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CAYLEY INCLUSION GRAPH OF A GROUP

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Abstract. Let G be a group and let $L^*(G) = L(G) \setminus \{G\}$, where $L(G)$ is the subgroup lattice of G. We consider the inclusion digraph on $L^*(G)$ having $L^*(G)$ as its vertex set and for two distinct vertices H and K, there is an arc from H to K if and only if $H \subset K$. In this paper, we show that this digraph is isomorphic to the Cayley graphs of some semigroups. We call this digraph as *Cayley inclusion graph (CI graph) of* G and is denoted by $\overrightarrow{\mathcal{CI}}(G)$. We denote the underlying graph of $\overrightarrow{\mathcal{CI}}(G)$ by $\mathcal{CI}(G)$. Moreover, we study some properties of $\overrightarrow{\mathcal{CI}}(G)$ and classify all finite groups whose CI graph is planar. As a consequence, we show that some non-abelian groups can be determined by their CI graphs.

1. Introduction

Associating graphs to algebraic structures and studying the interplay between the associated graphs and the algebraic structures is one of the main approaches in algebraic graph theory. Cayley graph is one such graph associated with a group (res., semigroup), which has been studied extensively in the literature because of its various applications. For instance, see [**8**], [**11**], [**12**], and the survey article [**13**]. In particular, the Cayley graphs of semigroups are related to automata theory (see [**9**] and [**10**]). For a semigroup X and a subset S of X, the Cayley graph $\overrightarrow{Cay}(X, S)$ of X relative to S is defined as the digraph with vertex set X and edge set consisting of those pairs (x, y) such that $y = sx$, for some $s \in S$. If $S^{-1} = S$, then this graph is undirected. The set of all ideals of a ring forms semigroups under the operation of sum and product of ideals, respectively. In [**1**], [**2**], Afkhami et al defined Cayley

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graphs on these semigroups and studied their properties. Motivated by these, in this paper, we define a Cayley graph on some class of subgroups of a group.

Let G be a group and $L(G)$ be its subgroup lattice. Let $L^*(G) = L(G) \setminus G$ and $L'(G) = L(G) \setminus \{e\}$, where e denotes the identity element in G. Then $L^*(G)$ forms a semigroup with respect to intersection of subgroups and $L'(G)$ forms a semigroup with respect to join of subgroups. So we can define the corresponding Cayley graphs $\overrightarrow{Cay}(L^*(G), L^*(G))$ and $\overrightarrow{Cay}(L'(G), L'(G))$. It is easy to see that $\overrightarrow{Cay}(L^*(G), L^*(G))$ and $\overrightarrow{Cay}(L'(G), L'(G))$ are isomorphic.

Now we consider the inclusion digraph on $L^*(G)$ having $L^*(G)$ as its vertex set and for two distinct vertices H and K, there is an arc from H to K if and only if $H \subset K$. Note that for $H, K \in L^*(G)$, $H \subset K$ if and only if $H = H \cap K$. It follows that this graph is same as $\overrightarrow{Cay}(L^*(G), L^*(G))$ and hence it is isomorphic to $\overrightarrow{Cay}(L'(G), L'(G))$. Thus, the inclusion graph on $L^*(G)$ is the Cayley graph on the semigroups $L^*(G)$ and $L'(G)$. We call this graph as *Cayley inclusion graph (CI graph) of* G and is denoted by $\overrightarrow{\mathcal{CI}}(G)$. We denote the underlying graph of $\overrightarrow{\mathcal{CI}}(G)$ by $\mathcal{CI}(G)$.

Note that in [**6**], Devi and Rajkumar studied several properties of the (undirected) inclusion graph on $L(G) \setminus \{G, \{e\}\}$.

In this paper, we use the notations and definitions of graph theory as in [**7**]. K_n denote the complete graph on n vertices. $K_{m,n}$ denote the complete bipartite graph on $m + n$ vertices whose each of the two partite sets are having m and n vertices, respectively. For a vertex v in given directed graph, we denote its indegree and outdegree by $deg_{in}(v)$ and $deg_{out}(v)$, respectively.

2. Some basic properties of CI graphs

Here we state the following result whose proof directly follows from the definition of the Cayley inclusion graph of a group.

PROPOSITION 2.1. *Let G be a finite group. Then*

(i) $\overrightarrow{\mathcal{CI}}(G)$ *has no directed cycle and so girth*$(\overrightarrow{\mathcal{CI}}(G)) = \infty$*;*

(ii) $\overrightarrow{\mathcal{CI}}(G)$ *has no directed Hamiltonian cycle or closed directed Eulerian trail;*

(iii) $\overrightarrow{\mathcal{CI}}(G)$ *is weakly connected but not strongly connected;*

(iv) *There is no subdigraph D of $\overrightarrow{\mathcal{CI}}(G)$ such that D is strongly connected;*

(v) $\overrightarrow{\mathcal{CI}}(G)$ *is unilaterally connected if and only if $G \cong \mathbb{Z}_{p^n}$, where p is a prime, $n \geqslant 1$;*

(vi) *A maximal unilateral subdigraph D of $\overrightarrow{\mathcal{CI}}(G)$ is a maximal chain in the subgroup lattice of G;*

(vii) *For a subgroup H of G, $deg_{out}(H)$ in $\overrightarrow{\mathcal{CI}}(G)$ is the number of proper subgroups of G which contains H and $deg_{in}(H)$ in $\overrightarrow{\mathcal{CI}}(G)$ is the number of proper subgroups of H.*

(viii) *$deg_{out}(H) = 0$ in $\overrightarrow{\mathcal{CI}}(G)$ if and only if H is a maximal subgroup of G and $deg_{in}(H) = 0$ in $\overrightarrow{\mathcal{CI}}(G)$ if and only if $H = \{e\}$;*

(ix) *For a finite group G of order $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$, where p_i's are distinct primes, the maximum length among all paths in $\overrightarrow{\mathcal{CI}}(G)$ is at most $\alpha_1 + \alpha_2 + \cdots + \alpha_m$; If G is solvable, the bound is sharp.*

3. Planarity of CI graphs

In this section, for a given group G, we consider the graph $\mathcal{CI}(G)$, which is the underlying graph of $\overrightarrow{\mathcal{CI}}(G)$. We prove the following theorem in which we classified all the finite groups for which $\mathcal{CI}(G)$ is planar. For this, we use the well known characterization for planar graphs: A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.

Theorem 3.1. *Let G be a finite group and p, q, r be distinct primes. Then $\mathcal{CI}(G)$ is planar if and only if G is isomorphic to one of the following :*

(i) $\mathbb{Z}_{p^n}$ $(n = 1, 2, 3, 4)$, $\mathbb{Z}_{p^m q}$ $(n = 1, 2, 3)$, $\mathbb{Z}_{pqr}$, $\mathbb{Z}_{p^m} \times \mathbb{Z}_p$ $(m = 1, 2)$,

(ii) M_{p^3}, $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$, $\mathbb{Z}_p \rtimes \mathbb{Z}_q$, A_4, Q_3,

(iii) $\langle a, b \mid a^{p^2} = b^q = 1, ab = ba^{\alpha} \rangle$, *where* $\alpha^q \equiv 1 (mod\ p^2)$ *and* $p > q$,

(iv) $\langle a, b, c \mid a^p = b^p = c^q = 1, c^{-1}ac = ba, c^{-1}bc = a, c^{-1}(ab)c = ab \rangle$, *where* $p > q$,

(v) $\langle a, b \mid a^{p^2} = b^q = 1, a^{-1}ba = b^{\mu} \rangle$, *where* $\mu^p \equiv 1 (mod\ q)$ *and* $p < q$,

(vi) $\langle a, b \mid a^{p^2} = b^q = 1, a^{-1}ba = b^{\mu} \rangle$, *where* $\mu^p \equiv 1 (mod\ q)$ *and* $p < q$,

(vii) $\langle a, b \mid a^p = b^p = c^q = 1, ab = ba, ac = ca, b^{-1}cb = c^{\mu} \rangle$, *where* $\mu^{p^2} \equiv 1 (mod\ q)$ *and* $p < q$,

(viii) $\langle a, b, c \mid a^p = b^p = c^q = 1, ab = ba, ca = ac^{\mu}, cb = bc^{\nu} \rangle$, *where* $2 < p < q$, $\nu, \mu \neq 1$ *and* $\mu \neq \nu$.

As a consequence of the above result, we prove the following:

Corollary 3.1. *Let G be a finite group and p, q be distinct primes.*

(1) *The following are equivalent:*
  (i) $G \cong \mathbb{Z}_p, \mathbb{Z}_{p^2}, \mathbb{Z}_p \times \mathbb{Z}_p$ *or* $\mathbb{Z}_{pq}$;
  (ii) $\mathcal{CI}(G)$ *is a tree;*
  (iii) $\mathcal{CI}(G)$ *is a star.*

(2) $\mathcal{CI}(G)$ *is a path if and only if* $G \cong \mathbb{Z}_p, \mathbb{Z}_{p^2}$ *or* $\mathbb{Z}_{pq}$.

As an application of this investigation, in the next result we show that some non-abelian groups can be determined by their CI graphs.

Corollary 3.2. *Let G be a finite group and p, q be distinct primes.*

(i) *If G' is a group such that $\mathcal{CI}(G') \cong \mathcal{CI}(G)$, where G is one of the following groups:* $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$, $\langle a, b \mid a^p = b^p = c^q = 1, ab = ba, ac = ca, b^{-1}cb = c^{\mu} \rangle$, *where* $\mu^{p^2} \equiv 1 (mod\ q)$ *and* $p < q$, $\langle a, b, c \mid a^p = b^p = c^q = 1, ab = ba, ca = ac^{\mu}, cb = bc^{\nu} \rangle$, *where* $2 < p < q$, $\nu, \mu \neq 1$ *and* $\mu \neq \nu$, A_4 *and* Q_3, *then* $G' \cong G$.

(ii) *If G' is a group such that $\mathcal{CI}(G') \cong \mathcal{CI}(\mathbb{Z}_p \rtimes \mathbb{Z}_q)$, then $G' \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$ or $\mathbb{Z}_p \times \mathbb{Z}_p$; If G' is nonabelian, then $G' \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$.*

(iii) *If G' is a group such that $\mathcal{CI}(G') \cong \mathcal{CI}(M_{p^3})$, then $G' \cong M_{p^3}$ or $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$; If G' is nonabelian, then $G' \cong M_{p^3}$.*

We start with the following result whose proof directly follows from the definition.

LEMMA 3.1. *Let G be a group and H be its subgroup. Then $\mathcal{CI}(H)$ is a subgraph of $\mathcal{CI}(G)$. If H is a normal subgroup of G, then $\mathcal{CI}(G/H)$ is isomorphic (as a graph) to a subgraph of $\mathcal{CI}(G)$.*

COROLLARY 3.3. *If H is a subgroup of G such that $\mathcal{CI}(H)$ or $\mathcal{CI}(G/H)$ is nonplanar, then $\mathcal{CI}(G)$ is nonplanar.*

LEMMA 3.2. *If G is a group and H is a proper subgroup of G such that H contains a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime, then $\mathcal{CI}(G)$ is nonplanar.*

PROOF. We know that $\mathbb{Z}_p \times \mathbb{Z}_p$ contains at least 3 subgroups of order p, let them be N_1, N_2, N_3. Then $\mathcal{CI}(G)$ contains $K_{3,3}$ as a subgraph with the bipartition $\{H, \mathbb{Z}_p \times \mathbb{Z}_p, \{e\}\}$, $\{N_1, N_2, N_3\}$. □

The dihedral group of order $2n, n \geqslant 3$ is given by $D_n = \langle a, b \mid a^n = b^2 = 1, ab = ba^{-1}\rangle$. In the following Lemma, we characterize the values of n, for which the graph $\mathcal{CI}(D_n)$ is planar.

LEMMA 3.3. *$\mathcal{CI}(D_n)$ is planar if and only if n is a prime or square of a prime.*

PROOF. Suppose n is a prime, then D_n contains a cyclic subgroup of order n, and n subgroups of order 2. It follows that $\mathcal{CI}(D_n) \cong K_{1,n+1}$, which is planar.

Suppose $n = p^2$, where p is a prime. Then $H_i := \langle a^i b\rangle$, $L_i := \langle a^p, a^i b\rangle$, where $0 \leqslant i \leqslant p^2 - 1$ $N := \langle a\rangle$ and $M := \langle a^p\rangle$ are the only subgroups of D_n. A planar embedding of $\mathcal{CI}(D_n)$ is given in Figure 1(a).

Suppose n is neither a prime nor a square of a prime, $n = m_1 m_2$, $m_1 \neq m_2$, then $\mathcal{CI}(D_n)$ contains a subgraph given in Figure 1(b), which is a subdivision of $K_{3,3}$ and hence $\mathcal{CI}(D_n)$ is nonplanar. □

PROPOSITION 3.1. *If G is a finite solvable group whose length of the composition series is at least 5, then $\mathcal{CI}(G)$ is nonplanar.*

PROOF. In this case, there is a chain of at least 5 normal subgroups of G. So they forms K_5 as a subgraph of $\mathcal{CI}(G)$ and hence $\mathcal{CI}(G)$ is nonplanar. □

PROPOSITION 3.2. *If G is a solvable group whose order has at least 4 distinct prime divisors, then $\mathcal{CI}(G)$ is nonplanar.*

PROOF. Let p, q, r, s be any four distinct prime factors of $|G|$. Since G is solvable, there is a Sylow basis of G containing P, Q, R, S, where P, Q, R, S are $p, q, r, s-$Sylow subgroups of G, respectively. Then $\mathcal{CI}(G)$ contains $K_{3,3}$ as a subgraph with bipartition $\{PQ, PQR, PQS\}, \{P, Q, \{e\}\}$ and hence $\mathcal{CI}(G)$ is nonplanar. □

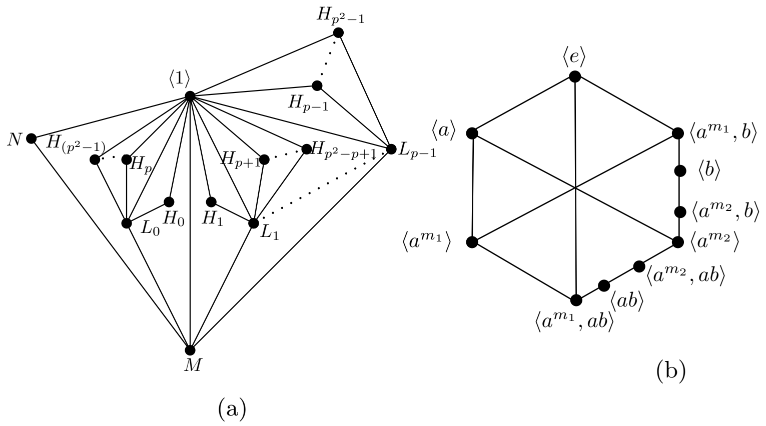


FIGURE 1. (a) $\mathcal{CI}(D_{p^2})$, (b) A subgraph of $\mathcal{CI}(D_n), n \neq p$ or p^2

By Propositions 3.1 and 3.2, it is enough to consider solvable groups whose orders are p^n $(n \leqslant 4)$, $p^m q$ $(m \leqslant 3)$, p^2q^2, pqr, p^2qr, where p, q, r are distinct primes.

PROPOSITION 3.3. *Let G be a group of order p^n, where p is a prime $n \geqslant 1$. Then $\mathcal{CI}(G)$ is planar if and only if G is isomorphic to $\mathbb{Z}_{p^n}, (n \leqslant 4)$, $\mathbb{Z}_p \times \mathbb{Z}_p$, $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$, M_{p^3}, $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$.*

PROOF. By Lemma 3.1, it is enough to consider the groups of order $p^n, n \leqslant 4$. It is easy to see that $\mathcal{CI}(\mathbb{Z}_p) \cong K_1$, $\mathcal{CI}(\mathbb{Z}_{p^2}) \cong K_{1,2}$, $\mathcal{CI}(\mathbb{Z}_p \times \mathbb{Z}_p) \cong K_{1,p+1}$ and $\mathcal{CI}(\mathbb{Z}_{p^3}) \cong K_3$, which are all planar.

$\mathcal{CI}(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$ is given in Figure 2(a), which is planar; $\mathcal{CI}(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$ contains a subgraph given in Figure 3(a), which is a subdivision of $K_{3,3}$ and hence $\mathcal{CI}(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$ is nonplanar; $\mathcal{CI}(M_{p^3})$ is given in Figure 2(b), which is planar. If $G \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$, then $M_i := \langle ac^i\rangle$, $L_i^j := \langle a(bc^j)^i\rangle$, $B_i := \langle bc^i\rangle$, $K := \langle a\rangle$, $C := \langle c\rangle$, $N_i := \langle a, bc^i\rangle$, where $0 \leqslant i, j \leqslant p-1$ and $H := \langle a, c\rangle$ are the only subgroups of G. The structure of $\mathcal{CI}(G)$ is given in Figure 2(c), which is planar.

Now, we consider the groups of order p^4. Suppose $G \cong \mathbb{Z}_{p^4}$, then $\mathcal{CI}(G) \cong K_4$, which is planar.

Suppose G is a noncyclic group of order p^4, then G must have a noncyclic subgroup of order p^3, say H. For, suppose all the subgroups of order p^3 in G are cyclic, then let $H_1, H_2, \ldots, H_k$ be such subgroups of G. Then $|H_i \cap H_j| = p^2$, $1 \leqslant i, j \leqslant k, i \neq j$. Since each $H_s, 1 \leqslant s \leqslant k$ is cyclic, it contains a unique subgroup of order p^2. Thus all $H_s, 1 \leqslant s \leqslant k$ contain a common subgrop of order p^2. Hence G contains a unique subgroup of order p^2, which implies that G is cyclic, which is a contradiction to our hypothesis. Then H contains a subgroup K isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. So by Lemma 3.2, $\mathcal{CI}(G)$ is nonplanar. $\square$

PROPOSITION 3.4. *Let G be a group of order pq, where p, q are distinct primes. Then $\mathcal{CI}(G)$ is planar.*

PROOF. It is easy to verify that $\mathcal{CI}(\mathbb{Z}_{pq}) \cong K_{1,2}$ and $\mathcal{CI}(\mathbb{Z}_p \rtimes \mathbb{Z}_q) \cong K_{1,p+1}$, which are planar. $\square$

PROPOSITION 3.5. *Let G be a group of order p^2q, where p, q are distinct primes. Then $\mathcal{CI}(G)$ is planar if and only if G is isomorphic to one of the following groups:*

(i) $\mathbb{Z}_{p^2q}$,

(ii) $\langle a, b \mid a^{p^2} = b^q = 1, ab = ba^{\alpha}\rangle$, *where* $\alpha^q \equiv 1(mod\ p^2)$ *and* $p > q$,

(iii) $\langle a, b, c \mid a^p = b^p = c^q = 1, c^{-1}ac = ba, c^{-1}bc = a, c^{-1}(ab)c = ab\rangle$, *where* $p > q$,

(iv) $\langle a, b \mid a^{p^2} = b^q = 1, a^{-1}ba = b^{\mu}\rangle$, *where* $\mu^p \equiv 1(mod\ q)$ *and* $p < q$,

(v) $\langle a, b \mid a^{p^2} = b^q = 1, a^{-1}ba = b^{\mu}\rangle$, *where* $\mu^p \equiv 1(mod\ q)$ *and* $p < q$,

(vi) $\langle a, b \mid a^p = b^p = c^q = 1, ab = ba, ac = ca, b^{-1}cb = c^{\mu}\rangle$, *where* $\mu^{p^2} \equiv 1(mod\ q)$ *and* $p < q$,

(vii) $\langle a, b, c \mid a^p = b^p = c^q = 1, ab = ba, ca = ac^{\mu}, cb = bc^{\nu}\rangle$, *where* $2 < p < q$, $\nu, \mu \neq 1$ *and* $\mu \neq \nu$.

PROOF. To prove this result, we use the classification of groups of order p^2q given in [**5**, p.202-215].

Case 1: Suppose G is abelian.

If $G_1 \cong \mathbb{Z}_{p^2q}$, then $\mathcal{CI}(G)$ is given in Figure 2(d), which is planar. If $G_2 \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q$, then $\mathcal{CI}(G_2)$ contains a subgraph given in Figure 3(b), which is a subdivision of $K_{3,3}$ and hence $\mathcal{CI}(G_2)$ is nonplanar.

Case 2: Suppose G is nonabelian and $p > q$, we further divide this into following subcases.

Subcase 2a: $G_3 := \langle a, b \mid a^{p^2} = b^q = 1, ab = ba^{\alpha}\rangle$, where $\alpha^q \equiv 1(mod\ p^2)$. Here $H_i := \langle ba^i\rangle$, where $0 \leqslant i \leqslant p^2 - 1$, $L_j := \langle a^p, ba^j\rangle$, where $0 \leqslant j \leqslant p - 1$, $N := \langle a\rangle$ and $M := \langle a^p\rangle$ are the only proper subgroups of G_3. Then $\mathcal{CI}(G_3)$ is given in Figure 2(e), which is planar.

Subcase 2b: $G_4 := \langle a, b, c \mid a^p = b^p = c^q = 1, ab = ba, ac = ca, c^{-1}bc = b^{\mu}\rangle$, where $\mu \neq 1$. Then $\mathcal{CI}(G_4)$ contains a graph given in Figure 3(c), which is a subdivision of $K_{3,3}$ and hence $\mathcal{CI}(G_4)$ is nonplanar.

Subcase 2c: $G_5 := \langle a, b, c \mid a^p = b^p = c^q = 1, ab = ba, ac = ca^{\mu}, c^{-1}bc = b^{\mu}\rangle$, where $\mu^q \equiv 1(mod\ p)$. Then $\mathcal{CI}(G_5)$ contains a graph given in Figure 3(d), which is a subdivision of $K_{3,3}$ and hence $\mathcal{CI}(G_5)$ is nonplanar.

Subcase 2d: $G_6 := \langle a, b, c \mid a^p = b^p = c^q = 1, ab = ba, ac = ca^{\mu}, c^{-1}bc = b^{\nu}\rangle$, where $\mu^q \equiv 1(mod\ p)$, $\nu^q \equiv 1(mod\ p)$ and $\mu \neq \nu$. Then $\mathcal{CI}(G_6)$ contains a graph given in Figure 3(e), which is a subdivision of $K_{3,3}$ and hence $\mathcal{CI}(G_6)$ is nonplanar.

Subcase 2e: $G_7 := \langle a, b, c \mid a^p = b^p = c^q = 1, c^{-1}ac = ba, c^{-1}bc = a, c^{-1}(ab)c = ab\rangle$. Then $\mathcal{CI}(G_7)$ is given in Figure 2(f), which is planar.

Case 3: Suppose G is nonabelian and $p < q$ but $(p, q) \neq (2, 3)$, we further divide this case into the following subcases.

Subcase 3a: $G_8 := \langle a, b \mid a^{p^2} = b^q = 1, a^{-1}ba = b^\mu\rangle$, where $\mu^p \equiv 1(mod\ q)$. Then $\mathcal{CI}(G_8)$ is given in Figure 2(g), which is planar.

Subcase 3b: $G_9 := \langle a, b \mid a^{p^2} = b^q = 1, a^{-1}ba = b^\mu\rangle$, where $\mu^p \equiv 1(mod\ q)$. Here $H_i := \langle b^{-i}a^pb^i\rangle$, $L_i := \langle b^{-i}ab^i\rangle$, where $0 \leqslant i \leqslant q-1$ and $K := \langle a^p, b\rangle$ are the only proper subgroups of G_9. Then $\mathcal{CI}(G_9)$ is given in Figure 2(h), which is planar.

Subcase 3c: $G_{10} := \langle a, b \mid a^p = b^p = c^q = 1, ab = ba, ac = ca, b^{-1}cb = c^\mu\rangle$, where $\mu^{p^2} \equiv 1(mod\ q)$. Here $L_i^j := \langle a^{-i}cb^ja^i\rangle$, $M_i := \langle b, a^{-i}ca^i\rangle$, $0 \leqslant i \leqslant p-1, 0 \leqslant j \leqslant q-1$, $K := \langle b\rangle$, $A := \langle a\rangle$ and $H := \langle a, b\rangle$ are the only proper subgroups of G_{10}. Then $\mathcal{CI}(G_{10})$ is given in Figure 2(i), which is planar.

Subcase 3d: $G_{11} := \langle a, b, c \mid a^p = b^p = c^q = 1, ab = ba, ca = ac^\mu, cb = bc^\nu\rangle$, where $p > 2$, $\nu \neq 1$, $\mu \neq 1$ and $\mu \neq \nu$. Here $H_i := c^{-i}\langle a, b\rangle c^i$, $K_i^j := \langle c^{-i}a^jbc^i\rangle$, $L_i := \langle c^{-i}ac^i\rangle$, where $0 \leqslant i \leqslant q-1, 0 \leqslant j \leqslant p-1$ and $N := \langle c\rangle$ are the only proper subgroups of G_{11}. Then $\mathcal{CI}(G_{11})$ is given in Figure 2(j), which is planar.

Case 4: Suppose G is a nonabelian group of order 12.

A_4, Q_3 and D_6 are the only non-isomorphic nonabelian groups of order 12. $\mathcal{CI}(A_4)$ and $\mathcal{CI}(Q_3)$ are given in Figure 2(k), 2(l) respectively, which are planar. If $G \cong D_6$, then by Lemma 3.3, $\mathcal{CI}(G)$ is nonplanar.

Combining all the above cases together, we get the proof. $\square$

PROPOSITION 3.6. *Let G be a group of order p^3q, where p, q are distinct primes. Then $\mathcal{CI}(G)$ is planar if and only if $G \cong \mathbb{Z}_{p^3q}$.*

PROOF. Suppose G has a noncyclic $p-$Sylow subgroup P, then P contains a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Then by Lemma 3.2, $\mathcal{CI}(G)$ is nonplanar. So, we assume that all $p-$Sylow subgroups are cyclic.

Case 1: Suppose both $p, q-$Sylow subgroups are unique, then $G \cong \mathbb{Z}_{p^3q}$. $\mathcal{CI}(G)$ is given in Figure 2(n), which is planar.

Case 2: Suppose $p-$Sylow subgroup is unique and $q-$Sylow subgroup is not unique.

Using the classification of groups given in [**18**, p.227-228], we can easily see that there is only one such group $G \cong \langle a, b \mid a^{p^3} = 1 = b^q, ab = ba^\alpha\rangle$, where $\alpha^q \equiv 1(mod\ p^3)$.

Let $\overline{a} = a^p$. Then $\overline{a}b = b\overline{a}^\alpha$. Here $H := \langle \overline{a}, b \mid \overline{a}^{p^2} = 1 = b^q = 1, \overline{a}b = b\overline{a}^\alpha\rangle$, where $\alpha^q \equiv 1(mod\ p^2)$ is a subgroup of G. Then $\mathcal{CI}(G)$ contains $K_{3,3}$ as a subgraph with bipartition $\{H, \langle\overline{a}^p\rangle, \langle e\rangle\}, \{\langle\overline{a}^p, b\rangle, \langle\overline{a}^p, b\overline{a}\rangle, \langle\overline{a}\rangle\}$.

Case 3: Suppose $q-$Sylow subgroup is unique and $p-$Sylow subgroup is not unique. Using the classification of groups given in [**18**, p.218-220], we can easily see that there are three such groups, which we deal in the following.

Subcase 3a: $G_1 := \langle a, b \mid a^{p^3} = 1 = b^q, ba = ab^\alpha\rangle$, $\alpha^p \equiv 1(mod\ q)$.

Since $p-$Sylow subgroup is not unique, there are atleast three $p-$Sylow subgroups, let them be P_1, P_2, P_3. Then by [**18**, p.218], $\langle a^p\rangle$ is the only subgroup of order p^2. Then $\langle a^p\rangle$ and $\langle a^{p^2}\rangle$ are subgroups of all P_i's. Thus $\mathcal{CI}(G_1)$ contains $K_{3,3}$ as a subgraph with bipartition $\{P_1, P_2, P_3\}, \{\langle a^p\rangle, \langle a^{p^2}\rangle, \{e\}\}$.

Subcase 3b: $G_2 := \langle a, b \mid a^{p^3} = 1 = b^q, ba = ab^{\alpha}\rangle$, $\alpha^{p^2} \equiv 1(mod\ q)$.

Here $H := \langle \overline{a}, b \mid \overline{a}^{p^2} = 1 = b^q, b\overline{a} = \overline{a}b^{\beta}\rangle$, where $\beta = \alpha^p$ and so $\beta^p \equiv 1(mod\ q)$ is a subgroup of G_2. Then $\mathcal{CI}(G_2)$ contains $K_{3,3}$ as a subgraph with bipartition $\{H, \langle\overline{a}^p\rangle, \langle e\rangle\}$, $\{\langle\overline{a}^p\rangle, \langle b^{-1}\overline{a}b\rangle, \langle b^{-2}\overline{a}b^2\rangle\}$.

Subcase 3c: $G_3 := \langle a, b \mid a^{p^3} = 1 = b^q, ba = ab^{\alpha}\rangle$, $\alpha^{p^3} \equiv 1(mod\ q)$.

Here $H := \langle \overline{a}, b \mid \overline{a}^{p^2} = 1 = b^q, b\overline{a} = \overline{a}b^{\beta}\rangle$, where $\beta = \alpha^p$ and so $\beta^{p^2} \equiv 1(mod\ q)$ is a subgroup of G_3. Then $\mathcal{CI}(G_3)$ contains $K_{3,3}$ as a subgraph with bipartition $\{H, \langle\overline{a}^p, b\rangle, \langle e\rangle\}$, $\{\langle\overline{a}^p\rangle, \langle b^{-1}\overline{a}b\rangle, \langle b^{-2}\overline{a}b^2\rangle\}$.

Case 4: Suppose both $p, q-$Sylow subgroups are not unique.

Then by [**18**, p.255-256], $(p, q) = (2, 3)$, that is $|G| = 24$. By [**4**, p.104], $G := \langle a, b \mid a^4 = b^3 = 1, (ab)^2 = 1\rangle$ is the only group of order 24, which has neither unique $2-$Sylow subgroup nor unique $3-$Sylow subgroup. But G has a noncyclic subgroup of order 8. Thus there is no group of order 24 which has all $2-$Sylow subgroup are cyclic and both $2-$Sylow subgroups and $3-$Sylow subgroups are not unique.

The proof follows by combining all the above cases. □

Proposition 3.7. *If G is a group of order p^2q^2, where p, q are distinct primes, then $\mathcal{CI}(G)$ is nonplanar.*

Proof. We divide the proof into several cases.

Case 1: Let G be abelian. Suppose $G \cong \mathbb{Z}_{p^2q^2}$, then $\mathcal{CI}(G)$ is given in Figure 3(j), which is a subdivision of $K_{3,3}$ and so nonplanar. Suppose G is not cyclic, then G contains a proper subgroup H which contains either $\mathbb{Z}_p \times \mathbb{Z}_p$ or $\mathbb{Z}_q \times \mathbb{Z}_q$. So by Lemma 3.2, $\mathcal{CI}(G)$ is nonplanar.

Case 2: If G is nonabelian and $(p, q) \neq (2, 3)$, then $G \cong P \rtimes Q$, where P is a $p-$Sylow subgroup and Q is a $q-$Sylow subgroup of G.

Subcase 2a: $P \cong \mathbb{Z}_{p^2}$ and $Q \cong \mathbb{Z}_{q^2}$. Then $\mathcal{CI}(G)$ contains $K_{3,3}$ as subgraph with the bipartition $\{\{e\}, \langle b^q\rangle, \langle a^p\rangle\}$ and $\{\langle a^p, b\rangle, \langle a, b^q\rangle, \langle a^p, b^q\rangle\}$ and hence $\mathcal{CI}(G)$ is nonplanar.

Subcase 2b: $P \cong \mathbb{Z}_{p^2}$ and $Q \cong \mathbb{Z}_q \times \mathbb{Z}_q$. Since P is normal and is cyclic, the subgroup of P of order p is normal, let it be A. Thus AQ is a subgroup of order pq^2, which contains a subgroup $\mathbb{Z}_q \times \mathbb{Z}_q$. So by Lemma 3.2, $\mathcal{CI}(G)$ is nonplanar.

Subcase 2c: $P \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and $Q \cong \mathbb{Z}_{q^2}$ or $\mathbb{Z}_q \times \mathbb{Z}_q$. Here P is normal. Let Q' be a subgroup of order q of G. Then PQ' is a subgroup of order p^2q, which contains a subgroup $\mathbb{Z}_p \times \mathbb{Z}_p$. Then by Lemma 3.2, $\mathcal{CI}(G)$ is nonplanar.

Case 3: If $n = (2, 3)$, that is, G is a group of order 36, then there are 10 non-abelain groups of order 36, which we consider in the following cases:

(i) $G \cong D_{18}$. Then by Lemma 3.3, $\mathcal{CI}(G)$ is nonplanar.

(ii) $G \cong S_3 \times S_3$. Then $S_3 \times \langle(12)\rangle$ is a subgroup of G which contains a subgroup $\langle(12)\rangle \times \langle(12)\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. So by Lemma 3.2, $\mathcal{CI}(G)$ is nonplanar.

(iii) $G \cong \mathbb{Z}_3 \times A_4$. Then $\mathcal{CI}(G)$ contains $K_{3,3}$ with the bipartition $\{\{e\} \times A_4, \mathbb{Z}_3 \times \langle(12)(34), (13)(24)\rangle, \{e\} \times \langle(12)(34), (13)(24)\rangle\}$, $\{\{e\} \times \langle(12)(34)\rangle, \{e\} \times \langle(14)(23)\rangle, \{e\} \times \{e\}\}$ and hence $\mathcal{CI}(G)$ is nonplanar.

(iv) $G \cong \mathbb{Z}_6 \times S_3$. Then $Z_6 \times \langle(12)\rangle$ is a subgroup of G which contains a subgroup $\langle 3\rangle \times \langle(12)\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. So by Lemma 3.2, $\mathcal{CI}(G)$ is nonplanar.

(v) $G \cong \mathbb{Z}_9 \rtimes \mathbb{Z}_4 = \langle a, b \mid a^9 = b^4 = 1, b^{-1}ab = a^i, i^4 \equiv 1(mod\ 9)\rangle$. Then $\mathcal{CI}(G)$ contains $K_{3,3}$ with the bipartition $\{\langle a, b^2\rangle, \langle a^3, b\rangle, \langle a^3, b^2\rangle\}$, $\{\langle a^3\rangle, \langle b^2\rangle, \{e\}\}$ and hence $\mathcal{CI}(G)$ is nonplanar.

(vi) $G \cong \mathbb{Z}_3 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$. Then $\mathcal{CI}(G)$ contains $K_{3,3}$ as subgraph with the bipartition $\{\mathbb{Z}_3 \rtimes \mathbb{Z}_4, \langle(0,2)\rangle, \langle(0,0)\rangle\}$, $\{\langle(0,1)\rangle, \langle(1,1)\rangle, \langle(2,1)\rangle\}$ and hence $\mathcal{CI}(G)$ is nonplanar.

(vii) $G \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$. In this case, we have two nonisomorphic groups. We can show that $\mathcal{CI}(G)$ is nonplanar for both groups by using the similar arguement in Subcase(2c).

(viii) $G \cong (\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$. Then G contains a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ whose $\mathcal{CI}$ graph is nonplanar and hence $\mathcal{CI}(G)$ is also nonplanar.

(ix) $G \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_9$. This case is also similar to the Subcase(2c), and so $\mathcal{CI}(G)$ is nonplanar.

Proof follows by combining all the above cases together. □

Proposition 3.8. *Let G be a group of order pqr, where p, q, r are distinct primes. Then $\mathcal{CI}(G)$ is planar if and only if $G \cong \mathbb{Z}_{pqr}$.*

Proof. We use the classification of groups of order pqr given in [**5**, p.215]. Assume that $p > q > r$.

Case 1: $G_1 := \mathbb{Z}_{pqr}$. Then $\mathcal{CI}(G_1)$ is given in Figure 2(m), which is planar.

Case 2: $G_2 := \langle a, b, c \mid a^p = b^q = c^r = 1, ab = ba, ac = ca, c^{-1}bc = b^\mu\rangle$, where r divides $p - 1$ and $\mu \neq 1$. Then $\mathcal{CI}(G_2)$ contains a subgraph given in Figure 3(f), which is a subdivision of $K_{3,3}$ and so $\mathcal{CI}(G_2)$ is nonplanar.

Case 3: $G_3 := \langle a, b, c \mid a^p = b^q = c^r = 1, ab = ba, bc = cb, c^{-1}ac = a^\mu\rangle$, where r divides $q - 1$ and $\mu \neq 1$. This group is obtained by interchanging a and b in the group G_2 mentioned in the case 2, and hence $\mathcal{CI}(G_3)$ is also nonplanar.

Case 4: $G_4 := \langle a, b, c \mid a^p = b^q = c^r = 1, ab = ba, ab = ba^\mu, bc = cb^\nu\rangle$, where r divides both $p - 1$ and $q - 1$ and $\nu, \mu \neq 1$. Then $\mathcal{CI}(G_4)$ contains a subgraph given in Figure 3(g), which is a subdivision of $K_{3,3}$ and so $\mathcal{CI}(H_4)$ is nonplanar.

Case 5: $G_5 := \langle a, b, c \mid a^p = b^q = c^r = 1, ac = ca, bc = cb, b^{-1}ab = a^\mu\rangle$. This group is obtained by replacing a, b, c by c, a, b respectively, in the group G_2 mentioned in Case 2. Hence $\mathcal{CI}(G_5)$ is also nonplanar.

Case 6: $G_6 := \langle a, b, c \mid a^p = b^q = c^r = 1, ab = ba^\mu, bc = cb\rangle$. Then $\mathcal{CI}(G_6)$ contains a subgraph given in Figure 2(h), which is a subdivision of $K_{3,3}$ and so $\mathcal{CI}(G_6)$ is nonplanar.

The proof follows by combining all the above cases. □

Proposition 3.9. *If G is a solvable group of order p^2qr, where p, q, r are distinct primes. Then $\mathcal{CI}(G)$ is nonplanar.*

Proof. Here G has a normal subgroup of order p^2q. Suppose there is a subgroup of order p^2q contains a subgroup $P \cong \mathbb{Z}_p \times \mathbb{Z}_p$, then by Lemma 3.2, $\mathcal{CI}(G)$ is nonplanar.

Now assume that all subgroups of order p^2q contains only cyclic subgroups of order p^2. Then these subgroups are isomorphic to one of G_1, G_3, G_8, G_9, mentioned in the proof of Proposition 3.5.

Case 1: Suppose there is a subgroup H of G which is isomorphic to one of G_3, G_8, G_9.

If $H \cong G_3$, then $\mathcal{CI}(G)$ contains $K_{3,3}$ as a subgraph with bipartition $\{H, \langle a^p\rangle, \{e\}\}$ and $\{\langle a^p, b\rangle, \langle a^p, ba\rangle, \langle a^p, ba^2\rangle\}$. If $H \cong G_8$, then $\mathcal{CI}(G)$ contains $K_{3,3}$ as a subgraph with bipartition $\{H, \langle a^p\rangle, \{e\}\}$ and $\{\langle a\rangle, \langle b^{-1}ab\rangle, \langle b^{-2}ab^2\rangle\}$. If $H \cong G_9$, then $\mathcal{CI}(G)$ contains $K_{3,3}$ as a subgraph with bipartition $\{H, \langle a^p, b\rangle, \{e\}\}$ and $\{\langle a^p\rangle, \langle b^{-1}a^pb\rangle, \langle b^{-2}a^pb^2\rangle\}$. So in this case $\mathcal{CI}(G)$ is nonplanar.

Case 2: Suppose that all subgroups of order p^2q in G are cyclic, that is isomorphic to G_1.

Then H is a normal subgroup of G such that $H \cong \mathbb{Z}_{p^2q}$. So the subgroup of order p^2 in H is normal in G. Thus $p-$Sylow subgroup is unique cyclic group in G and so all the Sylow subgroups of G are cyclic.

Suppose G is a nilpotent group, then $G \cong \mathbb{Z}_{p^2qr}$. Thus $\mathcal{CI}(G)$ is given in Figure 3(i), which is nonplanar. Suppose that G is non nilpotent. If $P = \langle x\rangle \cong \mathbb{Z}_{p^2}$ and $Q = \langle y\rangle$ and $R = \langle z\rangle$. Then by [**15**, Theorem 10.1.10], G is supersolvable and by [**3**, Theorem 2], G is a CLT group, it has a subgroup of K of order pqr, where $K = \langle z\rangle \rtimes \langle y^{g_1}, x^{pg_2}\rangle$, for some $g_1, g_2 \in G$. Thus K is a noncyclic subgroup of order pqr. So by Proposition 3.8, $\mathcal{CI}(H)$ is nonplanar and hence $\mathcal{CI}(G)$ is nonplanar.

The proof follows by combining all the above cases. □

Proposition 3.10. *If G is a finite nonsolvable group, then $\mathcal{CI}(G)$ is nonplanar.*

Proof. It is well known that any non-solvable group has a simple group as a sub-quotient and every simple group has a minimal simple group as a sub-quotient. So by Corollary 3.3, it is enough to show that $\mathcal{CI}$ graph of every minimal simple group is nonplanar. We use the J. G. Thompson's classification of minimal simple groups given in [**17**] and check this condition for this list of groups.

Case 1: $G \cong L_2(q^p)$.

If $p = 2$, then the only nonsolvable group is $L_2(4)$ and $L_2(4) \cong A_5$. Then G contains a subgroup $H_1 \cong A_4$. Consequently, $\mathcal{CI}(G)$ contains $K_{3,3}$ as subgraph with the bipartition $\{H_1, \langle(12)(34), (14)(23)\rangle, \langle(1)\rangle\}$, $\{\langle(12)(34)\rangle, \langle(14)(23)\rangle, \langle(13)(24)\rangle\}$ and hence $\mathcal{CI}(G)$ is nonplanar.

If $p > 2$, then $L_2(q^p)$ contains a subgroup isomorphic to $\mathbb{Z}_p^q$ whose $\mathcal{CI}$ graph is nonplanar and hence $\mathcal{CI}(G)$ is nonplanar.

Case 2: $G \cong L_3(3)$. Then G contains a subgroup isomorphic to $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$. So by Lemma 3.2, $\mathcal{CI}(G)$ is nonplanar.

Case 3: $G \cong L_2(p)$. Then G contains a subgroup D_{p-1} or D_{p+1} according as $p \equiv 1(\text{mod } 4)$ or $p \equiv 3(\text{mod } 4)$. Suppose $p \geqslant 7$, $p-1$ and $p+1$ are even and $p-1, p+1 \geqslant 6$. Then by Lemma 3.3, $\mathcal{CI}(G)$ is nonplanar. Suppose $p = 3$ or 5, then G contains a subgroup $H_1 \cong D_4$. Now consider the subgroups of H_1, $H_2 := \langle a^2 \rangle$, $H_3 := \langle a^2, b \rangle$, $H_4 := \langle a^2, ab \rangle$, $H_5 := \langle a \rangle$, $H_6 := \{e\}$. Then $\mathcal{CI}(G)$ contains $K_{3,3}$ as a subgraph with bipartition $\{H_1, H_2, H_6\}$, and $\{H_3, H_4, H_5\}$.

Case 4: $G \cong Sz(2^q)$. Then G has a subgroup isomorphic to $\mathbb{Z}_2^q$, $q \geqslant 3$. Then by Lemma 3.2, $\mathcal{CI}(G)$ is nonplanar. □

PROOF OF THEOREM 3.1.

Combining Propositions 3.1–3.10 proved so far in this section, we obtain the proof. □

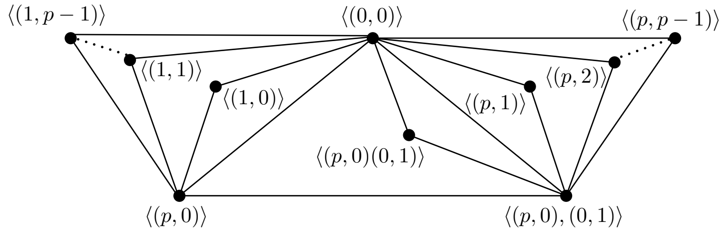
PROOF OF COROLLARY 3.1.

(1) $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are obvious. Now we prove $(iii) \Rightarrow (i)$. Consider a finite group G such that $\mathcal{CI}(G)$ is a star. Since any star is a planar graph, G must be one of the groups given in Theorem 3.1. Among the groups listed in Theorem 3.1, $\mathbb{Z}_p, \mathbb{Z}_{p^2}, \mathbb{Z}_p \times \mathbb{Z}_p$ or $\mathbb{Z}_{pq}$ are the only groups whose CI graph is a star.

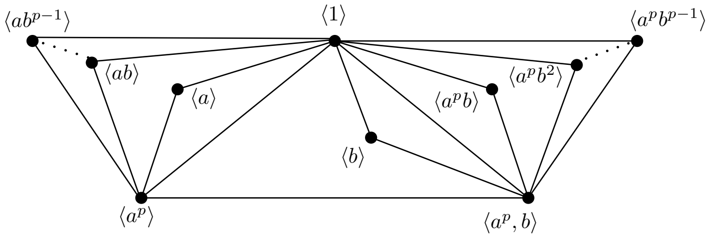
(2) Since any path is a planar graph, G must be a group listed in Theorem 3.1. Among all these groups, $\mathbb{Z}_p$, $\mathbb{Z}_{p^2}$, $\mathbb{Z}_{pq}$ are the only groups whose CI graphs is a path. □

PROOF OF COROLLARY 3.2. If G' is a group such that $\mathcal{CI}(G') \cong \mathcal{CI}(G)$, where G is one of the groups listed in Theorem 3.1, then G' must be finite and $\mathcal{CI}(G')$ is also planar.

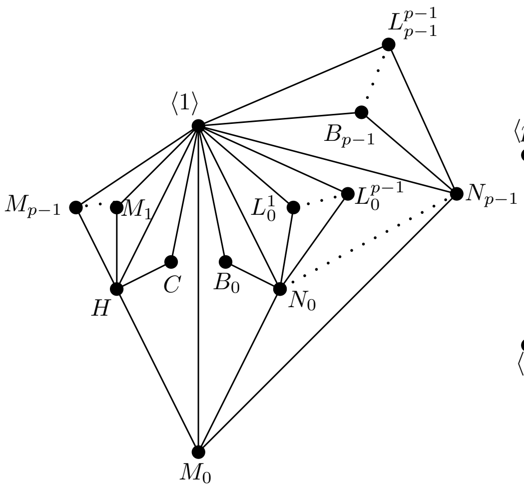
Now we prove part (i) and the proofs of the remaining parts are similar to this. If G is one of the groups listed in (i), then by Figures 2(c), 2(i), 2(j), 2(k), 2(l), we see that $\mathcal{CI}(G)$ is unique for each of these groups. So it follows that $G' \cong G$. □



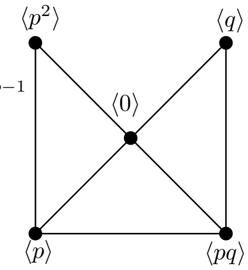
(a)


$\langle ab^{p-1}\rangle$
$\langle 1\rangle$
$\langle a^pb^{p-1}\rangle$
$\langle ab\rangle$
$\langle a\rangle$
$\langle a^pb\rangle$
$\langle a^pb^2\rangle$
$\langle b\rangle$
$\langle a^p\rangle$
$\langle a^p, b\rangle$

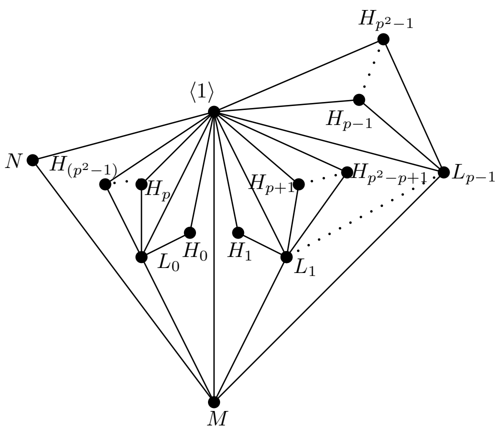
(b)


L_{p-1}^{p-1}
$\langle 1\rangle$
B_{p-1}
M_{p-1}
M_1
L_0^1
L_0^{p-1}
N_{p-1}
C
B_0
H
N_0
M_0

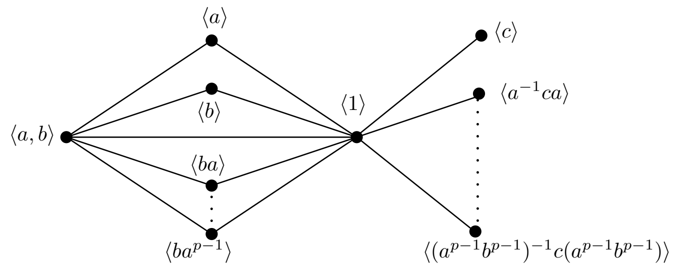
(c)


$\langle p^2\rangle$
$\langle q\rangle$
$\langle 0\rangle$
$\langle p\rangle$
$\langle pq\rangle$

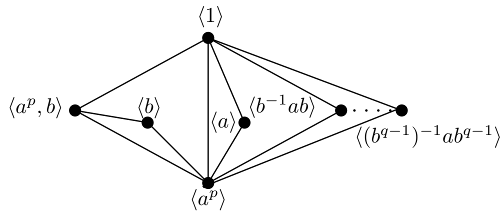
(d)


H_{p^2-1}
$\langle 1\rangle$
H_{p-1}
N
$H_{(p^2-1)}$
H_p
H_{p+1}
H_{p^2-p+1}
L_{p-1}
H_0
H_1
L_0
L_1
M

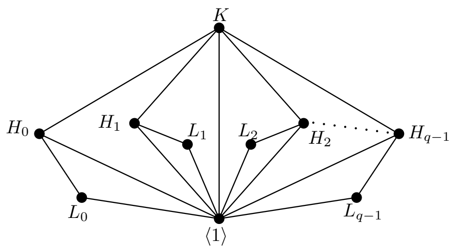
(e)


$\langle a\rangle$
$\langle b\rangle$
$\langle a,b\rangle$
$\langle 1\rangle$
$\langle ba\rangle$
$\langle ba^{p-1}\rangle$
$\langle c\rangle$
$\langle a^{-1}ca\rangle$
$\langle (a^{p-1}b^{p-1})^{-1}c(a^{p-1}b^{p-1})\rangle$

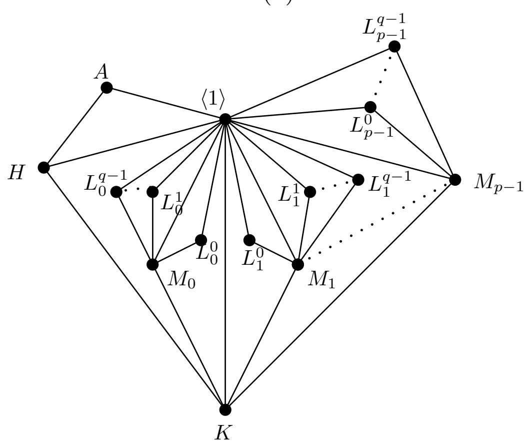
(f)


$\langle 1\rangle$
$\langle a^p,b\rangle$
$\langle b\rangle$
$\langle a\rangle$
$\langle b^{-1}ab\rangle$
$\langle (b^{q-1})^{-1}ab^{q-1}\rangle$
$\langle a^p\rangle$

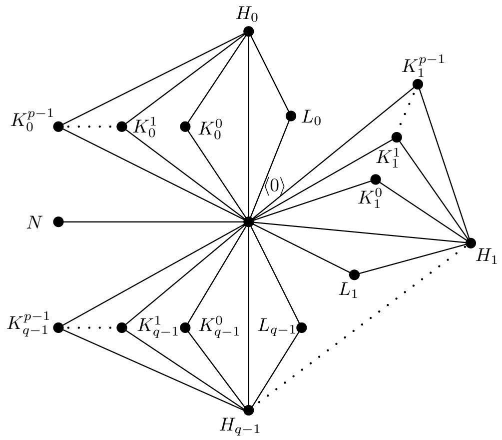
(g)


K
H_0
H_1
L_1
L_2
H_2
H_{q-1}
L_0
L_{q-1}
$\langle 1\rangle$

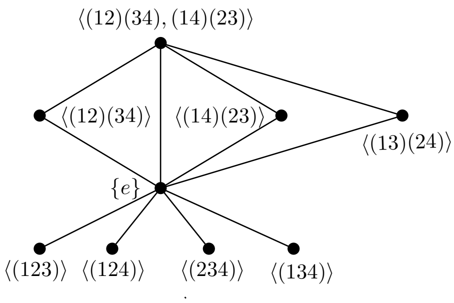
(h)


L_{p-1}^{q-1}
A
$\langle 1\rangle$
L_{p-1}^{0}
H
L_0^{q-1}
L_0^1
L_1^1
L_1^{q-1}
M_{p-1}
L_0^0
L_1^0
M_0
M_1
K

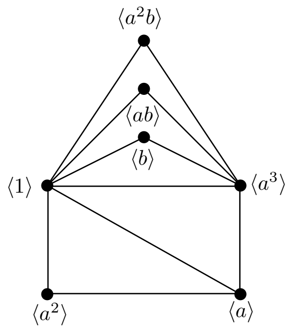
(i)


H_0
K_1^{p-1}
K_0^{p-1}
K_0^1
K_0^0
L_0
K_1^1
$\langle 0 \rangle$
K_1^0
N
H_1
L_1
K_{q-1}^{p-1}
K_{q-1}^1
K_{q-1}^0
L_{q-1}
H_{q-1}

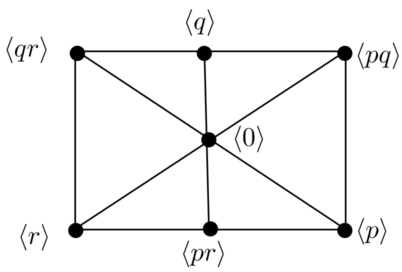
(j)


$\langle (12)(34), (14)(23) \rangle$
$\langle (12)(34) \rangle$
$\langle (14)(23) \rangle$
$\langle (13)(24) \rangle$
$\{e\}$
$\langle (123) \rangle$
$\langle (124) \rangle$
$\langle (234) \rangle$
$\langle (134) \rangle$

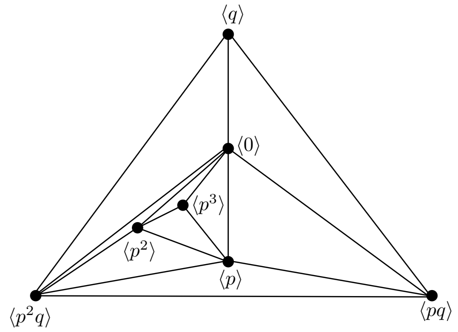
(k)


$\langle a^2b \rangle$
$\langle ab \rangle$
$\langle b \rangle$
$\langle 1 \rangle$
$\langle a^3 \rangle$
$\langle a^2 \rangle$
$\langle a \rangle$

(l)

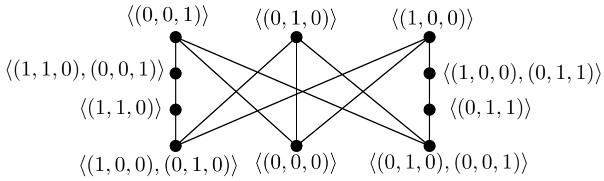

$\langle q \rangle$
$\langle qr \rangle$
$\langle pq \rangle$
$\langle 0 \rangle$
$\langle r \rangle$
$\langle p \rangle$
$\langle pr \rangle$

(m)

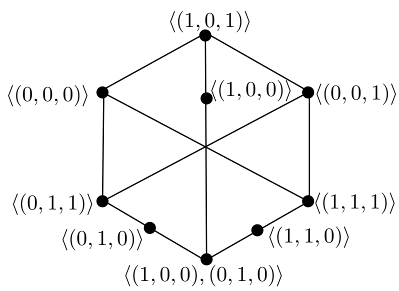


(n)

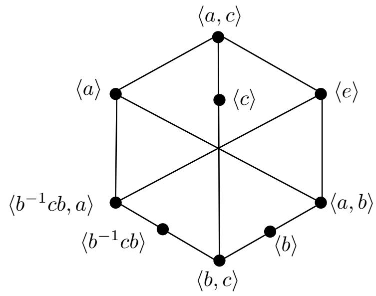
FIGURE 2. (a) $\mathcal{CI}(\mathbb{Z}_{p^2}\times\mathbb{Z}_p)$, (b) $\mathcal{CI}(M_{p^3})$, (c) $\mathcal{CI}((\mathbb{Z}_p\times\mathbb{Z}_p)\rtimes\mathbb{Z}_p)$, (d) $\mathcal{CI}(\mathbb{Z}_{p^2q})$, (e) $\mathcal{CI}(G_3)$, (f) $\mathcal{CI}(G_7)$, (g) $\mathcal{CI}(G_8)$, (h) $\mathcal{CI}(G_9)$, (i) $\mathcal{CI}(G_{10})$, (j) $\mathcal{CI}(G_{11})$, (k) $\mathcal{CI}(A_4)$, (l) $\mathcal{CI}(Q_3)$, (m) $\mathcal{CI}(\mathbb{Z}_{pqr})$, (n) $\mathcal{CI}(\mathbb{Z}_{p^3q})$.



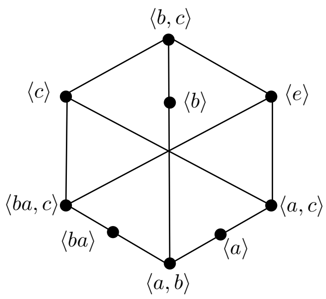
(a)



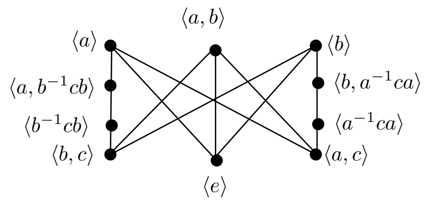
(b)



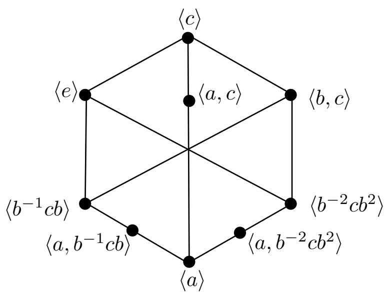
(c)


$\langle b, c\rangle$
$\langle c\rangle$
$\langle b\rangle$
$\langle e\rangle$
$\langle ba, c\rangle$
$\langle a, c\rangle$
$\langle ba\rangle$
$\langle a\rangle$
$\langle a, b\rangle$

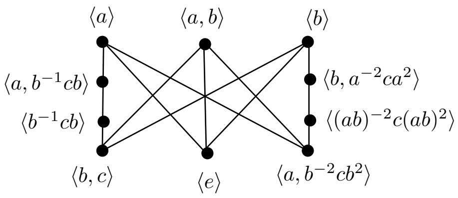
(d)


$\langle a\rangle$
$\langle a, b\rangle$
$\langle b\rangle$
$\langle a, b^{-1}cb\rangle$
$\langle b, a^{-1}ca\rangle$
$\langle b^{-1}cb\rangle$
$\langle a^{-1}ca\rangle$
$\langle b, c\rangle$
$\langle a, c\rangle$
$\langle e\rangle$

(e)


$\langle c\rangle$
$\langle e\rangle$
$\langle a, c\rangle$
$\langle b, c\rangle$
$\langle b^{-1}cb\rangle$
$\langle b^{-2}cb^2\rangle$
$\langle a, b^{-1}cb\rangle$
$\langle a, b^{-2}cb^2\rangle$
$\langle a\rangle$

(f)


$\langle a\rangle$
$\langle a, b\rangle$
$\langle b\rangle$
$\langle a, b^{-1}cb\rangle$
$\langle b, a^{-2}ca^2\rangle$
$\langle b^{-1}cb\rangle$
$\langle (ab)^{-2}c(ab)^2\rangle$
$\langle b, c\rangle$
$\langle e\rangle$
$\langle a, b^{-2}cb^2\rangle$

(g)

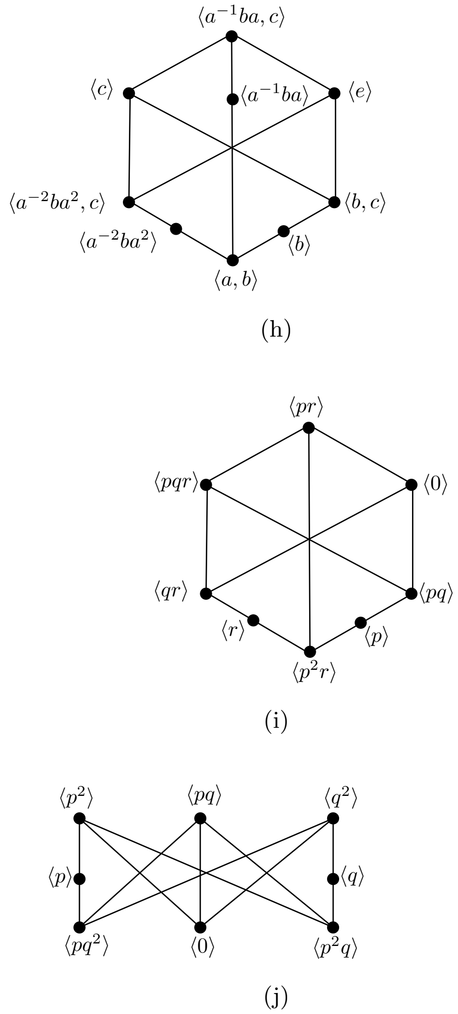


FIGURE 3. (a) A subgraph of $\mathcal{CI}(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$, (b) A subgraph of $\mathcal{CI}(G_2)$, (c) A subgraph of $\mathcal{CI}(G_4)$, (d) A subgraph of $\mathcal{CI}(G_5)$, (e) A subgraph of $\mathcal{CI}(G_6)$, (f) $\mathcal{CI}(H_2)$, (g) A subgraph of $\mathcal{CI}(\mathbb{Z}_{p^2qr})$, (h) $\mathcal{CI}(\mathbb{Z}_{p^2q^2})$, (i) $\mathcal{CI}(\mathbb{Z}_{p^2qr})$, (j) $\mathcal{CI}(\mathbb{Z}_{p^2q^2})$.

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