_k+1}), \end{aligned} \tag{2.21}$$

where

$$\begin{aligned} M(x_{2m_k}, x_{2n_k+1}) = \max\{&p(Sx_{2m_k}, Tx_{2n_k+1}), p(Sx_{2m_k}, Ax_{2m_k}), \\ &p(Tx_{2n_k+1}, Bx_{2n_k+1}), \\ &\tfrac{1}{2}[p(Sx_{2m_k}, Bx_{2n_k+1}) + p(Tx_{2n_k+1}, Ax_{2m_k})]\} \end{aligned}$$

$$= \max\{p(y_{2m_k-1}, y_{2n_k}), p(y_{2m_k-1}, y_{2m_k}), p(y_{2n_k}, y_{2n_k+1}), \frac{1}{2}[p(y_{2m_k-1}, y_{2n_k+1}) + p(y_{2n_k}, y_{2m_k})]\}.$$

Now, by using (2.15), (i), (ii), (iii) and (iv), we have

$$\lim_{k\to\infty} M(x_{2m_k}, x_{2n_k+1}) = \max\{\frac{\epsilon}{2}, 0, 0, \frac{1}{2}[\frac{\epsilon}{2} + \frac{\epsilon}{2}]\} = \frac{\epsilon}{2}.$$

On letting $k \to \infty$ in (2.21), we get $\frac{\epsilon}{2} \leqslant \beta(\frac{\epsilon}{2})\frac{\epsilon}{2} < \frac{\epsilon}{2}$, which is a contradiction. Therefore $\{y_{2n}\}$ is Cauchy. Thus by Proposition 2.1, $\{y_n\}$ is a Cauchy sequence in (X, p^s). Hence we have $\lim\limits_{n,m\to\infty} p^s(y_n, y_m) = 0$. Now, from Lemma 1.1, it follows that $\{y_n\}$ is a Cauchy sequence in (X, p).

Suppose $T(X)$ is complete. Since $y_{2n} = Ax_{2n} = Tx_{2n+1}$, it follows that $\{y_{2n}\} \subseteq T(X)$ is a Cauchy sequence in the complete metric space $(T(X), p^s)$, it follows that $\{y_{2n}\}$ converges in $(T(X), p^s)$, and $\{y_{2n}\}$ converges to u(say) in $T(X)$. Thus, $\lim\limits_{n\to\infty} p^s(y_{2n}, u) = 0$ for some $u \in T(X)$. That is, $y_{2n} \to u = Tt \in T(X)$ for some $t \in X$. Since $\{y_n\}$ is Cauchy in X and $y_n \to u$ as $n \to \infty$. From Lemma 1.2, we get

$p(u, u) = \lim\limits_{n\to\infty} p(y_{2n+1}, u) = \lim\limits_{n\to\infty} p(y_{2n}, y_{2n+1}) = \lim\limits_{n,m\to\infty} p(y_n, y_m) = 0$. We now show that for each $n \geqslant 1$ either

(2.22) (a): $\frac{1}{2}p(y_{2n-1}, y_{2n}) \leqslant p(y_{2n-1}, u)$ (or) (b): $\frac{1}{2}p(y_{2n}, y_{2n+1}) \leqslant p(y_{2n}, u)$

holds. On the contrary, suppose that

$$\tfrac{1}{2}p(y_{2n-1}, y_{2n}) > p(y_{2n-1}, u) \text{ and } \tfrac{1}{2}p(y_2n, y_{2n+1}) > p(y_{2n}, u) \text{ for some } n \geqslant 1.$$

Then, by (P_4) we have

$$p(y_{2n-1}, y_{2n}) \leqslant p(y_{2n-1}, u) + p(u, y_{2n}) - p(u, u) < \tfrac{1}{2}[p(y_{2n-1}, y_{2n}) + p(y_{2n}, y_{2n+1})] \leqslant p(y_{2n-1}, y_{2n}),$$

which is a contradiction. Therefore (2.22) holds.

Subcase (a): Suppose $\frac{1}{2}p(y_{2n-1}, y_{2n}) \leqslant p(y_{2n-1}, u)$. Suppose $Bt \neq u$. Since

$$\begin{aligned}\tfrac{1}{2}\min\{p(Sx_{2n}, Ax_{2n}), p(Tt, Bt)\} &\leqslant \tfrac{1}{2}p(Sx_{2n}, Ax_{2n})\\ &= \tfrac{1}{2}p(y_{2n-1}, y_{2n})\\ &\leqslant p(y_{2n-1}, u) = p(Sx_{2n}, Tt),\end{aligned}$$

it follows from the inequality (1.2), we have

$$p(Ax_{2n}, Bt) \leqslant \beta(M(x_{2n}, t))M(x_{2n}, t), \tag{2.23}$$

where $[M(x_{2n}, t) =$

$\max\{p(Sx_{2n}, Tt), p(Sx_{2n}, Ax_{2n}), p(Tt, Bt), \frac{1}{2}[p(Sx_{2n}, Bt) + p(Tt, Ax_{2n})]\}$.

On letting $n \to \infty$ and using $\lim\limits_{n\to\infty} Sx_{2n} = \lim\limits_{n\to\infty} Ax_{2n} = u$, we get

$$\begin{aligned}\lim_{n\to\infty} M(x_{2n}, t) &= \max\{p(u, Tt), p(u, u), p(Tt, Bt), \tfrac{1}{2}[p(u, Bt) + p(Tt, u)]\}\\ &= \max\{p(u, Tt), p(u, u), p(u, Bt), \tfrac{1}{2}[p(u, Bt) + p(u, u)]\} = p(u, Bt).\end{aligned}$$

On letting $n \to \infty$ in (2.23), we obtain

$$p(u, Bt) \leqslant \beta(p(u, Bt))p(u, Bt) < p(u, Bt),$$

which is a contradiction. Hence $Bt = u = Tt$. Since the pair (B,T) is weakly compatible, it follows that $Bu = BTt = TBt = Tu$.

Suppose $Bu \neq u$. We have $\frac{1}{2}\min\{p(Sx_{2n}, Ax_{2n}), p(Tu, Bu)\} \leqslant p(Sx_{2n}, Tu)$. From the inequality (1.2), we get

$$p(Ax_{2n}, Bu) \leqslant \beta(M(x_{2n}, u))M(x_{2n}, u), \tag{2.24}$$

where

$$M(x_{2n}, u) = \max\{p(Sx_{2n}, Tu), p(Sx_{2n}, Ax_{2n}), p(Tu, Bu), \frac{1}{2}[p(Sx_{2n}, Bu) + p(Tu, Ax_{2n})]\}.$$

On letting $n \to \infty$ and using $\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Ax_{2n} = u$, we get

$$\begin{aligned}\lim_{n\to\infty} M(x_{2n}, u) &= \max\{p(u, Tu), p(u, u), p(Tu, Bu), \tfrac{1}{2}[p(u, Bu) + p(Tu, u)]\}\\ &= \max\{p(u, Tu), p(u, u), p(u, Bu), \tfrac{1}{2}[p(u, Bu) + p(u, u)]\}\\ &= p(u, Bu).\end{aligned}$$

On letting $n \to \infty$ in (2.24), we obtain

$$p(u, Bu) \leqslant \beta(p(u, Bu))p(u, Bu) < p(u, Bu),$$

it is a contradiction. Hence $Bu = u = Tu$. Therefore u is a common fixed point of B and T. Thus, by Proposition 2.2, we get that u is the unique common fixed point of A, B, S and T.

Subcase (b): Suppose $\frac{1}{2}p(y_{2n}, y_{2n+1}) \leqslant p(y_{2n}, u)$. On proceeding as in Subcase (a), it follows that u is a unique common fixed point of A, B, S and T.

Case (ii): Suppose $y_{2m} = y_{2m+1}$ for some m. Assume that $y_{2m+1} \neq y_{2m+2}$. We have

$$M(x_{2m+2}, x_{2m+1}) = \max\{p(y_{2m+1}, y_{2m}), p(y_{2m+1}, y_{2m+2}), p(y_{2m}, y_{2m+1}), \frac{1}{2}[p(y_{2m+1}, y_{2m+1}) + p(y_{2m}, y_{2m+2})]\}.$$

From (P_2), we have

$$p(y_{2m+1}, y_{2m}) = p(y_{2m+1}, y_{2m+1}) \leqslant p(y_{2m+1}, y_{2m+2}).$$

Then, we have

$$\begin{aligned}\tfrac{1}{2}[p(y_{2m+1}, y_{2m+1}) + p(y_{2m}, y_{2m+2})] &\leqslant \tfrac{1}{2}[p(y_{2m}, y_{2m+1}) + p(y_{2m+1}, y_{2m+2})]\\ &\leqslant \tfrac{1}{2}[p(y_{2m+1}, y_{2m+2}) + p(y_{2m+1}, y_{2m+2})]\\ &= p(y_{2m+1}, y_{2m+2}).\end{aligned}$$

Hence $M(x_{2m+2}, x_{2m+1}) = p(y_{2m+1}, y_{2m+2})$. Since

$$\begin{aligned}\tfrac{1}{2}\min\{p(Sx_{2m+2}, Ax_{2m+2}), p(Tx_{2m+1}, Bx_{2m+1})\} &\leqslant p(Tx_{2m+1}, Bx_{2m+1})\\ &= p(Sx_{2m+2}, Tx_{2m+1}),\end{aligned}$$

it follows from the inequality (1.2), we have

$$\begin{aligned}p(y_{2m+2}, y_{2n+1}) &= p(Ax_{2m+2}, Bx_{2m+1})\\ &\leqslant \beta(M(x_{2m+2}, x_{2m+1}))M(x_{2m+2}, x_{2m+1})\\ &= \beta(p(y_{2m+2}, y_{2n+1}))p(y_{2m+2}, y_{2n+1}) < p(y_{2m+2}, y_{2n+1}),\end{aligned}$$

which is a contradiction. Hence $y_{2m+1} = y_{2m+2}$. On continuing this process, it follows that $y_n = y_{n+k}$ for all $k = 1, 2, 3, \ldots$. Thus $\{y_n\}$ is Cauchy.

The rest of the proof follows as in Case (i). □

3. Corollaries and examples

In this section, we draw some corollaries from the main results of Section 2 and provide examples in support of our results.

From Theorem 2.1, we have the following corollaries.

Corollary 3.1. *Let (X, p) be partial metric space and let f and g be selfmaps of X. Assume that there exist $\varphi \in \Psi$, $\phi \in \Phi$, and $F \in \mathcal{F}$ such that*

$$\varphi(p(fx, fy)) \leqslant F(\varphi(p(gx, gy)), \phi(p(gx, gy))) \text{ for all } x, y \in X.$$

If $f(X) \subseteq g(X)$, the pair (f, g) is weakly compatible and $g(X)$ is a complete subspace of X then f and g have a unique fixed point in X.

Corollary 3.2. *Let (X, p) be partial metric space and let f and g be selfmaps of X. Assume that there exist $\varphi \in \Psi$, $\phi \in \Phi$, and $F \in \mathcal{F}$ such that*

$$\varphi(p(fx, fy)) \leqslant F(\varphi(p(gy, fy)\tfrac{1+p(gx,fx)}{1+p(gx,gy)}), \phi(p(gy, fy)\tfrac{1+p(gx,fx)}{1+p(gx,gy)}))$$

for all $x, y \in X$. If $f(X) \subseteq g(X)$, the pair (f, g) is weakly compatible and $g(X)$ is a complete subspace of X then f and g have a unique fixed point in X.

Putting $T = f$ and g is the identity map on X in Theorem 2.1, we have the following.

Corollary 3.3. (Theorem 3.1, [**20**]) *Let (X, p) be a complete partial metric space and $T : X \to X$ ba a selfmap such that there exist a pair of functions $\varphi \in \Psi$, $\phi \in \Phi$, and $F \in \mathcal{C}$ such that*

$$\varphi(p(Tx, Ty)) \leqslant \max\{F(\varphi(p(x, y)), \phi(p(x, y))), \\ F(\varphi(p(y, Ty)\tfrac{1+p(x,Tx)}{1+p(x,y)}), \phi(p(y, Ty)\tfrac{1+p(x,Tx)}{1+p(x,y)}))\}$$

for all $x, y \in X$. Then T has a unique fixed point in X.

In Theorem 2.2, if $A = B = f$ and $S = T = g$, we have the following corollary.

Corollary 3.4. *Let (X, p) be a partial metric space, and let f, g be selfmaps of X. Assume that there exists $\beta \in \mathfrak{F}$ such that*

$\frac{1}{2}\min\{p(gx, fx), p(gy, fy)\} \leqslant p(gx, gy) \Rightarrow p(fx, fy) \leqslant \beta(M(x, y))M(x, y)$ *for all $x, y \in X$, where*

$$M(x, y) = \max\{p(gx, gy), p(gx, fx), p(gy, fy), \frac{1}{2}[p(gx, fy) + p(gy, fx)]\}.$$

If

(i) *$f(X) \subseteq g(X)$, $g(X)$ is a complete subspace of X, and*

(ii) *the pair (f, g) is weakly compatible,*

then f and g have a unique common fixed point in X.

The following is an example in support of Theorem 2.1.

EXAMPLE 3.1. Let $X = [0,1]$ and $p(x,y) = \begin{cases} 0 & \text{if } x = y \\ \max\{x,y\} & \text{if } x \neq y, \end{cases}$ for all $x, y \in X$. Then (X,p) is a partial metric space. Define $f, g : X \to X$ by $f(x) = \frac{x^2}{4}$, $g(x) = \frac{x^2}{2}$. Define $F(s,t) = \frac{99}{100}s$, φ, $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(t) = \frac{3}{4}t$, $\phi(t) = \frac{t}{3}$ for all $t \geqslant 0$. Clearly $f(X) \subseteq g(X)$ and the pair (f,g) is weakly compatible. Without loss of generality we assume $x \geqslant y$ Here $\varphi(p(fx,fy)) = \frac{3x^2}{16}$; $\varphi(p(gx,gy)) = \frac{3x^2}{8}$; $\phi(p(gx,gy)) = \frac{x^2}{6}$ $p(gx,fx) = \frac{x^2}{2}$; $p(gy,fy) = \frac{y^2}{2}$; $\varphi(p(gy,fy)\frac{1+p(gx,fx)}{1+p(gx,gy)}) = \frac{3y^2}{8}$. Now,

$$\begin{aligned}\varphi(p(fx,fy)) = \tfrac{3x^2}{16} &\leqslant \tfrac{99}{100}\tfrac{3x^2}{8} \\ &= \max\{F(\varphi(p(gx,gy)), \phi(p(gx,gy))), \\ &\qquad F(\varphi(p(gy,fy)\tfrac{1+p(gx,fx)}{1+p(gx,gy)}), \phi(p(gy,fy)\tfrac{1+p(gx,fx)}{1+p(gx,gy)}))\}\end{aligned}$$

Therefore f and g satisfy all the hypotheses of Theorem 2.1 and 0 is the unique common fixed point.

The following is an example in support of Theorem 2.2.

EXAMPLE 3.2. Let $X = [0,1]$ and $p(x,y) = \max\{x,y\}$ for all $x, y \in X$. Then (X,p) is a partial metric space. Define selfmaps A, B, S and T on X by

$$A(x) = \tfrac{x^2}{2},\ B(x) = \tfrac{x^2}{3},\ S(x) = \tfrac{x}{4}(5-x) \text{ and } T(x) = \tfrac{x}{5}(6-x).$$

Define $\beta : [0,\infty) \to [0,1)$ by $\beta(t) = \frac{1+t}{1+2t}$, $t \geqslant 0$.
Clearly $\beta \in \mathfrak{F}$. Also, clear that $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$. The pairs (A,S) and (B,T) are weakly compatible. Without loss of generality, we assume that $x \geqslant y$.

$$\begin{aligned}\tfrac{1}{2}\min\{p(Sx,Ax), p(Ty,By)\} &= \tfrac{1}{2}\min\{\max\{\tfrac{x}{4}(5-x), \tfrac{x^2}{2}\}, \max\{\tfrac{y}{5}(6-y), \tfrac{y^2}{3}\}\} \\ &= \tfrac{1}{2}\min\{\tfrac{x}{4}(5-x), \tfrac{y}{5}(6-y)\} \\ &= \tfrac{1}{2}\tfrac{y}{5}(6-y) \\ &\leqslant \max\{\tfrac{x}{4}(5-x), \tfrac{y}{5}(6-y)\} = p(Sx,Ty).\end{aligned}$$

Here
$p(Ax,By) = \max\{\frac{x^2}{2}, \frac{y^2}{2}\} = \frac{x^2}{2}$, $p(Sx,Ty) = \frac{x}{4}(5-x)$,
$p(Sx,Ax) = \frac{x}{4}(5-x)$, $p(Ty,By) = \frac{y}{5}(6-y)$, $p(Sx,By) = \frac{x}{4}(5-x)$,
$p(Ty,Ax) = \max\{\frac{y}{5}(6-y), \frac{x^2}{2}\}$ and $\frac{1}{2}[p(Sx,By) + p(Ty,Ax)] \leqslant \frac{x}{4}(5-x)$.
Therefore

$$\begin{aligned}M(x,y) &= \max\{p(Sx,Ty), p(Sx,Ax), p(Ty,By), \\ &\qquad \tfrac{1}{2}[p(Sx,By) + p(Ty,Ax)]\} \\ &= \tfrac{x}{4}(5-x).\end{aligned}$$

We now consider

$$p(Ax,By) = \frac{x^2}{2} \leqslant \beta(\frac{x}{4}(5-x))\frac{x}{4}(5-x) = \beta(M(x,y))M(x,y).$$

Therefore A, B, S and T satisfy all the hypotheses of Theorem 2.2 and 0 is the unique common fixed point of A, B, S and T.

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