JOURNAL OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Vol. 9(2)(2019), 277-292
DOI: 10.7251/JIMVI1902277B

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

P_1- ALMOST DISTRIBUTIVE FUZZY LATTICES

Assaye A. Berhanu, Alemneh Mihret, and Tefera Gerima

ABSTRACT. The concept of P_1-Almost Distributive Fuzzy Lattice as an extension of P_1-Almost Distributive Lattice is introduced and we prove basic properties a bout P_1-ADFL. Necessary and sufficient conditions for a P_0-ADFL to become P_1-ADFL are investigated.

1. Introduction

U. M. Swamy and G. C. Rao in [**11**] introduced the concept of an Almost Distributive Lattice (ADL) as a common abstraction of existing lattice and ring theoretic generalization of a Boolean algebra and observed that the set $PI(R)$ of all principal ideals of an ADL $(R, \vee, \wedge, 0, m)$ with a maximal element m,form a distributive lattice. G. Epstein and A. Horn in [**4**] introduced the concept of a P_1-lattice. Later in [**12**] T. Traczyk studied and explored its properties P_1-lattice has good application in computer and logic theory and the notion of a P_1-ADL was introduced by G. C. Rao, A. Mihret and N. Kakumar in [**5**].

The concept of fuzzy set was introduced by Zadeh in [**13**] and this concept was adapted by Goguen in [**5**] and Sanchez in [**10**] use to define and study fuzzy relations. In this paper we use fuzzy partial order relation defined in [**3**] and the idea of fuzzy lattice in [**3**] to extend some important properties of $P_1 - ADL$ to $P_1 - ADFL$'.

2. Preliminaries

DEFINITION 2.1. An algebra $(R, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an ADL if it satisfies the following axioms:

2010 *Mathematics Subject Classification.* 06D99; 06D15.
Key words and phrases. Almost Distributive Lattice (ADL), P_0-ADL,P_1- ADL, Principal ideal of an ADL, P_0- Almost Distributive Fuzzy Lattice (P_0- ADFL), P_1- Almost Distributive Fuzzy Lattice (P_1 - ADFL).

(1) $a \vee 0 = a$.
(2) $0 \wedge a = 0$.
(3) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$.
(4) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.
(5) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.
(6) $(a \vee b) \wedge b = b$, for all $a, b, c \in R$.

THEOREM 2.2 ([**9**]). *Let m be a maximal element in an ADL R and $a \in R$. Then the following are equivalent :*

(1) *m is maximal element of a poset $(R, \leqslant)$.*
(2) *$a \wedge m = a$.*
(3) *$a \vee m = m$.*
(4) *$a \vee m$ is maximal.*

DEFINITION 2.3. ([**12**]) Let R be an ADL with a maximal element m and

$$B(R) = \{a \in R | a \wedge b = 0 \text{ and } a \vee b \text{ is maximal for some } b \in R\}.$$

Then $(B(R), \vee, \wedge)$ is a relatively complemented ADL and it is called the Birkhoff center of R.

LEMMA 2.4 ([**9**]). *Let $(R, \vee, \wedge, 0)$ be an ADL. Then for any $a, b, c \in R$, the following conditions hold:*

(1) *$a \vee b = a \Leftrightarrow a \wedge b = b$.*
(2) *$a \vee b = b \Leftrightarrow a \wedge b = a$.*
(3) *$\bigwedge$ is associative.*
(4) *$a \wedge b \wedge c = b \wedge a \wedge c$.*
(5) *$(a \vee b) \wedge c = (b \vee a) \wedge c$.*
(6) *$a \wedge b = 0 \Leftrightarrow b \wedge a = 0$.*
(7) *$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.*
(8) *$a \wedge (a \vee b) = a$, $(a \wedge b) \vee b = b$ and $a \vee (b \wedge a) = a$.*
$a \wedge a = a$ and $a \vee a = a$.
(9) *$0 \vee a = a$ and $a \wedge 0 = 0$.*

DEFINITION 2.5. ([**9**]) Let R be an ADL with 0. Then for any $a, b \in R$, define $a \leqslant b$ if and only if $a \wedge b = a$ or equivalently $a \vee b = b$. Then $(R, \leqslant)$ is a poset.

DEFINITION 2.6. ([**7**]) Let R be an ADL with 0. Then a unary operation $\star$ on R is called a pseudo-complementation on R if, for $a, b \in R$:

(1) $a \wedge a^\star = 0$.
(2) $a \wedge b = 0 \Rightarrow a^\star \wedge b = b$.
(3) $(a \vee b)^\star = a^\star \wedge b^\star$.

So that $a^\star$ is called a pseudo-complement of $a \in R$ and R is called a pseudo-complemented ADL.An element a in R is said to be dense if $a^\star = 0$.

DEFINITION 2.7. ([**7**]) A pseudo-complemented ADL $(R, \vee, \wedge, \star, 0, m)$ is called a stone ADL if, for any $a \in R$, $a^\star \vee a^{\star\star} = 0^\star$.

DEFINITION 2.8. ([**9**]) Let R be an ADL with 0. A non empty subset I of R is an ideal of R, if it satisfy the following conditions:

(1) $a, b \in I \Rightarrow a \vee b \in I$.
(2) $a \in I$ and $x \in R \Rightarrow a \wedge x \in R$.

THEOREM 2.9 ([**9**]). *Let R be an ADL with 0. Then for any $a, b \in R$, we have the following :*

(1) $(a] = \{a \wedge x | x \in R\}$.
(2) $a \in (b] \Leftrightarrow b \wedge a = a$
$\Leftrightarrow (a] \subseteq (b] \Leftrightarrow a \wedge x \leqslant b \wedge x$, *for all* $x \in R$.

LEMMA 2.10 ([**9**]). *For any $a, b \in R$, the following hold:*

(1) $(a] \cap (b] = (a \wedge b] = (b \wedge a]$.
(2) $(a] \vee (b] = (a \vee b] = (b \vee a]$.

DEFINITION 2.11. ([**7**]) Let $(R, \vee, \wedge, 0, m)$ be an ADL with Birkhoff center $B(R)$. R is called a psuedo-suplemented ADL if, for each $x \in R$, there exists $b \in B(R)$ such that

(1) $x \wedge b = b$.
(2) If $c \in B(R)$ such that $x \wedge c = c$, then $b \wedge c = c$.

In this case, $b \wedge m$ is uniquely determined by x and is denoted by $!x$. We call $!x$ the pseudo-supplement of x.

DEFINITION 2.12. ([**7**]) Let R be a bounded distributive lattice and $B(R)$ the center of R. Achain base of R is a finite sequence $0 = e_0, e_1, ..., e_{n-2}, e_{n-1} = 1$ such that R is generated by $B(R) \cup \{e_0, e_1, ..., e_{n-1} = 1\}$.

If R has a chain base, then R is called a P_0- lattice.

DEFINITION 2.13. ([**7**]) Let R be an ADL with 0 and maximal elements. Then R is called a P_0- ADL if $(PI(R), \vee, \wedge, 0, R)$ is a P_0- lattice.

DEFINITION 2.14. ([**3**]) Let X be a set. A function $A : X \times X \rightarrow [0, 1]$ is said to be fuzzy partial order relation if it satisfies the following condition;

(1) $A(x, x) = 1, \forall x \in X$ that is A is reflexive.
(2) $A(x, y) > 0$, and $A(y, x) > 0$ implies that $x = y$.That is A is antisymmetric.
(3) $A(x, z) \geqslant sup_{y \in X} min[A(x, y), A(y, z)] > 0$. That is A is transitive.

If A is a fuzzy partial order relation in a set X, then (X, A) is called a fuzzy partial order relation or fuzzy poset.

DEFINITION 2.15. ([**3**]) Let (X, A) be a fuzzy poset. Then (X, A) is a fuzzy lattice if and only if $x \vee y$, and $x \wedge y$ exists for all $x, y \in X$.

DEFINITION 2.16. ([**3**]) Let (X, A) be a fuzzy lattice. Then (X, A) is distributive if and only if
$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, and $(x \vee y) \wedge (x \vee z) = x \vee (y \wedge z)$, for all $x, y, z \in X$.

DEFINITION 2.17. ([**6**]) Let(X, A) be a fuzzy lattice and $Y \subseteq X$. Then Y is an ideal of (X, A):

(1) If $x \in X$,$y \in Y$ and $A(x, y) > 0$, then $x \in Y$.
(2) If $x, y \in Y$,then $x \vee y \in Y$.

DEFINITION 2.18. ([**6**]) Let (X, A) be a fuzzy lattice and $x \in X$. Then the set determined by $\downarrow x = \{y \in X : A(y, x) > 0\}$ is called principal ideal of (X, A) generated by x. The family of all ideals of a fuzzy lattice (X, A) will be denoted by $I(X)$.

DEFINITION 2.19. Let $(R, \vee, \wedge, 0, m)$ be an ADL with a maximal element m, suppose $\rightarrow$ is a binary operation on R satisfying the following axioms for $a, b, c \in R$:

(1) $a \rightarrow a = m$
(2) $(a \rightarrow b) \wedge b = b$
(3) $a \wedge (a \rightarrow b) = a \wedge b \wedge m$
(4) $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$
(5) $(a \vee b) \rightarrow c = (a \rightarrow c) \wedge (b \rightarrow c)$,then $(R, \vee, \wedge, \rightarrow, 0, m)$ is Heyting almost distributive lattice.

DEFINITION 2.20. ([**5**]) A P_1- lattice $(R; e_0, e_1, ..., e_{n-1})$ is a P_0- lattice together with a chain base such that $e_{i+1} \rightarrow e_i = e_i$. It follows that $e_i \rightarrow e_j = e_j$, for $i > j$ and $e_i \rightarrow e_j = 1$, for $i \leqslant j$.

DEFINITION 2.21. ([**5**]) A P_1- ADL is a P_0- ADL $(R; e_0, e_1, ..., e_{n-1})$ such that $(e_{i+1} \rightarrow e_i) \wedge m = e_i \wedge m$, for every i.

LEMMA 2.22 ([**5**]). *If $(R; e_0, e_1, ..., e_{n-1})$ is a P_1- ADL, then the following holds:*

(1) $e_i \rightarrow e_j = e_j$, *for* $i > j$
(2) $e_i \rightarrow e_j = m$, *for* $i < j$.

THEOREM 2.23 ([**8**]). *Let $(R; e_o, e_1, ..., e_{n-1})$ be a P_0- ADL with Birkhoff center $B(R)$. Then the following are equivalent:*

(1) $e_i \rightarrow e_j$ *exists for* $0 \leqslant i,j \leqslant n - 1$
(2) *R is a Heyting ADL*
(3) *R is an R-ADL.*

DEFINITION 2.24. An ADL R with maximal element m is called an $R - ADL$, if for any $a, b \in R$, $[(a \rightarrow b) \vee (b \rightarrow a)] \wedge m = m$.

THEOREM 2.25 ([**7**]). *Every element in a P_O- ADL has both monotone and disjoint representation.*

DEFINITION 2.26. ([**2**]) Let (R, A) be an ADFL with maximal elements. Then (R, A) is called P_0- ADFL if $(PI(R), A)$ with maximal element R is a P_0- fuzzy lattice.

THEOREM 2.27. *Let (R, A) be an ADFL and let $((R, A); e_0, e_1, ..., e_{n-1})$ is a P_0- ADFL with center $B_A(R)$ and $R' = [e_i, e_j]$, where $e_i \leqslant e_j$. Then*

$$((R', A); e_{i+1}, e_{i+2}, ..., e_j)$$

is a P_0- *ADFL with center* $B_A(R^{'}) = \{[e_i \vee (b_i \wedge e_j)] \wedge m, b_i \in B_A(R)\}$.

DEFINITION 2.28. ([**1**]) Let $(R, \vee, \wedge, 0)$ be an algebra of type $(2, 2, 0)$ and we call (R, A) is an Almost Distributive Fuzzy Lattice(ADFL) if the following condition satisfied:

(1) $A(a, a \vee 0) = A(a \vee 0, a) = 1$.
(2) $A(0, 0 \wedge a) = A(0 \wedge a, 0) = 1$.
(3) $A((a \vee b) \wedge c, (a \wedge c) \vee (b \wedge c)) = A((a \wedge c) \vee (b \wedge c), (a \vee b) \wedge c) = 1$.
(4) $A(a \wedge (b \vee c), (a \wedge b) \vee (a \wedge c)) = A((a \wedge b) \vee (a \wedge c), a \wedge (b \vee c)) = 1$.
(5) $A(a \vee (b \wedge c), (a \vee b) \wedge (a \vee c)) = A((a \vee b) \wedge (a \vee c), a \vee (b \wedge c)) = 1$.
(6) $A((a \vee b) \wedge b, b) = A(b, (a \vee b) \wedge b) = 1$, for all $a, b, c \in R$.

Throughout this paper we write (R, A) for an ADFL and R an ADL $(R, \vee, \wedge, 0)$ with maximal element ,$(PI(R), A)$ is a principal ideal fuzzy lattice of an ADFL (R, A) and $B_A(PI(R))$ the Birkhoff center of a principal ideal $(PI(R), A)$. In (R, A) and $(PI(R), A)$ A represents $A : R \times R \to [0, 1]$ and $A : PI(R) \times PI(R) \to [0, 1]$ respectively $x \in (R, A) \Leftrightarrow x \in R$ and $(a] \in (PI(R), A) \Leftrightarrow (a] \in PI(R)$.

3. P_1- Almost Distributive Fuzzy Lattice

DEFINITION 3.1. Let (R, A) be an ADFL with maximal element m and let $\to$ be a binary operation on (R, A) satisfy the following axioms for $a, b, c \in R$:

(1) $A(a \to a, m) = A(m, a \to a) = 1$
(2) $A(b, (a \to b) \wedge b) > 0$
(3) $A(a \wedge (a \to b), a \wedge b \wedge m) = A(a \wedge b \wedge m, a \wedge (a \to b)) = 1$
(4) $A(a \to (b \wedge c), (a \to b) \wedge (a \to c)) = A((a \to b) \wedge (a \to c), a \to (b \wedge c)) = 1$
(5) $A((a \vee b) \to c, (a \to c) \wedge (b \to c)) = A((a \to c) \wedge (b \to c), (a \vee b) \to c) = 1$.

Then (R, A) is called a Heyting Almost Distributive Fuzzy Lattice.

DEFINITION 3.2. An R-ADFL is a Heyting ADFL in which

$$A(m, [(a \to b) \vee (b \to a)] \wedge m) > 0$$

holds for all $a, b \in R$.

LEMMA 3.3. *Let* (R, A) *be an ADFL with a maximal element* m *and* $B_A(R)$ *be the Birkhoff center of* (R, A)*. Then for each maximal element* m *and for* $x, y \in R$ *the following holds:*

(1) *If* $x \to y$ *exists and* $b \in B_A(R)$*, then*
$$A((b \wedge x) \to y, b^m \vee (x \to y)) = A(b^m \vee (x \to y), (b \wedge x) \to y) = 1.$$
(2) *If* $x \to y$ *exists and* $c \in B_A(R)$*, then*
$$A(x \to (c \vee y), (c \wedge m) \vee (x \to y)) = A((c \wedge m) \vee (x \to y), x \to (c \vee y)) = 1.$$
(3) *If* $x \to y$ *exists and* $b, c \in B_A(R)$*, then*
$$A((b \wedge x) \to (c \vee y), b^m \vee (c \wedge m) \vee (x \to y)) = A(b^m \vee (c \wedge m) \vee (x \to y), (b \wedge x) \to (c \vee y)) = 1,$$
where b^m *is the complement of* $b \wedge m$ *in* $[0, m]$.

THEOREM 3.4. *An ADFL (R, A) is a Heyting ADFL if and only if for any $a, b \in R$, there exist $t \in R$ such that :*
H_1: $A(a \wedge t, b \wedge a \wedge t) > 0$ and
H_2: If $s \in R$ and $A(a \wedge s, b \wedge a \wedge s) > 0$.
then $A(s, t \wedge s) > 0$.

PROOF. Assume (R, A) is a Heyting ADFL.
(1) Let $a, b \in R$ for $t \in R$, $b \wedge a \wedge t = a \wedge t$, since $b \wedge a = a$. Imply that $b \wedge a \wedge t \leqslant a \wedge t$ and $a \wedge t \leqslant b \wedge a \wedge t$ by definition of equality. Hence $A(a \wedge t, b \wedge a \wedge t) > 0$. Therefore H_1 holds.
(2) For $a, b \in R$, there exist $s \in R$ such that $b \wedge a \wedge s = a \wedge s$, then $t \wedge s = s$. Imply that $b \wedge a \wedge s \leqslant a \wedge s$ and $a \wedge s \leqslant b \wedge a \wedge s$. So that $A(a \wedge s, b \wedge a \wedge s) > 0$ and $t \wedge s = s$ which imply that $t \wedge s \leqslant s$ and $s \leqslant t \wedge s$. Hence $A(s, t \wedge s) > 0$. Therefore $A(a \wedge s, b \wedge a \wedge s) > 0$,then $A(s, t \wedge s) > 0$, for all $a, b \in R$ and $s \in R$.
Conversely, for any $a, b \in R$, we have
H_1:$A(a \wedge t, b \wedge a \wedge t) > 0$
H_2: If $s \in R$ and $A(a \wedge s, b \wedge a \wedge s) > 0$, then $A(s, s \wedge s) > 0$.
Let m be a maximal element of(R, A). Then $t \wedge m = a \to b$ exists. Now, for any $a, b \in R$, we have

$$A(a \wedge m, \vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m)) = A(\vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m), a \wedge m) = 1 \text{ and}$$

$$A(b \wedge m, \vee_{i=1}^{n-1}(c_i \wedge e_i \wedge m) = A(\vee_{i=1}^{n-1}(c_i \wedge e_i \wedge m), b \wedge m) = 1$$

Write $b \wedge m = \wedge_{j=1}^{n-1}(c_j \vee e_{j-1}) \wedge m$. Then

$$A(b \wedge m, \wedge_{j=1}^{n-1}(c_j \vee e_{j-1}) \wedge m) = A(\wedge_{j=1}^{n-1}(c_j \vee e_{j-1}) \wedge m, b \wedge m) = 1.$$

Let i and j be fixed. So that

$$A(b_i \wedge e_i \to c_j \vee e_{j-1}, [b_i^m \vee (c_j \wedge m)] \vee (e_i \to e_{j-1}) \\ = A([b_i^m \vee (c_j \wedge m) \vee (e_i \to e_{j-1}), b \wedge e_i \to c_j \vee e_{j-1}) = 1)$$

and hence

$$A((a \to b) \wedge m, \vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m) \to \wedge_{j=1}^{n-1}(c_j \vee e_{j-1}) \wedge m) = \\ A(\vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m) \to \wedge_{j=1}^{n-1}(c_j \vee e_{j-1}) \wedge m), (a \to b) \wedge m)) = 1$$

Thus
A($\vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m) \to \wedge_{j=1}^{n-1}(c_j \vee e_{j-1}) \wedge m, \wedge_{i=1}^{n-1} \wedge_{j=1}^{n-1} \{[b_i^m \vee (c_j \wedge m)] \vee (e_i \to e_{j-1})\} \wedge m)$
$= A(\vee_{i=1}^{n-1} \wedge_{j=1}^{n-1} [(b_i \wedge e_i) \to (c_i \vee e_{j-1})] \wedge m, \wedge_{i=1}^{n-1} \wedge_{j=1}^{n-1} \{[b_i^m \vee (c_j \wedge m)] \vee (e_i \to e_{j-1})\} \wedge m)$
$= A(\wedge_{i=1}^{n-1} \wedge_{j=1}^{n-1} \{[b_i^m \vee (c_j \wedge m)] \vee (e_i \to e_{j-1})\} \wedge m, \wedge_{i=1}^{n-1} \wedge_{j=1}^{n-1} \{[b_i^m \vee (c_j \wedge m)] \vee (e_i \to e_{j-1})\} \wedge m) = 1$.
So that we get
$A((a \to b) \wedge m, \wedge_{i=1}^{n-1} \wedge_{j=1}^{n-1} \{[b_i^m \vee (c_i \wedge m)] \vee (e_i \to e_{j-1})\} \wedge m)$
$= A(\wedge_{i=1}^{n-1} \wedge_{j=1}^{n-1} \{[b_i^m \vee (c_j \wedge m)] \vee (e_i \to e_{j-1})\} \wedge m, (a \to b) \wedge m) = 1$.
Imply that $e_i \to e_{j-1}$ exists. Therefore (R, A) is a Heyting ADFL. □

COROLLARY 3.5. *Let (R, A) be an ADFL with maximal element m and Birkhoff center $B_A(R)$ and $a, b \in R$. If $A(a, b \wedge a) > 0$, then $A(m, (a \to b) \wedge m) > 0$.*

LEMMA 3.6. *Let (R, A) be an ADFL and $\star$ be a pseudo-complementation on (R, A) with Birkhoff center $B_A(R)$. Then*

$$A((b \wedge x)^\star \wedge m, (b^m \vee x^\star) \wedge m) = A((b^m \vee x^\star) \wedge m, (b \wedge x)^\star \wedge m) = 1$$

for $b \in B_A(R)$,$x \in (R, A)$.

COROLLARY 3.7. *Let $((R, A); e_1, e_2, ..., e_{n-1})$ be a P_0- ADFL. Let $x \in R$ and*

$$A(x \wedge m, \vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m)) = A(\vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m), x \wedge m) = 1$$

be a monotone representation of x. Then

$$A(\vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m), \wedge_{i=1}^{n-1}(b_i \vee e_{i-1}) \wedge m) = A(\wedge_{i=1}^{n-1}(b_i \vee e_{i-1}) \wedge m, \vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m)) = 1.$$

More over, if $\star$ is pseudo- complementation on (R, A), then

$$A(\wedge_{i=1}^{n-1}(b_i^m \vee e_{i-1}^\star), x^\star) = A(x^\star, \wedge_{i=1}^{n-1}(b_i^m \vee e_{i-1}^\star)) = 1.$$

THEOREM 3.8. *Let $((R, A); e_0, e_1, ..., e_{n-1})$ be a P_0- ADFL with Birkhoff center $B_A(R)$. Then the following are equivalent.*

(1) *$e_i \to e_j$ exists, for $0 \leqslant i,j \leqslant n-1$.*
(2) *(R, A) is a Heyting ADFL.*
(3) *(R, A) is an R-ADFL.*

PROOF. (1) $\Rightarrow$ (2) Suppose $e_i \to e_j$ exists for $0 \leqslant i,j \leqslant n-1$. Then $e_i \to e_{j-1}$ exists for $0 \leqslant i,j \leqslant n-1$. Let $x, y \in R$ and $A(x \wedge m, \vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m)) = A(\vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m), x \wedge m) = 1$ and $A(y \wedge m, \vee_{i=1}^{n-1}(c_i \wedge e_i \wedge m))$=$A(\vee_{i=1}^{n-1}(c_i \wedge e_i \wedge m), y \wedge m) = 1$ be mono.reps. of x and y respectively. Thus for fixed i,j and $b_i, c_j \in B_A(R)$,$(b_i \wedge e_i) \to (c_j \vee e_{j-1})$ exists and equal to $[b_i^m \vee (c_j \wedge m)] \vee (e_i \to e_{j-1})$.

$A((x \to y) \wedge m, \wedge_{i,j}[b_i^m \vee (c_j \wedge m)] \vee (e_i \to e_{j-1}) \wedge m)$
$=A(\vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m) \to \wedge_{j=1}^{n-1}(c_j \vee e_{j-1}) \wedge m, \wedge_{i,j}[b_i^m \vee (c_j \wedge m)] \vee (e_i \to e_{j-1}) \wedge m)$
$= A(\vee_{i=1}^{n-1} \wedge_{j=1}^{n-1}[(b_i \wedge e_i) \to (c_j \vee e_{j-1})] \wedge m, \wedge_{i,j}[b_i^m \vee (c_j \wedge m)] \vee (e_i \to e_{j-1}) \wedge m)$
$=A(\wedge_{i=1}^{n-1} \wedge_{j=1}^{n-1}([b_i^m \vee (c_j \wedge m)] \vee (e_i \to e_{j-1}) \wedge m, \wedge_{i,j}[b_i^m \vee (c_j \wedge m) \vee (e_i \to e_{j-1}) \wedge m)$
$=A(\wedge_{i,j}[b_i^m \vee (c_j \wedge m)] \vee (e_i \to e_{j-1}) \wedge m, \wedge_{i,j}[b_i^m \vee (c_j \wedge m)] \vee (e_i \to e_{j-1}) \wedge m) = 1.$

Hence

$A((x \to y) \wedge m, \wedge_{i,j}\{[b_i^m \vee (c_j \wedge m)] \vee (e_i \to e_{j-1})\} \wedge m)$
$= A(\wedge_{i,j}\{[b_i^m \vee (c_j \wedge m)] \vee (e_i \to e_{j-1})\} \wedge m, (x \to y) \wedge m) = 1.$

Thus (R, A) is a Heyting ADFL.

To prove(2) $\Rightarrow$ (3), suppose (R, A) is a Heyting ADFL. Let $x, y \in R$ such that

$$A(x \wedge m, \vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m)) = A(\vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m), x \wedge m) = 1 \text{ and}$$

$$A(y \wedge m, \vee_{i=1}^{n-1}(c_i \wedge e_i \wedge m)) = A(\vee_{i=1}^{n-1}(c_i \wedge e_i \wedge m), y \wedge m) = 1$$

be mono.reps of x and y respectively. Since

$A((x \to y) \wedge m, \wedge_{i=1}^{n-1} \wedge_{j=1}^{n-1} \{[b_i^m \vee (c_j \wedge m)] \vee (e_i \to e_{j-1})\} \wedge m)$
$= A(x \to y) \wedge m, \wedge_{i=1}^{n-1} \wedge_{j=1}^{n-1} \{[b_i^m \vee (c_j \wedge m)] \vee (e_i \to e_{j-1})\} \wedge m) = 1$ and

$A((y \to x) \wedge m, \wedge_{i=1}^{n-1} \wedge_{j=1}^{n-1} \{[c_i^m \vee (b_j \wedge m)] \vee (e_i \to e_{j-1})\} \wedge m)$
$= A((y \to x) \wedge m, \wedge_{i=1}^{n-1} \wedge_{j=1}^{n-1} \{[c_i^m \vee (b_j \wedge m)] \vee (e_i \to e_{j-1})\} \wedge m) = 1.$
Now, again
$A((x \to y) \wedge m, \wedge_{i=1}^{n-1} \wedge_{j=1}^{i} \{[b_i^m \vee (c_j \wedge m)] \vee (e_i \to e_{j-1})\} \wedge m)$
$= A(\wedge_{i=1}^{n-1} \wedge_{j=1}^{n-1} \{[b_i^m \vee (c_j \wedge m)] \vee (e_i \to e_{j-1})\} \wedge m,$
$\wedge_{i=1}^{n-1} \wedge_{j=1}^{i} \{[b_i^m \vee (c_j \wedge m)] \vee (e_i \to e_{j-1})\} \wedge m)$
$= A(\wedge_{i=1}^{n-1} \wedge_{j=1}^{i} \{[b_i^m \vee (c_j \wedge m)] \vee (e_i \to e_{j-1})\} \wedge$
$\wedge_{j=i+1}^{n-1} \{[b_i^m \vee (c_j \wedge m)] \vee (e_i \to e_{j-1})\} \wedge m,$
$\wedge_{i=1}^{n-1} \wedge_{j=1}^{i} \{[b_i^m \vee (c_j \wedge m)] \vee (e_i \to e_{j-1})\} \wedge m)$
$= A(\wedge_{i=1}^{n-1} \wedge_{j=1}^{i} \{[b_i^m \vee (c_j \wedge m)] \vee (e_i \to e_{j-1})\} \wedge m,$
$\wedge_{i=1}^{n-1} \wedge_{j=1}^{i} \{[b_i^m \vee (c_j \wedge m)] \vee (e_i \to e_{j-1})\} \wedge m) = 1.$
Hence

$$A((x \to y) \wedge m, \wedge_{i=1}^{n-1} \wedge_{j=1}^{i} \{[b_i^m \vee (c_j \wedge m)] \vee (e_i \to e_{j-1})\} \wedge m) = 1,$$

since $e_{j-1} \wedge e_i = e_i$, for $j > i$. We get

$$A((e_i \to e_{j-1}) \wedge m, m) = A(m, (e_i \to e_{j-1}) \wedge m) = 1.$$

So that
$A((x \to y) \wedge m, \wedge_{i=1}^{n-1} \wedge_{j=1}^{i} (b_i^m \vee (c_j \wedge m)) \vee (e_i \to e_{j-1}) \wedge m)$
$\geqslant A((x \to y) \wedge m, \wedge_{i=1}^{n-1} \wedge_{j=1}^{i} (b_i^m \vee c_j) \wedge m)$
$\geqslant A(\wedge_{i=1}^{n-1} \wedge_{j=1}^{i} (b_i^m \vee c_j) \wedge m, \wedge_{i=1}^{n-1} (b_i^m \vee c_i) \wedge m)$
$= A(\wedge_{i=1}^{n-1} [b_i^m \vee \wedge_{j=1}^{i} (c_j \wedge m)], \wedge_{i=1}^{n-1} (b_i^m \vee c_i) \wedge m)$
$= A(\wedge_{i=1}^{n-1} (b_i^m \vee c_i) \wedge m, \wedge_{i=1}^{n-1} (b_i^m \vee c_i) \wedge m) = 1.$
Hence $A((x \to y) \wedge m, \wedge_{i=1}^{n-1} (b_i^m \vee c_i) \wedge m) > 0$. Similarly

$$A(\wedge_{i=1}^{n-1} (b_{i=1}^m \vee c_i) \wedge m, (x \to y) \wedge m) > 0.$$

In the same manner we have $A(\wedge_{i=1}^{n-1} (c_i^m \vee b_j) \wedge m, (y \to x) \wedge m) > 0$, which imply

$$A([(x \to y) \vee (y \to x)] \wedge m, \wedge_{i=1}^{n-1} \wedge_{j=1}^{n-1} (b_i^m \vee c_j \vee c_i^m \vee b_j) \wedge m) > 0.$$

Hence

$$A(\wedge_{i=1}^{n-1} \wedge_{j=1}^{n-1} (b_i^m \vee c_j \vee c_i^m \vee b_j) \wedge m, [(x \to y) \vee (y \to x)] \wedge m) > 0.$$

Now, if $i > j$, then $A(b_j \vee b_i^m, b_i \vee b_i^m = m) > 0$. If $i \leqslant j$, then $A(c_j \vee c_i^m, c_i \vee c_i^m = m) > 0$. Hence, for all $i, j \in \{1, 2, ..., n-1\}$, $A(b_i^m \vee c_j \vee c_i^m \vee b_j, m) > 0$. Hence $A(m, [(x \to y) \vee (y \to x)] \wedge m) > 0$. Therefore (R, A) is an R- ADFL.

To prove (3) $\Rightarrow$ (1), assume (R, A) be an R- ADFL. So that
$A((x \to y) \wedge m, \wedge_{i=1}^{n-1} \wedge_{j=1}^{n-1} \{[b_i^m \vee (c_j \wedge m)] \vee (e_i \to e_{j-1})\} \wedge m) =$
$A(\wedge_{i=1}^{n-1} \wedge_{i=1}^{n-1} \{[b_i^m \vee (c_j \wedge m)] \vee (e_i \to e_{j-1})\} \wedge m, (x \to y) \wedge m) = 1$, and
$A((y \to x) \wedge m, \wedge_{i=1}^{n-1} \wedge_{j=1}^{n-1} \{[c_i^m \vee (b_j \wedge m)] \vee (e_i \to e_{j-1})\} \wedge m)$

$= A(\wedge_{i=1}^{n-1} \wedge_{j=1}^{n-1} \{[(c_i^m \vee (b_j \wedge m)] \vee (e_i \to e_{j-1})\} \wedge m, (y \to x) \wedge m) = 1.$
Therefore $e_i \to e_{j-1}$ exists. □

DEFINITION 3.9. For a P_0- fuzzy lattice $((R, A); e_0, e_1, ..., e_{n-1})$ together with a chain base such that $A(e_i, e_{i+1} \to e_i) > 0$, for $0 \leqslant i \leqslant n-1$ we say that it is a P_1-fuzzy lattice.

DEFINITION 3.10. A P_1- ADFL is a P_0- ADFL $((R, A); e_0, e_1, ..., e_{n-1})$ such that $A(e_i \wedge m, (e_{i+1} \to e_i) \wedge m) > 0$, for $0 \leqslant i \leqslant n-1$.

THEOREM 3.11. *Let (R, A) be an ADFL with maximal element m. (R, A) is a P_1-ADFL if and only if $(PI(R), A)$ is a P_1- fuzzy lattice.*

PROOF. Suppose (R, A) is P_1- ADFL. Then there exist a chain base $\{e_1, e_2, ..., e_{n-1}\}$ such that $((R, A); e_1, e_2, ..., e_{n-1})$ is a P_0- ADFL and $A((e_{i+1} \to e_i) \wedge m, e_i \wedge m) > 0$ and $A(e_i \wedge m, (e_{i+1} \to e_i) \wedge m) > 0$, for $0 \leqslant i \leqslant n-2$. Thus $((PI(R), A); (e_0]_A, (e_1]_A, ..., (e_{n-1}]_A)$ is a P_0- fuzzy lattice. Since $(e_{i+1} \to e_i) \wedge m = e_i \wedge m$. So that

$$[(e_{i+1}]_A \to (e_i]_A] \wedge (m]_A = ((e_{i+1} \to e_i) \wedge m]_A = (e_i \wedge m]_A = (e_i]_A,$$

since m is maximal. Hence $(e_{i+1}]_A \to (e_i]_A \subseteq (e_i]_A$.

Similarly ,$(e_i]_A \subseteq (e_{i+1}]_A \to (e_i]_A$. Thus $((PI(R), A); (e_0]_A, (e_1]_A, ..., (e_{n-1}]_A)$ is a P_1- fuzzy lattice.

Conversely, suppose $((PI(R), A); (e_0]_A, (e_1]_A, ..., (e_{n-1}]_A)$ with maximal element R is a P_1- fuzzy lattice. Then there exist a finite sequence $(0]_A = (e_0]_A \subseteq (e_1]_A \subseteq .... \subseteq (e_{n-1}]_A = R$ such that $((PI(R), A); (e_0]_A, (e_1]_A, ..., (e_{n-1}]_A)$ is a P_0- fuzzy lattice. So that $(e_{i+1}]_A \to (e_i]_A \subseteq (e_i]_A$ and $(e_i]_A \subseteq (e_{i+1}]_A \to (e_i]_A$, for $0 \leqslant i \leqslant n-2$. $(e_{i+1}]_A \to (e_i]_A = (e_i]_A$. Imply that $e_{i+1} \to e_i = e_i$ and we have $e_1 \leqslant e_2 \leqslant ... \leqslant e_{n-1}$. Hence $((R, A); e_0, e_1, ..., e_{n-1})$ is a P_0- ADFL and hence $((R, A); e_0, e_1, ..., e_{n-1})$ is a P_1- ADFL. □

LEMMA 3.12. *Let $((R, A); e_0, e_1, ..., e_{n-1})$ be a P_1- ADFL. Then*

$$A(e_j \wedge m, (e_i \to e_j) \wedge m) > 0.$$

COROLLARY 3.13. *Let $((R, A); e_0, e_1, ..., e_{n-1})$ be a P_1- ADFL. Then (R, A) is a Heyting ADFL.*

THEOREM 3.14. *If $((R, A); e_0, e_1, ..., e_{n-1})$ is a P_0- ADFL with Birkhoff center $B_A(R)$ and (R, A) is a Heyting ADFL, then there exists a chain base $\{f_0, f_1, ..., f_{n-1}\}$ such that $((R, A); f_0, f_1, f_2, ..., f_{n-1})$ is a P_1- ADFL.*

PROOF. Let $((R, A); e_0, e_1, e_2, ..., e_{n-1})$ be a P_0-ADFL and (R, A) be a Heyting ADFL. Then there exist a chain base $\{f_0, f_1, ..., f_{n-1}\}$ such that $f_1 \wedge m$ is dense in (R, A), we show by induction on n.

If n= 1, we get the result by taking $A(0 = e_0, f_0) = A(f_0, 0 = e_0) = 1$.

Assume the result holds for n-1.

Let $(R', A) = ([e_1 \wedge m, m], A)$. Then by Theorem 2.27, $((R', A); e_0, e_1, ..., e_{n-1})$ is a P_0- ADFL. Since $(R', A) \subseteq (R, A)$ and (R, A) a Heyting ADFL, we get

$(R^{'}, A)$ is a Heyting ADFL. Hence by induction hypothesis, there exist a chain base $\{f_1, f_2, ..., f_{n-1}\}$ such that $((R^{'}, A); f_1, f_2, ..., f_{n-1})$ is a P_1- ADFL. Since $\{0 = e_0 = f_0, f_1, f_2, ..., f_{n-1}\}$ is a chain base of (R, A). Therefore $((R^{'}, A)$; f_1, vf_2, ..., $f_{n-1})$ is a P_0- ADFL. Since $((R^{'}, A); f_1, f_2, ..., f_{n-1})$ is a P_1- ADFL. We get $A((f_{i+1} \to f_i) \wedge m, f_i \wedge m) = A(f_i \wedge m, (f_{i+1} \to f_i) \wedge m) = 1$, for $1 \leqslant i \leqslant n-1$ in $(R^{'}, A)$. We prove that for $1 \leqslant i \leqslant n-2$,

$$A((f_{i+1} \to f_i) \wedge m, f_i \wedge m) = A(f_i \wedge m, (f_{i+1} \to f_i) \wedge m) = 1$$

in (R, A). Suppose $t \in R$ and $A(f_{i+1} \wedge t, f_i \wedge f_{i+1} \wedge t) > 0$. Since $A(f_i, f_{i+1}) > 0$ we have $f_i \leqslant f_{i+1}$ and $f_i \wedge f_{i+1} = f_i$. We get $A(f_{i+1} \wedge t, f_i \wedge t) > 0$, but $A(f_i \wedge t, f_{i+1} \wedge t) > 0$. So that $f_i \wedge t = f_{i+1} \wedge t$ by antisymmetry property of A. Hence $A(f_i \wedge t, f_{i+1} \wedge t) = A(f_{i+1} \wedge t, f_i \wedge t) = 1$. We need to show $A(t, f_i \wedge t) > 0$. Let $A(s, (f_i \vee t) \wedge m) = A((f_i \vee t) \wedge m, s) = 1$ and hence $s \in R^{'}$. Now,

$A(f_i \wedge s, (f_i \wedge m) \vee (f_{i+1} \wedge t \wedge m) = A(f_i \wedge (f_i \vee t) \wedge m, (f_i \wedge m) \vee (f_{i+1} \wedge t \wedge m))$
$= A([(f_i \wedge f_i) \vee (f_i \wedge t)] \wedge m, (f_i \wedge m) \vee (f_{i+1} \wedge t \wedge m))$
$= A([f_i \vee (f_i \wedge t)] \wedge m, (f_i \wedge m) \vee (f_{i+1} \wedge t \wedge m))$ since $f_i \wedge f_i = f_i$
$= A((f_i \wedge m) \vee (f_i \wedge t \wedge m), (f_i \wedge m) \vee (f_{i+1} \wedge t \wedge m))$
$=A((f_i \wedge m) \vee (f_i \wedge t \wedge m), (f_i \wedge m) \vee (f_i \wedge t \wedge m)) = 1$.
$A(f_{i+1} \wedge s, (f_i \wedge m) \vee (f_{i+1} \wedge t \wedge m))$=$A(f_{i+1} \wedge (f_i \vee t) \wedge m, f_{i+1} \wedge s)$
$= A([(f_{i+1} \wedge f_i) \vee (f_{i+1} \wedge t)] \wedge m, f_{i+1} \wedge s)$
$A([f_i \vee (f_{i+1} \wedge t)] \wedge m, [f_i \vee (f_{i+1} \wedge t)] \wedge m)$
$= A((f_i \wedge m) \vee (f_{i+1} \wedge t \wedge m), (f_i \wedge m) \vee (f_{i+1} \wedge t \wedge m)) = 1$.
Imply that $f_i \wedge s = f_{i+1} \wedge s$, since $f_i \wedge t = f_{i+1} \wedge t$. Hence $A(f_i \wedge s, f_{i+1} \wedge s) = A(f_{i+1} \wedge s.f_i \wedge s) = 1$.
$A(f_i \wedge f_{i+1} \wedge s, s)$= $A(f_i \wedge [(f_i \wedge m) \vee (f_i \wedge t \wedge m], s)$
=$A(f_i \wedge [(f_i \vee (f_i \wedge t) \wedge m], s)$
=$A(f_i \wedge [(f_i \vee f_i) \wedge (f_i \vee t) \wedge m], s)$
$= A(f_i \wedge [f_i \wedge (f_i \vee t) \wedge m], s)$
$= A((f_i \vee t) \wedge m, s) = A(s, s) = 1$.
Hence $A(s, f_i \wedge f_{i+1} \wedge s)$= $A(f_i \wedge f_{i+1} \wedge s, s) = 1$. We have $A((f_{i+1} \to f_i) \wedge s, s)$ =$A(s, (f_{i+1} \to f_i) \wedge s) = 1$. So that $A(s, f_i \wedge s) > 0$. Therefore $A(f_i \wedge (f_i \vee t) \wedge m, (f_i \vee t) \wedge m)$=$A((f_i \vee t) \wedge m, f_i \wedge (f_i \vee t) \wedge m) = 1$. $A((f_i \vee t) \wedge m, [f_i \vee (f_i \wedge t)] \wedge m)$= $A(f_i \wedge (f_i \vee t) \wedge m, [f_i \vee (f_i \wedge t)] \wedge m)$
=$A([(f_i \wedge f_i) \vee (f_i \wedge t)] \wedge m, [f_i \vee (f_i \wedge t)] \wedge m)$
$= A([f_i \vee (f_i \wedge t)] \wedge m, [f_i \vee (f_i \wedge t)] \wedge m) = 1$.
Imply that $f_i \wedge t = t$. Hence $A(t, f_i \wedge t) > 0$. So that $A((f_{i+1} \to f_i) \wedge m, f_i \wedge m)$ =$A(f_i \wedge m, (f_{i+1} \to f_i) \wedge m) = 1$ in (R, A). for $1 \leqslant i \leqslant n-2$. Since f_1 is dense in (R, A),$A(f_1 \to f_0, f_0) = A(f_0, f_1 \to f_0) = 1$. Hence $((R, A); f_0, f_1, ..., f_{n-1})$ is a P_1- ADFL. □

THEOREM 3.15. *If* $((R, A); e_{j0}, e_{j1}, ..., e_{n_j-1})$ *is a* P_1- *ADFL, for* $j \in J$, $(R, A) = (\Pi_{j\in J} R_j, A)$, $n = max\{n_j\}$ *and* e_{jk} *is defined to be* $e_j(n_j - 1)$ *i.e*

$e_{jk} = e_{j(n_j-1)}$ *for* $k \geqslant n_j$*,then* $((\Pi_{j\in J} R_j, A); e_0, e_1, ..., e_{n-1})$ *is a* P_1- *ADFL, where* $e_i = (e_{ji}, j \in J)$.

Proof. Let $((R_j, A); e_{j0}, e_{j1}, ..., e_{j(n_k)})$ is a P_1- ADFL, where $e_i = (e_{ji}, j \in J), i \in J$. Then for each $j \in J$,

$$A((e_{j(i+1)} \to e_{i+1}) \wedge m, e_{i+1} \wedge m) = A(e_{i+1} \wedge m, (e_{j(i+1)} \to e_{i+1}) \wedge m) = 1,$$

for $0 \leqslant i \leqslant n_j - 1$. Now, define $e_i = (e_{ji}, j \in J)$ for $0 \leqslant i \leqslant n-1$, where $e_{ji} = e_j(n_j - 1)$, for $n_j \leqslant i \leqslant n-1$. Then by $((R, A); e_0, e_1, ..., e_{n-1})$ is a P_0- ADFL and $A(e_{i+1} \wedge m, (e_{j(i+1)} \to e_{i+1}) \wedge m) > 0$, for $0 \leqslant i \leqslant n-1$. Fix i, for $0 \leqslant i \leqslant n-1$, we need to show $A(e_i \wedge m, (e_{i+1} \to e_i) \wedge m) > 0$. Since

$A((e_i \wedge t \wedge m)_j, e_{ji} \wedge m)$=$A((e_i \wedge e_{i+1} \wedge t \wedge m)_j, e_{ji} \wedge m)$

=$A((e_{ji} \wedge e_{j(i+1)} \wedge t_j \wedge m), e_{ji} \wedge m) \leqslant A(e_{ji} \wedge m, e_{ji} \wedge m) = 1$

imply $(e_i \wedge t \wedge m)_j \leqslant e_{ji} \wedge m$, for $j \in J$. Hence $A((e_i \wedge t \wedge m)_j, e_{ji} \wedge m) > 0$. Therefore $A(e_{ji} \wedge t_j \wedge m, e_{ji} \wedge m) > 0$. Thus $A(t_j \wedge m, e_{j(i+1)} \to e_{ji} = e_{ji} \wedge m) > 0$ for all $j \in J$. so that $A(t \wedge m, e_i \wedge m) > 0$. Imply that $A(t, e_i) > 0$ and hence $A(t, e_i \wedge t) > 0$. Therefore $A(e_i, e_{i+1} \to e_i) > 0$. Hence $A(e_i \wedge m, (e_{i+1} \to e_i) \wedge m) > 0$, for $0 \leqslant i \leqslant n-1$. Thus $((R, A); e_0, e_1, ..., e_{n-1})$ is a P_1- ADFL. □

Lemma 3.16. *Let* (R, A) *be an ADFL with Birkhoff center* $B_A(R)$ *and* b_1, b_2, ..., b_{n-1}, c_1, c_2, ..., c_{n-1} *be elements in* $B_A(R)$ *such that* $A(b_i \wedge b_j, 0) > 0$*, for* $i \neq j$ *and* $A(c_{i+1} \wedge m, c_i \wedge c_{i+1} \wedge m) > 0$*, for* $1 \leqslant i \leqslant n-2$*. Then*

$$A(\vee_{i=1}^{n-1}(b_i \wedge c_i \wedge m), \wedge_{i=1}^{n-1}(b_i^m \vee c_i) \wedge m) = A(\wedge_{i=1}^{n-1}(b_i^m \vee c_i) \wedge m, \vee_{i=1}^{n-1}(b_i \wedge c_i \wedge m)) = 1$$

where b_i^m *is the complement of* $b_i \wedge m$ *in* $[0, m]$ *and*

$$A(b_0 \wedge m, \wedge_{i=1}^{1} b_i^m) = A(\wedge_{i=1}^{1} b_{i=1}^m, b_0 \wedge m) = 1 \text{ and } A(c_0, m) = A(m, c_0) = 1.$$

Definition 3.17. Let $((R, A); e_0, e_1, ..., e_{n-1})$ be a P_1- ADFL. The chain base $\{e_0, e_1, ..., e_{n-1}\}$ is called strictly increasing if $A(e_0 \wedge m, e_1 \wedge m) > 0$, $A(e_1 \wedge m, e_2 \wedge m) > 0, ...,$ and $A(e_{n-2} \wedge m, e_{n-1} \wedge m) > 0$.

Theorem 3.18. *Let* $((R, A); e_0, e_1, ..., e_{n-1})$ *be a* P_1- *ADFL. Then*

(1) [i] $A(0 = e_0 \wedge m, e_1 \wedge m) > 0$*,* $A(e_1 \wedge m, e_2 \wedge m) > 0$*,...,*

$$A(e_p \wedge m = e_{p-1} \wedge m, e_{n-1} \wedge m = m) = A(e_{n-1} \wedge m = m, e_p \wedge m = e_{p-1} \wedge m) = 1,$$

for some $p \geqslant 1$ *and* (R, A) *has order* p*.*

[ii] $e_{i+1} \wedge m$ *is the smallest dense element in* $[e_i \wedge m, m]$*, for* $0 \leqslant i \leqslant p-2$*.*

[iii] $\{e_0, e_1, ..., e_{n-1}\}$ *is the unique strictly increasing chain base in* (R, A) *satisfying* (ii) *such that* $((R, A); e_0, e_1, ..., e_{p-1})$ *is a* P_1- *ADFL.*

(2) *If* $A(x \wedge m, \vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m)) = A(\vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m), x \wedge m) = 1$ *is a mono.rep. of* x*, then*

$$A([(x \to e_i) \to e_i] \wedge m, (b_{i+1}^m \vee e_i) \wedge m) = A((b_{i+1}^m \vee e_i) \wedge m, [(x \to e_i) \to e_i] \wedge m) = 1,$$

for $0 \leqslant i \leqslant n-1$*.*

(3) *If*

$$A(x \wedge m, \vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m) = A(\vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m), x \wedge m) = 1$$

is a disjoint representation of x and

$$A(y\wedge m,\vee_{i=1}^{n-1}(c_i\wedge e_i\wedge m))=A(\vee_{i=1}^{n-1}(c_i\wedge e_i\wedge m),y\wedge m)=1$$

is a mono.reps. of x, then

$$A((x\to y)\wedge m,(y\wedge m)\vee[\vee_{i=1}^{n-1}(b_i\wedge c_i\wedge m)]=$$
$$A((y\wedge m)\vee[\vee_{i=1}^{n-1}(b_i\wedge c_i\wedge m),(x\to y)\wedge m)=1$$

where $A(b_0\wedge m,\wedge_{i=1}^{1}b_i^m)=A(\wedge_{i=1}^{1}b_i^m,b_0\wedge m)=1$, and $A(c_0,m)=A(m,c_0)=1$.

Proof. Let $((R,A);e_0,e_1,...,e_{n-1})$ be a P_1- ADFL.

(1) Let p be the smallest positive integer such that

$$A(e_{p-1}\wedge m,e_p\wedge m)=A(e_p\wedge m,e_{p-1}\wedge m)=1.$$

Now, since $e_{p-1}\wedge m=(e_p\to e_{p-1})\wedge m=m$. Hence

$$A(e_{p-1}\wedge m,m)=A(m,e_{p-1}\wedge m)=1.$$

Imply that

$$A(e_{p-1}\wedge m,e_p\wedge m)=A(e_p\wedge m,e_{p-1}\wedge m)=1,\ ...\ ,A(m,e_{p-1}\wedge m)>0.$$

Hence (R,A) has order p and $\{e_0,e_1,...,e_{p-1}\}$ is strictly increasing chain base such that $((R,A);e_0,e_1,...,e_{p-1})$ is a P_1- ADFL. Now, since $(e_{i+1}\to e_i)\wedge m=e_i\wedge m$ in $[e_i\wedge m,m]$ imply that

$$A(e_i\wedge m,(e_{i+1}\to e_i)\wedge m)=A((e_{i+1}\to e_i)\wedge m,e_i\wedge m)=1$$

in $([e_i\wedge m,m],A)$ where $e_{i+1}\wedge m$ is dense in $[e_i\wedge m,m]$. Suppose f is dense in $[e_i\wedge m,m]$, for $0\leqslant i\leqslant p-2$. We know that $([e_i\wedge m,m],A)$ is a P_0- fuzzy lattice with $\{e_i\wedge m,e_{i+1}\wedge m,...,e_{p-1}\wedge m=m)\}$ as a chain base. Let

$$A(f\wedge m,\vee_{k=i+1}^{p-1}(b_k\wedge e_k\wedge m))=A(\vee_{k=i+1}^{p-1}(b_k\wedge e_k\wedge m),f\wedge m)=1$$

be a mono.reps. of f where $b_k\in B_A([e_i\wedge m,m])$. Then
$A(e_i\wedge m,b_{i+1}^m)=A(f^\star\wedge m,b_{i+1}^m)=A(\wedge_{k=i+1}^{p-1}(b_i^m\vee e_k^\star)\wedge m,b_{i+1}^m)$ by corollary 3.6
$=A((b_{i+1}^m\vee e_{i+1}^\star)\wedge(b_{i+2}^m\vee e_{i+2}^\star)\wedge...\wedge(b_{p-1}^m\vee e_{p-1}^\star)\wedge m,b_{i+1}^m)$
$\geqslant A([b_{i+1}^m\vee(e_{i+1}^\star\wedge...\wedge e_{p-1}^\star)]\wedge m,b_{i+1}^m)=A((b_{i+1}^m\vee e_{i+1}^\star)\wedge m,b_{i+1}^m)$
$=A(b_{i+1}^m,b_{i+1}^m)=1.$
Hence $A(b_{i+1}^m,e_i\wedge m)$=$A(e_i\wedge m,b_{i+1}^m)=1$ since b_{i+1}^m is in $[e_i\wedge m,m]$. Now,
$A(m,b_{i+1}\wedge m)=A((b_{i+1}\vee b_{i+1}^m)\wedge m,b_{i+1}\wedge m)=A((b_{i+1}\vee e_i\wedge m)\wedge m,b_{i+1}\wedge m)$
$=A((b_{i+1}\vee e_i)\wedge m,b_{i+1}\wedge m)=A((b_{i+1}\wedge m)\vee(e_i\wedge m),b_{i+1}\wedge m)$
$=A(b_{i+1}\wedge m,b_{i+1}\wedge m)=1$, since $e_i\wedge m$ is the zero element in $[e_i\wedge m,m]$.
Hence $A(m,b_{i+1}\wedge m)$ =$A(b_{i+1}\wedge m,m)=1>0$. Again,
$A(e_{i+1}\wedge m,f\wedge m)=A(b_{i+1}\wedge e_{i+1}\wedge m,f\wedge m)$
$\leqslant A((b_{i+1}\wedge e_{i+1}\wedge m)\vee\vee_{k=i+2}^{p-1}(b_k\wedge e_k\wedge m),f\wedge m)$
$=A(\vee_{k=i+1}^{p-1}(b_k\wedge e_k\wedge m),f\wedge m)=A(f\wedge m,f\wedge m)=1>0.$
Therefore $A(e_{i+1}\wedge m,f\wedge m))$= $A(f\wedge m,e_{i+1}\wedge m)=1$. Thus $e_{i+1}\wedge m$ is the smallest dense element in $[e_i\wedge m,m]$. Now, to prove the uniqueness, suppose $\{e_0.e_1,...,e_{n-1}\}$

and $\{f_1, f_2, ..., f_{n-1}\}$ with $\{A(e_0, e_1) > 0, A(e_1, e_2) > 0, ..., A(e_{p-2}, e_{p-1}) > 0\}$ and $\{A(f_0, f_1) > 0, A(f_1, f_2) > 0, ..., A(f_{k-2}, f_{k-1}) > 0\}$ be strictly increasing chain base for (R, A) satisfying (ii). Clearly $A(e_0 \wedge m, f_0 \wedge m)= A(f_0 \wedge m, e_0 \wedge m) = 1$. Assume that $A(e_i \wedge m, f_i \wedge m)= A(f_i \wedge m, e_i \wedge m) = 1$, for $0 \leqslant i \leqslant p-1$. Then $e_{i+1} \wedge m$, $f_{i+1} \wedge m$ are smallest dense elements in $[e_i \wedge m, m] = [f_i \wedge m, m]$. Thus we get $A(e_{i+1} \wedge m, f_{i+1} \wedge m) = A(f_{i+1} \wedge m, e_{i+1} \wedge m) = 1$. Hence by induction, $p = k$ and $A(e_i \wedge m, f_i \wedge m) = A(f_i \wedge m, e_i \wedge m) = 1$, for $0 \leqslant i \leqslant p-1$. Therefore $\{A(0 = e_0, e_1) > 0, A(e_1, e_2) > 0, ..., A(e_{p-2}, e_{p-1}) > 0\}$ is strictly increasing chain base for (R, A) such that $((R, A); e_0, e_1, ..., e_{p-1})$ is a P_1- ADFL.

(2) Let $x \in R$ and

$$A(x \wedge m, \vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m) = A(\vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m), x \wedge m) = 1$$

be a monotone representation of x. Now,

$A((x \to e_i) \wedge m, (b_{i+1}^m \vee e_i) \wedge m) = (\vee_{j=1}^{n-1}(b_j \wedge e_j \wedge m)) \to e_i \wedge m, (b_{i+1}^m \vee e_i) \wedge m)$
$= A((\vee_{j=1}^{n-1}[(b_j \wedge e_j) \to e_i)] \wedge m, (b_{i+1}^m \vee e_i) \wedge m)$
$= A(\wedge_{j=1}^{n-1}(b_j^m \vee e_j) \to e_i)) \wedge m, (b_{i+1}^m \vee e_i) \wedge m))$
$= A((\wedge_{j=1}^{n-1}(b_j^m \vee (e_j \to e_i)) \wedge m, (b_{i+1}^m \vee e_i) \wedge m)$
$= A(\wedge_{j=1}^{i}[(b_j^m \vee (e_j \to e_i) \wedge (\wedge_{j=i+1}^{n-1} b_j^m \vee (e_j \to e_i), (b_{i+1}^m \vee e_i) \wedge m)$
$= A(\wedge_{j=i+1}^{n-1}(b_j^m \vee (e_i \wedge m), (b_{i+1}^m \vee e_i) \wedge m)$ since $b_j \geqslant b_{j+1} \Rightarrow b_{j+1}^m \geqslant b_j^m$
$= A((\wedge_{j=i+1}^{n-1}(b_j^m \vee (e_i \wedge m), (b_{i+1}^m \vee e_i) \wedge m)$
$= A((\wedge_{j=i+1}^{n-1} b_j^m \vee e_i) \wedge (\wedge_{j=i+1}^{n-1} b_j^m \vee m), (b_{i+1}^m \vee e_i) \wedge m)$
$= A((\wedge_{j=i+1}^{n-1} b_j^m \vee e_i) \wedge m, (b_{i+1}^m \vee e_i) \wedge m)$, since $b_j^m \in [e_i \wedge m, m]$.
$= A((b_{i+1}^m \vee e_i) \wedge m, (b_{i+1}^m \vee e_i) \wedge m) = 1.$

Hence

$A((x \to e_i) \wedge m, (\wedge_{j=i+1}^{n-1} b_j^m \vee e_i) \wedge m) = A(\wedge_{j=i+1}^{n-1} b_j^m \vee e_i) \wedge m, (x \to e_i) \wedge m) = 1.$

So that we have

$$A((x \to e_i) \wedge m, (b_{i+1}^m \vee e_i) \wedge m) = A((b_{i+1}^m \vee e_i, (x \to e_i) \wedge m) = 1,$$

since $\wedge_{j=i+1}^{n-1} b_j^m = b_{i+1}^m$. Again,

$A([(x \to e_i) \to e_i] \wedge m, (b_{i+1}^m \vee e_i) \wedge m)$
$= A([(b_{i+1}^m \vee e_i) \to e_i] \wedge m, (b_{i+1}^m \vee e_i) \wedge m)$
$= A([(b_{i+1}^m \to e_i) \wedge (e_i \to e_i)] \wedge m, (b_{i+1}^m \vee e_i) \wedge m)$
$= A([b_{i+1}^m \to e_i) \wedge m] \wedge m, (b_{i+1}^m \vee e_i) \wedge m)$ since $e_i \to e_i = m$
$= A((b_{i+1}^m \to e_i) \wedge m, (b_{i+1}^m \vee e_i) \wedge m)$
$= A((b_{i+1}^m \wedge e_{n-1} \to e_i) \wedge m, (b_{i+1}^m \vee e_i) \wedge m)$, since $b_{i+1}^m = b_{i+1}^m \wedge e_{n-1}$
$= A([b_{i+1}^m \vee (e_{n-1} \to e_i) \wedge m, (b_{i+1}^m \vee e_i) \wedge m)$,
since $b_{i+1}^m \wedge e_{n-1} \to e_i = b_{i+1}^m \vee (e_{n-1} \to e_i)$
$= A((b_{i+1}^m \wedge m) \vee (e_{n-1} \to e_i) \wedge m), (b_{i+1}^m \vee e_i) \wedge m)$

$= A((b^m_{i+1} \wedge m) \vee (e_i \wedge m), (b^m_{i+1} \vee e_i) \wedge m)$
$= A((b^m_{i+1} \vee e_i) \wedge m, (b^m_{i+1} \vee e_i) \wedge m) = 1$, for $0 \leqslant i \leqslant n-1$.
Hence
$A([(x \to e_i) \to e_i] \wedge m, (b^m_{i+1} \vee e_i) \wedge m) = A((b^m_{i+1} \vee e_i) \wedge m, [(x \to e_i) \to e_i)] \wedge m) = 1.$

(3) Let $x, y \in R$ and

$$A(x \wedge m, \vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m)) = A(\vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m), x \wedge m) = 1$$

be a disjoint representation of x and

$$A(y \wedge m, \vee_{j=1}^{n-1}(c_j \wedge e_j \wedge m) = A(\vee_{j=1}^{n-1}(c_j \wedge e_j \wedge m), y \wedge m) = 1$$

be a mono.reps. of y. Then
$A((x \to y) \wedge m, \wedge_{i=1}^{n-1}[b^m_i \vee (e_i \to y)] \wedge m)$
$= A([\vee_{i=1}^{n-1}(b_i \wedge e_i \wedge m] \to (y \wedge m), \wedge_{i=1}^{n-1}[b^m_i \vee (e_i \to y)] \wedge m)$
$= A(\vee_{i=1}^{n-1}[(b_i \wedge e_i) \to y] \wedge m, \wedge_{i=1}^{n-1}[b^m_i \vee (e_i \to y)] \wedge m)$
$= A(\wedge_{i=1}^{n-1}[b^m_i \vee (e_i \to y)] \wedge m, \wedge_{i=1}^{n-1}[b^m_i \vee (e_i \to y)] \wedge m) = 1.$
Hence

$$A((x \to y) \wedge m, \wedge_{i=1}^{n-1}[b^m_i \vee (e_i \to y)] \wedge m)$$
$$= A(\wedge_{i=1}^{n-1}[b^m_i \vee (e_i \to y)] \wedge m, (x \to y) \wedge m) = 1.$$

Again,
$A((e_i \to y) \wedge m, \vee_{j=1}^{i-1}(c_j \wedge e_j \wedge m) \vee (c_i \wedge m))$
$= A([e_i \to \vee_{j=1}^{n-1}[c_j \wedge e_j)] \wedge m, \vee_{j=1}^{i-1}(c_j \wedge e_j \wedge m) \vee (c_i \wedge m))$
$= A(\vee_{j=1}^{n-1}[e_i \to (c_j \wedge e_j)] \wedge m, \vee_{j=1}^{i-1}(c_j \wedge e_j \wedge m) \vee (c_i \wedge m))$
$= A(\vee_{j=1}^{n-1}([(e_i \to c_j) \wedge (e_i \to e_j)] \wedge m, \vee_{j=1}^{i-1}(c_j \wedge e_j \wedge m) \vee (c_i \wedge m)$
$= A(\vee_{j=1}^{i-1}([(e_i \to c_j) \wedge (e_i \to e_j)] \wedge m) \vee (\vee_{j=i}^{n-1}[(e_i \to c_j) \wedge ((e_i \to e_j)] \wedge m, \vee_{j=1}^{i-1}(c_j \wedge e_j \wedge m) \vee (c_i \wedge m))$
$A(\vee_{j=1}^{i-1}([(e_i \to c_j) \wedge (e_j \wedge m)] \vee (\vee_{j=i}^{n-1}[(e_i \to c_j) \wedge m]), \vee_{j=1}^{i-1}(c_j \wedge e_j \wedge m) \vee (c_i \wedge m)),$
since $(e_i \to e_j) \wedge m = e_j \wedge m$, for $j < i$ and $(e_i \to e_i) \wedge m = m \wedge m = m$
$= A(\vee_{j=1}^{i-1}(c_j \wedge e_j \wedge m) \vee (\vee_{j=i}^{n-1}(c_j \wedge m)), \vee_{j=1}^{i-1}(c_j \wedge e_j \wedge m) \vee (c_i \wedge m))$
$= A(\vee_{j=1}^{i-1}(c_j \wedge e_j \wedge m) \vee (c_i \wedge m), \vee_{j=1}^{i-1}(c_j \wedge e_j \wedge m) \vee (c_i \wedge m)) = 1.$
Hence
$A((e_i \to y) \wedge m, \vee_{j=1}^{i-1}(c_j \wedge e_i \wedge m) \vee (c_i \wedge m))$
$= A(\vee_{j=1}^{i-1}(c_j \wedge e_j \wedge m) \vee (e_i \wedge m), (e_i \to y) \wedge m) = 1.$
Now,
$A((y \vee c_i) \wedge m, \vee_{j=1}^{i-1}(c_j \wedge e_j \wedge m))$
$= A([\vee_{j=1}^{n-1}(c_j \wedge e_j \wedge m)] \vee (c_i \wedge m), \vee_{j=1}^{i-1}(c_j \wedge e_j \wedge m) \vee (c_i \wedge m))$
$= A([(c_1 \wedge e_1) \vee (c_2 \wedge e_2) \vee ...(c_i \wedge e_i) \vee ....(c_{n-1} \wedge e_{n-1}) \vee c_i] \wedge m, \vee_{j=1}^{i-1}(c_j \wedge e_j \wedge m) \vee (c_i \wedge m))$

$= A([(c_1 \wedge e_1) \vee (c_2 \wedge e_2) \vee ... \vee c_i] \wedge m, \vee_{j=1}^{i-1}(c_j \wedge e_j \wedge m) \vee (c_i \wedge m))$
$= A(\vee_{j=1}^{i-1}(c_j \wedge e_j \wedge m) \vee (c_i \wedge m), \vee_{j=1}^{i-1}(c_j \wedge e_j \wedge m) \vee (c_i \wedge m)) = 1.$
Hence

$$A((y \vee c_i) \wedge m, \vee_{j=1}^{i-1}(c_j \wedge e_j \wedge m) \vee (c_i \wedge m))$$
$$= A(\vee_{j=1}^{i-1}(c_j \wedge e_j \wedge m) \vee (c_i \wedge m), (y \vee c_i) \wedge m) = 1.$$

Therefore

$$A(e_i \to y) \wedge m, (y \vee c_i) \wedge m) = A((y \vee c_i) \wedge m, (e_i \to y) \wedge m) = 1 \text{ and}$$

$A((x \to y) \wedge m{=}\wedge_{j=1}^{n-1}(b_i^m \vee (e_i \to y) \wedge m, (y \wedge m) \vee \vee_{i=1}^{n-1}(b_i \wedge c_i \wedge m))$
$= A(\wedge_{i=1}^{n-1}(b_i^m \vee (y \vee c_i) \wedge m), (y \wedge m) \vee \vee_{i=1}^{n-1}(b_i \wedge c_i \wedge m))$
$= A((y \wedge m) \vee \wedge_{i=1}^{n-1}(b_i^m \vee c_i) \wedge m, (y \wedge m) \vee \vee_{i=1}^{n-1}(b_i \wedge c_i \wedge m))$
$= A((y \wedge m) \vee \vee_{i=1}^{n-1}(b_i \wedge c_i \wedge m), (y \wedge m) \vee \vee_{i=1}^{n-1}(b_i \wedge c_i \wedge m)) = 1.$
Thus
$A((x \to y) \wedge m, (y \wedge m) \vee \vee_{i=1}^{n-1}(b_i \wedge c_i \wedge m)$
$= A((y \wedge m) \vee \vee_{i=1}^{n-1}(b_i \wedge c_i \wedge m), (x \to y) \wedge m) = 1$ where

$$A(b_0 \wedge m, \wedge_{i=1}^{1} b_i^m) = A(\wedge_{i=1}^{1} b_i^m, b_o \wedge m) = 1$$

and $A(c_0, m) = A(m, c_0) = 1.$ □

References

[1] A. Berhanu, G. Yohannes and T. Bekalu. Almost distributive fuzzy lattice. *International Journal of Mathematics and its Application*, **5**(1C)(2017), 307–316.

[2] A. Berhanu and T. Gerima. P_0- Almost distributive fuzzy lattice. (communication for publication)

[3] I. Chon. Fuzzy partial order relations and fuzzy lattices. *Korean J. Math.*, **17**(4)(2009), 361–374.

[4] G. Epstein and A. Horn. Chain based lattices. *Pacific J. Math.*, 55(1)(1974), 65–84.

[5] J. A. Goguen. L-fuzzy set. *J. Math. Anal. Appl.*, **18**(1)(1967), 145–174.

[6] I. Mezzomo, B. C. Bedregal and R. H. N. Santiago. On fuzzy ideals of fuzzy lattice. IEEE International Conference on Fuzzy Systems (2012), 1-5. DOI: 10.1109/FUZZ-IEEE.2012.6251307 [*Second Brazilian Congress on fuzzy system* -2012, 657–671.

[7] G. C. Rao and M. Alamneh. P_0- Almost distributive lattice. *Southeast Asian Bull. Math.*, **42**(3)(2018), 437–452

[8] G. C. Rao, A. Mihret and N. Kakumar. p_1- Almost distributive lattice. *International Journal of Mathematical Archive*, **4**(2)(2013), 1–11.

[9] G. C. Rao. *Almost distributive lattice*. Ph.D. Thesis, Andhra University, Waltair, 1980.

[10] E. Sanchez. Resolution of composite fuzzy relation equation. *Information and Control,*, **30**(1)(1976),38–48.

[11] U. M. Swamy and G. C. Rao. Almost distributive lattices. *J. Austral Math. Soc. (series A)*, **31**(1)(1981),78–91.

[12] T. Traczyk. Axioms and properties of post. *Colloq. Math.*, **10**(2)(1963), 193–200.

[13] L. A. Zadeh. Fuzzy sets. *Information and Control*, **8**(3)(1965), 338–353

Received by editors 06.12.2017; Revised version 01.12.2018; Available online 20.05.2019.

A. Berhanu: Bahir Dar University, College of Science, Department of Mathematics, Bahir Dar, Ethiopia

E-mail address: berhanu$_a$*ssaye@yahoo.com*

A. Mihret: Bahir Dar University, College of Science, Department of Mathematics, Bahir Dar, Ethiopia

E-mail address: mihretmahlet@yahoo.com

T. Gerima: Wollo Unversity, College of Natural Science, Department of Mathematics, Dessie, Ethiopia

E-mail address: gerima233@gmail.com