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# ON $*$-IDEALS AND KERNEL IDEALS IN PSEUDO-COMPLEMENTED ALMOST SEMILATTICES

**G. Nanaji Rao, S. Sujatha Kumari**

Abstract. The concepts of ideal quotient, extended ideal, contracted ideal are introduced in $ASL$ $L$ and proved some basic properties of these concepts. Obtained the set $I^*(L)$ of all $*$-ideals of a $*$-commutative $PCASL$ $L$ is a complete lattice with respect to set inclusion and proved that the centre of $I^*(L)$ is trivial. Derived a one-to-one correspondence between set of all extended ideals and set of all contracted ideals in $ASLs$. Also, proved the set $KI(L)$ of all kernel ideals in a $*$-commutative $PCASL$ $L$ in which $x \leqslant x^{**}$, for all $x \in L$ is a complete implicative lattice and the residuals in the lattice $KI(L)$ coincides with the corresponding residuals in the lattice $I^*(L)$. We established an isomorphism between the centre of $KI(L)$ and the Boolean algebra $S(L)$ of all closed elements in $*$-commutative $PCASL$ $L$ in which $x \leqslant x^{**}$, for all $x \in L$.

## 1. Introduction

The concept of pseudo-complementation $*$ on an $ASL$ with 0 was introduced by Nanaji Rao and Sujatha Kumari [**2**], and proved some basic properties of pseudo-complementation $*$. Also, proved that the pseudo-complementation on an $ASL$ is equationally definable. They observed that an $ASL$ can have more than one pseudo-complementation. Infact, they proved that a one-to-one correspondence between set of all pseudo-complementations on an $ASL$ $L$ and the set of all maximal elements in $L$. Also, Nanaji Rao and Sujatha Kumari [**3**], introduced the concepts of kernel ideal, $*$-ideal and $*$-congruence in a $*$-commutative $PCASL$ $L$ and derived necessary and sufficient condition for an $ASL$ congruence to become a $*$-congruence. They established the smallest $*$-congruence with given

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kernel ideal and largest $*$-congruence with given kernel ideal and characterized the largest $*$-congruence in terms of smallest $*$-congruence and the $*$-congruence $\psi$ on $*$-commutative $PCASL$ $L$ defined by $(x, y) \in \psi$ if and only if $x^{**} = y^{**}$.

In this paper we introduced the concepts of ideal quotient in $ASL$ $L$ and proved some basic properties of ideal quotients in $L$. We observed that the set $I^*(L)$ of all $*$-ideals of $*$-commutative $PCASL$ $L$ is a complete lattice with respect to set inclusion and proved that the centre of $I^*(L)$ is trivial. Also, we introduced the concepts of extended ideal and contracted ideal in $ASL$ $L$ and proved some basic properties of these concepts. We established a one-to-one correspondence between set of all extended ideals and set of all contracted ideals in $ASLs$. We proved that the set $KI(L)$ of all kernel ideals in a $*$-commutative $PCASL$ $L$ in which $x \leqslant x^{**}$, for all $x \in L$ is a complete implicative lattice and proved the residuals in the lattice $KI(L)$ coincides with the corresponding residuals in the lattice $I^*(L)$. Finally, we proved that the centre of $KI(L)$ is isomorphic with the Boolea algebra $S(L)$ of all closed elements in $*$-commutative $PCASL$ $L$.

## 2. Preliminaries

In this section we collect few important definitions and results which are already known and which will be used more frequently in the paper.

Definition 2.1. Let $(P, \leqslant)$ be a poset and $S$ be a non-empty subset of $P$. Then

(1) An element $a$ in $P$ is called a lower bound of $S$ if $a \leqslant x$ for all $x \in S$.
(2) An element $a$ in $P$ is called an upper bound of $S$ if $x \leqslant a$ for all $x \in S$.
(3) An element $a$ in $P$ is called the greatest lower bound (g.l.b or infimum) of $S$ if $a$ is a lower bound of $S$ and $b \in P$ such that $b$ is a lower bound of $S$, then $b \leqslant a$.
(4) An element $a$ in $P$ ia called the least upper bound (l.u.b or supremum) of $S$ if $a$ is an upper bound of $S$ and $b \in P$ such that $b$ is an upper bound of $S$, then $a \leqslant b$.

Definition 2.2. A poset $(L, \leqslant)$ is called a complete lattice if, every non-empty subset of $L$ has both l.u.b. and g.l.b. in $L$.

Definition 2.3. An almost semilattice(ASL) is an algebra $(L, \circ)$ where $L$ is a non-empty set and $\circ$ is a binary operation on $L$, satisfies the following conditions:

(1) $(x \circ y) \circ z = x \circ (y \circ z)$ (Associative Law)
(2) $(x \circ y) \circ z = (y \circ x) \circ z$ (Almost Commutative Law)
(3) $x \circ x = x$, for all $x, y, z \in L$ (Idempotent Law).

Theorem 2.1. *Let $L$ be an $ASL$. Define a relation $\leqslant$ on $L$ by $a \leqslant b$ if and only if $a \circ b = a$. Then $\leqslant$ is a partial ordering on $L$.*

Theorem 2.2. *Let $L$ be an $ASL$. Then for any $a, b \in L$, we have the following:*

(1) *$a \circ b \leqslant b$.*
(2) *$a \circ b = b \circ a$ whenever $a \leqslant b$.*

Definition 2.4. An $ASL$ with 0 is an algebra $(L, \circ, 0)$ of type $(2, 0)$ satisfies the following axioms:

(1) $(x \circ y) \circ z = x \circ (y \circ z)$ (*Associative Law*)
(2) $(x \circ y) \circ z = (y \circ x) \circ z$ (*Almost Commutative Law*)
(3) $x \circ x = x$ (*Idempotent Law*)
(4) $0 \circ x = 0$, for all $x, y, z \in L$.

Theorem 2.3. *Let $L$ be an $ASL$ with 0. Then for any $a, b \in L$, we have the following:*

(1) $a \circ 0 = 0$.
(2) $a \circ b = 0$ *if and only if* $b \circ a = 0$.
(3) $a \circ b = b \circ a$ *whenever* $a \circ b = 0$.

Definition 2.5. Let $L$ be an $ASL$. Then an element $m \in L$ is said to be unimaximal if $m \circ x = x$, for all $x \in L$.

Theorem 2.4. *Let $(L, \circ)$ be an $ASL$. Then for any $a, b \in L$ with $a \leqslant b$, we have $a \circ c \leqslant b \circ c$ and $c \circ a \leqslant c \circ b$, for all $c \in L$.*

Definition 2.6. A non-empty subset $I$ of an $ASL$ $L$ is said to be an ideal if $x \in I$ and $a \in L$, then $x \circ a \in I$.

Corollary 2.1. *Let $L$ be an $ASL$ and $I$ be an ideal of $L$. Then, for any $a, b \in L$, $a \circ b \in I$ if and only if $b \circ a \in I$.*

Theorem 2.5. *Let $S$ be a non-empty subset of an $ASL$ $L$. Then $(S] = \{(\circ_{i=1}^{n} s_i) \circ x : x \in L, s_i \in S$ where $1 \leqslant i \leqslant n$ and $n$ is a positive integer $\}$ is the smallest ideal of $L$ containing $S$.*

Lemma 2.1. *Let $L$ be an $ASL$ and $a \in L$. Then $(a] = \{a \circ x : x \in L\}$ is an ideal of $L$.*

Note that, for any $a$ in an $ASL$ $L$, $(a]$ is called the principal ideal generated by $a$.

Lemma 2.2. *Let $L$ be an $ASL$ and $a, b \in L$. Then $a \in (b]$ if and only if $a = b \circ a$.*

Lemma 2.3. *Intersection of any two ideals of an $ASL$ $L$ is again an ideal.*

Theorem 2.6. *The set $I(L)$ of all ideals of an $ASL$ $L$ is a distributive lattice with respect to set inclusion.*

Lemma 2.4. *Let $L$ be an $ASL$ and for any $a, b \in L$, $(a \circ b] = (a] \cap (b] = (b] \cap (a] = (b \circ a]$.*

It can be easily verified that the set $PI(L)$ of all principal ideals of an $ASL$ $L$ is a semilattice with respect to set inclusion.

Theorem 2.7. *Let $L$ be an $ASL$. Then the following conditions are equivalent:*

(1) *The intersections of any family of ideals is non-empty.*
(2) *The intersections of any family of ideals is again an ideal.*
(3) *The lattice $I(L)$ has least element.*

(4) *The lattice* $I(L)$ *is complete.*
(5) *The semilattice* $PI(L)$ *has least element.*
(6) $L$ *has a minimal element.*

Definition 2.7. Let $L$ and $L^{'}$ be two $ASLs$ with zero elements 0 and $0^{'}$ respectively. Then a mapping $f : L \to L^{'}$ is called an ASL homomorphism if it satisfies the following conditions:

(1) $f(a \circ b) = f(a) \circ f(b)$, for all, $a, b \in L$
(2) $f(0) = 0^{'}$.

Definition 2.8. A proper ideal $P$ of an $ASL$ $L$ is said to be a prime ideal if for any $x, y \in L$, $x \circ y \in P$ implies that either $x \in P$ or $y \in P$.

Definition 2.9. Let $(L, \circ, 0)$ be an $ASL$ with zero. Then a unary operation $a \mapsto a^*$ on $L$ is said to be pseudo-complementation on $L$ for any $a, b \in L$, it satisfies the following conditions:

(1) $a \circ b = 0 \Rightarrow a^* \circ b = b$
(2) $a \circ a^* = 0$.

Theorem 2.8. *Let* $L$ *be an* $ASL$ *with* 0. *Then a unary operation* $* : L \to L$ *is a pseudo-complementation on* $L$ *if and only if it satisfies the following conditions:*

(1) $a^* \circ b = (a \circ b)^* \circ b$
(2) $0^* \circ a = a$
(3) $0^{**} = 0$.

Note that, if $L$ is an $ASL$ with pseudo-complementation $*$, then we say that $L$ is a pseudo-complemented ASL and is denoted by PCASL.

Remark 1. *Whether* $*$ *elements commutes are not, is not known so far in pseudo-complementated ASL with pseudo-complementation* $*$*, investigation is going on.*

Definition 2.10. Let $(L, \circ, 0)$ be a pseudo-complemented $ASL$, with pseudo-complementation $*$. Then $L$ is said to be $*$-commutative if $a^* \circ b^* = b^* \circ a^*$, for all $a, b \in L$.

Lemma 2.5. *Let* $L$ *be a* $PCASL$. *Then for any* $a, b \in L$, *we have the following:*

(1) $0^* \circ a = a$
(2) $0^*$ *is unimaximal*
(3) $a^{**} \circ a = a$
(4) $a$ *is unimaximal* $\Rightarrow a^* = 0$
(5) $0^{**} = 0$.

Theorem 2.9. *Let* $L$ *be a* $*$*-commutative* $PCASL$. *Then for any* $a, b \in L$, *we have the following:*

(1) $a \leqslant b \Rightarrow b^* \leqslant a^*$
(2) $a^{***} = a^*$
(3) $a^* \leqslant b^* \Leftrightarrow b^{**} \leqslant a^{**}$.

THEOREM 2.10. *Let $L$ be a $*$-commutative $PCASL$. Then for any $a, b \in L$, we have the following:*

(1) $(a \circ b)^{**} = a^{**} \circ b^{**}$
(2) $(a \circ b)^* = (b \circ a)^*$
(3) $a^*, b^* \leqslant (a \circ b)^*$.

DEFINITION 2.11. An ideal $I$ of a $PCASL$ $L$ is said to be a kernel ideal if $I$ is the kernel of a $*$-congruence on $L$.

THEOREM 2.11. *An ideal $I$ of a $*$-commutative $PCASL$ $L$ is a kernel ideal of $L$ if and only if for any $i, j \in I$ implies $(i^* \circ j^*)^* \in I$.*

COROLLARY 2.2. *An ideal $I$ of a $*$-commutative $PCASL$ $L$ is a kernel ideal if and only if*

(i) $i \in I \Rightarrow i^{**} \in I$

(ii) $i, j \in I \Rightarrow \exists\ k \in I$ *such that* $i^* \circ j^* = k^*$.

THEOREM 2.12. *Let $L$ be a $*$-commutative $PCASL$ and let $I$ be a kernel ideal of $L$. Then the smallest $*$-congruence with kernel $I$ is given by*

$$(x, y) \in R_I \text{ if and only if } i^* \circ x = i^* \circ y, \text{ for some } i \in I.$$

DEFINITION 2.12. An ideal $I$ of a $PCASL$ $L$ is said to be a $*$-ideal if $i \in I$, then $i^{**} \in I$.

## 3. Ideal Quotients in ASLs

Recall that if $L$ is an $ASL$, then the set $I(L)$ of all ideals in $L$ for a distributive lattice with respect to set inclusion and $I(L)$ is a complete lattice with respect to set inclusion provided $L$ has minimal element. It can be easily seen that $I^*(L)$ of all $*$-ideals of a $*$-commutative PCASL $L$ is a complete lattice with respect to set inclusion. In this section we introduce the concept of ideal quotient $(I : J)$ of any two ideals $I, J$ of an ASL $L$ and prove some basic properties of ideal quotients.

DEFINITION 3.1. Let $L$ be an $ASL$ and $I, J$ be ideals of $L$. Then define $(I : J) = \{x \in L : x \circ j \in I,\ for\ all\ j \in J\}$.

THEOREM 3.1. *Let $L$ be an $ASL$ and $I$ be an ideal of $L$. Then for any ideal $J$ of $L, (I : J)$ is an ideal of $L$ containing $I$.*

PROOF. Suppose $I$ is a ideal. Since $0 \in (I : J)$, $(I : J)$ is a non-empty subset of $L$. Let $x \in (I : J)$ and $a \in L$. Then $x \circ j \in I$, for all $j \in J$. This implies $(x \circ j) \circ a \in I$, for all $j \in J$. It follows that $a \circ (x \circ j) \in I$, for all $j \in J$ and hence $(x \circ a) \circ j \in I$, for all $j \in J$. Thus $x \circ a \in (I : J)$. Therefore $(I : J)$ is an ideal of $L$. Let $x \in I$ and $j \in J$. Then $x \circ j \in I$. It follows that $x \in (I : J)$. Therefore $I \subseteq (I : J)$. □

Note that, if $I = (0]$, then $(I : J) = ((0] : J) = (0 : J)$ is denoted by $J^0$.

THEOREM 3.2. *Let $L$ be an $ASL$ and $I, J, K$ be ideals of $L$. Then we have the following:*

(1) $(I : J) \cap K \subseteq I$.

(2) $I \subseteq J \Rightarrow (I : K) \subseteq (J : K)$.

(3) $I \subseteq J \Rightarrow (K : J) \subseteq (K : I)$.

(4) $((I : J) : K) = (I : J \cap K) = ((I : K) : J)$

(5) $(\bigcap_{i=1}^{n} I_i : J) = \bigcap_{i=1}^{n} (I_i : J)$.

(6) $(I : \bigcup_{i=1}^{n} J_i) = \bigcap_{i=1}^{n} (I : J_i)$.

Proof. (1) Let $t \in (I : J) \cap J$. Then $t \in (I : J)$ and $t \in J$. It follows that $t \circ j \in I$ for all, $j \in J$ and $t \in J$. Inparticular $t = t \circ t \in I$. Thus $(I : J) \cap J \subseteq I$. Suppose $I \subseteq J$. Let $x \in (I : K)$. Then $x \circ k \in I$, for all, $k \in K$. It follows that $x \circ k \in J$, for all $k \in K$. Therefore $x \in (J : K)$. Thus $(I : K) \subseteq (J : K)$.

(2) Suppose $I \subseteq J$. Let $x \in (K : J)$. Then $x \circ j \in K$, for all $j \in J$. It follows that $x \circ i \in K$, for all $i \in I$, since $I \subseteq J$. Therefore $x \in (K : I)$. Hence $(K : J) \subseteq (K : I)$.

(3) Let $t \in ((I : J) : K)$. Then $t \circ s \in (I : J)$, for all $s \in K$. It follows that $(t \circ s) \circ j \in I$, for all $j \in J$, for all $s \in K$. Now, let $a \in J \cap K$. Then $a \in J$ and $a \in K$. It follows that $t \circ a = t \circ (a \circ a) \in I$. Therefore $t \in (I : J \cap K)$. Hence $((I : J) : K) \subseteq (I : J \cap K)$. On the other hand, let $t \in (I : J \cap K)$ and $s \in K$. Let $j \in J$. Then $(t \circ s) \circ j = t \circ (s \circ j) \in I$, since $s \circ j \in J \cap K$. Therefore $t \in ((I : J) : K)$. Henec $(I : J \cap K) \subseteq ((I : J) : K)$. Thus $(I : J \cap K) = ((I : J) : K)$. Since $(I : J \cap K) = (I : K \cap J), (I : K \cap J) = ((I : K) : J)$.

(4) Suppose $t \in L$. Then

$$\begin{aligned} t \in (\bigcap_{i=1}^{n} I_i : J) &\Leftrightarrow t \circ j \in \bigcap_{i=1}^{n} I_i, \text{ for all } j \in J \\ &\Leftrightarrow t \circ j \in I_i, \text{ for all } i, \text{ and for all } j \in J \\ &\Leftrightarrow t \in (I_i : J), \text{ for all } i \\ &\Leftrightarrow t \in \bigcap_{i=1}^{n} (I_i : J). \end{aligned}$$

Therefore $(\bigcap_{i=1}^{n} I_i : J) = \bigcap_{i=1}^{n} (I_i : J)$.

(5) Suppose $t \in L$. Then

$$\begin{aligned} t \in (I : \bigcup_{i=1}^{n} J_i) &\Leftrightarrow t \circ j \in I, \text{ for all } j \in \bigcup_{i=1}^{n} J_i \\ &\Leftrightarrow t \circ j \in I,\ for\ all\ j \in J_i\ and\ for\ all\ \text{i} \\ &\Leftrightarrow t \in (I : J_i),\ for\ all\ \text{i} \\ &\Leftrightarrow t \in \bigcap_{i=1}^{n} (I : J_i). \end{aligned}$$

Therefore $(I : \bigcup_{i=1}^{n} J_i) = \bigcap_{i=1}^{n} (I : J_i)$ □

In the following we prove that if $I$ is a $*$-ideal and $J$ is any ideal of an $ASL$ $L$, then $(I : J)$ is a $*$-ideal.

THEOREM 3.3. *Let $L$ be a $*$-commutative PCASL and $I$ be a $*$-ideal of $L$. Then for any ideal $J$ of $L$, $(I : J)$ is a $*$-ideal of $L$.*

PROOF. Let $i \in (I : J)$. Then $i \circ j \in I$, for all $j \in J$. This implies $(i \circ j)^{**} \in I$, for all $j \in J$. It follows that $i^{**} \circ j^{**} \in I$, for all $j \in J$. Now, consider $i^{**} \circ j = i^{**} \circ (j^{**} \circ j) = (i^{**} \circ j^{**}) \circ j \in I$, for all $j \in J$. Hence $i^{**} \circ j \in I$, for all $j \in J$. Thus $i^{**} \in (I : J)$. Therefore $(I : J)$ is a $*$-ideal of $L$. □

COROLLARY 3.1. *Let $L$ be a $*$-commutative PCASL. Then for any $I, J \in I^*(L), (I : J)$ is a $*$-ideal.*

Next, we prove that, if $I$ is a $*$-ideal and $J$ is any ideal in a $*$-commutative PCASL L, $(I : J) = \{x \in L : (x^{**}] \cap J \subseteq I\}$. For this, first we prove the following.

LEMMA 3.1. *Let $L$ be a $*$-commutative PCASL. Then for any $I \in I^*(L)$ and $J \in I(L), \{x \in L : (x^{**}] \cap J \subseteq I\}$ is a $*$-ideal.*

PROOF. Put $H = \{x \in L : (x^{**}] \cap J \subseteq I\}$. Since $(0^{**}] \cap J = (0] \cap J = (0] \subseteq I, 0 \in H$. Hence $H$ is non-empty subset of $L$. Let $x \in H$ and $a \in L$. Then we have $(x^{**}] \cap J \subseteq I$. Now, consider $((x \circ a)^{**}] \cap J = (x^{**} \circ a^{**}] \cap J = ((x^{**}] \cap (a^{**}]) \cap J = ((a^{**}] \cap (x^{**}]) \cap J = (a^{**}] \cap ((x^{**}] \cap J) \subseteq (x^{**}] \cap J \subseteq I$. Therefore $x \circ a \in H$. Hence $H$ is an ideal of $L$. Let $x \in H$. Then $(x^{**}] \cap J \subseteq I$. Now, consider $((x^{**})^{**}] \cap J = (x^{****}] \cap J = (x^{**}] \cap J \subseteq I$. Therefore $x^{**} \in H$. Thus $H$ is a $*$-ideal of $L$. □

LEMMA 3.2. *Let $L$ be a $*$-commutative PCASL. Then for any $I \in I^*(L)$ and $J$ is any ideal of $L, (I : J)$ is the largest $*$-ideal with the property $(I : J) \cap J \subseteq I$.*

PROOF. Clearly, $(I : J)$ is a $*$-ideal of $L$. Let $x \in (I : J) \cap J$. Then $x \in (I : J)$ and $x \in J$. Hence $x \circ j \in I$, *for all* $j \in J$ and $x \in J$. Inparticular $x = x \circ x \in I$. Therefore $(I : J) \cap J \subseteq I$. Suppose $K \in I^*(L)$ such that $K \cap J \subseteq I$. Now, we shall prove that $K \subseteq (I : J)$. Let $x \in K$. Then we have $x \circ j \in K$ for all, $j \in J$. Also, we have $j \circ x \in J$, for all, $j \in J$. It follows that $x \circ j \in J$ for all, $j \in J$. Hence $x \circ j \in K \cap J$ for all, $j \in J$. This implies $x \circ j \in I$ for all, $j \in J$. Therefore $x \in (I : J)$. Thus $K \subseteq (I : J)$. Hence $(I : J)$ is the largest $*$-ideal with the property $(I : J) \cap J \subseteq I$. □

Now, we prove the following.

THEOREM 3.4. *Let $L$ be a $*$-commutative PCASL and $I, J \in I^*(L)$. Then $(I : J) = \{x \in L : (x^{**}] \cap J \subseteq I\}$.*

PROOF. Put $H = \{x \in L : (x^{**}] \cap J \subseteq I\}$. Then clearly, $H$ is a $*$-ideal. Since $(I : J)$ is the largest $*$-ideal with the property that $(I : J) \cap J \subseteq I$, it is enough to prove that $H$ is the largest $*$-ideal with the property that $(I : J) \cap J \subseteq I$. Let $x \in H \cap J$. Then $x \in H$ and $x \in J$. It follows that $(x^{**}] \cap J \subseteq I$. Since $x^* \circ x = 0, x^{**} \circ x = x$. This implies $x \in (x^{**}]$. It follows that $x \in (x^{**}] \cap J \subseteq I$. Therefore $x \in I$. Hence $H \cap J \subseteq I$. Now, suppose $K \in I^*(L)$ such that $K \cap J \subseteq I$. Let $x \in K$. Then $x^{**} \in K$. This implies $(x^{**}] \subseteq K$. It follows that $(x^{**}] \cap J \subseteq K \cap J$. Since $K \cap J \subseteq I, (x^{**}] \cap J \subseteq I$. This implies $x \in H$. Therefore

$K \subseteq H$. Hence $H$ is the largest $*$-ideal with the property that $(I : J) \cap J \subseteq I$. Thus $(I : J) = \{x \in L : (x^{**}] \cap J \subseteq I\}$. □

COROLLARY 3.2. *The centre of $I^*(L)$ is trivial.*

PROOF. Suppose $I^*(L)$ is a complemented lattice. Suppose $I \in I^*(L)$ such that $I$ is complemented. Then there exists $J \in I^*(L)$ such that $I \cap J = (0]$ and $I \cup J = L$. Now, we shall prove that $J = I^0$. Let $x \in J$ and $i \in I$. Then $x \circ i \in J \cap I = (0]$ and hence $x \circ i = 0$. Therefore $x \in I^0$. Hence $J \subseteq I^0$. Now, let $x \in I^0$. Then $x \circ i = 0$, for all $i \in I$. Since, $x \in I^0 \subseteq L = I \cup J$, either $x \in I$ or $x \in J$. If $x \in I$ then we get $x = x \circ x = 0$. Hence $x \in J$. It follows that $I^0 \subseteq J$. Thus $J = I^0$. Therefore the complement of $I$ is uniquely determined by $I^0$. Now, we have $0^* \in L = I \cup I^0$. Therefore either $0^* \in I$ or $0^* \in I^0$. If $0^* \in I$, then $I = L$. Now, if $0^* \in I^0$, then $I = (0]$, since if $x \in I$ then $0^* \circ x = 0$ and hence $x = 0$. Thus the centre of $I^*(L)$ is trivial. □

## 4. Extended Ideals and Contracted Ideals in ASLs

In this section we introduce the concepts of extended and contracted ideals in *ASLs* and prove some basic properties of these concepts. Also, prove that, if $f : L \to M$ is an ASL homomorphism then the set of all cotracted ideals in $L$ is bijective with the set of all extended ideals in $M$. Now, we begin this section with the following definition.

DEFINITION 4.1. Let $f : L \to M$ be an $ASL$ homomorphism and let $I$ be an ideal of $L$. Then the ideal genereated by $f(I)$ is calld an extended ideal and is denoted by $I^e$.

It can be easily seen that

$$I^e = \{(\circ_{i=1}^n f(x_i)) \circ y : x_i \in I, y \in M, 1 \leqslant i \leqslant n, n \in Z^+\}.$$

Also, seen that, $I^e$ is the smallest ideal containing $f(I)$. In the following we prove some basic properties of extended ideals.

THEOREM 4.1. *Let $f : L \to M$ be an ASL homomorphism and let $I_1, I_2$ be ideals of $L$. Then we have the following:*

(1) $I_1 \subseteq I_2 \Rightarrow {I_1}^e \subseteq {I_2}^e$
(2) $(I_1 \cup I_2)^e = I_1^e \cup I_2^e$
(3) $(I_1 \cap I_2)^e = I_1^e \cap I_2^e$
(4) $(I_1 : I_2)^e \subseteq (I_1^e : I_2^e)$

PROOF. (1) Suppose $I_1 \subseteq I_2$ and $t \in I_1^e$. Then $t = (\circ_{i=1}^n f(x_i)) \circ y$, where $x_i \in I_1, y \in M$. Therefore $t = (\circ_{i=1}^n f(x_i)) \circ y$, where $x_i \in I_2, y \in M$. Hence $t \in I_2^e$. Thus $I_1^e \subseteq I_2^e$.

(2) We have $I_1, I_2 \subseteq I_1 \cup I_2$. Therefore by (1), we get ${I_1}^e \subseteq (I_1 \cup I_2)^e, {I_2}^e \subseteq (I_1 \cup I_2)^e$ and hence ${I_1}^e \cup {I_2}^e \subseteq (I_1 \cup I_2)^e$. Conversely, suppose $t \in (I_1 \cup I_2)^e$. Then $t = (\circ_{i=1}^n f(x_i)) \circ y$, where $x_i \in I_1 \cup I_2, y \in M$. This implies $t = (\circ_{i=1}^n f(x_i)) \circ y$, where $x_i \in I_1$ or $x_i \in I_2, y \in M$. It follows that $t \in {I_1}^e \cup {I_2}^e$. Hence $(I_1 \cup I_2)^e \subseteq {I_1}^e \cup {I_2}^e$. Thus $(I_1 \cup I_2)^e = {I_1}^e \cup {I_2}^e$.

(3) We have $I_1 \cap I_2 \subseteq I_1, I_2$. Therefore by (1), we get $(I_1 \cap I_2)^e \subseteq I_1^e, I_2^e$ and hence $(I_1 \cap I_2)^e \subseteq I_1^e \cap I_2^e$. Conversely, suppose $t \in I_1^e \cap I_2^e$. Then $t \in I_1^e$ and $I_2^e$. It follows that $t = (\circ_{i=1}^n f(x_i)) \circ y$, where $x_i \in I_1, y \in M$ and $t = (\circ_{i=1}^m f(w_i)) \circ z$, where $w_i \in I_2, z \in M$. Now, $t = t \circ t = ((\circ_{i=1}^n f(x_i)) \circ y) \circ ((\circ_{i=1}^m f(w_i)) \circ z) = f(\circ_{i=1}^n x_i \circ \circ_{i=1}^m w_i) \circ (y \circ z) \in (I_1 \cap I_2)^e$ since $\circ_{i=1}^n x_i \circ \circ_{i=1}^n w_i \in I_1 \cap I_2, y \circ z \in M$. Therefore $t \in (I_1 \cap I_2)^e$. Hence $I_1^e \cap I_2^e \subseteq (I_1 \cap I_2)^e$. Thus $(I_1 \cap I_2)^e = I_1^e \cap I_2^e$.

(4) Let $t \in (I_1 : I_2)^e$. Then $t = (\circ_{i=1}^n f(x_i)) \circ y$, where $x_i \in (I_1 : I_2), y \in M$. It follows that $x_i \circ i_2 \in I_1$, for all $i_2 \in I_2$. Now, let $s \in I_2^e$. Then $s = (\circ_{i=1}^m \circ f(w_i)) \circ z$, where $z \in M, w_i \in I_2$. Now, consider $t \circ s = ((\circ_{i=1}^n f(x_i)) \circ y) \circ ((\circ_{i=1}^m f(w_i)) \circ z) = f(\circ_{i=1}^n x_i \circ \circ_{i=1}^m w_i) \circ (y \circ z) \in I_1^e$ since $\circ_{i=1}^n x_i \circ \circ_{i=1}^m w_i \in I_1, y \circ z \in M$. Therefore $t \circ s \in I_1^e$. Hence $t \in (I_1^e : I_2^e)$. Thus $(I_1 : I_2)^e \subseteq (I_1^e : I_2^e)$. □

Definition 4.2. Let $L, M$ be *PCASLs*. Then a mapping $f : L \to M$ is said to be PCASL homomorphism, if $f$ is an ASL homomorphism and $f(a^*) = (f(a))^*$, for all $a \in L$.

Theorem 4.2. *Let $f : L \to M$ be a PCASL homomorphism and $I$ be a $*$-ideal of $L$. Then $I^e$ is a $*$-ideal of $M$.*

Proof. Suppose $I$ is a $*$-ideal of $L$. Now, let $x \in I^e$. Then $x = (\circ_{i=1}^n f(x_i)) \circ y$, where $x_i \in I, y \in M$. This implies $x_i^{**} \in I$. Now, consider $x^{**} = ((\circ_{i=1}^n f(x_i)) \circ y)^{**} = (\circ_{i=1}^n f(x_i))^{**} \circ y^{**} = (\circ_{i=1}^n f(x_i)^{**}) \circ y^{**} = (\circ_{i=1}^n f(x_i^{**})) \circ y^{**} \in I^e$ since $x_i^{**} \in I$, for all $i$ and $y^{**} \in M$. Hence $x^{**} \in I^e$. Thus $I^e$ is a $*$-ideal of $M$. □

Next, we introduce the concept of contracted ideal in a *ASL* $L$ and prove some basic properties of contracted ideals.

Lemma 4.1. *Let $L$ and $M$ be ASLs with zero and let $f : L \to M$ be an ASL homomorphism. If $J$ is an ideal of $M$, then $f^{-1}(J)$ is an ideal of $L$. Inparticular if $J$ is a prime ideal of $M$, then so is $f^{-1}(J)$.*

Proof. We have $f^{-1}(J) = \{x \in L : f(x) \in J\}$. Since $0 = f(0) \in J, 0 \in f^{-1}(J)$. Hence $f^{-1}(J)$ is a non-empty subset of $L$. Let $x \in f^{-1}(J)$ and $a \in L$. Then $f(x) \in J$ and $f(a) \in f(L)$. It follows that $f(x) \circ f(a) \in J$. This implies $f(x \circ a) \in J$. Therefore $x \circ a \in f^{-1}(J)$. Hence $f^{-1}(J)$ ia an ideal of $L$. Suppose $J$ is a prime ideal of $B$. Let $x, y \in A$ such that $x \circ y \in f^{-1}(J)$. Then $f(x \circ y) \in J$. This implies $f(x) \circ f(y) \in J$. Therefore either $f(x) \in J$ or $f(y) \in J$, since $J$ is prime ideal. It follows that $x \in f^{-1}(J)$ or $y \in f^{-1}(J)$. Thus $f^{-1}(J)$ is a prime ideal of $L$. □

Definition 4.3. Let $L, M$ be *ASLs* with zero and $f : L \to M$ be an ASL homomorphism. If $J$ is an ideal of $M$, then the ideal $f^{-1}(J)$ is called the contracted ideal of $J$ and is denoted by $J^c$.

In the following, we prove some basic properties of contracted ideals in *ASLs*.

Theorem 4.3. *Let $L, M$ be ASLs with zero and let $f : L \to M$ be an ASL homomorphism. Then for any ideals $J_1, J_2$ of $M$, we have the following:*

(1) $J_1 \subseteq J_2 \Leftrightarrow {J_1}^c \subseteq {J_2}^c$

(2) $(J_1 \cup J_2)^c = J_1^c \cup J_2^c$
(3) $(J_1 \cap J_2)^c = J_1^c \cap J_2^c$
(4) *If $f$ is an epimorphism then* $(J_1 : J_2)^c = (J_1^c : J_2^c)$.

PROOF. (1) Suppose $J_1 \subseteq J_2$. Now, let $x \in J_1^c = f^{-1}(J_1)$. Then $f(x) \in J_1$ and hence $f(x) \in J_2$. Therefore $x \in f^{-1}(J_2) = {J_2}^c$. Hence ${J_1}^c \subseteq {J_2}^c$.

(2) We have $J_1, J_2 \subseteq J_1 \cup J_2$.Therefore by (1), we get ${J_1}^c \subseteq (J_1 \cup J_2)^c, {J_2}^c \subseteq (J_1 \cup J_2)^c$. Hence ${J_1}^c \cup {J_2}^c \subseteq (J_1 \cup J_2)^c$. Conversely, suppose $t \in (J_1 \cup J_2)^c = f^{-1}(J_1 \cup J_2)$. This implies $f(t) \in (J_1 \cup J_2)$. It follows that $f(t) \in J_1$ or $f(t) \in J_2$. Therefore $t \in f^{-1}(J_1)$ or $t \in f^{-1}(J_2)$ and hence $t \in J_1^c$ or $t \in J_2^c$. Therefore $t \in J_1^c \cup J_2^c$. Hence $(J_1 \cup J_2)^c \subseteq J_1^c \cup J_2^c$. Thus $(J_1 \cup J_2)^c = J_1^c \cup J_2^c$.

(3) We have $J_1 \cap J_2 \subseteq J_1$ and $J_1 \cap J_2 \subseteq J_2$. Therefore by (1), we get $(J_1 \cap J_2)^c \subseteq J_1^c$ and $(J_1 \cap J_2)^c \subseteq J_2^c$. Therefore $(J_1 \cap J_2)^c \subseteq J_1^c \cap J_2^c$. Conversely, suppose $t \in J_1^c \cap J_2^c$. Then $t \in J_1^c$ and $t \in J_2^c$. It follows that $t \in f^{-1}(J_1)$ and $t \in f^{-1}(J_2)$. This implies $f(t) \in J_1$ and $f(t) \in J_2$. Therefore $f(t) \in J_1 \cap J_2$. This implies $t \in f^{-1}(J_1 \cap J_2)$. Hence $t \in (J_1 \cap J_2)^c$. Therefore $J_1^c \cap J_2^c \subseteq (J_1 \cap J_2)^c$. Hence $(J_1 \cap J_2)^c = J_1^c \cap J_2^c$.

(4) Suppose $f$ is an epimorphism. Now, let $t \in (J_1 : J_2)^c = f^{-1}(J_1 : J_2)$. Then $f(t) \in (J_1 : J_2)$. It follows that $f(t) \circ j_2 \in J_1$, for all $j_2 \in J_2$. Now, let $i_2 \in J_2^c = f^{-1}(J_2)$. Then $f(i_2) \in J_2$. It follows that $f(t) \circ f(i_2) \in J_1$. This implies $f(t \circ i_2) \in J_1$. Therefore $(t \circ i_2) \in f^{-1}(J_1)$. It follows that $(t \circ i_2) \in J_1^c$, for all $i_2 \in J_2^c$. Therefore $t \in (J_1^c : J_2^c)$. Hence $(J_1 : J_2)^c \subseteq (J_1^c : J_2^c)$. Conversely, suppose $t \in (J_1^c : J_2^c)$. Then $t \circ i_2 \in J_1^c$, for all $i_2 \in J_2^c$. It follows that $f(t \circ i_2) \in J_1$, for all $i_2 \in J_2^c$. Therefore $f(t) \circ f(i_2) \in J_1$, for all $f(i_2) \in J_2$. Now, let $j_2 \in J_2$. Then there exists $t_2 \in L$ such that $f(t_2) = j_2 \in J_2$. It follows that $f(t) \circ f(t_2) \in J_1$ and hence $f(t) \circ j_2 \in J_1$, for all $j_2 \in J_2$. It follows that $f(t) \in (J_1 : J_2)$. Therefore $t \in f^{-1}(J_1 : J_2) = (J_1 : J_2)^c$. Therefore $t \in (J_1 : J_2)^c$. Hence $(J_1^c : J_2^c) \subseteq (J_1 : J_2)^c$. Thus $(J_1 : J_2)^c = (J_1^c : J_2^c)$. □

THEOREM 4.4. *Let $f : L \to M$ be PCASL homomorphism and let $J$ be a $*$-ideal of $M$. Then $J^c$ is a $*$-ideal of $L$.*

PROOF. Suppose $J$ is a $*$-ideal of $M$. Then we have $J^c = \{x \in L : f(x) \in J\}$ is an ideal of $L$. Let $x \in J^c$. Then $x \in f^{-1}(J)$. This implies $f(x) \in J$. It follows that $(f(x))^{**} \in J$. Therefore $f(x^{**}) \in J$. Hence $x^{**} \in f^{-1}(J) = J^c$. Thus $J^c$ is a $*$-ideal of $L$. □

THEOREM 4.5. *Let $f : L \to M$ be PCASL homomorphism and $J$ be a kernel ideal of $M$. Then $J^c$ is a kernel ideal of $L$.*

PROOF. Suppose $J$ is a kernel ideal of $M$. Now, let $x, y \in J^c = f^{-1}(J)$. Then $f(x) \in J, f(y) \in J$. It follows that $(f(x)^* \circ f(y)^*)^* \in J$. Therefore $(f(x^*) \circ f(y^*))^* \in J$. It follows that $(f(x^* \circ y^*))^* \in J$. This implies $f((x^* \circ y^*)^*) \in J$. Therefore $(x^* \circ y^*)^* \in f^{-1}(J) = J^c$. Hence $(x^* \circ y^*)^* \in J^c$. Thus $J^c$ is a kernel ideal of $L$. □

In the following we characterize extended ideals and contracted ideals in *ASLs* and prove that there is a bijection between set of all contracted ideals and the set of all extended ideals. For, this first we need the following.

THEOREM 4.6. *Let $L, M$ be ASLs and $f : L \to M$ be an ASL homomorphism and let $I$ be an ideal of $L$, $J$ be an ideal of $M$. Then we have the following:*

(1) $I \subseteq I^{ec}$
(2) $J \supseteq J^{ce}$
(3) $I^e = I^{ece}$
(4) $J^c = J^{cec}$

PROOF. (1) Let $t \in I$. Then $f(t) \in f(I) \subseteq I^e$. This implies $f(t) \in I^e$. It follows that $t \in f^{-1}(I^e) = I^{ec}$. Thus $I \subseteq I^{ec}$.

(2) Let $t \in J^{ce}$. Then $t = (\circ_{i=1}^{n} f(x_i)) \circ y$, where $x_i \in J^c$, for all $i$ and $y \in M$. Now, $x_i \in J^c = f^{-1}(J)$, for all, $i$. Hence $f(x_i) \in J$, for all $i$. It followes that $f(x_i) \circ y \in J$, for all $i$. Therefore $t = (\circ_{i=1}^{n} f(x_i)) \circ y \in J$. Thus $J^{ce} \subseteq J$.

(3) From (1), we get $I \subseteq I^{ec}$. It follows that $I^e \subseteq I^{ece}$. Again, put $J = I^e$ in (2), we get $I^e \supseteq I^{ece}$. Hence $I^e = I^{ece}$.

(4) From (2), we get $J^{ce} \subseteq J$. It followes that $J^{cec} \subseteq J^c$. Again, put $I = J^c$ in (1), we get $J^c \subseteq J^{cec}$. Therefore $J^c = J^{cec}$. □

THEOREM 4.7. *Let $L, M$ be ASLs and let $f : L \to M$ be an ASL homomorphism. Let $C$ be the set of all contracted ideals in $L$ and let $E$ be the set of all extended ideals in $M$. Then we have the following:*

(1) $C = \{I \in I(L) : I = I^{ec}\}$
(2) $E = \{J \in I(M) : J = J^{ce}\}$
(3) *The mapping $g : C \to E$ defined by $g(I) = I^e$ is a bijection and its invers mapping $h : E \to C$ defined by $h(J) = J^c$.*

PROOF. (1) Let $I \in C$. Then $I = J^c$, for some ideal $J$ of $M$. Now, considetr $I^{ec} = (J^c)^{ec} = J^{cec} = J^c = I$. Therefore $I^{ec} = I$. Conversely, suppose $I = I^{ec}$. Now, $I = I^{ec} = (I^e)^c$ and $I^e$ is an ideal of $M$. It follows that $I$ is a contracted ideal. Therefore $I \in C$.

(2) Let $J \in E$. Then $J = I^e$, for some ideal $I$ of $L$. Now, consider $J^{ce} = (I^e)^{ce} = I^{ece} = I^e = J$. Therefore $J^{ce} = J$. Conversely. suppose $J = J^{ce}$. Now, $J = J^{ce} = (J^c)^e$ and $J^c$ is an ideal of $L$. It follows that $J$ is an extended ideal. Therefore $J \in C$.

(3) We have $g : C \to E$ defined by $g(I) = I^e$, for all $I \in C$. Clearly, $g$ is both well defined and one-one. Let $J \in E$. Then $J = J^{ce}$. Now, $J^c = J^{cec} = (J^c)^{ec}$. Therefore $J^c \in C$. Now, consider $g(J^c) = J^{ce} = J$. Therefore $g(J^c) = J$. Hence $g$ is onto. Hence $g$ is bijection. Let $J \in E$. Then $J = J^{ce}$. Now, consider $(g \circ h)(J) = g(h(J)) = g(J^c) = (J^c)^e = J^{ce} = J = I_d(J)$, for all $J \in E$ where $I_d$ is the identity map on $E$. Therefore $g \circ h = I_d$. Similarly, we can prove that $h \circ g = I_d$. Hence $h \circ g = I_d = g \circ h$. Thus $h$ is the inverse of $g$. □

## 5. The lattices of kernel ideals and $*$-ideals

In this section, we observe that join of any two kernel ideals need not be a kernel ideal by means of example. We prove that the set $KI(L)$ of all kernel ideals in a $*$-commutative $PCASL$ $L$ in which $x \leqslant x^{**}$, for all $x \in L$ is a complete implicative lattice and prove that the residuals in the lattice $KI(L)$ coincides with the corresponding residuals in the lattice $I^*(L)$ of all $*$-ideals in $L$. If $L$ is a $*$-commutative PCASL in which $x \leqslant x^{**}$, for all $x \in L$, then we prove that the centre of $KI(L)$ is isomorphic with the Boolean algebra $S(L)$ of all closed elements in $L$.

It can be easily observed that if $L$ is a pseudo-complemented distributive lattice, then the set of all complemented elements in $L$ is sub lattice of $L$ and for any $x, y \in L, (x] \vee (y] = (x \vee y]$. Also, easily verified that an ideal $I$ of pseudo-complemented distributive lattice is a kernel ideal if and only if $I$ is a $*$-ideal. It followes that in a pseudo-complemented distributive lattice $L$, join of any two kernel ideals is again a kernel ideal if and only if $L$ is Stone lattice. First, we give an example of join of two kernel ideals is not a kernel ideal in an $ASL$.

Example 5.1. Let $A = \{0, a\}$ and $B = \{0, b_1, b_2\}$ are two discrete $ASLs$. Let $L = A \times B = \{(0, 0), (0, b_1), (0, b_2), (a, 0), (a, b_1), (a, b_2)\}$. Define a binary operation $\circ$ on $L$ as follows:

| $\circ$ | (0, 0) | (0, $b_1$) | (0, $b_2$) | (a, 0) | (a, $b_1$) | (a, $b_2$) |
|---|---|---|---|---|---|---|
| (0, 0) | (0, 0) | (0, 0) | (0, 0) | (0, 0) | (0, 0) | (0, 0) |
| (0, $b_1$) | (0, 0) | (0, $b_1$) | (0, $b_2$) | (0, 0) | (0, $b_1$) | (0, $b_2$) |
| (0, $b_2$) | (0, 0) | (0, $b_1$) | (0, $b_2$) | (0, 0) | (0, $b_1$) | (0, $b_2$) |
| (a, 0) | (0, 0) | (0, 0) | (0, 0) | (a, 0) | (a, 0) | (a, 0) |
| (a, $b_1$) | (0, 0) | (0, $b_1$) | (0, $b_2$) | (a, 0) | (a, $b_1$) | (a, $b_2$) |
| (a, $b_2$) | (0, 0) | (0, $b_1$) | (0, $b_2$) | (a, 0) | (a, $b_1$) | (a, $b_2$) |

Then clearly, $(L, \circ)$ is an $ASL$. Now, define a unary operation $*$ on $L$, by $(0, 0)^* = (a, b_1)$, $(0, b_1)^* = (0, b_2)^* = (a, 0)$, $(a, 0)^* = (0, b_1)$ and $(a, b_1)^* = (a, b_2)^* = (0, 0)$. Then clearly, $*$ is a pseudo-complementation on $L$. Now, put $I = \{(0, 0), (a, 0)\}, J = \{(0, 0), (0, b_1), (0, b_2)\}$. Then clearly, $I$ and $J$ are kernel ideals. Now $I \vee J = I \cup J = \{(0, 0), (a, 0), (0, b_1), (0, b_2)\}$ which is not a kernel ideal, since $(a, 0), (0, b_1) \in I \cup J$, $((a, 0)^* \circ (0, b_1)^*)^* = ((0, b_1) \circ (a, 0))^* = (0, 0)^* = (a, b_1) \notin I \cup J$.

Now, returning to the case of pseudo-complemented almost semilattice $L$. Note that in $PCASL$, $x \nleqslant x^{**}$ in general. For, consider the following example.

Example 5.2. Let L = { 0, a, b, c}. Now, define binary operation $\circ$ on $L$ as follows:

| $\circ$ | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | a | a |
| b | 0 | a | b | c |
| c | 0 | a | b | c |

Then clearly $(L, \circ)$ is an $ASL$. Also, if we define $0^* = b$ and $x^* = 0$, for all $x(\neq 0) \in L$, then clearly $L$ is a $PCASL$. In this ASL, we have $c \circ c^{**} = c \circ (c^*)^* = c \circ 0^* = c \circ b = b \neq c$ and hence $c \nleq c^{**}$.

However, we prove that, if $x \leqslant x^{**}$ for all $x$ in $*$-commutative $PCASL$, then the set of all kernel ideals in $L$ form a complete implicative lattice and residuals in the lattice $KI(L)$ coincides with the corresponding residuals in the lattice $I^*(L)$. Recall that if $I$ is a kernel ideal of a $*$-commutative PCASL $L$, then $R_I = \{(x, y) \in L \times L : i^* \circ x = i^* \circ y,\ for\ some\ i \in I\}$ is the smallest $*$-congruence on $L$ with kernel $I$. First we prove the following.

LEMMA 5.1. *Let $L$ be a $*$-commutative $PCASL$ and $I$ be a kernel ideal of $L$. If $a, b \in I$, then $(a, b) \in R_I$.*

PROOF. Suppose $a, b \in I$. Since $I$ is a kernel ideal, $(a^* \circ b^*)^* \in I$. Again, since $a \circ a^* \circ b^* = 0$ and $b \circ a^* \circ b^* = 0$. It follows that $a^* \circ b^* \circ a = 0$ and $a^* \circ b^* \circ b = 0$. This implies $(a^* \circ b^*)^{**} \circ a = 0$ and $(a^* \circ b^*)^{**} \circ b = 0$. It follows that $(a^* \circ b^*)^*)^* \circ a = 0$ and $(a^* \circ b^*)^*)^* \circ b = 0$. Therefore $(a^* \circ b^*)^*)^* \circ a = (a^* \circ b^*)^*)^* \circ b$ and $(a^* \circ b^*)^* \in I$. Hence $(a, b) \in R_I$. □

Note that if $L$ is a $*$-commutative $PCASL$ and $x \in L$, then the congruence class of $x$ with respect to the congruence relation $R_I$ is denoted by $x/R_I$ and hence $x/R_I = \{y \in L : (x, y) \in R_I\}$.

LEMMA 5.2. *Let $L$ be a $*$-commutative $PCASL$ and $I$ be a kernel ideal of $L$. Then $L/R_I = \{x/R_I : x \in L\}$ is a $*$-commutative $PCASL$, under induced operations on $L$.*

PROOF. Suppose $x/R_I, y/R_I \in L/R_I$. Now, define a binary operation $\underline{\circ}$ and a unary operation $*$ on $L/R_I$ by $x/R_I \,\underline{\circ}\, y/R_I = (x \circ y)/R_I$ and $(x/R_I)^* = x^*/R_I$. Then clearly, operations $\underline{\circ}$ and $*$ are well defined. Also, clearly $(L/R_I, \underline{\circ})$ is an ASL. Let $x/R_I \in L/R_I$. Now, consider $x/R_I \,\underline{\circ}\, (x/R_I)^* = x/R_I \,\underline{\circ}\, x^*/R_I = (x \circ x^*)/R_I = 0/R_I$. Suppose $y/R_I \in L/R_I$ such that $x/R_I \,\underline{\circ}\, y/R_I = 0/R_I$. Then $(x \circ y)/R_I = 0/R_I$. It follows that $(x \circ y, 0) \in R_I$. This implies $i^* \circ x \circ y = i^* \circ 0 = 0$, for some $i \in I$. Therefore $i^* \circ x \circ y = 0$. This implies $x \circ i^* \circ y = 0$. It follows that $x^* \circ i^* \circ y = i^* \circ y$. Therefore $i^* \circ x^* \circ y = i^* \circ y$. Hence $(x^* \circ y, y) \in R_I$. It follows that $(x^* \circ y)/R_I = y/R_I$. This implies $x^*/R_I \,\underline{\circ}\, y/R_I = y/R_I$. Therefore $(x/R_I)^* \circ y/R_I = y/R_I$. Hence $L/R_I$ is a PCASL. Clearly, $L/R_I$ is $*$-commutative. □

Now, we prove that the set $KI(L)$ is a complete implicative lattice.

THEOREM 5.1. *Let $L$ be a $*$-commutative $PCASL$ in which $x \leqslant x^{**}$, for all $x \in L$.Then ordered by set inclusion, $KI(L)$ forms a complete implicative lattice in which the operations are as follows: If $\{I_\alpha : \alpha \in \Delta\}$ is any family of kernel ideals of $L$, then*

$$\bigwedge_{\alpha \in \Delta} I_\alpha = inf_{KI(L)}\{I_\alpha : \alpha \in \Delta\} = \bigcap_{\alpha \in \Delta} I_\alpha,$$

$$\bigvee_{\alpha\in\Delta} I_\alpha = sup_{KI(L)}\{I_\alpha : \alpha \in \Delta\}$$
$$= \{x \in L : (\exists \alpha_1, \alpha_2, ..., \alpha_n \in \Delta)(\exists x_i \in I_{\alpha_i}), x \leqslant (\circ_{i=1}^n x_i^*)^*\}$$

*and residuals in $KI(L)$ coinsides with the corresponding residuals in $I^*(L)$.*

Proof. Clearly, $KI(L)$ is a poset with respect to set inclusion. Suppose $S = \{I_\alpha : \alpha \in \Delta\}$ is a non-empty subset of $KI(L)$. Then clearly, $\bigcap_{\alpha\in\Delta} I_\alpha$ is the greatest lower bound of $S$. Since $0 \in \bigvee_{\alpha\in\Delta} I_\alpha$, $\bigvee_{\alpha\in\Delta} I_\alpha \neq \emptyset$. Let $x \in \bigvee_{\alpha\in\Delta} I_\alpha$ and $t \in L$. Then there exists $\alpha_1, \alpha_2, ......, \alpha_n \in \Delta$, and there exists $x_i \in I_{\alpha_i}$, such that $x \leqslant (\circ_{i=1}^n x_i^*)^*$. This implies $t \circ x \leqslant x \leqslant (\circ_{i=1}^n x_i^*)^*$. It follows that $(t\circ x)^{**} \leqslant (\circ_{i=1}^n x_i^*)^{***}$. Hence $(x\circ t)^{**} \leqslant (\circ_{i=1}^n x_i^*)^*$. It follows that $x\circ t \leqslant (\circ_{i=1}^n x_i^*)^*$. Therefore $x\circ t \in \bigvee_{\alpha\in\Delta} I_\alpha$. Hence $\bigvee_{\alpha\in\Delta} I_\alpha$ is a ideal of $L$. Let $x, y \in \bigvee_{\alpha\in\Delta} I_\alpha$. Then there exists $\alpha_1, \alpha_2, ......, \alpha_n \in \Delta$, and there exists $x_i, y_i \in I_{\alpha_i}$, such that $x \leqslant (\circ_{i=1}^n x_i^*)^*$ and $y \leqslant (\circ_{i=1}^n y_i^*)^*$ . Now, since $x_i, y_i \in I_{\alpha_i}, I_{\alpha_i}$ is a kernel ideal, there exists $z_i \in I_{\alpha_i}$ such that $x_i^* \circ y_i^* = z_i^*$, for $i = 1, 2, ...., n$. This implies $(\circ_{i=1}^n x_i^*)^{**} \leqslant x^*$ and $(\circ_{i=1}^n y_i^*)^{**} \leqslant y^*$. It follows that $(\circ_{i=1}^n x_i^*)^{**} \circ (\circ_{i=1}^n y_i^*)^{**} \leqslant x^* \circ y^*$. Therefore $(x^* \circ y^*)^* \leqslant ((\circ_{i=1}^n x_i^*)^{**} \circ (\circ_{i=1}^n y_i^*)^{**})^* = ((\circ_{i=1}^n x_i^*) \circ (\circ_{i=1}^n y_i^*))^* = (\circ_{i=1}^n (x_i^* \circ y_i^*))^*$. Therefore $(x^* \circ y^*)^* \leqslant (\circ_{i=1}^n z_i^*)^*$. Thus $(x^* \circ y^*)^* \in \bigvee_{\alpha\in\Delta} I_\alpha$. Therefore $\bigvee_{\alpha\in\Delta} I_\alpha$ is a kernel ideal of $L$. Hence $\bigvee_{\alpha\in\Delta} I_\alpha \in KI(L)$. Clearly, $\bigvee_{\alpha\in\Delta} I_\alpha$ is an upper bound of $S$. Suppose $K \in KI(L)$ such that $K$ is an upper bound of $S$. Then $I_\alpha \subseteq K$, for all $I_\alpha \in S$. Let $x \in \bigvee_{\alpha\in\Delta} I_\alpha$. Then there exists $\alpha_1, \alpha_2, ......, \alpha_n \in \Delta$, and there exists $x_i \in I_{\alpha_i}$, such that $x \leqslant (\circ_{i=1}^n x_i^*)^*$. Since $I_{\alpha_i} \subseteq K$, for all $i, x_i \in K$, for all $i$. It follows that $(\circ_{i=1}^n x_i^*)^* \in K$. This implies $x \in K$. Therefore $\bigvee_{\alpha\in\Delta} I_\alpha \subseteq K$. Hence $\bigvee_{\alpha\in\Delta} I_\alpha$ is the least upper bound of $S$. Thus $(KI(L), \subseteq)$ is a complete lattice.

Now, we shall prove that $KI(L)$ is an implicative lattice. That is enough to prove that $KI(L)$ satisfies infinite meet distributive law. Let $\{I_\alpha : \alpha \in \Delta\}$ be a non-empty subset of $KI(L)$ and $I \in KI(L)$. Now, we shall prove that $I \cap (\bigvee_{\alpha\in\Delta} I_\alpha) = \bigvee_{\alpha\in\Delta} (I \cap I_\alpha)$. Let $x \in I \cap (\bigvee_{\alpha\in\Delta} I_\alpha)$. Then $x \in I$ and $x \in \bigvee_{\alpha\in\Delta} I_\alpha$. Then there exists $\alpha_1, \alpha_2, ......, \alpha_n \in \Delta$, and there exists $x_i \in I_{\alpha_i}$, such that $x \leqslant (\circ_{i=1}^n x_i^*)^*$. We have $x \leqslant x^{**}$ and $x \leqslant (\circ_{i=1}^n x_i^*)^*$. It follows that $x \leqslant x^{**} \circ (\circ_{i=1}^n x_i^*)^*$. This implies $x \leqslant x^{**} \circ sup_{S(L)}\{x_i^{**} : 1 \leqslant i \leqslant n\}$. Therefore $x \leqslant sup_{S(L)}\{x^{**} \circ x_i^{**} : 1 \leqslant i \leqslant n\}$. It follows that $x \leqslant (\circ_{i=1}^n (x \circ x_i)^*)^*$ and $x \circ x_i \in I \cap I_{\alpha_i}$, *for all* $i$. Hence $x \in \bigvee_{\alpha\in\Delta} (I \cap I_\alpha)$. Thus $I \cap (\bigvee_{\alpha\in\Delta} I_\alpha) \subseteq \bigvee_{\alpha\in\Delta} (I \cap I_\alpha)$. Clearly, $\bigvee_{\alpha\in\Delta} (I \cap I_\alpha) \subseteq I \cap (\bigvee_{\alpha\in\Delta} I_\alpha)$. Therefore $I \cap (\bigvee_{\alpha\in\Delta} I_\alpha) = \bigvee_{\alpha\in\Delta} (I \cap I_\alpha)$. Thus $KI(L)$ is a implicative lattice.

Now, we shall prove that residuals in $KI(L)$ is coincide with the corresponding residuals in $I^*(L)$. Let $I, J \in KI(L)$ and let $x, y \in (I : J)$. Then $(x^{**}] \cap J \subseteq I$ and $(y^{**}] \cap J \subseteq I$. Let $j \in ((x^* \circ y^*)^*] \cap J$. Then $j \in ((x^* \circ y^*)^*]$ and $j \in J$. It follows that $j = (x^* \circ y^*)^* \circ j$ and $j \in J$. This implies $x^* \circ y^* \circ j = x^* \circ y^* \circ (x^* \circ y^*)^* \circ j = 0$. It follows that $y^{**} \circ j \circ x^* = j \circ x^*$. Since $y^{**} \in (y^{**}]$, $y^{**} \circ j \circ x^* \in (y^{**}]$. Therefore $j \circ x^* \in (y^{**}]$ and $j \circ x^* \in J$. Hence $j \circ x^* \in (y^{**}] \cap J \subseteq I$. Thus $j \circ x^* \in I$. Again, we have $j \circ x^{**} \in (x^{**}]$. Therefore $j \circ x^{**} \in (x^{**}]$ and $j \in J$ and hence $j \circ x^{**} \in (x^{**}] \cap J \subseteq I$. Thus $j \circ x^{**} \in I$. We have $j \circ x^* \in I$ and $0 \in I$. It follows that $(j \circ x^*, 0) \in R_I$. This implies $(j \circ x^*)/R_I = 0/R_I$. It follows that $j/R_I \underline{\circ}\, x^*/R_I = 0/R_I$. Therefore $x^*/R_I \underline{\circ}\, j/R_I = 0/R_I$. Hence $(x^*/R_I)^* \underline{\circ}\, j/R_I = j/R_I$. Therefore $x^{**}/R_I \underline{\circ}\, j/R_I = j/R_I$. Similarly, we can prove that $x^*/R_I \underline{\circ}\, j/R_I = j/R_I$. It follows that $(x^{**} \circ x^* \circ j)/R_I = j/R_I$. This implies $0/R_I = j/R_I$. Therefore $(j, 0) \in R_I$. Hence $i^* \circ j = i^* \circ 0$, for some $i \in I$. This implies $i^* \circ j = 0$. It follows that $i^{**} \circ j = j$. Since $I$ is a kernel ideal, $i^{**} \in I$. This implies $i^{**} \circ j \in I$. Hence $j \in I$. Therefore $((x^* \circ y^*)^*] \cap J \subseteq I$. Hence $(x^* \circ y^*)^* \in (I : J)$. Thus the residuals in $KI(L)$ coincides with the corresponding residuals in $I^*(L)$. □

COROLLARY 5.1. *Let $L$ be a ∗-commutative PCASL in which $x \leqslant x^{**}$, for all $x \in L$. Then every element in $KI(L)$, has at most one complement.*

Finally, we prove that the centre of $KI(L)$ is isomorphic to the Boolean algebra $S(L)$. First we prove that a kernel ideal $J$ of ∗-commutative PCASL in which $x \leqslant x^{**}$ for all $x \in L$ is in the centre of $KI(L)$ if and only if $J$ is a principal ideal. For this, we need the following. Recall that if $x$ is in Boolean algebra $S(L)$ of all closed elements in ∗-commutative PCASL, then $(x]$ is a kernel ideal. Now, we prove the converse.

LEMMA 5.3. *Let $L$ be a ∗-commutative PCASL in which $x \leqslant x^{**}$, for all $x \in L$. If a principal ideal $I = (x]$ is a kernel ideal then $x$ is in the Boolean algebra $S(L)$.*

PROOF. Suppose $(x]$ is a kernel ideal. Then $(x]$ is a ∗-ideal. Since $x \in (x], x^{**} \in (x]$. It follows that $x^{**} = x \circ x^{**} = x^{**} \circ x = x$. Therefore $x$ is in the Boolean algebra $S(L)$. □

Now, we have the following lemma whose proof is straightforward.

LEMMA 5.4. *Let $L$ be a ∗-commutative PCASL in which $x \leqslant x^{**}$, for all $x \in L$. Then the following conditions are equivalent:*

(1) *Every ideal of $L$ is a kernel ideal.*
(2) *Every principal ideal of $L$ is a kernel ideal.*
(3) *$L$ is a Boolean algebra.*

LEMMA 5.5. *Let $L$ be a ∗-commutative PCASL in which $x \leqslant x^{**}$, for all $x \in L$. Then for any $J \in KI(L)$, the complement of $J$ is $J^0 = (0 : J)$ in $KI(L)$.*

PROOF. Suppose $J$ is kernel ideal. Now, $J^0 = \{x \in L : x \circ j = 0,\ \text{for all } j \in J\}$. Let $x, y \in J^0$. Then $x \circ j = 0, y \circ j = 0,\ \text{for all } j \in J$. It follows that $x^* \circ j = j, y^* \circ j = j$, for all $j \in J$. Therefore $j = x^* \circ j \circ y^* \circ j = (x^* \circ y^*) \circ j$. Therefore $j = (x^* \circ y^*) \circ j$. Now, consider $(x^* \circ y^*)^* \circ j = (x^* \circ y^*)^* \circ (x^* \circ y^*) \circ j = 0$, for all $j \in J$. Hence $(x^* \circ y^*)^* \in J^0$. Thus $J^0$ is a kernel ideal of $L$. Suppose $I$ is a complement of $J$ in $KI(L)$. Now, we shall prove that $I = (0 : J)$. Let $t \in I, j \in J$. Then $t \circ j \in I$ and $j \circ t \in J$. This implies $t \circ j \in J$. It follows that $t \circ j \in I \cap J = (0]$. Therefore $t \circ j \in (0]$. This implies $t \circ j = 0$. Therefore $t \in (0 : J)$. Hence $I \subseteq (0 : J)$. Conversely, suppose $t \in (0 : J)$. Then $t \circ j = 0$, for all $j \in J$. We have $I \vee J = L$. Now, $0^* \in L = I \vee J$. Then $0^* \leqslant (x^* \circ y^*)^*$, for some $x \in I, y \in J$. This implies $0^* = 0^* \circ (x^* \circ y^*)^* = (x^* \circ y^*)^*$. This implies $0^* \circ t = (x^* \circ y^*)^* \circ t$. Therefore $t = (x^* \circ y^*)^* \circ t = (x^* \circ y^* \circ t)^* \circ t$. Now, we have $t \circ j = 0$, for all $j \in J$. Inparticular $t \circ y = 0$. This implies $y \circ t = 0$. It followes that $y^* \circ t = t$. Therefore $(x^* \circ y^*)^* \circ t = (x^* \circ t)^* \circ t = (x^*)^* \circ t = x^{**} \circ t$. Hence $t \in I$. Therefore $(0 : J) \subseteq I$. Thus $(0 : J) = I$. □

THEOREM 5.2. *Let $L$ be a $*$-commutative PCASL in which $x \leqslant x^{**}$, for all $x \in L$ and $J$ be a kernel ideal of $L$. Then $J$ is in the centre of $KI(L)$ iff $J$ is principal ideal.*

PROOF. Suppose $J$ is in the centre of $KI(L)$. Then we have $J$ is complemented. Therefore $J$ has unique complement, namely, $(0 : J)$. Then $J \cap (0 : J) = (0]$ and $J \vee (0 : J) = L$. Since $0^* \in L, 0^* \leqslant (x^* \circ y^*)^*$, for some $x \in J, y \in (0 : J)$. Now, since $x \in J, x^{**} \in J$. Therefore $(x^{**}] \subseteq J$. Conversely, let $j \in J$. Then we have $(0 : J) = \{z \in L : (z^{**}] \cap J \subseteq (0]\} = \{z \in L : (z^{**}] \cap J = (0]\}$. Since $y \in (0 : J), (y^{**}] \cap J = (0]$. Now, we have $y^{**} \in (y^{**}]$ and hence $y^{**} \circ j \in (y^{**}]$. On the other hand, since $j \in J, y^{**} \circ j \in J$. Therefore $y^{**} \circ j \in (y^{**}] \cap J = (0]$. This implies $y^{**} \circ j = 0$. It follows that $y^{***} \circ j = j$. Hence $y^* \circ j = j$. Now, we have $j \in J$. Consider,

$$\begin{aligned} j^{**} &= 0^* \circ j^{**} \\ &= (x^* \circ y^*)^* \circ j^{**} \\ &= j^{**} \circ (x^* \circ y^*)^* \\ &= j^{**} \circ (x^* \circ y^*)^{***} \\ &= j^{**} \circ ((x^* \circ y^*)^*)^{**} \\ &= (j \circ (x^* \circ y^*)^*)^{**} \\ &= ((x^* \circ y^*)^*) \circ j)^{**} \\ &= ((x^* \circ y^* \circ j)^* \circ j)^{**}. \end{aligned}$$

Again, we have,

$$\begin{aligned} & y^* \circ j = j \\ \Rightarrow\quad & x^* \circ j = x^* \circ (y^* \circ j) \\ \Rightarrow\quad & (x^* \circ j)^* = (x^* \circ (y^* \circ j))^* \\ \Rightarrow\quad & (x^* \circ j)^* \circ j = (x^* \circ (y^* \circ j))^* \circ j \\ \Rightarrow\quad & x^{**} \circ j = (x^* \circ y^* \circ j)^* \circ j. \end{aligned}$$

Therefore $j^{**} = ((x^* \circ y^* \circ j)^* \circ j)^{**} = (x^{**} \circ j)^{**} = x^{**} \circ j^{**}$. Hence $j^{**} \in (x^{**}]$. Now, we have $j = j^{**} \circ j \in (x^{**}]$. Therefore $j \in (x^{**}]$. Hence $J \subseteq (x^{**}]$. Therefore $J = (x^{**}]$. Thus $J$ is principal ideal.

Conversely, suppose $J$ is a principal ideal. Then $J = (a]$, for some $a \in L$. We have $a \in J$ and $J$ is kernel ideal and hence is a $*$-ideal. Therefore $a^{**} \in J$. It follows that $(a] \subseteq (a^{**}] \subseteq J = (a]$. Therefore $J = (a^{**}]$. Since $a \leqslant a^{**}$, $(a] \subseteq (a^{**}]$. Now, consider

$$
\begin{aligned}
(0 : J) &= \{z \in L : (z^{**}] \cap J \subseteq (0]\} \\
&= \{z \in L : (z^{**}] \cap J = (0]\} \\
&= \{z \in L : (z^{**}] \cap (a^{**}] = (0]\} \\
&= \{z \in L : (z^{**} \circ a^{**}] = (0]\} \\
&= \{z \in L : ((z \circ a)^{**}] = (0]\} \\
&= \{z \in L : (z \circ a)^{**} = 0\} \\
&= \{z \in L : z \circ a = 0\} \\
&= \{z \in L : a \circ z = 0\} \\
&= \{z \in L : a^{*} \circ z = z\} \\
&= (a^{*}].
\end{aligned}
$$

Therefore $(0 : J) = (a^*]$. Now, $J \cap (0 : J) = (a^{**}] \cap (a^*] = ((a^{**} \circ a^*)] = (0]$ and $J \vee (0 : J) = (a^{**}] \vee (a^*] = \{x \in L : x \leqslant (t^* \circ s^*)^*$, where $t \in (a^{**}], s \in (a^*]\} = \{x \in L : x \leqslant 0^*\} = L$. Hence $J$ is complemented. Thus $J$ is in the centre of $KI(L)$. □

Finally, we prove the following theorem.

THEOREM 5.3. *Let $L$ be a $*$-commutative PCASL in which $x \leqslant x^{**}$, for all $x \in L$. Then the centre of $KI(L)$ is isomorphic to $S(L)$.*

PROOF. Suppose $B(KI(L))$ is the Boolean centre of $KI(L)$. Now, define $\psi : B(KI(L)) \to S(L)$ as follows: for any $I \in B(KI(L))$, we have $I = (x]$, for some $x \in S(L)$. Then there exists $x, y \in S(L)$ such that $I = (x]$ and $J = (y]$. Now, $I = J \Leftrightarrow (x] = (y] \Leftrightarrow x = y \Leftrightarrow \psi(I) = \psi(J)$. Therefore $\psi$ is well defined and one-one. let $x \in S(L)$. Then we have $(x]$ is a kernel ideal. Then by theorem 5.2, $(x]$ is in the centre of $KI(L)$. Now, $\psi(x] = x$. Thus $\psi$ is onto and hence $\psi$ is bijection. Now, we shall prove that $\psi$ is homomorphism. Let $I, J \in B(KI(L))$. Then we have $I, J$ are kernel ideals. Then there exists $x, y \in S(L)$ such that $I = (x]$ and $J = (y]$. Now, $I \cap J = (x] \cap (y] = (x \circ y]$ and $I \cap J$ is a kernel ideal. Now, consider $\psi(I \cap J) = \psi((x \circ y]) = x \circ y = \psi(I) \circ \psi(J)$. Let $t \in ((x^* \circ y^*)^*]$. This implies $t = (x^* \circ y^*)^* \circ t$. It follows that $t \in I \vee J$. Hence $((x^* \circ y^*)^*] \subseteq I \vee J$. Thus $I \vee J = ((x^* \circ y^*)^*] = (x \veebar y]$. Therefore $\psi$ is an homomorphism. Now, $\psi((0]) = 0$ and $\psi(L) = \psi((0^*])$. Thus the centre of $KI(L)$ is isomorphic to $S(L)$. □

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G. Nanaji Rao: Department of Mathematics, Andhra University, Visakhapatnam-530003, India.

*E-mail address*: nani6us@yahoo.com, drgnanajirao.math@auvsp.edu.in

S. Sujatha Kumari: Department of Mathematics, Andhra University, Visakhapatnam-530003, India,

*E-mail address*: sskmaths9@gmail.com