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HOSOYA POLYNOMIAL AND WIENER INDEX OF GENERALIZED xyz-POINT-LINE TRANSFORMATION GRAPHS $T^{xyz}(G)$

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ABSTRACT. The Wiener index $W(G)$ of a graph G is the sum of distances between all (unordered) pairs of vertices of G. In this paper, we obtain the Hosoya polynomial and Wiener index of generalized *xyz-point-line transformation* graphs $T^{xyz}(G)$ in terms of order and size of respective transformation graphs. Further, we study few transformation graphs in detail and obtain bounds for them in terms of order and size of the underline graphs.

1. Introduction

In this paper, we are concerned with simple, connected, nontrivial and undirected finite graph with n vertices and m edges. Let G be such a graph with the vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and the edge set $E(G) = \{e_1, e_2, ..., e_m\}$. The distance between two vertices v_i and v_j is denoted by $d_G(v_i, v_j)$ is the length of the shortest path between the vertices v_i and v_j in G. The shortest v_i - v_j path

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is called geodesic. The diameter of G, denoted by diam(G), is the length of any longest geodesic. The degree of a vertex v_i in G is the number of edges incident with v_i and is denoted by $d_G(v_i) = deg(v_i)$.

The complement $\overline{G}$ of a graph G is a graph whose vertex set is $V(G)$ in which two vertices are adjacent if and only if they are not adjacent in G. Therefore $\overline{G}$ has n vertices and $\frac{n(n-1)}{2} - m$ edges. The line graph $L(G)$ of a graph G is the graph with vertex set as the edge set of G and two vertices of $L(G)$ are adjacent whenever the corresponding edges in G have a vertex in common. The subdivision graph $S(G)$ of a graph G whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if one is a vertex of G and other is an edge of G incident with it. The partial complement of subdivision graph $\overline{S}(G)$ of a graph G whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if one is a vertex of G and the other is an edge of G not incident with it.

In this paper, we denote P_n, C_n, K_n, $K_{a,b}$, S_n, $S_{a,b}$, W_n, T_n and for a path, a cycle, a complete graph, a complete bipartite graph, a star, a bistar, a wheel and a tree respectively. For undefined terms and notations refer [**15, 18**].

A topological index is a numerical parameter mathematically derived from the graph structure. It is a graph invariant, thus it does not depend on the labeling or pictorial representation of the graph. The topological indices play an important role in chemical graph theory.

One of the oldest and most thoroughly studied distance based topological index is Wiener index [**25**] and it has numerous chemical applications which was introduced by an American physical chemist H. Wiener in 1947.

The Wiener index (or Wiener number) [**25**] of a graph G, denoted by $W(G)$ is the sum of distances between all (unordered) pairs of vertices of G.

$$W(G) = \sum_{i<j} d_G(v_i, v_j).$$

The Wiener index of a graph belongs to the molecular structure-descriptors called topological indices, which are used for the design of molecules with desired properties [**19**]. Its mathematical properties are well established.

The Wiener polarity index [**25**] of a graph G, denoted by $W_p(G)$, is equal to the number of unordered pairs of vertices of distance three in G.

$$W_p(G) = |\{(u, v)/d_G(u, v) = 3\}|.$$

In [**25**], Wiener used a linear formula involving $W(G)$ and $W_p(G)$ to obtain the boiling points t_B of the paraffins, that is

$$t_B = aW(G) + bW_p(G) + c$$

where a, b and c are constants for a given isomeric group.

In the year 1988, Hosoya [**16**] introduced a new distance based graph polynomial called Wiener polynomial. For more details refer [**1, 9, 12, 13, 17, 23**]. The Wiener polynomial of a connected graph G is denoted by $W(G; q)$ and is defined by,

$$W(G;q) = \sum_{i<j} q^{d_G(v_i,v_j)}$$

where q is a parameter. Nowadays, the majority of researchers uses the name Hosoya polynomial insted of Wiener polynomial.

The relation between Wiener polynomial and Wiener index is,

$$W(G) = \frac{d}{dq}(W(G;q))\Big|_{q=1}. \tag{1.1}$$

Hence, we can derive the expression for the Wiener index of G from that of the Hosoya polynomial of G. We denote the number of unordered pairs of vertices of distance four in G by $W_F(G)$.

2. Generalized xyz-point-line transformation graphs $T^{xyz}(G)$

Obtaining a new graph from the given graph by using incidence and adjacency relationship between the elements of a graph G is known as graph transformation. Let $\mathcal{G}$ denote the set of simple graphs. Various important results in graph theory have been obtained by considering some function $\mathcal{F} : \mathcal{G} \to \mathcal{G}$ by establishing these operations on certain graph parameters. Perhaps one of the first important transformation graph of a graph is its complement[**15**]. The line graph [**26**], subdivision graph [**15**] middle graph [**14**], jump graph [**10**], semitotal-point graph[**21**] and the total graph [**8**] are the graphs which received most attention in the literature. In [**7**], Wu and Meng generalized the concept of total graph and were termed total transformation graphs. In [**5**] Basavanagoud et al. generalized the concept of semitotal point graph. Deng et al. [**11**] introduced graph transformations $T^{(xyz)}$:$\mathcal{G} \to \mathcal{G}$ depending on the parameters $x, y, z \in \{0, 1, +, -\}$ which induce 64 *xyz-transformation graphs.* In [**2**], Basavanagoud studied the basic properties of the *xyz-transformation graphs* by calling them *xyz-point-line transformation graphs* by changing the notion of *xyz-transformations* of a graph G as $T^{xyz}(G)$ to avoid confusion between different transformation graphs.

Since there are 64 distinct 3 - permutations of $\{0, 1, +, -\}$. Thus they obtained 64 kinds of generalized *xyz-point-line transformation graphs.* The vertex v_i' of $T^{xyz}(G)$ corresponding to a vertex v_i of G is referred to as *point-vertex* and vertex e_i' of $T^{xyz}(G)$ corresponding to an edge e_i of G is referred to as *line-vertex.* In Figures 1, 2, 3 and 4 self-explanatory examples of $T^{xyz}(G)$ graphs are depicted, dark circles represents the *point-vertices* and light circles represents the *line-vertices* of $T^{xyz}(G)$.

Definition 2.1. ([**11**]) Given a graph G with vertex set $V(G)$ and edge set $E(G)$ and three variables $x, y, z \in \{0, 1, +, -\}$, the *xyz-point-line transformation graphs* $T^{xyz}(G)$ of G is the graph with vertex set $V(T^{xyz}(G)) = V(G) \cup E(G)$ and the edge set $E(T^{xyz}(G)) = E((G)^x) \cup E((L(G))^y) \cup E(W)$ where $W = S(G)$ if $z = +$, $W = \overline{S}(G)$ if $z = -$, W is the graph with $V(W) = V(G) \cup E(G)$ and with no edges if $z = 0$ and W is the complete bipartite graph with parts $V(G)$ and $E(G)$ if $z = 1$.

In the literature, the *xyz-point-line transformation graphs* $T^{00+}(G)$, $T^{0++}(G)$, $T^{+++}(G)$ and $T^{-++}(G)$ are known as Subdivision graph [**15**], Middle graph [**14**], Total graph [**8**] and Quasi-total graph [**22**] respectively. The Wiener index of these graphs can be found in [**3**], [**4**] and [**6**].

Before proceeding to our results, we wish to make a note that *xyz-point-line transformation graphs* with $z = 0$ yields all disconnected graphs. Hence one can not determine the Hosoya polynomial and Wiener index of these graphs as connectedness is essential. Therefore we start results with *xyz-point-line transformation graphs* with $z = +$.

The following theorems are useful for proving our results.

THEOREM 2.1 ([**24**]). *Let G be a graph of order n and size m. Then $W(G) = n^2 - n - m$ if and only if $diam(G) \leqslant 2$.*

THEOREM 2.2 ([**20**]). *The Hosoya polynomial satisfies the following conditions:*

(i) *$deg(W(G;q))$ equals the diameter of G.*
(ii) *$[q^o]W(G;q)$ =0.*
(iii) *$[q^1]W(G;q) = |E(G)|$, where $E(G)$ is an edge set of G.*
(iv) *$W(G;1) = \binom{|V(G)|}{2}$, where $V(G)$ is the vertex set of G.*
(v) *$W'(G;1) = W(G)$.*

3. Results on xyz-point-line transformation graphs $T^{xyz}(G)$ with $z = +$

In this section, we present the results on the graph $T^{+1+}(G)$ as an example to obtain Hosoya polynomial and Wiener index of other transformation graphs. The results obtained for the graph $T^{+1+}(G)$ can also be obtained for other transformation graphs in the same techniques.

DEFINITION 3.1. The *xyz-point-line transformation graph* $T^{+1+}(G)$ of a graph G is a graph whose vertex set is $V(G) \cup E(G)$ and any two vertices in $T^{+1+}(G)$ are adjacent if and only if they correspond to two adjacent vertices of G or two adjacent or two not adjacent edges of G or to a vertex and an edge incident with it in G. Hence, the order and size of $T^{+1+}(G)$ are $n+m$ and $\frac{1}{2}m(m+5)$ respectively.

THEOREM 3.1 ([**2**]). *For any graph G, $T^{+1+}(G)$ is connected if and only if G contains no isolated vertex.*

THEOREM 3.2. *For any graph G, $diam(T^{+1+}(G)) \leqslant 3$ with equality holds if and only if $diam(G) > 2$.*

PROOF. For $u, v \in V(G)$, if u and v are adjacent in G, then they are adjacent in $T^{+1+}(G)$. Suppose u and v are not adjacent in G. If there exists a vertex w which is adjacent to both u and v in G, then u', w', v' is a path of length two in $T^{+1+}(G)$. Otherwise, there exists a pair of edges e_1 and e_2 incident with u and v respectively in G such that u', e_1', e_2', v' is a path of length three in $T^{+1+}(G)$.

For $e_1, e_2 \in E(G)$, since, K_m is an induced subgraph of $T^{+1+}(G)$ with vertex set $E(G)$. Therefore all the edge-points are adjacent to each other in $T^{+1+}(G)$.

For $u \in V(G)$ and $e \in E(G)$, if u and e are incident in G, then they are adjacent in $T^{+1+}(G)$. Otherwise, there exists an edge e_1 which is incident with u in G such that u', e_1', e' is a path of length two in $T^{+1+}(G)$.

Now suppose that $diam(T^{+1+}(G)) = 3$. Then there exists a pair of vertices $u', v' \in V(T^{+1+}(G))$ such that $d_{T^{+1+}(G)}(u', v') = 3$. By the above argument, we see that $u, v \in V(G)$, which are not adjacent and are not adjacent to a common vertex in G. Therefore $d_G(u, v) > 2$. Hence, $diam(G) > 2$.

Conversely, if $diam(G) > 2$, then for the vertices u and v incident with e_1 and e_2 in G respectively such that u', e_1', e_2, v' is a path of length three in $T^{+1+}(G)$. Thus $diam(T^{+1+}(G)) = 3$ □

The Wiener polarity index of $T^{+1+}(G)$ of some standard classes of graphs are given in the following corollary which are immediate from the proof of the Theorem 3.2.

Corollary 3.1. *The following holds*
(i) $W_p(T^{+1+}(T_n)) = k - 1$, *where k is the number of pendant vertices.*
(ii) $W_p(T^{+1+}(P_n)) = \binom{n-2}{2}$.
(iii) $W_p(T^{+1+}(S_n)) = 0$.
(iv) $W_p(T^{+1+}(C_n)) = \begin{cases} \frac{n(n-5)}{2} & if\ n > 5, \\ 0 & otherwise. \end{cases}$
(v) $W_p(T^{+1+}(K_n)) = 0$.
(vi) $W_p(T^{+1+}(K_{a,b})) = 0$.
(vii) $W_p(T^{+1+}(W_n)) = 0$.

Theorem 3.3. *For any graph G of order n and size m,*

$$\begin{aligned} W(T^{+1+}(G); q) &= W_p(T^{+1+}(G))q^3 \\ &\quad + \left(\frac{1}{2}n(n + 2m - 1) - 3m - W_p(T^{+1+}(G)\right) q^2 \\ &\quad + \left(\frac{1}{2}m(m + 5)\right) q. \end{aligned}$$

and

$$W(T^{+1+}(G)) = W_p(T^{+1+}(G)) + n(n - 1) + \frac{1}{2}m(4n + m - 7).$$

Proof. By definition of Hosoya polynomial of a graph, we have

$$W(T^{+1+}(G); q) = \sum_{u,v \in V(T^{+1+}(G))} q^{d_{T^{+1+}(G)}(u,v)}$$

,

$$\begin{aligned} W(T^{+1+}(G); q) &= \sum_{u,v \in V(G)} q^{d_{T^{+1+}(G)}(u,v)} + \sum_{u,v \in E(G)} q^{d_{T^{+1+}(G)}(u,v)} \\ &\quad + \sum_{u \in V(G), v \in E(G)} q^{d_{T^{+1+}(G)}(u,v)} \end{aligned}$$

$$
\begin{aligned}
W(T^{+1+}(G);q) &= \sum_{u,v\in V(G),u\sim v} q^{d_{T^{+1+}(G)}(u,v)} + \sum_{u,v\in V(G),u\nsim v} q^{d_{T^{+1+}(G)}(u,v)} \\
&\quad + \sum_{u,v\in E(G)} q^{d_{T^{+1+}(G)}(u,v)} + \sum_{u\in V(G),v\in E(G),u\sim v} q^{d_{T^{+1+}(G)}(u,v)} \\
&\quad + \sum_{u\in V(G),v\in E(G),u\nsim v} q^{d_{T^{+1+}(G)}(u,v)} \\
&= \sum_{u,v\in V(G),u\sim v} q^{d_{T^{+1+}(G)}(u,v)} \\
&\quad + \sum_{u,v\in V(G),u\nsim v,d_G(u,v)\geqslant 3} q^{d_{T^{+1+}(G)}(u,v)} \\
&\quad + \sum_{u,v\in V(G),u\nsim v,d_G(u,v)<3} q^{d_{T^{+1+}(G)}(u,v)} \\
&\quad + \sum_{u,v\in E(G)} q^{d_{T^{+1+}(G)}(u,v)} \\
&\quad + \sum_{u\in V(G),v\in E(G),u\sim v} q^{d_{T^{+1+}(G)}(u,v)} \\
&\quad + \sum_{u\in V(G),v\in E(G),u\nsim v} q^{d_{T^{+1+}(G)}(u,v)}
\end{aligned}
$$

By proof of the Theorem 3.2, we have

$$
\begin{aligned}
W(T^{+1+}(G);q) &= mq + W_p(T^{+1+}(G))q^3 + \left(\binom{n}{2} - m - W_p(T^{+1+}(G))\right) q^2 \\
&\quad + \binom{m}{2} q + 2mq + (m(n-2))q^2 \\
&= W_p(T^{+1+}(G))q^3 \\
&\quad + \left(\frac{1}{2}n(n+2m-1) - 3m - W_p(T^{+1+}(G))\right) q^2 \\
&\quad + \frac{1}{2}m(m+5).
\end{aligned}
$$

From Eq. (1.1), the Wiener index of $T^{+1+}(G)$ is

$$
\begin{aligned}
W(T^{+1+}(G)) &= \frac{d}{dt}(W(T^{+1+}(G);q))\Big|_{q=1} \\
&= W_p(T^{+1+}(G)) + n(n-1) + \frac{1}{2}m(4n+m-7).
\end{aligned}
$$

□

Corollary 3.2. *For any tree T_n of order n other than star and path,*

$$
W(T^{+1+}(T_n);q) = (k-1)q^3 + \left(\frac{3}{2}(n-1)(n-2) - k + 1\right) q^2 + \frac{1}{2}(n-1)(n+4)q
$$

and

$$W(T^{+1+}(T_n)) = \frac{1}{2}(n-1)(7n-8) + k - 1.$$

PROOF. By Theorem 3.3, Corollary 3.1 (i) and a fact that $m = n-1$ for tree T_n the result follows. □

COROLLARY 3.3. *For any path P_n of order n,*

$$W(T^{+1+}(P_n);q) = \binom{n-2}{2}q^3 + \left(\frac{3}{2}(n-1)(n-2) - \binom{n-2}{2}\right)q^2 + \frac{1}{2}(n-1)(n+4)q$$

and

$$W(T^{+1+}(P_n)) = \frac{1}{2}(n-1)(7n-8) + \binom{n-2}{2}.$$

PROOF. By Theorem 3.3, Corollary 3.1 (ii) and a fact that $m = n-1$ for path P_n the result follows. □

COROLLARY 3.4. *For any star S_n of order n,*

$$W(T^{+1+}(S_n);q) = \frac{3}{2}(n-1)(n-2)q^2 + \frac{1}{2}(n-1)(n+4)q$$

and

$$W(T^{+1+}(S_n)) = \frac{1}{2}(n-1)(7n-8).$$

PROOF. By Theorem 3.3, Corollary 3.1 (iii) and a fact that $m = n-1$ for star S_n the result follows. □

COROLLARY 3.5. *For any cycle C_n of order n,*

$$W(T^{+1+}(C_n);q) = \begin{cases} \frac{1}{2}n(3n-7)q^2 + \frac{1}{2}n(n+5)q & \text{if } 3 \leqslant n \leqslant 5, \\ \frac{n(n-5)}{2}q^3 + n(n-1)q^2 + \frac{1}{2}n(n+5)q & \text{otherwise.} \end{cases}$$

and

$$W(T^{+1+}(C_n);q) = \begin{cases} \frac{1}{2}n(7n-9) & \text{if } 3 \leqslant n \leqslant 5, \\ n(4n-7) & \text{otherwise.} \end{cases}$$

PROOF. By Theorem 3.3, Corollary 3.1 (iv) and a fact that $m = n$ for cycle C_n the result follows. □

COROLLARY 3.6. *For any complete graph K_n of order n,*

$$W(T^{+1+}(K_n);q) = \frac{1}{2}n(n-1)(n-2)q^2 + \frac{1}{8}n(n-1)(n^2-n+10)q$$

and

$$W(T^{+1+}(K_n)) = \frac{1}{8}n(n-1)(n^2+7n-6).$$

PROOF. By Theorem 3.3, Corollary 3.1 (v) and a fact that $m = \frac{n(n-1)}{2}$ for complete graph K_n the result follows. □

Corollary 3.7. *For any complete bipartite graph $K_{a,b}$,*

$$W(T^{+1+}(K_{a,b});q) = \left(\frac{1}{2}(a+b)(a+b-1) + ab(a+b-3)\right)q^2 + \frac{1}{2}ab(ab+5)q$$

and

$$W(T^{+1+}(K_{a,b})) = (a+b)(a+b-1) + \frac{1}{2}ab(4a+4b+ab-7).$$

Proof. By Theorem 3.3, Corollary 3.1 (vi) and the facts $n = a+b, m = ab$ for complete bipartite graph $K_{a,b}$ the result follows. □

Corollary 3.8. *For any wheel W_n of order n,*

$$W(T^{+1+}(W_n);q) = \frac{1}{2}(n-1)(5n-12)q^2 + (n-1)(2n+3)q$$

and

$$W(T^{+1+}(W_n)) = (n-1)(7n-9).$$

Proof. By Theorem 3.3, Corollary 3.1 (vii) and a fact that $m = 2(n-1)$ for wheel W_n the result follows. □

Lemma 3.1. *For any connected graph G of order n,*

$$\frac{1}{2}(n-1)(7n-8) \leqslant W(T^{+1+}(G)) \leqslant \frac{1}{8}n(n-1)(n^2+7n-6).$$

Upper bound attains if G is a complete graph and lower bound attains if G is a star.

Proof. The generalized *xyz-point-line transformation graph* $T^{+1+}(G)$ has $n+m$ vertices and $\frac{1}{2}m(m+5)$ edges. Any graph G of order n has maximum number of edges if and only if $G \cong K_n$ and $T^{+1+}(G)$ has maximum number of vertices if and only if $G \cong K_n$, where G is a graph of order n.

We know that Wiener index of a graph G increases when new vertices are added to the graph and $T^{+1+}(K_n)$ has maximum number of vertices compared with any other $T^{+1+}(G)$, where G is a graph of order n. Therefore, $W(T^{+1+}(G)) \leqslant W(T^{+1+}(K_n))$. From Corollary 3.6, $W(T^{+1+}(K_n)) = \frac{1}{8}n(n-1)(n^2+7n-6)$. So,

$$W(T^{+1+}(G)) \leqslant \frac{1}{8}n(n-1)(n^2+7n-6) \tag{3.1}$$

with equality in Eq. (3.1) if and only if $G \cong K_n$.

Any graph G of order n has minimum number of edges if and only if $G \cong T_n$ and $T^{+1+}(S_n)$ has minimum number of vertices compared to any other $T^{+1+}(G)$, where G is a graph of order n. Therefore, $W(T^{+1+}(S_n)) \leqslant W(T^{+1+}(G))$. From Corollary 3.4, $W(T^{+1+}(S_n)) = \frac{1}{2}(n-1)(7n-8)$. Therefore,

$$\frac{1}{2}(n-1)(7n-8) \leqslant W(T^{+1+}(G)) \tag{3.2}$$

with equality in Eq. (3.2) if and only if $G \cong S_n$.
From Eqs. (3.1) and (3.2), we have

$$\frac{1}{2}(n-1)(7n-8) \leqslant W(T^{+1+}(G)) \leqslant \frac{1}{8}n(n-1)(n^2+7n-6).$$

Upper bound attains if G is a complete graph and lower bound attains if G is a star. □

The order and size of *xyz-point-line transformation graphs* [**2**] with $z = +$ are listed in Table 1.

TABLE 1

Transformation Graph	**Order**	**Size**
$T^{10+}(G)$	$n+m$	$\frac{1}{2}n(n-1)+2m$
$T^{+0+}(G)$	$n+m$	$3m$
$T^{-0+}(G)$	$n+m$	$\frac{1}{2}n(n-1)+m$
$T^{01+}(G)$	$n+m$	$\frac{1}{2}m(m+3)$
$T^{11+}(G)$	$n+m$	$\frac{1}{2}(n(n-1)+m(m+3))$
$T^{+1+}(G)$	$n+m$	$\frac{1}{2}m(m-1)+3m$
$T^{-1+}(G)$	$n+m$	$\frac{1}{2}(n(n-1)+m(m+1))$
$T^{0++}(G)$	$n+m$	$m+\frac{1}{2}\sum\limits_{i=1}^{n} d_G(v_i)^2$
$T^{1++}(G)$	$n+m$	$\frac{1}{2}(n(n-1)+\sum\limits_{i=1}^{n} d_G(v_i)^2)-m$
$T^{+++}(G)$	$n+m$	$2m+\frac{1}{2}\sum\limits_{i=1}^{n} d_G(v_i)^2$
$T^{-++}(G)$	$n+m$	$\frac{1}{2}(n(n-1)+\sum\limits_{i=1}^{n} d_G(v_i)^2)$
$T^{0-+}(G)$	$n+m$	$\frac{1}{2}(m(m+1)-\sum\limits_{i=1}^{n} d_G(v_i)^2)+2m$
$T^{1-+}(G)$	$n+m$	$\frac{1}{2}(n(n-1)+m(m+5)-\sum\limits_{i=1}^{n} d_G(v_i)^2$
$T^{+-+}(G)$	$n+m$	$\frac{1}{2}(m(m+1)-\sum\limits_{i=1}^{n} d_G(v_i)^2)+3m$
$T^{--+}(G)$	$n+m$	$\frac{1}{2}(n(n-1)+m(m+1)-\sum\limits_{i=1}^{n} d_G(v_i)^2)+m$

The diameters of *xyz-point-line transformation graphs* $T^{xyz}(G)$ with $z = +$ can be found in [**2**]. As we have the formula for finding the Wiener index of a graph G which has the diameter $\leqslant 3$, we restrict the diameters of *xyz-point-line transformation graphs* to 3 by giving condition on order of graph G or on the graph itself (which is suitable). To avoid the routine work of calculating the diameter we present that of *xyz-point-line transformation graphs* $T^{xyz}(G)$ with $z = +$ in Table 2. The Wiener polarity index of the various *xyz-point-line transformation graphs* $T^{xyz}(G)$ with $z = +$ are given in the following observation:

OBSERVATION 3.1. Let G be a connected graph with n vertices and m edges. Then

(i) $W_p(T^{10+}(G)) = \frac{1}{2}(m(m+1) - \sum\limits_{i=1}^{n} d_G(v_i)^2)$.

(ii) $W_p(T^{-0+}(G)) = \frac{1}{2}(m(m+1) - \sum\limits_{i=1}^{n} d_G(v_i)^2)$.

Table 2

Transformation Graph	**Condition**	$\boldsymbol{diam}(T^{xyz}(G))$
$T^{10+}(G)$	For any $n > 1$	$\leqslant 3$
$T^{+0+}(G)$	$diam(G) \leqslant 3$	$\leqslant 3$
$T^{-0+}(G)$	For any $n > 1$	$\leqslant 3$
$T^{01+}(G)$	For any $n > 1$	$\leqslant 3$
$T^{11+}(G)$	For any $n > 1$	$\leqslant 2$
$T^{+1+}(G)$	For any $n > 1$	$\leqslant 3$
$T^{+0+}(G)$ (Semitotal point graph)	$diam(G) \leqslant 3$	$\leqslant 3$
$T^{-1+}(G)$	For any $n > 1$	$\leqslant 2$
$T^{0++}(G)$ (Middle graph)	$diam(G) \leqslant 2$	$\leqslant 3$
$T^{1++}(G)$ (Quasi-vertex total graph)	For any $n > 1$	$\leqslant 3$
$T^{+++}(G)$ (Total graph)	$diam(G) \leqslant 3$, C_{2k+1}, ($k > 2 \in N$) is not induced subgraph of G	$\leqslant 3$
$T^{-++}(G)$ (Quasi total graph)	For any $n > 4$	$\leqslant 3$
$T^{0-+}(G)$	For any $n > 1$	$\leqslant 4$
$T^{1-+}(G)$	For any $n > 1$	$\leqslant 2$
$T^{+-+}(G)$	For any $n > 1$	$\leqslant 3$
$T^{--+}(G)$	For any $n > 4$	$\leqslant 2$

(iii) $W_p(T^{01+}(G)) = \frac{1}{2}n(n-1) - m$.

(iv) $W_p(T^{+0+}(G)) = W_p(G) + \frac{1}{2}(m^2 - \sum\limits_{i=1}^{n} d_G(v_i)^2)$.

(v) $W_p(T^{0++}(G)) = n(n-1) - m$.

(vi) $W_p(T^{+++}(G)) = 2W_p(G)$.

(vii) $W_p(T^{-++}(G)) = \frac{1}{2}(m^2 - \sum\limits_{i=1}^{n} d_G(v_i)^2)$.

(viii) $W_p(T^{+-+}(G)) = n(n-1) - m$.

(ix) $W_p(T^{1++}(G)) = W_p(L(G))$.

Theorem 3.4. *Let G be a graph of order n and size m with $diam(G) \leqslant 3$. Then*

$$W(G;q) = W_p(G)q^3 + \left(\binom{n}{2} - W_p(G) - m\right)q^2 + mq$$

and

$$W(G) = 3W_p(G) + 2\left(\binom{n}{2} - m - W_p(G)\right) + m.$$

PROOF. Let G be a graph of order n and size m with $diam(G) \leqslant 3$. Then by definition of Hosoya polynomial, we have

$$W(G;q) = \sum_{u,v \in V(G)} q^{d_G(u,v)}$$

and by Theorem 2.2, the highest power of polynomial is equal to the diameter of G. Let $A_i(G) = |\{(u,v)/d_G(u,v) = i\}|$. Therefore the expected Hosoya polynomial for G is

$$W(G;q) = \sum_{i=1}^{3} A_i(G)q^i.$$

By definition of $A_i(G)$, we have

$$A_1(G) = m,\ A_3(G) = W_p(G) \text{ and } A_2(G) = \tbinom{n}{2} - m - W_p(G).$$

Therefore,

$$W(G;q) = W_p(G)q^3 + \left(\binom{n}{2} - W_p(G) - m\right) q^2 + mq.$$

From Eq. (1.1), the Wiener index for G is

$$\begin{aligned} W(G) &= \frac{d}{dq}(W(G;q))\Big|_{q=1} \\ &= 3W_p(G) + 2\left(\binom{n}{2} - m - W_p(G)\right) + m \\ &= n(n-1) - m + W_p(G) \end{aligned}$$

□

By using the Wiener polarity index from Observation 3.1, the order and size of the respective graphs given in Table 2, we have the following corollaries which are immediate from Theorem 3.4.

COROLLARY 3.9. *For any graph G of order n and size m,*

$$\begin{aligned} W(T^{10+}(G);q) &= \frac{1}{2}\left(m(m+1) - \sum_{i=1}^{n} d_G(v_i)^2\right) q^3 \\ &\quad + \left(m(n-3) + \frac{1}{2}\sum_{i=1}^{n} d_G(v_i)^2\right) q^2 + \left(\frac{1}{2}n(n-1) + 2m\right) q \end{aligned}$$

and

$$W(T^{10+}(G)) = \frac{1}{2}\left(n(n-1) + m(4n+3m-5) - \sum_{i=1}^{n} d_G(v_i)^2\right).$$

COROLLARY 3.10. *For any graph G of order n and size m,*

$$\begin{aligned} W(T^{-0+}(G);q) &= \frac{1}{2}\left(m(m+1)-\sum_{i=1}^{n} d_G(v_i)^2\right)q^3 \\ &\quad+\left(m(n-2)+\frac{1}{2}\sum_{i=1}^{n} d_G(v_i)^2\right)q^2+\left(\frac{1}{2}n(n-1)+m\right)q \end{aligned}$$

and

$$W(T^{-0+}(G))=\frac{1}{2}\left(n(n-1)+m(4n+3m-3)-\sum_{i=1}^{n} d_G(v_i)^2\right).$$

COROLLARY 3.11. *For any connected graph G of order n and size m,*

$$W(T^{01+}(G);q)=\frac{1}{2}\left(n(n-1)-m\right)q^3+m(n-1)q^2+\frac{1}{2}m(m+3)q$$

and

$$W(T^{01+}(G))=\frac{1}{2}\left(3n(n-1)+m(4n+m-7)\right).$$

COROLLARY 3.12. *For any graph G of order n and size m,*

$$W(T^{11+}(G);q)=m(n-2)q^2+\left(\frac{1}{2}(n(n-1)+m(m+3))\right)q$$

and

$$W(T^{11+}(G))=\frac{1}{2}\big(n(n-1)+m(4n+m-5)\big).$$

COROLLARY 3.13. *For any graph G of order n and size m,*

$$\begin{aligned} W(T^{1-+}(G);q) &= \left(m(n-3)+\frac{1}{2}\sum_{i=1}^{n} d_G(v_i)^2\right)q^2 \\ &\quad+\left(\frac{1}{2}(n(n-1)+m(m+5)-\sum_{i=1}^{n} d_G(v_i)^2\right)q \end{aligned}$$

and

$$W(T^{1-+}(G))=\frac{1}{2}\left(n(n-1)+m(4n+m-7)+\sum_{i=1}^{n} d_G(v_i)^2\right).$$

COROLLARY 3.14. *For any connected graph G of order n and size m with $diam(G)\leqslant 3$,*

$$\begin{aligned} W(T^{+0+}(G);q) &= \left(W_p(G)+\frac{1}{2}\bigg(m(m+1)+\sum_{i=1}^{n} d_G(v_i)^2\bigg)\right)q^3 \\ &\quad+\left(\binom{n+m}{2}-W_p(G)-\frac{1}{2}\bigg(m(m+1)+\sum_{i=1}^{n} d_G(v_i)^2\bigg) \right.\\ &\quad\left.-3m\right)q^2+3mq \end{aligned}$$

and

$$W(T^{+0+}(G)) = W_p(G) + 2\binom{n+m}{2} + \frac{1}{2}\left(m(m+1) + \sum_{i=1}^{n} d_G(v_i)^2\right) - 3m.$$

COROLLARY 3.15. *For any connected graph G of order n and size m with $diam(G) \leqslant 2$,*

$$\begin{aligned} W(T^{0++}(G);q) &= \big(n(n-1)-m\big)q^3 \\ &+ \left(\binom{n+m}{2} - n(n-1) - \frac{1}{2}\sum_{i=1}^{n} d_G(v_i)^2\right)q^2 \\ &+ \left(m + \frac{1}{2}\sum_{i=1}^{n} d_G(v_i)^2\right)q \end{aligned}$$

and

$$W(T^{0++}(G)) = n(n-1) + 2\binom{n+m}{2} - \frac{3}{2}\sum_{i=1}^{n} d_G(v_i)^2 - 2m.$$

COROLLARY 3.16. *For any graph G of order n and size m,*

$$\begin{aligned} W(T^{1++}(G);q) &= W_pL(G)q^3 \\ &+\Bigg[\binom{n+m}{2} - W_pL(G) \\ &-\frac{1}{2}\left(n(n-1) + \sum_{i=1}^{n} d_G(v_i)^2\right) + m\Bigg]q^2 \\ &+ \left(\frac{1}{2}\left(n(n-1) + \sum_{i=1}^{n} d_G(v_i)^2\right) - m\right)q \end{aligned}$$

and

$$W(T^{1++}(G)) = W_pL(G) - \frac{1}{2}\left(n(n-1) + \sum_{i=1}^{n} d_G(v_i)^2\right) + m.$$

COROLLARY 3.17. *For any graph $G \ncong C_{2k+1}$ of order n and size m with $diam(G) \leqslant 3$ where $k > 2 \in N$,*

$$\begin{aligned} W(T^{+++}(G);q) &= 2W_p(G)q^3 + \Bigg(\binom{n+m}{2} - 2W_p(G) \\ &-2m - \frac{1}{2}\sum_{i=1}^{n} d_G(v_i)^2\Bigg)q^2 \\ &+ \left(2m + \frac{1}{2}\sum_{i=1}^{n} d_G(v_i)^2\right)q \end{aligned}$$

and

$$W(T^{+++}(G)) = 2W_p(G) + 2\binom{n+m}{2} - \frac{1}{2}\sum_{i=1}^{n} d_G(v_i)^2 - 2m.$$

COROLLARY 3.18. *For any graph G of order $n > 4$ and size m,*

$$\begin{aligned}W(T^{-++}(G);q) &= \left(\frac{1}{2}\left(m^2 - \sum_{i=1}^{n} d_G(v_i)^2\right)\right)q^3\\&+\left(\binom{n+m}{2} - \frac{1}{2}(m^2 + n(n-1))\right)q^2\\&+\left(m + \frac{1}{2}\sum_{i=1}^{n} d_G(v_i)^2\right)q\end{aligned}$$

and

$$W(T^{-++}(G)) = 2\binom{n+m}{2} - n(n-1) - m + \frac{1}{2}\left(m^2 - 5\sum_{i=1}^{n} d_G(v_i)^2\right).$$

COROLLARY 3.19. *For any graph G of order n and size m,*

$$\begin{aligned}W(T^{+-+}(G);q) &= \big(n(n-1) - m\big)q^3\\&+\left(\binom{n+m}{2} - n(n-1) - \frac{1}{2}\left(m(m+1) - \sum_{i=1}^{n} d_G(v_i)^2\right)\right.\\&\left.-2m\right)q^2 + \left(m + \frac{1}{2}\sum_{i=1}^{n} d_G(v_i)^2\right)q\end{aligned}$$

and

$$W(T^{+-+}(G)) = 2\binom{n+m}{2} + n(n-1) - m(m+7) + \frac{3}{2}\sum_{i=1}^{n} d_G(v_i)^2.$$

COROLLARY 3.20. *For any graph G of order $n > 4$ and size m,*

$$\begin{aligned}W(T^{--+}(G);q) &= \left(\binom{n+m}{2} - \frac{1}{2}\left(n(n-1) + m(m+1) - \sum_{i=1}^{n} d_G(v_i)^2\right)\right.\\&\left.-m\right)q^2 + \left(\frac{1}{2}\left(n(n-1) + m(m+1) - \sum_{i=1}^{n} d_G(v_i)^2\right) + m\right)q\end{aligned}$$

and

$$W(T^{--+}(G)) = 2\binom{n+m}{2} - \frac{1}{2}\left(n(n-1) + m(m+1) + \sum_{i=1}^{n} d_G(v_i)^2\right) - m.$$

COROLLARY 3.21. *For any connected graph G of order n and size m,*

$$W(T^{-1+}(G);q) = (m(n-1))q^2 + \left(\frac{1}{2}(n(n-1) + m(m+1))\right)q$$

and

$$W(T^{-1+}(G)) = \frac{(m+n)^2 + m(2n-3) - n}{2}.$$

4. Results on xyz-point-line transformation graphs $T^{xyz}(G)$ with $z = -$

In this section we present the results on the graph $T^{-0-}(G)$ as an example to obtain Hosoya polynomial and Wiener index of other transformation graphs. The results obtained for the graph $T^{-0-}(G)$ can also be obtained for other transformation graphs in the same manner.

DEFINITION 4.1. The *xyz-point-line transformation graph* $T^{-0-}(G)$ is a graph whose vertex set is $V(G) \cup E(G)$ and any two vertices in $T^{-0-}(G)$ are adjacent if and only if they correspond to two non adjacent vertices of G or to a vertex and an edge not incident with it in G. Therefore, the order and size of $T^{-0-}(G)$ are $n + m$ and $\frac{1}{2}n(n-1) + m(n-3)$ respectively.

THEOREM 4.1. *For any graph G, $T^{-0-}(G)$ is connected if and only if G is neither a star nor a triangle.*

PROOF. Suppose G is neither a star nor a triangle. Then we have the following cases.

Case 1. If G is connected, then it has at least three edges say e_1, e_2 and e_3 such that $G[\{e_1, e_2, e_3\}] \cong P_4$ or $P_3 \cup K_2$ or $3K_2$ and one can easily check that $T^{-0-}(G)$ of these graphs are connected. If any vertex $v \in V(G)$ which is not incident with any $e_i (1 \leqslant i \leqslant 3)$ in G, then v' is adjacent to e'_1, e'_2 and e'_3 in $T^{-0-}(G)$. Otherwise, any edge $e \in E(G)$ other than $e_i (1 \leqslant i \leqslant 3)$ is not incident with at least one vertex u of subgraph of G with the three edges $e_i (1 \leqslant i \leqslant 3)$ in G, then e' is adjacent to u' in $T^{-0-}(G)$. Thus $T^{-0-}(G)$ is connected.

Case 2. If G is not connected, then $\overline{G}$ is a connected induced subgraph of $T^{-0-}(G)$ and each line-vertex is adjacent with at least one point-vertex in $T^{-0-}(G)$ as each edge is not incident with at least one vertex in G. Therefore $T^{-0-}(G)$ is connected.

Conversely, if G is a star, then $T^{-0-}(G)$ has an isolated vertex which corresponds to the central vertex of G. Suppose G is a triangle. Then $T^{-0-}(G)$ is disconnected. $\square$

THEOREM 4.2. *For any graph $G(\ncong K_3, S_n)$ of order n, $diam(T^{-0-}(G)) \leqslant 4$ with equality holds if and only if $G \cong K_4, K_2 \cdot K_3$.*

PROOF. We prove this by the following cases.

Case 1. For $u, v \in V(G)$, if u and v are not adjacent in G, then they are adjacent in $T^{-0-}(G)$. Otherwise,

$$d_{T^{-0-}(G)}(u, v) = \begin{cases} 3 & \text{if } u \sim v \text{ in } G \text{ which covers all the edges of } G, \\ 2 & \text{otherwise.} \end{cases}$$

Case 2. For $e_1, e_2 \in E(G)$, if e_1 and e_2 are adjacent in G, then there exists a vertex v which is neither incident with e_1 nor e_2 such that e'_1, v', e'_2 is a path of length two in $T^{-0-}(G)$. Otherwise,

$$d_{T^{-0-}(G)}(e_1, e_2) = \begin{cases} 4 & if\ G \cong K_4, \\ 3 & if\ G \cong C_4, \\ 2 & otherwise. \end{cases}$$

Case 3. For $u \in V(G)$ and $e \in E(G)$, if u and e are not incident in G, then they are adjacent in $T^{-0-}(G)$. Suppose, u and e are incident in G. If $G \cong K_2 \cdot K_3$, then

$$d_{T^{-0-}(G)}(u,e) = \begin{cases} 4 & \textit{if } d_G(u) = 3 \textit{ and } e \textit{ is pendant edge in } G, \\ 3 & \textit{if } d_G(u) = 3 \textit{ and } e \textit{ is not a pendant edge in } G\ , \\ 2 & \textit{otherwise.} \end{cases}$$

Otherwise,

$$d_{T^{-0-}(G)}(u,e) = \begin{cases} 3 & if\ d_G(u) = n-1, \\ 2 & otherwise. \end{cases}$$

$\square$

The following corollaries are immediate from the proof of the Theorem 4.2.

Corollary 4.1. *For any tree $T_n (\ncong S_n)$ of order $n > 3$, $diam(T^{-0-}(T_n)) \leqslant 3$ with equality holds if and only if T_n is a double star.*

Corollary 4.2. *For any cycle C_n of order $n > 3$, $diam(T^{-0-}(C_n)) \leqslant 3$ with equality holds if and only if $n = 4$.*

Corollary 4.3. *For any complete graph K_n of order $n > 3$,*

$$diam(T^{-0-}(K_n)) \leqslant 4$$

with equality holds if and only if $n = 4$.

Corollary 4.4. *For any wheel W_n of order n, $diam(T^{-0-}(W_n)) \leqslant 4$ with equality holds if and only if $n = 4$.*

Corollary 4.5. *For any complete bipartite graph $K_{a,b}$ other than star,*

$$diam(T^{-0-}(K_{a,b})) \leqslant 3$$

with equality holds if and only if $a = b = 2$.

Corollary 4.6. *For given graph G, the number of pairs of vertices which are at distance 4 are denoted by $W_F(G)$ and*

$$W_F(T^{-0-}(G)) = \begin{cases} 1 & if\ G \cong K_2 \cdot K_3, \\ 3 & if\ G \cong K_4, \\ 0 & otherwise. \end{cases}$$

The Wiener polarity index of $T^{-0-}(G)$ of some standard graphs are given in following corollary.

Corollary 4.7. *The Wiener polarity index of $T^{-0-}(G)$ are:*

(i) $W_p(T^{-0-}(T_n)) = \begin{cases} 1 & if\ T_n \cong S_{a,b}, \\ 0 & otherwise. \end{cases}$

(ii) $W_p(T^{-0-}(C_n)) = \begin{cases} 2 & if\ n = 4, \\ 0 & otherwise. \end{cases}$

(iii) $W_p(T^{-0-}(K_n)) = n(n-1)$ *for all* $n \geqslant 5$.

(iv) $W_p(T^{-0-}(W_n)) = n - 1$ *for all* $n \geqslant 5$.

(v) $W_p(T^{-0-}(K_{a,b})) = \begin{cases} 2 & if\ a=b=2, \\ 0 & otherwise. \end{cases}$

THEOREM 4.3. *For any graph G of order $n>3$,*

$$\begin{aligned} W(T^{-0-}(G);q) &= W_F(T^{-0-}(G))q^4 + W_p(T^{-0-}(G))q^3 \\ &\quad + \left(\frac{1}{2}m(m+5) - W_p(T^{-0-}(G)) - W_F(T^{-0-}(G))\right)q^2 \\ &\quad + \left(\frac{1}{2}n(n-1) + m(n-3)\right)q \end{aligned}$$

and $W(T^{-0-}(G)) = 2W_F(T^{-0-}(G)) + W_p(T^{-0-}(G)) + \frac{1}{2}n(n-1) + m(n+m+2)$.

PROOF. Let G be a graph of order $n>3$. Then by Theorem 4.2,

$$diam(T^{-0-}(G)) \leqslant 4.$$

Therefore by definition of Hosoya polynomial, we have

$$W(T^{-0-}(G);q) = \sum_{u,v\in V(T^{-0-}(G))} q^{d_{T^{-0-}(G)}(u,v)}$$

and by the properties of Hosoya polynomial of a graph G, the highest power of polynomial is equal to the diameter of G. Let $A_i(G) = |\{(u,v)|d_G(u,v)=i\}|$. Therefore the expected Hosoya polynomial for $T^{-0-}(G)$ is

$$W(T^{-0-}(G);q) = \sum_{i=1}^{4} A_i(T^{-0-}(G))q^i.$$

Since, $A_1(T^{-0-}(G)) = \frac{1}{2}n(n-1) + m(n+m+2)$, $A_3(T^{-0-}(G)) = W_p(T^{-0-}(G))$, $A_4(T^{-0-}(G)) = W_F(T^{-0-}(G))$ and $A_2(T^{-0-}(G)) = \frac{1}{2}m(m+5) - W_p(T^{-0-}(G)) - W_F(T^{-0-}(G))$. Therefore,

$$\begin{aligned} W(T^{-0-}(G);q) &= W_F(T^{-0-}(G))q^4 + W_p(T^{-0-}(G))q^3 \\ &\quad + \left(\frac{1}{2}m(m+5) - W_p(T^{-0-}(G)) - W_F(T^{-0-}(G))\right)q^2 \\ &\quad + \left(\frac{1}{2}n(n-1) + m(n-3)\right)q. \end{aligned}$$

From Eq. (1.1), the Wiener index for $T^{-0-}(G)$ is

$$\begin{aligned} W(T^{-0-}(G)) &= \frac{d}{dq}(W(T^{-0-}(G);q))\Big|_{q=1} \\ &= 2W_F(T^{-0-}(G)) + W_p(T^{-0-}(G)) + \frac{1}{2}n(n-1) + m(n+m+2). \end{aligned}$$

□

Corollary 4.8. *For any tree T_n of order $n > 3$,*

$$W(T^{-0-}(T_n);q) = \begin{cases} q^3 + (\frac{1}{2}(n-1)(n+4)-1)q^2 + \frac{3}{2}(n-1)(n-2)q & if\ T_n \cong S_{a,b}, \\ \frac{1}{2}(n-1)(n+4)q^2 + \frac{3}{2}(n-1)(n-2)q & otherwise. \end{cases}$$

and $W(T^{-0-}(T_n)) = \begin{cases} \frac{1}{2}n(5n-3) & if\ T_n \cong S_{a,b}, \\ \frac{1}{2}(n-1)(5n+2) & otherwise. \end{cases}$

Proof. By Theorem 4.3, Corollaries 4.6, 4.7 (i) and a fact that $m = n-1$ for tree T_n the result follows. $\square$

Corollary 4.9. *For any path P_n of order $n > 4$, $W(T^{-0-}(P_n)) = \frac{1}{2}(n-1)(5n+2)$.*

Corollary 4.10. *For any tree T of order $n > 3$,*

$$W(T^{-0-}(T_n)) = W(T^{-0-}(S_{a,b})) - 1.$$

Corollary 4.11. *For any cycle C_n of order $n > 3$,*

$$W(T^{-0-}(C_n);q) = \begin{cases} 2q^3 + 16q^2 + 10q & if\ n = 4, \\ \frac{n}{2}(n+5)q^2 + \frac{n}{2}(3n-7)q & otherwise. \end{cases}$$

and $W(T^{-0-}(C_n)) = \begin{cases} 48 & if\ n = 4, \\ \frac{n}{2}(5n+3) & otherwise. \end{cases}$

Proof. By Theorem 4.3, Corollaries 4.6, 4.7 (ii) and a fact that $m = n$ for cycle C_n the result follows. $\square$

Corollary 4.12. *For any complete graph K_n of order $n > 3$,*

$$W(T^{-0-}(K_n);q) = \begin{cases} 3q^4 + 12q^3 + 18q^2 + 12q & if\ n = 4, \\ n(n-1)q^3 + \frac{1}{8}n(n^3 - 2n^2 + 3n - 2)q^2 & \\ +\frac{1}{2}n(n-1)(n-2)q & otherwise. \end{cases}$$

and $W(T^{-0-}(K_n)) = \begin{cases} 96 & if\ n = 4, \\ \frac{1}{4}n(n^3 + 9n - 10) & otherwise. \end{cases}$

Proof. By Theorem 4.3, Corollaries 4.6, 4.7 (iii) and a fact that $m = \frac{n(n-1)}{2}$ for complete graph K_n the result follows. $\square$

Corollary 4.13. *For any wheel W_n of order n,*

$$W(T^{-0-}(W_n);q) = \begin{cases} 3q^4 + 12q^3 + 18q^2 + 12q & if\ n = 4, \\ (n-1)q^3 + 2(n^2-1)q^2 + \frac{1}{2}(n-1)(5n-12)q & otherwise. \end{cases}$$

and

$$W(T^{-0-}(W_n)) = \begin{cases} 96 & if\ n = 4, \\ \frac{1}{2}(n-1)(13n+2) & otherwise. \end{cases}$$

Proof. By Theorem 4.3, Corollaries 4.6, 4.7 (iv) and a fact that $m = 2(n-1)$ for wheel W_n the result follows. $\square$

Corollary 4.14. *For any complete bipartite graph $K_{a,b}$ other than star of order n,*

$$W(T^{-0-}(K_{a,b});q) = \begin{cases} 2q^3 + 16q^2 + 10q & if\ a = b = 2, \\ \frac{1}{2}ab(ab+5)q^2 + (\frac{1}{2}(a+b)(a+b-1) & \\ +ab(a+b-3))q & otherwise. \end{cases}$$

$$and\, W(T^{-0-}(K_{a,b})) = \begin{cases} 48 & if\ a = b = 2, \\ \frac{1}{2}(a+b)(a+b-1) + ab(a+b+ab+2) & otherwise. \end{cases}$$

Proof. By Theorem 4.3, Corollaries 4.6, 4.7 (v) and the facts $n = a+b, m = ab$ for complete bipartite graph $K_{a,b}$ the result follows. □

Lemma 4.1. *For any graph G of order $n > 3$,*

$$\frac{1}{2}(n-1)(5n+2) \leqslant W(T^{-0-}(G)) \leqslant \frac{1}{4}n(n^3 + 9n - 10).$$

Upper bound attains if G is a complete graph other than K_4 and lower bound attains if G is tree other than double star.

Proof. The generalized *xyz-point-line transformation graph* $T^{-0-}(G)$ has $n+m$ vertices and $\frac{1}{2}n(n-1) + m(n-3)$ edges. Any graph G of order n has maximum number of edges if and only if $G \cong K_n$. Hence, $T^{-0-}(G)$ has maximum number of vertices if and only if $G \cong K_n$, where G is a graph of order n.

We know that, the Wiener index of a graph G increases when new vertices are added to the graph and $T^{-0-}(K_n)$ has maximum number of vertices compared with any other $T^{-0-}(G)$, where G is a graph of order n. Therefore, $W(T^{-0-}(G)) \leqslant W(T^{-0-}(K_n))$. From Corollary 4.12, $W(T^{-0-}(K_n)) = \frac{1}{4}n(n^3 + 9n - 10)$. So,

$$W(T^{-0-}(G)) \leqslant \frac{1}{4}n(n^3 + 9n - 10) \tag{4.1}$$

with equality in Eq. (4.1) if and only if $G \cong K_n (n \neq 4)$.

Any graph G of order n has minimum number of edges if and only if $G \cong T_n$ and $T^{-0-}(G)$ has minimum number of vertices if and only if $G \cong T_n (\ncong S_{a,b})$, where G is a graph of order n. So, $T^{-0-}(G)(\ncong T^{-0-}(S_{a,b}))$ has minimum number of vertices compared to any other $T^{-0-}(G)$, where G is a graph of order n. Therefore, $W(T^{-0-}(T_n)) \leqslant W(T^{-0-}(G))$. From Corollary 4.8, $W(T^{-0-}(T_n)) = \frac{1}{2}(n-1)(5n+2)$. Therefore,

$$\frac{1}{2}(n-1)(5n+2) \leqslant W(T^{-0-}(G)) \tag{4.2}$$

with equality in Eq. (4.2) if and only if $G \cong T_n (\ncong S_{a,b})$.
From Eqs. (4.1) and (4.2), we have

$$\frac{1}{2}(n-1)(5n+2) \leqslant W(T^{-0-}(G)) \leqslant \frac{1}{4}n(n^3 + 9n - 10).$$

Upper bound attains if G is a complete graph other than K_4 and lower bound attains if G is tree other than double star. □

TABLE 3

Transformation Graph	**Order**	**Size**
$T^{00-}(G)$	$n+m$	$m(n-2)$
$T^{01-}(G)$	$n+m$	$\binom{m}{2}+m(n-2)$
$T^{0+-}(G)$	$n+m$	$mn+\frac{1}{2}\sum\limits_{i=1}^{n} d_G(v_i)^2-3m$
$T^{0--}(G)$	$n+m$	$\binom{m+1}{2}-\frac{1}{2}\sum\limits_{i=1}^{n} d_G(v_i)^2+m(n-2)$
$T^{10-}(G)$	$n+m$	$\binom{n}{2}+m(n-2)$
$T^{11-}(G)$	$n+m$	$\binom{n}{2}+\binom{m}{2}+m(n-2)$
$T^{1+-}(G)$	$n+m$	$\binom{n}{2}+\frac{1}{2}\sum\limits_{i=1}^{n} d_G(v_i)^2+mn-3m$
$T^{1--}(G)$	$n+m$	$\binom{n}{2}+\binom{m+1}{2}-\frac{1}{2}\sum\limits_{i=1}^{n} d_G(v_i)^2+m(n-2)$
$T^{+0-}(G)$	$n+m$	$m(n-1)$
$T^{+1-}(G)$	$n+m$	$\binom{m}{2}+m(n-1)$
$T^{++-}(G)$	$n+m$	$\frac{1}{2}\sum\limits_{i=1}^{n} d_G(v_i)^2+m(n-2)$
$T^{+--}(G)$	$n+m$	$m+\binom{m+1}{2}-\frac{1}{2}\sum\limits_{i=1}^{n} d_G(v_i)^2$
$T^{-0-}(G)$	$n+m$	$\binom{n}{2}+mn-3m$
$T^{-1-}(G)$	$n+m$	$\binom{n}{2}+\binom{m}{2}+mn-3m$
$T^{-+-}(G)$	$n+m$	$\binom{n}{2}+\frac{1}{2}\sum\limits_{i=1}^{n} d_G(v_i)^2+m(n-2)$
$T^{---}(G)$	$n+m$	$\binom{n}{2}+\binom{m+1}{2}-\frac{1}{2}\sum\limits_{i=1}^{n} d_G(v_i)^2+mn-3m$

Order and size of *xyz-point-line transformation graphs* [**3**] in terms of order and size of original graphs (n, m) are given in Table 3.

As we have standard results to calculate Wiener index of graphs with diameter $\leqslant 3$, we restrict diameter of *xyz-point-line transformation graphs* $T^{xy-}(G)$ to either 3 or 4 by giving condition on number of vertices of graph G. For the sake of avoiding routine work of calculation we present the diameter of transformation graphs in Table 4.

THEOREM 4.4. *For any graph G of diameter $\leqslant 2$, the Hosoya polynomial of G is given by*

$$W(G;q) = (\tfrac{n(n-1)}{2} - m)q^2 + mq$$

where n and m are order and size of graph G.

PROOF. From the condition three of the Theorem 2.2 we can see that coefficient of q is always equal to the edge set, i.e, m. Since diameter is two, the remaining edges are given by $\frac{n(n-1)}{2} - m$, which is coefficient of q^2. $\square$

TABLE 4

Transformation Graph	**Condition on n**	**Diameter**
$T^{00-}(G)$	$n > 4$	3
$T^{01-}(G)$	$n > 4$	2
$T^{0+-}(G)$	$n > 4$	2
$T^{0--}(G)$	$n > 4$	3
$T^{10-}(G)$	$n > 3$	2
$T^{11-}(G)$	$n > 3$	2
$T^{1+-}(G)$	$n > 3$	2
$T^{1--}(G)$	$n > 3$	2
$T^{+0-}(G)$	$n > 3$	3
$T^{+1-}(G)$	$n > 2$	2
$T^{++-}(G)$	$n > 3$	2
$T^{+--}(G)$	$n > 3$	2
$T^{-0-}(G)$	$n > 4$	3
$T^{-1-}(G)$	$n > 4$	2
$T^{-+-}(G)$	$n > 4$	2
$T^{---}(G)$	$n > 4$	2

COROLLARY 4.15. *Let G be a graph of order $n > 3$. Then Hosoya polynomial of $T^{1y-}(G)$, where $y \in \{0, 1, +, -\}$ is given by*

$$W(T^{1y-}(G); q) = \left(\frac{n(n-1)}{2} - m\right) q^2 + mq \tag{4.3}$$

where, n and m are order and size of respective transformation graphs $T^{1y-}(G)$.

PROOF. By substituting the order and size of respective graphs from Table 3 in equation 4.3 we get the desired Hosoya polynomial of respective graphs. □

COROLLARY 4.16. *Let G be a graph of order $n > 4$ not a star. Then Hosoya polynomial of $T^{-y-}(G)$, where $y \in \{1, +, -\}$ is given by*

$$W(T^{-y-}(G); q) = \left(\frac{n(n-1)}{2} - m\right) q^2 + mq \tag{4.4}$$

where, n and m are order and size of respective transformation graphs.

COROLLARY 4.17. *For any graph G of order $n > 4$ not a star, Hosoya polynomial of $T^{01-}(G)$ and $T^{0+-}(G)$ is given by*

$$W(G; q) = \left(\frac{n(n-1)}{2} - m\right) q^2 + mq \tag{4.5}$$

where, n and m are order and size of respective transformation graphs.

COROLLARY 4.18. *For any graph G of order $n > 3$, Hosoya polynomial of $T^{++-}(G)$ and $T^{+--}(G)$ is given by*

$$W(T^{++-}(G); q) = \left(\frac{n(n-1)}{2} - m\right) q^2 + mq \tag{4.6}$$

where, n and m are order and size of respective transformation graphs.

COROLLARY 4.19. *For any graph G of order $n > 2$, Hosoya polynomial of $T^{+1-}(G)$ is given by*

$$W(T^{+1-}(G);q) = \left(\frac{n(n-1)}{2} - m\right) q^2 + mq \tag{4.7}$$

where, n and m are order and size of $T^{+1-}(G)$.

THEOREM 4.5. *For any graph G of diameter $\leqslant 3$, the Hosoya polynomial of G is given by*

$$W(G;q) = W_p q^3 + \left(\tfrac{n(n-1)}{2} - W_p - m\right) q^2 + mq.$$

PROOF. The coefficient of q is m is clear from condition(iii) of Theorem 2.2. Since diameter of the graph G is 3, let W_p be the number of such pairs of vertices which are at distance 3. Thus coefficient of q^3 is W_p. This implies that the remaining pair of vertices gives distance two. Therefore coefficient of q^2 is given by $\frac{n(n-1)}{2} - W_p - m$. □

OBSERVATION 4.1. Let G be a graph with n vertices and m edges. Then

(i) $W_p(T^{00-}(G)) = 2m$.
(ii) $W_p(T^{0--}(G)) = 2m$.
(iii) $W_p(T^{+0-}(G)) = k$, number of pendant vertices.

COROLLARY 4.20. *Let G be a graph of order $n > 4$. Then Hosoya polynomial of $T^{00-}(G)$ is given by*

$$W(T^{00-}(G);q) = 2m \cdot q^3 + \left(\frac{n(n-1)}{2} - 3m\right) q^2 + mq \tag{4.8}$$

where, n and m are order and size of transformation graphs $T^{00-}(G)$.

COROLLARY 4.21. *Let G be a graph of order $n > 4$. Then Hosoya polynomial of $T^{0--}(G)$ is given by*

$$W(T^{0--}(G);q) = 2m \cdot q^3 + \left(\frac{n(n-1)}{2} - 3m\right) q^2 + mq \tag{4.9}$$

where, n and m are order and size of transformation graphs $T^{0--}(G)$.

COROLLARY 4.22. *Let G be a graph of order $n > 4$. Then Hosoya polynomial of $T^{+0-}(G)$ is given by*

$$W(T^{+0-}(G);q) = k \cdot q^3 + \left(\frac{n(n-1)}{2} - k - m\right) q^2 + mq \tag{4.10}$$

where, n and m are order and size of transformation graphs $T^{+0-}(G)$ and k is number of pendant vertices.

Differentiating polynomials obtained above with respect to q at $q = 1$, we get Wiener index of desired *xyz-point-line transformation graphs*. As an example, we study the transformation graph $T^{-0-}(G)$ in detail, rest of the transformations can be studied in same fashion.

5. Results on xyz-point-line transformation graphs $T^{xyz}(G)$ with $z = 1$

THEOREM 5.1. *If G is a graph of order $n \geqslant 2$, then*

$$W(T^{xy1}(G); q) = Bq^2 + Aq \text{ and } W(T^{xy1}(G)) = 2B + A.$$

where A denotes the number of edges in transformation graph $T^{xy1}(G)$ and $B = \binom{n+m}{2} - A =$ the number of pairs of vertices with distance 2 in transformation graph $T^{xy1}(G)$.

PROOF. From the definition of Hosoya polynomial,

$$W(T^{xy1}(G); q) \quad = \sum_{u,v \in V(T^{xy1}(G))} q^{d_{T^{xy1}(G)}(u,v)}$$

Since $diam(T^{xy1}(G)) \leqslant 2$ for any graph G. Therefore, from Theorem 2.2, the highest power of q represents the diameter of graph and coefficient of q represent the number of edges of a graph. Thus the coefficient of $q^2 = B$.

$\implies W(T^{xy1}(G); q) = Bq^2 + Aq$

and by using Eq. (1.1) we get, $W(T^{xy1}(G)) = 2B + A$. □

The Wiener index of these graphs can also be obtained by using Theorem 2.1.

The order and size of the xyz-transformation graphs can be found in [**2**]. Using those we give the values of A and B in Table 5.

TABLE 5

Graph	**A**	**B**
$T^{001}(G)$	mn	$\frac{1}{2}[n(n-1) + m(m-1)]$
$T^{101}(G)$	$mn + \frac{n(n-1)}{2}$	$\frac{m(m-1)}{2}$
$T^{+01}(G)$	$m(n+1)$	$\frac{1}{2}[n(n-1) + m(m-3)]$
$T^{-01}(G)$	$(n-1)(\frac{n}{2} + m)$	$\frac{m(m+1)}{2})$
$T^{011}(G)$	$m(\frac{m-1}{2} + n)$	$\frac{n(n-1)}{2}$

6. Conclusion

In the present work, we obtained the Hosoya polynomial and Wiener index of the *xyz-point-line transformation graphs* $T^{xyz}(G)$ in terms of order and size of the parent graph by reducing the diameter of certain transformation graphs to 3 or 4. For future research it remains a task to obtain the Hosoya polynomial and Wiener index of the two graphs $T^{0-+}(G)$ and $T^{00+}(G)$. Since the diameter of these graphs increases as number of vertices in G increases. Also it is challenging to find Hosoya polynomial and Wiener index for the *xyz-point-line transformation graphs* $T^{xyz}(G)$ which have diameter > 3.

Graph	**A**	**B**
$T^{111}(G)$	$\frac{n(n-1)}{2}+\frac{m(m-1)}{2}+mn$	0
$T^{+11}(G)$	$\frac{m(m+1)}{2}+mn$	$\frac{n(n-1)}{2}-m$
$T^{-11}(G)$	$\frac{n(n-1)}{2}+\frac{m(m-3)}{2}+mn$	m
$T^{0+1}(G)$	$m(n-1)+\frac{1}{2}\sum\limits_{i=1}^{n} d_G(v_i)^2$	$\frac{n(n-1)}{2}+\frac{m(m+1)}{2}-\frac{1}{2}\sum\limits_{i=1}^{n} d_G(v_i)^2$
$T^{1+1}(G)$	$(n-1)(\frac{n}{2}+m)+\frac{1}{2}\sum\limits_{i=1}^{n} d_G(v_i)^2$	$\frac{m(m+1)}{2}-\frac{1}{2}\sum\limits_{i=1}^{n} d_G(v_i)^2$
$T^{++1}(G)$	$mn+\frac{1}{2}\sum\limits_{i=1}^{n} d_G(v_i)^2$	$\frac{n(n-1)}{2}+\frac{m(m-1)}{2}-\frac{1}{2}\sum\limits_{i=1}^{n} d_G(v_i)^2$
$T^{-+1}(G)$	$\frac{n(n-1)}{2}+m(n-2)+\frac{1}{2}\sum\limits_{i=1}^{n} d_G(v_i)^2$	$\frac{m(m+3)}{2}-\frac{1}{2}\sum\limits_{i=1}^{n} d_G(v_i)^2$
$T^{0-1}(G)$	$m(\frac{m+1}{2}+n)-\frac{1}{2}\sum\limits_{i=1}^{n} d_G(v_i)^2$	$\frac{n(n-1)}{2}-m+\frac{1}{2}\sum\limits_{i=1}^{n} d_G(v_i)^2$
$T^{1-1}(G)$	$\frac{n(n-1)}{2}+m(\frac{m+1}{2}+n)-\frac{1}{2}\sum\limits_{i=1}^{n} d_G(v_i)^2$	$-m+\frac{1}{2}\sum\limits_{i=1}^{n} d_G(v_i)^2$
$T^{+-1}(G)$	$m(\frac{m+3}{2}+n)-\frac{1}{2}\sum\limits_{i=1}^{n} d_G(v_i)^2$	$\frac{n(n-1)}{2}-2m+\frac{1}{2}\sum\limits_{i=1}^{n} d_G(v_i)^2$
$T^{--1}(G)$	$\frac{n(n-1)}{2}+m(\frac{m-1}{2}+n)-\frac{1}{2}\sum\limits_{i=1}^{n} d_G(v_i)^2$	$\frac{1}{2}\sum\limits_{i=1}^{n} d_G(v_i)^2$

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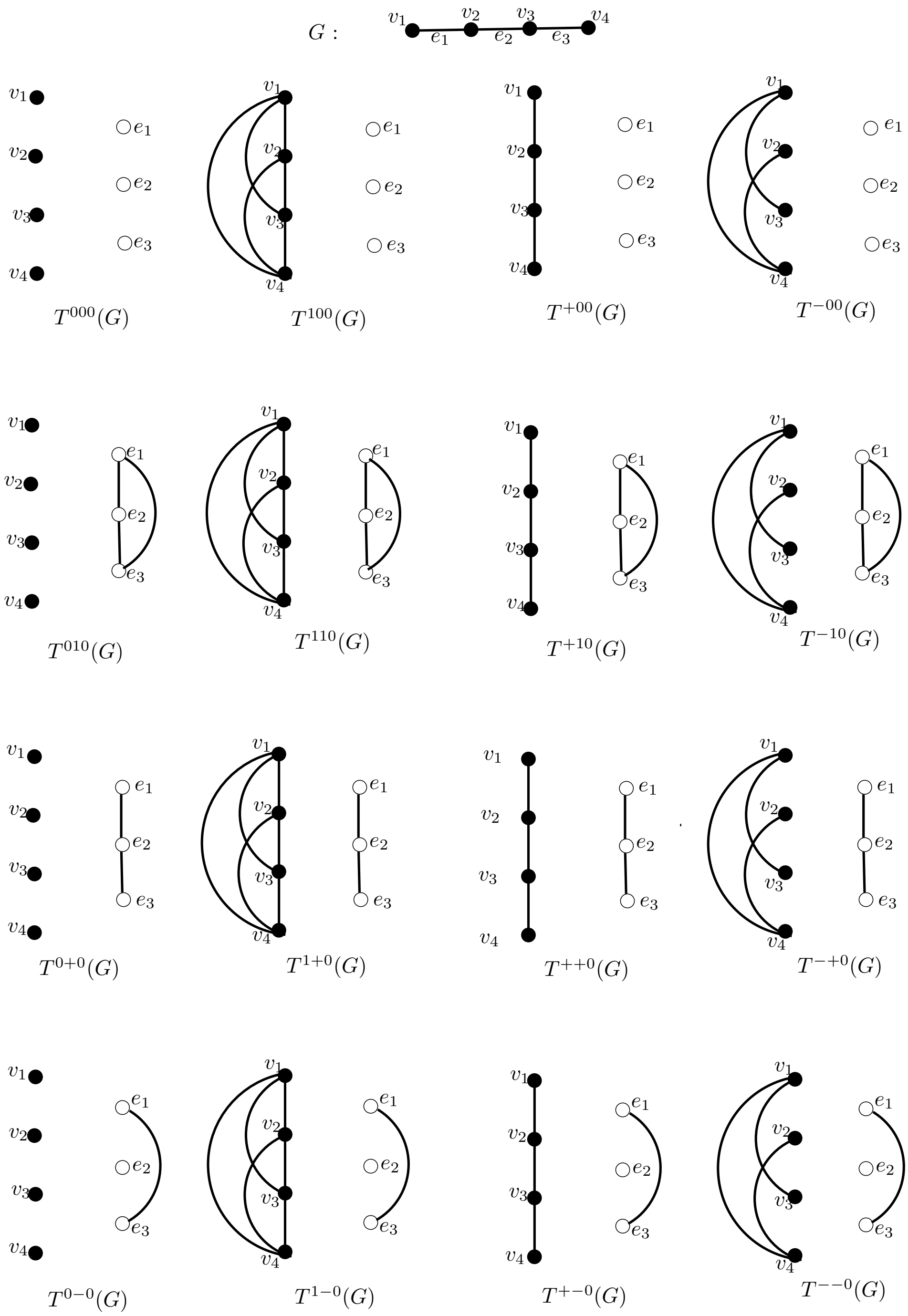
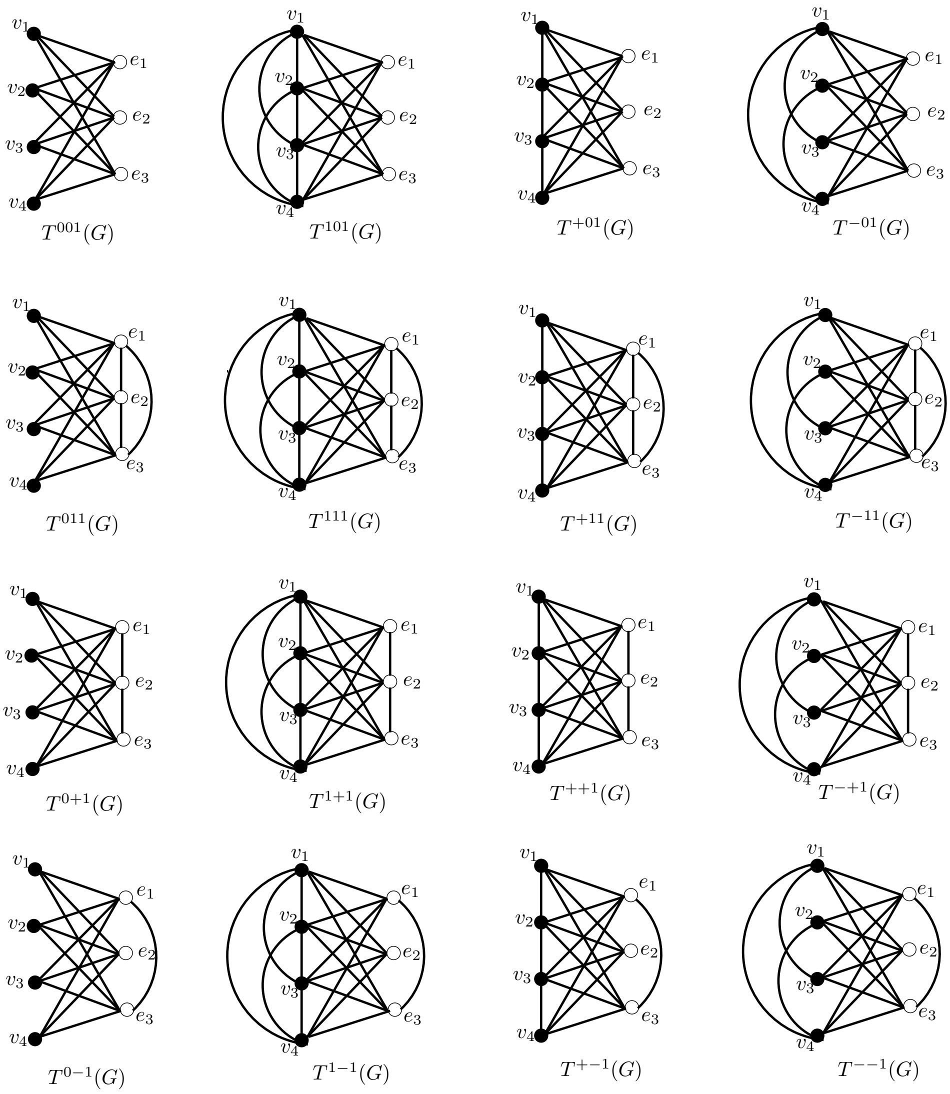


G :
v_1 v_2 v_3 v_4 e_1 e_2 e_3
$T^{000}(G)$
$T^{100}(G)$
$T^{+00}(G)$
$T^{-00}(G)$
$T^{010}(G)$
$T^{110}(G)$
$T^{+10}(G)$
$T^{-10}(G)$
$T^{0+0}(G)$
$T^{1+0}(G)$
$T^{++0}(G)$
$T^{-+0}(G)$
$T^{0-0}(G)$
$T^{1-0}(G)$
$T^{+-0}(G)$
$T^{--0}(G)$

FIGURE 1



v_1
v_2
v_3
v_4
e_1
e_2
e_3
$T^{001}(G)$
$T^{101}(G)$
$T^{+01}(G)$
$T^{-01}(G)$
$T^{011}(G)$
$T^{111}(G)$
$T^{+11}(G)$
$T^{-11}(G)$
$T^{0+1}(G)$
$T^{1+1}(G)$
$T^{++1}(G)$
$T^{-+1}(G)$
$T^{0-1}(G)$
$T^{1-1}(G)$
$T^{+-1}(G)$
$T^{--1}(G)$

FIGURE 2

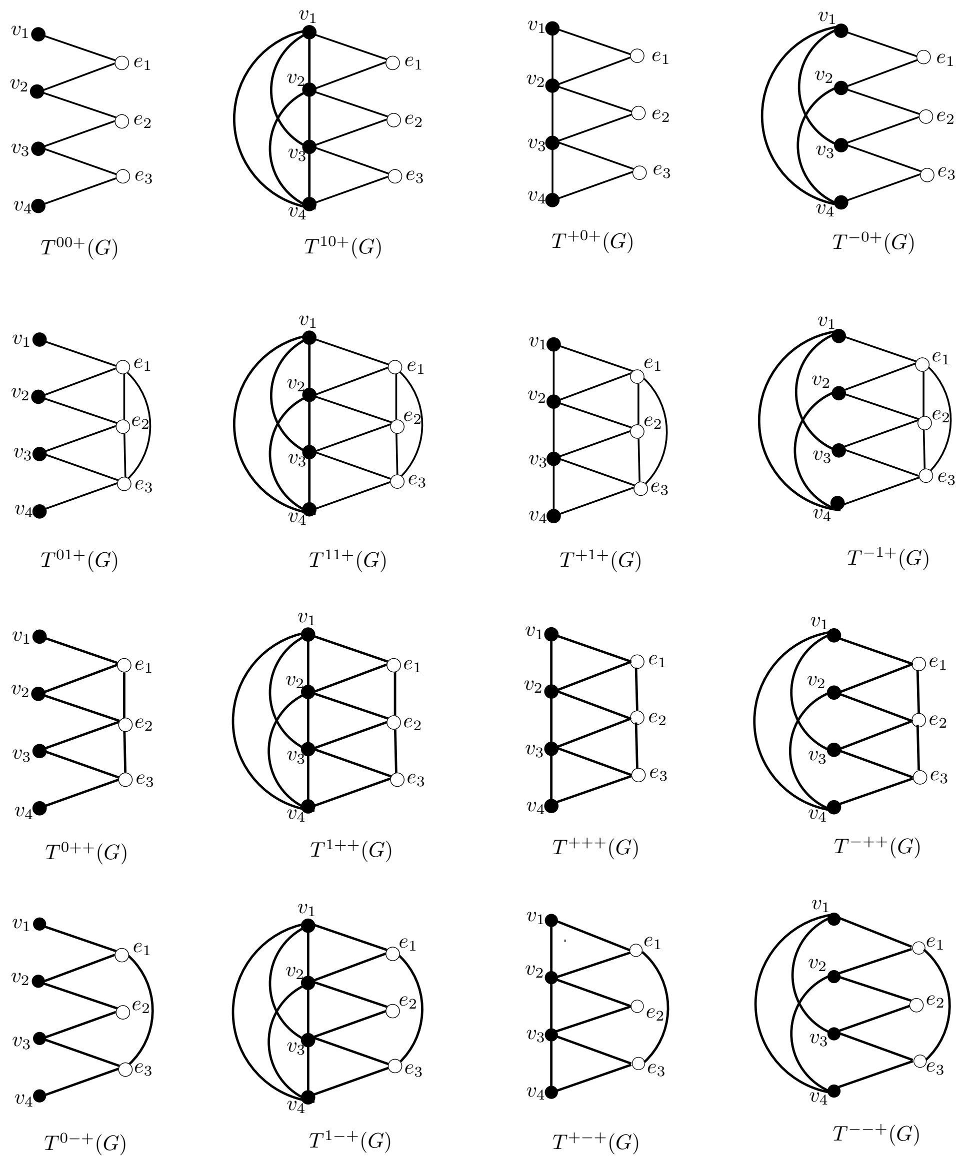


FIGURE 3

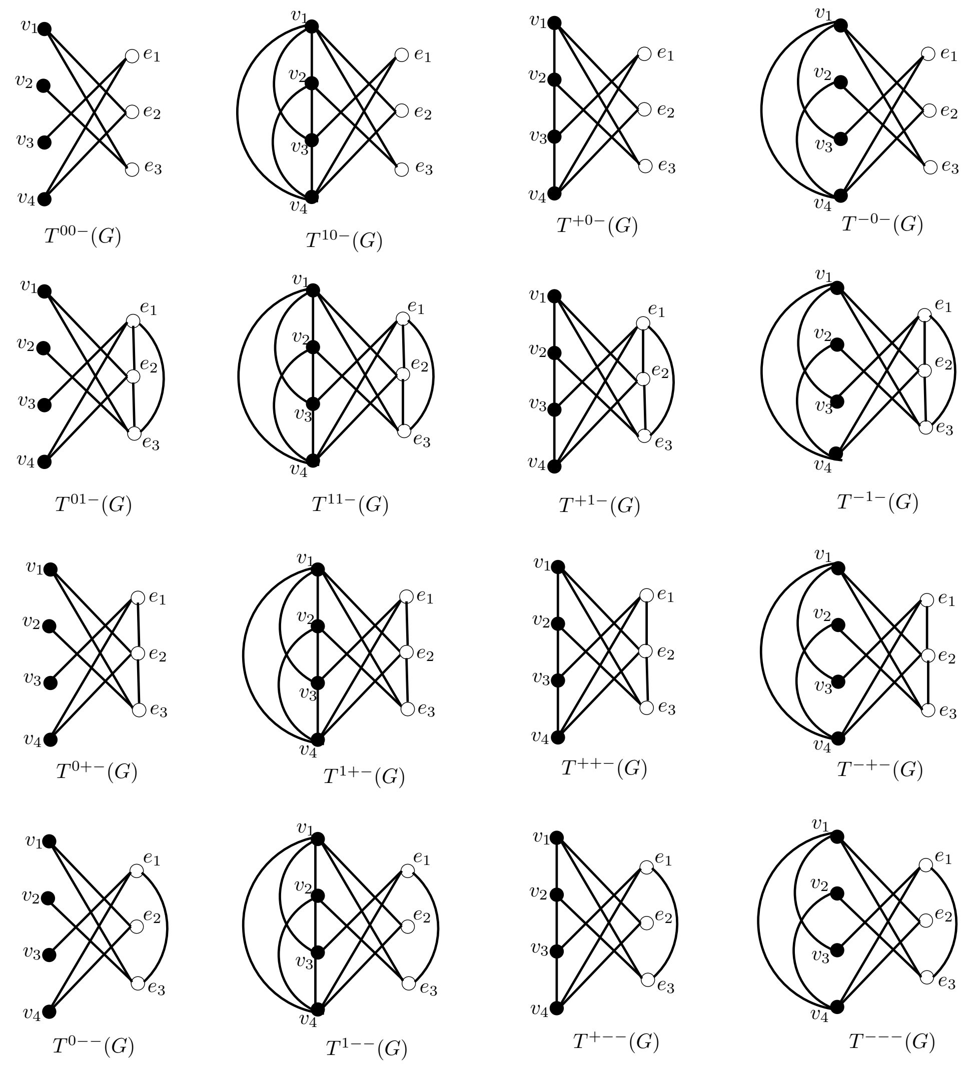


FIGURE 4