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TOTAL INFLUENCE NUMBER IN COMPLEMENTARY PRISMS

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Abstract. The graph labeling problem that appears in graph theory has a fast development recently. Numerous variations of labeling have been investigated in the literature. The total influence number is a new approach to the concept of graph labeling. The total influence number can be viewed as vertex labeling problems concerned with the sum of the labels. Although many vertex labeling problems concerning with the sum of all of the labels study to minimize the sum, the total influence number has the aim of maximizing the sum. This means that this parameter attempt to maximize the profit associated with each vertex. In this paper, we consider the total influence number in complementary prisms. We determined the total influence number of $G\overline{G}$ for specific graphs G.

1. Introduction

The field of graph theory plays a vital role in various fields. One of the main problems in graph theory is graph labeling. Typically, the problems can be described as follows: for a given graph, find an optimal way of labeling the vertices with distinct integers. If we want to find the labeling which minimizes the total 'length' sum of the edges, we have the minimum sum problems. In a similar way the so-called bandwidth and cutwidth problems are defined. Graph labelings were first introduced by A. Rosa in 1967 [**9**]. Labeled graphs are becoming an increasingly useful family of Mathematical Models for a broad range of applications.

Numerous variations of labeling have been investigated in the literature. Many graph labeling problems seek to find the smallest integer label required to satisfy certain constraints. Other problems seek to minimize the sum of all of the labels.

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In this paper, we study the total influence number as a graph parameter. The total influence number can be viewed as vertex labeling problems concerned with the sum of the labels. It is another type of the graph parameter known as the influence number.

The total influence number is a new approach to the concept of graph labeling, introduced by Daugherty and Daugherty et al. in [**4, 5**]. Although many vertex labeling problems are concerned with studying to minimize the sum of all of the labels, the influence and total influence numbers have the aim of maximizing the sum. Aytac and Ciftci studied the total influence number of some splitting graphs in [**1**]. We studied the total influence number of some complement graphs in [**2**].

Throughout this paper, the following notation will be used. Let $G = (V, E)$ be a simple undirected graph of order n. The vertex set and edge set of a graph is denoted by $V(G)$ and $E(G)$, respectively. It is assumed that $V(G)$ will be abbreviated V. For a vertex subset $S \subseteq V$, $\overline{S} = V - S$ denotes the complement of S with respect to V.

The shortest distance in G between two vertices u and v will be denoted $d(u, v)$. The diameter of G, denoted by $diam(G)$ is the largest distance between two vertices in $V(G)$ [**3, 6**]. For any vertex u, let $d(u, S) = \min\limits_{v \in S} d(u, v)$. Then $d(u, S) = 0$ if and only if $u \in S$

The total influence number of a vertex $v \in S$ is

$$\eta_T(v) = \sum_{u \in \overline{S}} \frac{1}{2^{d(u,v)}}.$$

The total influence number of a vertex subset S is

$$\eta_T(S) = \sum_{v \in S} \eta_T(v) = \sum_{v \in S} \sum_{u \in \overline{S}} \frac{1}{2^{d(u,v)}}.$$

The total influence number of a graph G is $\eta_T(G) = \max\limits_{S \subseteq V} \eta_T(S)$. A set S is called η_T-set if $\eta_T(S) = \eta_T(G)$ [**4, 5**].

The article proceeds as follows. In Section 2, basic results of literature on the total influence number of some special graphs are presented. Some results of the total influence number for complementary prisms are given in Section 3.

THEOREM 1.1 ([**10**]). *If f is continious on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.*

To find the absolute maximum and minimum values of a continious function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

2. Main Results On The Total Influence Number

In this section, we recall some of the known results with respect to total influence number.

THEOREM 2.1 ([**4**]). *For any graph G=(V,E), with vertex partitions V_1 and V_2 and a set $S \subseteq V$ let $S_1 = V_1 \cap S$, $S_2 = V_2 \cap S$, $\overline{S} = V - S$, $\overline{S}_1 = V_1 - S_1$ and $\overline{S}_2 = V_2 - S_2$. Then,*

$$\eta_T(S) = \eta_T(S_1, \overline{S}_1) + \eta_T(S_2, \overline{S}_1) + \eta_T(S_2, \overline{S}_2) + \eta_T(S_1, \overline{S}_2).$$

DEFINITION 2.1. ([**4**]) A vertex subset S is called an alternating set if and only if S is either (1) the empty set or (2) a maximal independent set such that $\exists u \in S \ni \forall v \in S$, $d(u, v) = 2k$ for some $k \in \mathbb{Z}$.

THEOREM 2.2 ([**4**]). *For a path P_n $(n > 1)$, a vertex subset S has maximum total influence if and only if it is a non-empty alternating set.*

THEOREM 2.3 ([**4**]). *The total influence number of*

(a) *the complete graph K_n is*

$$\eta_T(K_n) = \begin{cases} \frac{n^2}{8} & \text{if } n \text{ is even,} \\ \frac{n^2-1}{8} & \text{if } n \text{ is odd.} \end{cases}$$

(b) *the star $K_{1,n}$ is*

$$\eta_T(K_{1,n}) = \begin{cases} \frac{(n+2)^2}{16} & \text{if } n \text{ is even,} \\ \frac{(n+1)(n+3)}{16} & \text{if } n \text{ is odd.} \end{cases}$$

(c) *the double star $DS_{n,m}$ is*

$$\eta_T(DS_{n,m}) = \begin{cases} \frac{1}{16}n^2 + \frac{3}{8}n + \frac{1}{16}m^2 + \frac{3}{8}m + \frac{1}{16}nm + \frac{3}{4} & \text{if } n, m \text{ are even,} \\ \frac{1}{16}n^2 + \frac{3}{8}n + \frac{1}{16}m^2 + \frac{3}{8}m + \frac{1}{16}nm + \frac{11}{16} & \text{otherwise.} \end{cases}$$

(d) *the complete bipartite graph $K_{n,m}$ is*

$$\eta_T(K_{n,m}) = \begin{cases} \frac{mn}{2} & \text{if } n \geqslant \frac{m}{2}, \\ \frac{(2n+m)^2}{16} & \text{if } n < \frac{m}{2}, m \text{ is even,} \\ \frac{(2n+m+1)(2n+m-1)}{16} & \text{if } n < \frac{m}{2}, m \text{ is odd.} \end{cases}$$

(e) *the path P_n is*

$$\eta_T(P_n) = \begin{cases} \frac{(10)2^{-n}+6n-10}{9} & \text{if } n \text{ is even,} \\ \frac{(8)2^{-n}+6n-10}{9} & \text{if } n \text{ is odd.} \end{cases}$$

THEOREM 2.4 ([**2**]). *For a graph C_n with $n \geqslant 6$, the total influence number is*

$$\eta_T(C_n) = \begin{cases} \frac{2n}{3} + 2^{-\frac{n+1}{2}}(-\frac{1}{9} - n) - \frac{2}{9} & \text{if } n \text{ is odd and } \lceil \frac{n}{2} \rceil \text{ is even,} \\ \frac{2n}{3} + 2^{-\frac{n+1}{2}}(\frac{1}{9} - n) - \frac{2}{9} & \text{if } n \text{ and } \lceil \frac{n}{2} \rceil \text{ are odd,} \\ \frac{2n}{3} - \frac{2n}{3}2^{-\frac{n}{2}} & \text{if } n \text{ and } \frac{n}{2} \text{ are even,} \\ \frac{2n}{3} - \frac{5n}{6}2^{-\frac{n}{2}} & \text{if } n \text{ is even and } \frac{n}{2} \text{ is odd.} \end{cases}$$

THEOREM 2.5 ([**2**]). *The total influence number of* $W_{1,n}$ *with* $n \geqslant 6$ *is*

$$\eta_T(W_{1,n}) = \begin{cases} \frac{n^2+8n-1}{16} & \text{if } n \text{ is odd,} \\ \frac{n^2+8n}{16} & \text{if } n \text{ is even.} \end{cases}$$

THEOREM 2.6 ([**2**]). *The total influence number of* $\overline{tK_2}$ *is*

$$\eta_T(\overline{tK_2}) = \begin{cases} \frac{t^2}{2} & \text{if } t \text{ is even,} \\ \frac{2t^2-1}{4} & \text{if } t \text{ is odd.} \end{cases}$$

3. Total Influence Number Of Some Complementary Prism Graphs

In this section, we have calculated total influence number of complementary prisms of some known graph such as $K_n\overline{K_n}$, $K_{1,n-1}\overline{K_{1,n-1}}$, $K_{n,m}\overline{K_{n,m}}$, $tK_2\overline{tK_2}$.

DEFINITION 3.1. ([**7, 8**]) For a graph G, its complementary prism, denoted $G\overline{G}$, is formed from a copy of G and a copy of $\overline{G}$ by adding a perfect matching between corresponding vertices. For each $v \in V(G)$, let $\overline{v}$ denote the vertex v in the copy of $\overline{G}$. Formally, $G\overline{G}$ is formed from $G \cup \overline{G}$ by adding the edge $v\overline{v}$ for every $v \in V(G)$.

To aid the discussion of complementary prisms, we will use the following terminology: For vertex partitions V_1 and V_2 of $G\overline{G}$, let $V_1 = V(G)$ and $V_2 = V(\overline{G})$.

THEOREM 3.1. *For the complementary prism graph* $K_n\overline{K}_n$*, a set* S *is an* η_T*-set if and only if it contains exactly* $\lfloor \frac{n}{2} \rfloor$ *vertices from* V_1 *and* $\lceil \frac{n}{2} \rceil$ *vertices from* V_2 *or* $\lceil \frac{n}{2} \rceil$ *vertices from* V_1 *and* $\lfloor \frac{n}{2} \rfloor$ *vertices from* V_2*. Additionally, these vertices are not corresponding to each other. Furthermore,*

$$\eta_T(K_n\overline{K}_n) = \begin{cases} \frac{9n^2+8n}{32} & \text{if } n \text{ is even,} \\ \frac{9n^2+8n-1}{32} & \text{if } n \text{ is odd.} \end{cases}$$

PROOF. For a vertex subset S, let $x = |V_1 \cap S|$ and $y_1 + y_2 = |V_2 \cap S|$ and $f(x, y_1, y_2) := \eta_T(S)$, where y_1, y_2 are the number of vertices corresponding to x vertices and not corresponding to x vertices, respectively. Thus we have

$$\begin{aligned} f(x, y_1, y_2) &= \frac{1}{2}x(n-x) + \frac{1}{2}(x-y_1) + \frac{1}{4}(x-1)(x-y_1) + \frac{1}{4}x(n-x-y_2) \\ &+ \frac{1}{4}y_1(n-x) + \frac{1}{8}y_1(n-y_1-y_2) + \frac{1}{2}y_2 + \frac{1}{4}y_2(y_2-1) \\ &+ \frac{1}{4}y_2(n-x-y_2) + \frac{1}{8}y_2(n-y_1-y_2). \end{aligned} \tag{3.1}$$

Bounds are $0 \leqslant x \leqslant n$, $0 \leqslant y_1 \leqslant x$ and $0 \leqslant y_2 \leqslant n-x$. Solving the system $f_x(x, y_1, y_2) = 0$, $f_{y_1}(x, y_1, y_2) = 0$, $f_{y_2}(x, y_1, y_2) = 0$ does not give a solution. So we search to the maximum of $f(x, y_1, y_2)$ by looking at the boundaries of x, y_1, y_2.

Case 1. For $x = 0$, we maximize $f(0, y_1, y_2) = \frac{1}{4}y_2 - \frac{1}{4}y_1 - \frac{1}{4}y_1y_2 + \frac{3}{8}y_1n + \frac{3}{8}y_2n - \frac{1}{8}y_1^2 - \frac{1}{8}y_2^2$. Solving the system $f_{y_1}(x, y_1, y_2) = 0$, $f_{y_2}(x, y_1, y_2) = 0$ does not give a solution and we must seek the maximum of the function at the boundaries of y_1 and y_2.

Case 1.1. For $y_1 = 0$ ($y_1 = x = 0$), we maximize $f(0,0,y_2) = \frac{1}{8}y_2(3n - y_2 + 2)$. Solving $f_{y_2}(0,0,y_2) = 0$ gives $y_2 = \frac{3n+2}{2}$. But this value is outside the range $[0,n]$. From the boundaries of y_2, we get $f(0,0,0) = 0$ and $f(0,0,n) = \frac{n(n+1)}{4}$. Since $0 < |S| < 2n$, we ignore $f(0,0,0) = 0$.

Case 1.2. For $y_2 = 0$ and $y_2 = n - x = n$, since $0 \leqslant y_1 \leqslant x$ and $x = 0$, y_1 just takes the value 0. Thus we have same results as the boundaries of y_2 in Case 1.1.

In Case 1, the function is maximized at $y_1 = 0$ and $y_2 = n$.

Case 2. For $x = n$, we maximize $f(n, y_1, y_2) = -\frac{1}{8}y_1^2 - \frac{1}{4}y_1y_2 - \frac{1}{4}y_1n - \frac{1}{4}y_1 - \frac{1}{8}y_2^2 - \frac{1}{8}y_2n + \frac{1}{4}y_2 + \frac{1}{4}n^2 + \frac{1}{4}n$. Solving $f_{y_1}(n, y_1, y_2) = 0$, $f_{y_2}(n, y_1, y_2) = 0$ for y_1 and y_2 doesn't give a solution and we must seek the maximum of the function at the boundaries of y_1 and y_2.

Case 2.1. For $y_1 = 0$ and $y_1 = x = n$, since $0 \leqslant y_2 \leqslant n - x$ and $x = n$, y_2 just takes the value 0. So we have $f(n,0,0) = \frac{n(n+1)}{4}$ and $f(n,n,0) = 0$ at the boundaries of y_1, respectively.

Case 2.2. For $y_2 = 0$ ($y_2 = n - x = 0$), we maximize $f(n, y_1, 0) = -\frac{1}{8}(y_1 - n)(y_1 + 2n + 2)$. Solving $f_{y_1}(n, y_1, 0) = 0$ gives $y_1 = -\frac{n+2}{2}$. But this value is outside the range $[0,n]$. From the boundaries of y_1, we get same results as Case 2.1.

In Case 2, the function is maximized at $y_1 = 0$ and $y_2 = 0$.

Case 3. For $y_1 = 0$, $f(x, 0, y_2) = \frac{1}{4}y_2 + \frac{1}{4}x + \frac{3}{8}y_2n - \frac{1}{2}y_2x + \frac{3}{4}nx - \frac{1}{8}y_2^2 - \frac{1}{2}x^2$. Solving $f_x(x,0,y_2) = 0$, $f_{y_2}(x,0,y_2) = 0$ for x and y_2 doesn't give a solution and we must seek the maximum of the function at the boundaries of x and y_2.

Case 3.1 Examinations at $x = 0$ and $x = n$ are equivalent to Case 1.1 and Case 2.1, respectively.

Case 3.2 For $y_2 = 0$, we maximize $f(x,0,0) = \frac{1}{4}x(3n - 2x + 1)$. Solving $f_x(x,0,0) = 0$ gives $x = \frac{3n+1}{4} \in [0,n]$.

i. Let n be even and $n = 4k$. We have $f(\lfloor\frac{3n+1}{4}\rfloor, 0, 0) = f(\frac{3n}{4}, 0, 0) = \frac{9n^2+6n}{32}$ and $f(\lceil\frac{3n+1}{4}\rceil, 0, 0) = f(\frac{3n+4}{4}, 0, 0) = \frac{9n^2+6n-8}{32}$.

ii. Let n be even and $n \neq 4k$. We have $f(\lfloor\frac{3n+1}{4}\rfloor, 0, 0) = f(\frac{3n-2}{4}, 0, 0) = \frac{9n^2+6n-8}{32}$ and $f(\lceil\frac{3n+1}{4}\rceil, 0, 0) = f(\frac{3n+2}{4}, 0, 0) = \frac{9n^2+6n}{32}$.

iii. Let n be odd and $n = 4k + 1$. We have $f(\frac{3n+1}{4}, 0, 0) = \frac{9n^2+6n+1}{32}$.

iv. Let n be odd and $n \neq 4k + 1$. We have $f(\lfloor\frac{3n+1}{4}\rfloor, 0, 0) = f(\frac{3n-1}{4}, 0, 0) = \frac{9n^2+6n-3}{32}$ and $f(\lceil\frac{3n+1}{4}\rceil, 0, 0) = f(\frac{3n+3}{4}, 0, 0) = \frac{9n^2+6n-3}{32}$.

After examination at the boundaries of x, we get $f(0,0,0) = 0$ and $f(n,0,0) = \frac{n(n+1)}{4}$. Since $0 < |S| < 2n$, we can ignore $f(0,0,0) = 0$.

Case 3.3. For $y_2 = n - x$, we maximize $f(x, 0, n - x) = \frac{1}{4}n^2 + \frac{1}{8}nx + \frac{1}{4}n - \frac{1}{8}x^2$. Solving $f_x(x, 0, n - x) = 0$ gives $x = \frac{n}{2} \in [0,n]$.

i. Let n be even. $x = \frac{n}{2}$ is only valid. So we have $f(\frac{n}{2}, 0, \frac{n}{2}) = \frac{9n^2+8n}{32}$.

ii. Let n be odd. We need to try both $x = \lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$ and $x = \lceil \frac{n}{2} \rceil = \frac{n+1}{2}$. So we have $f(\frac{n-1}{2}, 0, \frac{n-1}{2}) = \frac{9n^2+8n-17}{32}$, $f(\frac{n-1}{2}, 0, \frac{n+1}{2}) = \frac{9n^2+8n-1}{32}$ and $f(\frac{n+1}{2}, 0, \frac{n-1}{2}) = \frac{9n^2+8n-1}{32}$. For $f(\frac{n+1}{2}, 0, \frac{n+1}{2})$, since $y_2 + x > n$ we can ignore this result.

After examination at the boundaries of x, we get $f(0,0,n) = f(n,0,0) = \frac{n(n+1)}{4}$.

In Case 3, the function is maximized at $y_2 = n - x$ and $x = \frac{n}{2}$.

Case 4. For $y_1 = x$, we maximize $f(x,x,y_2) = \frac{1}{4}y_2 + \frac{3}{8}y_2 n - \frac{3}{4}y_2 x + \frac{9}{8}nx - \frac{1}{8}y_2^2 - \frac{9}{8}x^2$. Solving the system $f_x(x,x,y_2) = 0$ and $f_{y_2}(x,x,y_2) = 0$ doesn't give a solution and we must seek the maximum of the function at the boundaries of x and y_2.

Case 4.1. Examinations at $x = 0$ and $x = n$ are equivalent to Case 1.1 and Case 2.1, respectively.

Case 4.2. For $y_2 = 0$, we maximize $f(x,x,0) = \frac{9}{8}x(n-x)$. Solving $f_x(x,x,0) = 0$ gives $x = \frac{n}{2} \in [0,n]$.

i. Let n be even. $x = \frac{n}{2}$ is only valid.
$f(\frac{n}{2}, \frac{n}{2}, 0) = \frac{9n^2}{32}$.

ii. Let n be odd. We need to try both $x = \lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$ and $x = \lceil \frac{n}{2} \rceil = \frac{n+1}{2}$. So we have $f(\frac{n-1}{2}, \frac{n-1}{2}, 0) = \frac{9n^2-9}{32}$, $f(\frac{n+1}{2}, \frac{n+1}{2}, 0) = \frac{9n^2-9}{32}$ and $f(\frac{n+1}{2}, \frac{n-1}{2}, 0) = \frac{9n^2+7}{32}$. For $f(\frac{n-1}{2}, \frac{n+1}{2}, 0)$, since $y_1 > x$, we can ignore this result.

At the boundaries of x, since $0 < |S| < 2n$, we don't obtain solution.

Case 4.3. For $y_2 = n - x$, we maximize $f(x,x,n-x) = \frac{1}{4}(n-x)(n+2x+1)$. Solving $f_x(x,x,n-x) = 0$ gives $x = \frac{n-1}{4} \in [0,n]$. We need to try both $x = \lfloor \frac{n-1}{4} \rfloor$ and $x = \lceil \frac{n-1}{4} \rceil$. Consequently we find the maximum value of this case,
if n is even:

$$\begin{cases} f(\frac{n}{4}, \frac{n-4}{4}, \frac{3n}{4}) = \frac{9n^2+6n+4}{32} & \text{if } n = 4k, \\ f(\frac{n+2}{4}, \frac{n-2}{4}, \frac{3n-2}{4}) = \frac{9n^2+6n+4}{32} & \text{if } n \neq 4k. \end{cases}$$

if n is odd:

$$\begin{cases} f(\frac{n-1}{4}, \frac{n-1}{4}, \frac{3n+1}{4}) = \frac{9n^2+6n+1}{32} & \text{if } n = 4k+1, \\ f(\frac{n+1}{4}, \frac{n-3}{4}, \frac{3n-1}{4}) = \frac{9n^2+6n+5}{32} & \text{if } n \neq 4k+1. \end{cases}$$

After examination at the boundaries of x, we have $f(0,0,n) = \frac{n(n+1)}{4}$ and $f(n,n,0) = 0$. Since $0 < |S| < 2n$, we can ignore $f(n,n,0) = 0$.

In Case 4, the function is maximized at $y_2 = n - x$ and $x = \frac{n-1}{4}$.

Case 5. For $y_2 = 0$, we maximize $f(x,y_1,0) = \frac{1}{4}x - \frac{1}{4}y_1 + \frac{3}{8}y_1 n - \frac{1}{2}y_1 x + \frac{3}{4}nx - \frac{1}{8}y_1^2 - \frac{1}{2}x^2$. Solving the system $f_x(x,y_1,0) = 0$ and $f_{y_1}(x,y_1,0) = 0$ doesn't give a solution so we must seek the maximum of the function at the boundaries of x and y_1. Examinations at $x = 0$, $x = n$, $y_1 = 0$, $y_1 = x$ are equivalent to Case 1.2, Case 2.2 and Case 3.2, Case 4.2, respectively.

Case 6. For $y_2 = n - x$, we maximize $f(x, y_1, n-x) = -\frac{1}{8}y_1^2 + \frac{1}{8}y_1 n - \frac{1}{4}y_1 x - \frac{1}{4}y_1 + \frac{1}{4}n^2 + \frac{1}{8}nx + \frac{1}{4}n - \frac{1}{8}x^2$. Solving the system $f_x(x, y_1, n-x) = 0$ and $f_{y_1}(x, y_1, n-x) = 0$ doesn't give a solution so we must seek the maximum of the function at the boundaries of x and y_1. Examinations at $x = 0$, $x = n$, $y_1 = 0$, $y_1 = x$ are equivalent to Case 1.2, Case 2.2, Case 3.3, Case 4.3, respectively.

From Case 1, Case 2, Case 3, Case 4, Case 5 and Case 6, the proof is completed. $\square$

THEOREM 3.2. *For the complementary prism graph $K_{1,n-1}\overline{K}_{1,n-1}$, a set S is an η_T-set if and only if it contains exactly $\lceil \frac{n}{2} \rceil$ vertices from V_1, exactly $\lfloor \frac{n}{2} \rfloor$ vertices from V_2 and center vertex of $K_{1,n-1}$ must be in S and its corresponding vertex in $\overline{K}_{1,n-1}$ must not be in S or is the complement of this set. Furthermore,*

$$\eta_T(K_{1,n-1}\overline{K}_{1,n-1}) = \frac{5n^2 + 3n}{16}.$$

PROOF. For the total influence set S, assume without loss of generality that the center vertex of $K_{1,n-1}$ is in S. Since its corresponding vertex in $\overline{K}_{1,n-1}$ only adjacent to center vertex, this vertex is not in S. Let $x = |V_1 \cap S|$ and $y_1 + y_2 = |V_2 \cap S|$, where y_1, y_2 are the number of vertices corresponding to x vertices and not corresponding to x vertices, respectively. By using definitions of x, y_1, y_2 and $f(x, y_1, y_2) := \eta_T(S)$, we have

$$\begin{aligned}
f(x, y_1, y_2) =& \frac{1}{2}(n-x) + \frac{1}{2} + \frac{1}{4}(n-1-y_1-y_2) + \frac{1}{4}(x-1)(n-x) \\
&+ \frac{1}{2}(x-1-y_1) + \frac{1}{4}(x-2)(x-1-y_1) + \frac{1}{4}(x-1)(n-x-y_2) \\
&+ \frac{1}{4}(x-1) + \frac{1}{4}y_1(n-x) + \frac{1}{8}y_1 + \frac{1}{2}y_1(n-1-y_1-y_2) + \frac{1}{2}y_2 \\
&+ \frac{1}{4}y_2(y_2-1) + \frac{1}{4}y_2(n-x-y_2) + \frac{1}{8}y_2 + \frac{1}{2}y_2(n-1-y_1-y_2).
\end{aligned}$$

Boundaries are $1 \leqslant x \leqslant n$, $0 \leqslant y_1 \leqslant x-1$ and $0 \leqslant y_2 \leqslant n-x$. Solving the system $f_x(x, y_1, y_2) = 0$, $f_{y_1}(x, y_1, y_2) = 0$ and $f_{y_2}(x, y_1, y_2) = 0$ doesn't give a solution so we must seek the maximum of the function at the boundaries of x, y_1 and y_2.

Case 1. For $x = 1$, we maximize $f(1, y_1, y_2) = \frac{3}{4}n - \frac{5}{8}y_2 - \frac{9}{8}y_1 - y_1 y_2 + \frac{3}{4}y_1 n + \frac{3}{4}y_2 n - \frac{1}{2}y_1^2 - \frac{1}{2}y_2^2 - \frac{1}{4}$. Solving the system $f_{y_1}(1, y_1, y_2) = 0$, $f_{y_2}(1, y_1, y_2) = 0$ doesn't give a solution so we must seek the maximum of the function at the boundaries of y_1 and y_2.

Case 1.1. For $y_1 = 0$ ($y_1 = x - 1 = 0$), we maximize $f(1, 0, y_2) = \frac{3}{4}n - \frac{5}{8}y_2 + \frac{3}{4}y_2 n - \frac{1}{2}y_2^2 - \frac{1}{4}$. Solving $f_{y_2}(1, 0, y_2) = 0$ gives $y_2 = \frac{6n-5}{8}$.

i. Let n be even and $n = 4k$. We need to try both $y_2 = \lfloor \frac{6n-5}{8} \rfloor = \frac{3n-4}{4}$ and $y_2 = \lceil \frac{6n-5}{8} \rceil = \frac{3n}{4}$. Thus we obtain $f(1, 0, \frac{3n-4}{4}) = \frac{9n^2+9n-4}{32}$ and $f(1, 0, \frac{3n}{4}) = \frac{9n^2+9n-8}{32}$.

ii. Let n be even and $n \neq 4k$. We need to try both $y_2 = \lfloor \frac{6n-5}{8} \rfloor = \frac{3n-6}{4}$ and $y_2 = \lceil \frac{6n-5}{8} \rceil = \frac{3n-2}{4}$. Thus we obtain $f(1,0,\frac{3n-6}{4}) = \frac{9n^2+9n-14}{32}$ and $f(1,0,\frac{3n-2}{4}) = \frac{9n^2+9n-2}{32}$.

iii. Let n be odd and $n = 4k+1$. We need to try both $y_2 = \lfloor \frac{6n-5}{8} \rfloor = \frac{3n-3}{4}$ and $y_2 = \lceil \frac{6n-5}{8} \rceil = \frac{3n+1}{4}$. Thus we obtain $f(1,0,\frac{3n-3}{4}) = \frac{9n^2+9n-2}{32}$ and $f(1,0,\frac{3n+1}{4}) = \frac{9n^2+9n-14}{32}$.

iv. Let n be odd and $n \neq 4k+1$. We need to try both $y_2 = \lfloor \frac{6n-5}{8} \rfloor = \frac{3n-5}{4}$ and $y_2 = \lceil \frac{6n-5}{8} \rceil = \frac{3n-1}{4}$. Thus we obtain $f(1,0,\frac{3n-5}{4}) = \frac{9n^2+9n-8}{32}$ and $f(1,0,\frac{3n-1}{4}) = \frac{9n^2+9n-4}{32}$.

After doing examinations at the boundaries of y_2, we get $f(1,0,0) = \frac{3n-1}{4}$ and $f(1,0,n-1) = \frac{2n^2+3n-1}{8}$.

Case 1.2. For $y_2 = 0$ and $y_2 = n-x = n-1$, since $0 \leqslant y_1 \leqslant x-1$ and $x=1$, y_1 just takes the value 0. Thus we have same results as at the boundaries of y_2 in Case 1.1.

In Case 1, the function is maximized at $y_1 = 0$ and $y_2 = \frac{6n-5}{8}$.

Case 2. For $x = n$, we maximize $f(n,y_1,y_2) = -\frac{1}{2}y_1^2 - y_1y_2 + \frac{1}{4}y_1n - \frac{5}{8}y_1 - \frac{1}{2}y_2^2 + \frac{1}{4}y_2n - \frac{1}{8}y_2 + \frac{1}{4}n^2 + \frac{1}{4}n$. Solving the system $f_{y_1}(n,y_1,y_2) = 0$ and $f_{y_2}(n,y_1,y_2) = 0$ doesn't give a solution so we must seek the maximum of the function at the boundaries of y_1 and y_2.

Case 2.1. For $y_1 = 0$ and $y_1 = x-1 = n-1$, since $0 \leqslant y_2 \leqslant n-x$ and $x = n$, y_2 just takes the value 0. Thus we get $f(n,0,0) = \frac{n^2+n}{4}$ and $f(n,n-1,0) = \frac{3n+1}{8}$ from boundaries of y_1.

Case 2.2. For $y_2 = 0$, ($y_2 = n-x = 0$) we maximize $f(n,y_1,0) = -\frac{1}{2}y_1^2 + \frac{1}{4}y_1n - \frac{5}{8}y_1 + \frac{1}{4}n^2 + \frac{1}{4}n$. Solving $f_{y_1}(n,y_1,0) = 0$ gives $y_1 = \frac{2n-5}{8} \in [0,n-1]$.

i. Let n be even and $n = 4k$. After trying both $y_1 = \lfloor \frac{2n-5}{8} \rfloor = \frac{n-4}{4}$ and $y_1 = \lceil \frac{2n-5}{8} \rceil = \frac{n}{4}$, we get $f(n,\frac{n-4}{4},0) = \frac{9n^2+3n+4}{32}$ and $f(n,\frac{n}{4},0) = \frac{9n^2+3n}{32}$.

ii. Let n be even and $n \neq 4k$. After trying both $y_1 = \lfloor \frac{2n-5}{8} \rfloor = \frac{n-6}{4}$ and $y_1 = \lceil \frac{2n-5}{8} \rceil = \frac{n-2}{4}$, we get $f(n,\frac{n-6}{4},0) = \frac{9n^2+3n-6}{32}$ and $f(n,\frac{n-2}{4},0) = \frac{9n^2+3n+6}{32}$.

iii. Let n be odd and $n = 4k+1$. After trying both $y_1 = \lfloor \frac{2n-5}{8} \rfloor = \frac{n-5}{4}$ and $y_1 = \lceil \frac{2n-5}{8} \rceil = \frac{n-1}{4}$, we get $f(n,\frac{n-5}{4},0) = \frac{9n^2+3n}{32}$ and $f(n,\frac{n-1}{4},0) = \frac{9n^2+3n+4}{32}$.

iv. Let n be odd and $n \neq 4k+1$. After trying both $y_1 = \lfloor \frac{2n-5}{8} \rfloor = \frac{n-3}{4}$ and $y_1 = \lceil \frac{2n-5}{8} \rceil = \frac{n+1}{4}$, we get $f(n,\frac{n-3}{4},0) = \frac{9n^2+3n+6}{32}$ and $f(n,\frac{n+1}{4},0) = \frac{9n^2+3n-6}{32}$.

We also get same results as Case 2.1. at the boundaries of y_1.

In Case 2, the function is maximized at $y_1 = \frac{2n-5}{8}$ and $y_2 = 0$.

Case 3. For $y_1 = 0$, we maximize $f(x,0,y_2) = \frac{1}{4}n - \frac{1}{8}y_2 + \frac{3}{4}y_2n - \frac{1}{2}y_2x + \frac{1}{2}nx - \frac{1}{2}y_2^2 - \frac{1}{4}x^2$. Solving the system $f_x(x,0,y_2) = 0$ and $f_{y_2}(x,0,y_2) = 0$ doesn't give a solution so we must seek the maximum of the function at the boundaries of x and y_2.

Case 3.1. Examinations at $x = 1$ and $x = n$ are equivalent to Case 1.1 and Case 2.1, respectively.

Case 3.2. For $y_2 = 0$, we maximize $f(x, 0, 0) = -\frac{1}{4}x^2 + \frac{1}{2}nx + \frac{1}{4}n$. Solving $f_x(x, 0, 0) = 0$ gives $x = n \in [0, n]$. Thus we get $f(n, 0, 0) = \frac{n^2+2n}{4}$ at $x = n$. We also get $f(1, 0, 0) = \frac{3n-1}{4}$ from the boundaries of x.

Case 3.3. For $y_2 = n-x$, we maximize $f(x, 0, n-x) = \frac{1}{4}n^2+\frac{1}{4}nx+\frac{1}{8}n-\frac{1}{4}x^2+\frac{1}{8}x$. Solving $f_x(x, 0, n-x) = 0$ gives $x = \frac{2n+1}{4} \in [1, n]$.

i. Let n be even. After trying both $x = \lfloor \frac{2n+1}{4} \rfloor = \frac{n}{2}$ and $x = \lceil \frac{2n+1}{4} \rceil = \frac{n+2}{2}$, we obtain $f(\frac{n}{2}, 0, \frac{n}{2}) = \frac{5n^2+3n}{16}$, $f(\frac{n}{2}, 0, \frac{n-2}{2}) = \frac{5n^2+3n-6}{16}$ and $f(\frac{n+2}{2}, 0, \frac{n-2}{2}) = \frac{5n^2+3n-2}{16}$. Additionally, we get $f(\frac{n+2}{2}, 0, \frac{n}{2})$ but we ignore this result, since $x+y_2 > n$.

ii. Let n be odd. After trying both $x = \lfloor \frac{2n+1}{4} \rfloor = \frac{n-1}{2}$ and $x = \lceil \frac{2n+1}{4} \rceil = \frac{n+1}{2}$, we obtain $f(\frac{n-1}{2}, 0, \frac{n-1}{2}) = \frac{5n^2+3n-4}{16}$, $f(\frac{n-1}{2}, 0, \frac{n+1}{2}) = \frac{5n^2+3n-2}{16}$ and $f(\frac{n+1}{2}, 0, \frac{n-1}{2}) = \frac{5n^2+3n}{16}$. Additionally, we get $f(\frac{n+1}{2}, 0, \frac{n+1}{2})$ but we ignore this result since $x + y_2 > n$.

We also examine at the boundaries of x. From this examinations, we get $f(1, 0, n-1) = \frac{2n^2+3n-1}{8}$ and $f(n, 0, 0) = \frac{n^2+n}{4}$.

In Case 3, the function is maximized at $y_2 = n - x$ and $x = \frac{2n+1}{4}$.

Case 4. For $y_1 = x - 1$, we maximize $f(x, x-1, y_2) = \frac{7}{8}y_2 - \frac{1}{2}n + \frac{7}{8}x + \frac{3}{4}y_2n - \frac{3}{2}y_2x + \frac{5}{4}nx - \frac{1}{2}y_2^2 - \frac{5}{4}x^2 + \frac{1}{8}$. Solving the system $f_x(x, x-1, y_2) = 0$ and $f_{y_2}(x, x-1, y_2) = 0$ doesn't give a solution so we must seek the maximum of the function at the boundaries of x and y_2.

Case 4.1. Examinations at $x = 1$ and $x = n$ are equivalent to Case 1.1 and Case 2.1, respectively.

Case 4.2. For $y_2 = 0$, we maximize $f(x, x-1, 0) = \frac{7}{8}x - \frac{1}{2}n + \frac{5}{4}nx - \frac{5}{4}x^2 + \frac{1}{8}$. Solving $f_x(x, x-1, 0) = 0$ gives $x = \frac{10n+7}{20} \in [1, n]$.

i. Let n be even. We need to try both $x = \lfloor \frac{10n+7}{20} \rfloor = \frac{n}{2}$ and $x = \lceil \frac{10n+7}{20} \rceil = \frac{n+2}{2}$ for x. We also need to try both $x - 1 = \frac{n-2}{2}$ and $x - 1 = \frac{n}{2}$ for y_1. By using these results, we obtain $f(\frac{n}{2}, \frac{n-2}{2}, 0) = \frac{5n^2-n+2}{16}$, $f(\frac{n+2}{2}, \frac{n-2}{2}, 0) = \frac{5n^2-n+6}{16}$ and $f(\frac{n+2}{2}, \frac{n}{2}, 0) = \frac{5n^2-n-4}{16}$. We also get $f(\frac{n}{2}, \frac{n}{2}, 0)$ but we can ignore this result since $y_1 > x - 1$.

ii. Let n be odd. We need to try both $x = \lfloor \frac{10n+7}{20} \rfloor = \frac{n-1}{2}$ and $x = \lceil \frac{10n+7}{20} \rceil = \frac{n+1}{2}$ for x. We also need to try both $x-1 = \frac{n-3}{2}$ and $x-1 = \frac{n-1}{2}$ for y_1. By using these results, we obtain $f(\frac{n-1}{2}, \frac{n-3}{2}, 0) = \frac{5n^2-n-10}{16}$, $f(\frac{n+1}{2}, \frac{n-3}{2}, 0) = \frac{5n^2-n+2}{16}$ and $f(\frac{n+1}{2}, \frac{n-1}{2}, 0) = \frac{5n^2-n+4}{16}$. We also get $f(\frac{n-1}{2}, \frac{n-1}{2}, 0)$ but we can ignore this result since $y_1 > x - 1$.

We also examine at the boundaries of x. From these examinations, we get $f(1, 0, 0) = \frac{3n-1}{4}$ and $f(n, n-1, 0) = \frac{3n+1}{8}$.

Case 4.3. For $y_2 = n - x$, we maximize $f(x, x-1, n-x) = \frac{1}{4}n^2 + \frac{3}{8}n - \frac{1}{4}x^2 + \frac{1}{8}$. Solving $f_x(x, x-1, n-x) = 0$ doesn't give a solution. After examinations the boundaries of x, we get $f(1, 0, n-1) = \frac{2n^2+3n-1}{8}$ and $f(n, n-1, 0) = \frac{3n+1}{8}$.

In Case 4, the function is maximized at $y_2 = 0$ and $x = \frac{10n+7}{20}$.

Case 5. For $y_2 = 0$, we maximize $f(x, y_1, 0) = \frac{1}{4}n - \frac{5}{8}y_1 + \frac{3}{4}y_1 n - \frac{1}{2}y_1 x + \frac{1}{2}nx - \frac{1}{2}y_1^2 - \frac{1}{4}x^2$. Solving the system $f_x(x, y_1, 0) = 0$ and $f_{y_1}(x, y_1, 0) = 0$ gives $x = \frac{2n+5}{4}$ and $y_1 = \frac{2n-5}{4}$.

i. Let n be even. We need to try both $\lfloor \frac{2n+5}{4} \rfloor = \frac{n+2}{2}$ and $\lceil \frac{2n+5}{4} \rceil = \frac{n+4}{2}$ for x, also both $\lfloor \frac{2n-5}{4} \rfloor = \frac{n-4}{2}$ and $\lceil \frac{2n-5}{4} \rceil = \frac{n-2}{2}$ for y_1. From these cases, we get $f(\frac{n+2}{2}, \frac{n-4}{2}, 0) = \frac{5n^2-n}{16}$, $f(\frac{n+2}{2}, \frac{n-2}{2}, 0) = \frac{5n^2-n+6}{16}$, $f(\frac{n+4}{2}, \frac{n-4}{2}, 0) = \frac{5n^2-n+4}{16}$ and $f(\frac{n+4}{2}, \frac{n-2}{2}, 0) = \frac{5n^2-n+2}{16}$.

ii. Let n be odd. We need to try both $\lfloor \frac{2n+5}{4} \rfloor = \frac{n+1}{2}$ and $\lceil \frac{2n+5}{4} \rceil = \frac{n+3}{2}$ for x, also both $\lfloor \frac{2n-5}{4} \rfloor = \frac{n-3}{2}$ and $\lceil \frac{2n-5}{4} \rceil = \frac{n-1}{2}$ for y_1. From these cases, we get $f(\frac{n+1}{2}, \frac{n-3}{2}, 0) = \frac{5n^2-n+2}{16}$, $f(\frac{n+1}{2}, \frac{n-1}{2}, 0) = \frac{5n^2-n+4}{16}$, $f(\frac{n+3}{2}, \frac{n-3}{2}, 0) = \frac{5n^2-n+6}{16}$ and $f(\frac{n+3}{2}, \frac{n-1}{2}, 0) = \frac{5n^2-n}{16}$.

Additionally, examinations at the boundaries of x and y_1 are equivalent to Case 1.2, Case 2.2 and Case 3.2, Case 4.2, respectively.

Case 6. For $y_2 = n - x$, we maximize $f(x, y_1, n-x) = \frac{3}{8}n - \frac{1}{8}y_1 - \frac{1}{8}x - \frac{1}{4}y_1 n + (\frac{1}{4}x - \frac{1}{4})(n-x) - \frac{1}{2}y_1(y_1 - x + 1) + \frac{1}{4}n^2$. Solving the system $f_x(x, y_1, n-x) = 0$ and $f_{y_1}(x, y_1, n-x) = 0$ doesn't give a solution. Examinations at the boundaries of x and y_1 are equivalent to Case 1.2, Case 2.2 and Case 3.3, Case 4.3, respectively.

From Case 1, Case 2, Case 3, Case 4, Case 5 and Case 6, the total influence number of $K_{1,n-1}\overline{K}_{1,n-1}$ is $\eta_T(K_{1,n-1}\overline{K}_{1,n-1}) = (5n^2 + 3n)/16$. □

THEOREM 3.3. *Consider the complementary prism graph $K_{n,m}\overline{K}_{n,m}$ with $n < m$, let S be a total influence set and $S_1 = V_1 \cap S$, $S_2 = V_2 \cap S$, $\overline{S}_1 = V_1 - S_1$, $\overline{S}_2 = V_2 - S_2$. Additionally V_1 and V_2 can be partitoned in two pieces $V_1^{(1)} = \{v_1, v_2, ..., v_n\}$, $V_1^{(2)} = \{v_{n+1}, v_{n+2}, ..., v_{n+m}\}$ and $V_2^{(1)} = \{\overline{v}_1, \overline{v}_2, ..., \overline{v}_n\}$, $V_2^{(2)} = \{\overline{v}_{n+1}, \overline{v}_{n+2}, ..., \overline{v}_{n+m}\}$, respectively. Let $x = |V_1^{(1)} \cap S|$, $y = |V_1^{(2)} \cap S|$, $\overline{x} = |V_2^{(1)} \cap S|$, $\overline{y} = |V_2^{(2)} \cap S|$. Then S is an η_T-set if and only if the following conditions or their complements are satisfied:*

n	m	x	y	$\overline{x}$	$\overline{y}$	
odd	*even*	0	$\frac{2n+m}{2}$	$\frac{n-1}{2}$	$\frac{m}{2}$	$n < \frac{m}{2}$
even	*even*	0	$\frac{2n+m}{2}$	$\frac{n}{2}$	$\frac{m}{2}$	
odd	*odd*	0	$\frac{2n+m+1}{2}$	$\frac{n-1}{2}$	$\frac{m-1}{2}$	
even	*odd*	0	$\frac{2n+m+1}{2}$	$\frac{n}{2}$	$\frac{m-1}{2}$	
odd	*odd*	0	m	$\frac{n-1}{2}$	$\frac{m-1}{2}$	$n \geqslant \frac{m}{2}$
even	*odd*	0	m	$\frac{n}{2}$	$\frac{m-1}{2}$	
odd	*even*	0	m	$\frac{n-1}{2}$	$\frac{m}{2}$	
even	*even*	0	m	$\frac{n}{2}$	$\frac{m}{2}$	

Furthermore,

if $n < \frac{m}{2}$*:*

$$\eta_T(K_{n,m}\overline{K}_{n,m}) = \begin{cases} \begin{cases} \frac{5m^2+9mn+4m+8n^2-4}{16} & \text{if } n \text{ is odd,} \\ \frac{5m^2+9mn+4m+8n^2-2n}{16} & \text{if } n \text{ is even,} \end{cases} & \text{if } m \text{ is even,} \\ \begin{cases} \frac{5m^2+9mn+4m+8n^2+2n-4}{16} & \text{if } n \text{ is odd,} \\ \frac{5m^2+9mn+4m+8n^2-1}{16} & \text{if } n \text{ is even,} \end{cases} & \text{if } m \text{ is odd.} \end{cases}$$

if $n \geqslant \frac{m}{2}$*:*

$$\eta_T(K_{n,m}\overline{K}_{n,m}) = \begin{cases} \begin{cases} \frac{4m^2+13mn+6m+4n^2-2n-5}{16} & \text{if } n \text{ is odd,} \\ \frac{4m^2+13mn+4m+4n^2}{16} & \text{if } n \text{ is even,} \end{cases} & \text{if } m \text{ is odd,} \\ \begin{cases} \frac{4m^2+13mn+4m+4n^2-4}{16} & \text{if } n \text{ is odd,} \\ \frac{4m^2+13mn+2m+4n^2+2n}{16} & \text{if } n \text{ is even,} \end{cases} & \text{if } m \text{ is even.} \end{cases}$$

PROOF. By Theorem (2.1), we can write following expression for $\eta_T(S)$.

$$\eta_T(S) = \eta_T(S_1, \overline{S}_1) + \eta_T(S_1, \overline{S}_2) + \eta_T(S_2, \overline{S}_1) + \eta_T(S_2, \overline{S}_2). \tag{3.2}$$

By Theorem (2.3), we know $\eta_T(S_1, \overline{S}_1)$ and values of x, y. We must calculate $\eta_T(S_2, \overline{S}_2)$ and values of $\overline{x}$, $\overline{y}$. Examining the influence of S_2 to $\overline{S}_2$ gives

$$f(\overline{x}, \overline{y}) = \frac{1}{2}\overline{x}(n-\overline{x}) + \frac{1}{8}\overline{x}(m-\overline{y}) + \frac{1}{2}\overline{y}(m-\overline{y}) + \frac{1}{8}\overline{y}(n-\overline{x}).$$

Bounds are $0 \leqslant \overline{x} \leqslant n$ and $0 \leqslant \overline{y} \leqslant m$. Solving the system $f_{\overline{x}}(\overline{x}, \overline{y}) = 0$ and $f_{\overline{y}}(\overline{x}, \overline{y}) = 0$ gives $\overline{x} = \frac{n}{2} \in [0, n]$ and $\overline{y} = \frac{n}{2} \in [0, m]$. $\overline{x} = \frac{n}{2}$ is only valid when n is even , we need to try both $\overline{x} = \frac{n+1}{2}$ and $\overline{x} = \frac{n-1}{2}$ when n is odd, and similarly for $\overline{y}$ when m is odd. For each case we calculate $\eta_T(S_2)$.

i. Let n, m be even. We have

$$f(\frac{n}{2}, \frac{m}{2}) = \frac{2m^2+mn+2n^2}{16}.$$

ii. Let n be even, m be odd. We have

$$f(\frac{n}{2}, \frac{m-1}{2}) = \frac{2m^2+mn+2n^2-2}{16}$$

$$f(\frac{n}{2}, \frac{m+1}{2}) = \frac{2m^2+mn+2n^2-2}{16}.$$

iii. Let n be odd, m be even. Since the equivalence of the complementary sets, this case is equivalent to ii.

iv. Let n, m be odd. We have

$$f(\frac{n-1}{2}, \frac{m-1}{2}) = \frac{2m^2+2n^2+mn-5}{16}$$

$$f(\frac{n-1}{2}, \frac{m+1}{2}) = \frac{2m^2+2n^2+mn-3}{16}$$

$$f(\frac{n+1}{2},\frac{m-1}{2})=\frac{2m^2+2n^2+mn-3}{16}$$

$$f(\frac{n+1}{2},\frac{m+1}{2})=\frac{2m^2+2n^2+mn-5}{16}.$$

We also examine at the boundaries of $\overline{x}$ and $\overline{y}$. Due to symmetry of $f(\overline{x},\overline{y})$, we can ignore searching along $\overline{x}=n$ and $\overline{y}=m$. Thus we consider two cases: first $\overline{x}=0$ and second $\overline{y}=0$.

Case 1. For $\overline{x}=0$, we maximize $f(0,\overline{y})=\frac{1}{8}n\overline{y}+\frac{1}{2}\overline{y}(m-\overline{y})$. Solving $f_{\overline{y}}(0,\overline{y})=0$ gives $\overline{y}=\frac{4m+n}{8}$. If $n\leqslant 4m$, $\frac{4m+n}{8}\in[0,m]$. We have four subcases depending on n and m. We calculate $\eta_T(S_2)$ for each case.

Case 1.1. Let n, m be even.

i. For $n=8k$, we have $f(0,\lfloor\frac{4m+n}{8}\rfloor)=f(0,\lceil\frac{4m+n}{8}\rceil)=f(0,\frac{4m+n}{8})$ $=\frac{16m^2+8mn+n^2}{128}$.

ii. For $n=8k+2$, we have $f(0,\lfloor\frac{4m+n}{8}\rfloor)=f(0,\frac{4m+n-2}{8})=\frac{16m^2+8mn+n^2-4}{128}$ and $f(0,\lceil\frac{4m+n}{8}\rceil)=f(0,\frac{4m+n+6}{8})=\frac{16m^2+8mn+n^2-36}{128}$.

iii. For $n=8k+4$, we have $f(0,\lfloor\frac{4m+n}{8}\rfloor)=f(0,\frac{4m+n-4}{8})=\frac{16m^2+8mn+n^2-16}{128}$ and $f(0,\lceil\frac{4m+n}{8}\rceil)=f(0,\frac{4m+n+4}{8})=\frac{16m^2+8mn+n^2-16}{128}$.

iv. For $n=8k+6$, we have $f(0,\lfloor\frac{4m+n}{8}\rfloor)=f(0,\frac{4m+n-6}{8})=\frac{16m^2+8mn+n^2-36}{128}$ and $f(0,\lceil\frac{4m+n}{8}\rceil)=f(0,\frac{4m+n+2}{8})=\frac{16m^2+8mn+n^2-4}{128}$.

Case 1.2. Let n be odd, m be even.

i. For $n=8k+1$, we have $f(0,\lfloor\frac{4m+n}{8}\rfloor)=f(0,\frac{4m+n-1}{8})=\frac{16m^2+8mn+n^2-1}{128}$ and $f(0,\lceil\frac{4m+n}{8}\rceil)=f(0,\frac{4m+n+7}{8})=\frac{16m^2+8mn+n^2-49}{128}$.

ii. For $n=8k+3$, we have $f(0,\lfloor\frac{4m+n}{8}\rfloor)=f(0,\frac{4m+n-3}{8})=\frac{16m^2+8mn+n^2-9}{128}$ and $f(0,\lceil\frac{4m+n}{8}\rceil)=f(0,\frac{4m+n+5}{8})=\frac{16m^2+8mn+n^2-25}{128}$.

iii. For $n=8k+5$, we have $f(0,\lfloor\frac{4m+n}{8}\rfloor)=f(0,\frac{4m+n-5}{8})=\frac{16m^2+8mn+n^2-25}{128}$ and $f(0,\lceil\frac{4m+n}{8}\rceil)=f(0,\frac{4m+n+3}{8})=\frac{16m^2+8mn+n^2-9}{128}$.

iv. For $n=8k+7$, we have $f(0,\lfloor\frac{4m+n}{8}\rfloor)=f(0,\frac{4m+n-7}{8})=\frac{16m^2+8mn+n^2-49}{128}$ and $f(0,\lceil\frac{4m+n}{8}\rceil)=f(0,\frac{4m+n+1}{8})=\frac{16m^2+8mn+n^2-1}{128}$.

Case 1.3. Let n be even and m be odd.

i. For $n=8k$, we have $f(0,\lfloor\frac{4m+n}{8}\rfloor)=f(0,\frac{4m+n-4}{8})=\frac{16m^2+8mn+n^2-16}{128}$ and $f(0,\lceil\frac{4m+n}{8}\rceil)=f(0,\frac{4m+n+4}{8})=\frac{16m^2+8mn+n^2-16}{128}$

ii. For $n=8k+2$, we have $f(0,\lfloor\frac{4m+n}{8}\rfloor)=f(0,\frac{4m+n-6}{8})=\frac{16m^2+8mn+n^2-36}{128}$ and $f(0,\lceil\frac{4m+n}{8}\rceil)=f(0,\frac{4m+n+2}{8})=\frac{16m^2+8mn+n^2-4}{128}$.

iii. For $n=8k+4$, we have $f(0,\lfloor\frac{4m+n}{8}\rfloor)=f(0,\lceil\frac{4m+n}{8}\rceil)=f(0,\frac{4m+n}{8})=\frac{16m^2+8mn+n^2}{128}$.

iv. For $n = 8k + 6$, we have $f(0, \lfloor \frac{4m+n}{8} \rfloor) = f(0, \frac{4m+n-2}{8}) = \frac{16m^2+8mn+n^2-4}{128}$ and $f(0, \lceil \frac{4m+n}{8} \rceil) = f(0, \frac{4m+n+6}{8}) = \frac{16m^2+8mn+n^2-36}{128}$.

Case 1.4. Let n be odd, m be odd.

i. For $n = 8k + 1$, we have $f(0, \lfloor \frac{4m+n}{8} \rfloor) = f(0, \frac{4m+n-5}{8}) = \frac{16m^2+8mn+n^2-25}{128}$ and $f(0, \lceil \frac{4m+n}{8} \rceil) = f(0, \frac{4m+n+3}{8}) = \frac{16m^2+8mn+n^2-9}{128}$.

ii. For $n = 8k + 3$, we have $f(0, \lfloor \frac{4m+n}{8} \rfloor) = f(0, \frac{4m+n-7}{8}) = \frac{16m^2+8mn+n^2-49}{128}$ and $f(0, \lceil \frac{4m+n}{8} \rceil) = f(0, \frac{4m+n+1}{8}) = \frac{16m^2+8mn+n^2-1}{128}$.

iii. For $n = 8k + 5$, we have $f(0, \lfloor \frac{4m+n}{8} \rfloor) = f(0, \frac{4m+n-1}{8}) = \frac{16m^2+8mn+n^2-1}{128}$ and $f(0, \lceil \frac{4m+n}{8} \rceil) = f(0, \frac{4m+n+7}{8}) = \frac{16m^2+8mn+n^2-49}{128}$.

iv. For $n = 8k + 7$, we have $f(0, \lfloor \frac{4m+n}{8} \rfloor) = f(0, \frac{4m+n-3}{8}) = \frac{16m^2+8mn+n^2-9}{128}$ and $f(0, \lceil \frac{4m+n}{8} \rceil) = f(0, \frac{4m+n+5}{8}) = \frac{16m^2+8mn+n^2-25}{128}$.

We also get $f(0, 0) = 0$ and $f(0, m) = \frac{mn}{8}$ from the boundaries of $\overline{y}$. Since $0 < |S_2| < n + m$, we can ignore $f(0, 0) = 0$.

Case 2. For $\overline{y} = 0$, we maximize $f(\overline{x}, 0) = \frac{1}{8}m\overline{x} + \frac{1}{2}\overline{x}(n-\overline{x})$. Solving $f_{\overline{x}}(\overline{x}, 0) = 0$ gives $\overline{x} = \frac{m+4n}{8}$. If $m \leqslant 4n$, $\frac{m+4n}{8} \in [0, n]$. We have four subcases depending on n and m. We calculate $\eta_T(S_2)$ for each case.

Case 2.1. Let n, m be even.

i. For $m = 8k$, we have $f(\lfloor \frac{m+4n}{8} \rfloor, 0) = f(\lceil \frac{m+4n}{8} \rceil, 0) = f(\frac{m+4n}{8}, 0)$ $= \frac{m^2+8mn+16n^2}{128}$.

ii. For $m = 8k + 2$, we have $f(\lfloor \frac{m+4n}{8} \rfloor, 0) = f(\frac{m+4n-2}{8}, 0) = \frac{m^2+8mn+16n^2-4}{128}$ and $f(\lceil \frac{m+4n}{8} \rceil, 0) = f(\frac{m+4n+6}{8}, 0) = \frac{m^2+8mn+16n^2-36}{128}$.

iii. For $m = 8k + 4$, we have $f(\lfloor \frac{m+4n}{8} \rfloor, 0) = f(\frac{m+4n-4}{8}, 0) = \frac{m^2+8mn+16n^2-16}{128}$ and $f(\lceil \frac{m+4n}{8} \rceil, 0) = f(\frac{m+4n+4}{8}, 0) = \frac{m^2+8mn+16n^2-16}{128}$.

iv. For $m = 8k + 6$, we have $f(\lfloor \frac{m+4n}{8} \rfloor, 0) = f(\frac{m+4n-6}{8}, 0) = \frac{m^2+8mn+16n^2-36}{128}$ and $f(\lceil \frac{m+4n}{8} \rceil, 0) = f(\frac{m+4n+2}{8}, 0) = \frac{m^2+8mn+16n^2-4}{128}$.

Case 2.2. Let n be odd, m be even.

i. For $m = 8k+1$, we have $f(\lfloor \frac{m+4n}{8} \rfloor, 0) = f(\frac{m+4n-1}{8}, 0) = \frac{m^2+8mn+16n^2-1}{128}$ and $f(\lceil \frac{m+4n}{8} \rceil, 0) = f(\frac{m+4n+7}{8}, 0) = \frac{m^2+8mn+16n^2-49}{128}$.

ii. For $m = 8k + 3$, we have $f(\lfloor \frac{m+4n}{8} \rfloor, 0) = f(\frac{m+4n-3}{8}, 0) = \frac{m^2+8mn+16n^2-9}{128}$ and $f(\lceil \frac{m+4n}{8} \rceil, 0) = f(\frac{m+4n+5}{8}, 0) = \frac{m^2+8mn+16n^2-25}{128}$.

iii. For $m = 8k + 5$, we have $f(\lfloor \frac{m+4n}{8} \rfloor, 0) = f(\frac{m+4n-5}{8}, 0) = \frac{m^2+8mn+16n^2-25}{128}$ and $f(\lceil \frac{m+4n}{8} \rceil, 0) = f(\frac{m+4n+3}{8}, 0) = \frac{m^2+8mn+16n^2-9}{128}$.

iv. For $m = 8k + 7$, we have $f(\lfloor \frac{m+4n}{8} \rfloor, 0) = f(\frac{m+4n-7}{8}, 0) = \frac{m^2+8mn+16n^2-49}{128}$ and $f(\lceil \frac{m+4n}{8} \rceil, 0) = f(\frac{m+4n+1}{8}, 0) = \frac{m^2+8mn+16n^2-1}{128}$.

Case 2.3. Let n be even and m be odd.

i. For $m = 8k$, we have $f(\lfloor \frac{m+4n}{8} \rfloor, 0) = f(\frac{m+4n-4}{8}, 0) = \frac{m^2+8mn+16n^2-16}{128}$ and $f(\lceil \frac{m+4n}{8} \rceil, 0) = f(\frac{m+4n+4}{8}, 0) = \frac{m^2+8mn+16n^2-16}{128}$

ii. For $m = 8k+2$, we have $f(\lfloor \frac{m+4n}{8} \rfloor, 0) = f(\frac{m+4n-6}{8}, 0) = \frac{m^2+8mn+16n^2-36}{128}$ and $f(\lceil \frac{4m+n}{8} \rceil, 0) = f(\frac{m+4n+2}{8}, 0) = \frac{m^2+8mn+16n^2-4}{128}$.

iii. For $m = 8k+4$, we have $f(\lfloor \frac{m+4n}{8} \rfloor, 0) = f(\lceil \frac{m+4n}{8} \rceil, 0) = f(\frac{m+4n}{8}, 0) = \frac{m^2+8mn+16n^2}{128}$.

iv. For $m = 8k+6$, we have $f(\lfloor \frac{m+4n}{8} \rfloor, 0) = f(\frac{m+4n-2}{8}, 0) = \frac{m^2+8mn+16n^2-4}{128}$ and $f(\lceil \frac{m+4n}{8} \rceil, 0) = f(\frac{m+4n+6}{8}, 0) = \frac{m^2+8mn+16n^2-36}{128}$.

Case 2.4. Let n be odd, m be odd.

i. For $m = 8k+1$, we have $f(\lfloor \frac{m+4n}{8} \rfloor, 0) = f(\frac{m+4n-5}{8}, 0) = \frac{m^2+8mn+16n^2-25}{128}$ and $f(\lceil \frac{m+4n}{8} \rceil, 0) = f(\frac{m+4n+3}{8}, 0) = \frac{m^2+8mn+16n^2-9}{128}$.

ii. For $m = 8k+3$, we have $f(\lfloor \frac{m+4n}{8} \rfloor, 0) = f(\frac{m+4n-7}{8}, 0) = \frac{m^2+8mn+16n^2-49}{128}$ and $f(\lceil \frac{m+4n}{8} \rceil, 0) = f(\frac{m+4n+1}{8}, 0) = \frac{m^2+8mn+16n^2-1}{128}$.

iii. For $m = 8k+5$, we have $f(\lfloor \frac{m+4n}{8} \rfloor, 0) = f(\frac{m+4n-1}{8}, 0) = \frac{m^2+8mn+16n^2-1}{128}$ and $f(\lceil \frac{m+4n}{8} \rceil, 0) = f(\frac{m+4n+7}{8}, 0) = \frac{m^2+8mn+16n^2-49}{128}$.

iv. For $m = 8k+7$, we have $f(\lfloor \frac{m+4n}{8} \rfloor, 0) = f(\frac{m+4n-3}{8}, 0) = \frac{m^2+8mn+16n^2-9}{128}$ and $f(\lceil \frac{m+4n}{8} \rceil, 0) = f(\frac{m+4n+5}{8}, 0) = \frac{m^2+8mn+16n^2-25}{128}$.

We also get $f(0,0) = 0$ and $f(n,0) = \frac{mn}{8}$ from the boundaries of $\overline{x}$. Since $0 < |S_2| < n+m$, we can ignore $f(0,0) = 0$.

The function $f(\overline{x}, \overline{y})$ is maximized at $\overline{x} = \frac{n}{2}$ and $\overline{y} = \frac{m}{2}$. And also we know the value of x and y by Theorem (2.3). By considering the value of x, y, $\overline{x}$, $\overline{y}$, we can write two functions for $K_{n,m}\overline{K}_{n,m}$. If $x = 0$ and $y + \overline{y} > m$, we will use first function, f_1. If $x = n$ and $y + \overline{y} < m$, we will use second function, f_2.

$$\begin{aligned}
f_1(x,y,\overline{x},\overline{y}) =& \frac{1}{4}y(m-y) + \frac{1}{2}yn + \frac{1}{2}(m-\overline{y}) + \frac{1}{4}(y-1)(m-\overline{y}) + \frac{1}{4}y(n-\overline{x}) \\
&+ \frac{1}{2}\overline{x}(n-\overline{x}) + \frac{1}{2}\overline{x} + \frac{1}{4}\overline{x}(n-1) + \frac{1}{4}\overline{x}(m-y) + \frac{1}{8}\overline{x}(m-\overline{y}) \\
&+ \frac{1}{2}\overline{y}(m-\overline{y}) + \frac{1}{2}(m-y) + \frac{1}{4}(m-y)(\overline{y}-1) + \frac{1}{4}\overline{y}n + \frac{1}{8}\overline{y}(n-\overline{x}).
\end{aligned}$$

$$\begin{aligned}
f_2(x,y,\overline{x},\overline{y}) =& \frac{1}{2}n(m-y) + \frac{1}{2}(n-\overline{x}) + \frac{1}{4}(n-1)(n-\overline{x}) + \frac{1}{4}n(m-\overline{y}) \\
&+ \frac{1}{2}\overline{x}(n-\overline{x}) + \frac{1}{4}\overline{x}(m-y) + \frac{1}{8}\overline{x}(m-\overline{y}) + \frac{1}{4}y(m-y) + \frac{1}{4}y(n-\overline{x}) \\
&+ \frac{1}{2}y + \frac{1}{4}y(m-\overline{y}-1) + \frac{1}{2}\overline{y} + \frac{1}{4}\overline{y}(m-y-1) + \frac{1}{8}\overline{y}(n-\overline{x}) \\
&+ \frac{1}{2}\overline{y}(m-\overline{y})
\end{aligned}$$

We substitute the value of x, y, $\overline{x}$, $\overline{y}$ into the these functions, depending on three subcases and compare them to maximize the summing unknow terms in the expression (3.2). These subcases are first, $n \geqslant \frac{m}{2}$; second, $n < \frac{m}{2}$ and m is even; last, $n < \frac{m}{2}$ and m is odd.

Case i. Let $n \geqslant \frac{m}{2}$.
$(x, y) = \{(0, m) \text{ or } (n, 0)\}$
$(\overline{x}, \overline{y}) = \{(\lfloor \frac{n}{2} \rfloor, \lfloor \frac{m}{2} \rfloor) \text{ or } (\lfloor \frac{n}{2} \rfloor, \lceil \frac{m}{2} \rceil) \text{ or } (\lceil \frac{n}{2} \rceil, \lfloor \frac{m}{2} \rfloor) \text{ or } (\lceil \frac{n}{2} \rceil, \lceil \frac{m}{2} \rceil).\}$

- Let n, m be odd.
  $f_1(0, m, \frac{n-1}{2}, \frac{m-1}{2}) = \frac{4m^2+13mn+6m+4n^2-2n-5}{16}$
  $f_1(0, m, \frac{n-1}{2}, \frac{m+1}{2}) = \frac{4m^2+13mn+2m+4n^2+2n-7}{16}$
  $f_1(0, m, \frac{n+1}{2}, \frac{m-1}{2}) = \frac{4m^2+13mn+2m+4n^2+2n+1}{16}$
  $f_1(0, m, \frac{n+1}{2}, \frac{m+1}{2}) = \frac{4m^2+13mn-2m+4n^2+6n-5}{16}$
- Let n be odd, m be even.
  $f_1(0, m, \frac{n-1}{2}, \frac{m}{2}) = \frac{4m^2+13mn+4m+4n^2-4}{16}$
  $f_1(0, m, \frac{n+1}{2}, \frac{m}{2}) = \frac{4m^2+13mn+4n^2+4n}{16}$
- Let n be even, m be odd.
  $f_1(0, m, \frac{n}{2}, \frac{m-1}{2}) = \frac{4m^2+13mn+4m+4n^2}{16}$
  $f_1(0, m, \frac{n}{2}, \frac{m+1}{2}) = \frac{4m^2+13mn+4n^2+4n-4}{den}$
- Let n, m even.
  $f_1(0, m, \frac{n}{2}, \frac{m}{2}) = \frac{4m^2+13mn+2m+4n^2+2n}{16}$

These sets are complements of sets which obtained by $x = n$. Thus we get same results from f_2 with $x = n$.

Case ii. Let m be even and $n < \frac{m}{2}$.
$(x, y) = \{(0, n + \frac{m}{2}) \text{ or } (n, \frac{m}{2} - n)\}$
$(\overline{x}, \overline{y}) = \{(\lfloor \frac{n}{2} \rfloor, \frac{m}{2}) \text{ or } (\lceil \frac{n}{2} \rceil, \frac{m}{2})\}$.

- Let n be odd.
  $f_1(0, \frac{2n+m}{2}, \frac{n-1}{2}, \frac{m}{2}) = \frac{5m^2+9mn+4m+8n^2-4}{16}$
  $f_1(0, \frac{2n+m}{2}, \frac{n+1}{2}, \frac{m}{2}) = \frac{5m^2+9mn+4m+8n^2-4n}{16}$
- Let n be even.
  $f_1(0, \frac{2n+m}{2}, \frac{n}{2}, \frac{m}{2}) = \frac{5m^2+9mn+4m+8n^2-2n}{16}$

These sets are complements of sets which obtained by $x = n$. Thus we get same results from f_2 with $x = n$.

Case iii. Let m be odd and $n < \frac{m}{2}$.
$(x, y) = \{(0, \lfloor n + \frac{m}{2} \rfloor) \text{ or } (0, \lceil n + \frac{m}{2} \rceil) \text{ or } (n, \lfloor \frac{m}{2} - n \rfloor) \text{ or } (n, \lceil \frac{m}{2} - n \rceil)\}$
$(\overline{x}, \overline{y}) = \{(\lfloor \frac{n}{2} \rfloor, \lfloor \frac{m}{2} \rfloor) \text{ or } (\lfloor \frac{n}{2} \rfloor, \lceil \frac{m}{2} \rceil) \text{ or } (\lceil \frac{n}{2} \rceil, \lfloor \frac{m}{2} \rfloor) \text{ or } (\lceil \frac{n}{2} \rceil, \lceil \frac{m}{2} \rceil)\}$.

- Let n be even.
  $f_1(0, \frac{2n+m-1}{2}, \frac{n}{2}, \frac{m-1}{2}) = \frac{5m^2+9mn+4m+8n^2-1}{16}$
  $f_1(0, \frac{2n+m-1}{2}, \frac{n}{2}, \frac{m+1}{2}) = \frac{5m^2+9mn+4m+8n^2-4n-1}{16}$
  $f_1(0, \frac{2n+m+1}{2}, \frac{n}{2}, \frac{m-1}{2}) = \frac{5m^2+9mn+4m+8n^2-1}{16}$
  $f_1(0, \frac{2n+m+1}{2}, \frac{n}{2}, \frac{m+1}{2}) = \frac{5m^2+9mn+4m+8n^2-4n-9}{16}$

- Let n be odd.
$f_1(0, \frac{2n+m-1}{2}, \frac{n-1}{2}, \frac{m-1}{2}) = \frac{5m^2+9mn+4m+8n^2+2n-8}{16}$
$f_1(0, \frac{2n+m-1}{2}, \frac{n-1}{2}, \frac{m+1}{2}) = \frac{5m^2+9mn+4m+8n^2-2n-6}{16}$
$f_1(0, \frac{2n+m-1}{2}, \frac{n+1}{2}, \frac{m-1}{2}) = \frac{5m^2+9mn+4m+8n^2-2n+2}{16}$
$f_1(0, \frac{2n+m-1}{2}, \frac{n+1}{2}, \frac{m+1}{2}) = \frac{5m^2+9mn+4m+8n^2-6n}{16}$
$f_1(0, \frac{2n+m+1}{2}, \frac{n-1}{2}, \frac{m-1}{2}) = \frac{5m^2+9mn+4m+8n^2+2n-4}{16}$
$f_1(0, \frac{2n+m+1}{2}, \frac{n-1}{2}, \frac{m+1}{2}) = \frac{5m^2+9mn+4m+8n^2-2n-10}{16}$
$f_1(0, \frac{2n+m+1}{2}, \frac{n+1}{2}, \frac{m-1}{2}) = \frac{5m^2+9mn+4m+8n^2-2n-2}{16}$
$f_1(0, \frac{2n+m+1}{2}, \frac{n+1}{2}, \frac{m+1}{2}) = \frac{5m^2+9mn+4m+8n^2-6n-12}{16}$

These sets are complements of sets which obtained by $x = n$. Thus we get same results from f_2 with $x = n$.

From Case i, Case ii and Case iii, the proof is completed. □

THEOREM 3.4. *For the complementary prism graph $tK_2\overline{tK_2}$, let S be a total influence set and $S_1 = V_1 \cap S$, $S_2 = V_2 \cap S$, $\overline{S}_1 = V_1 - S_1$, $\overline{S}_2 = V_2 - S_2$. Additionally, V_1 and V_2 can be partitioned in two pieces $V_1^{(1)} = \{v_1, v_3, ..., v_{2t-1}\}$, $V_1^{(2)} = \{v_2, v_4, ..., v_{2t}\}$ and $V_2^{(1)} = \{\overline{v}_1, \overline{v}_3, ..., \overline{v}_{2t-1}\}$, $V_2^{(2)} = \{\overline{v}_2, \overline{v}_4, ..., \overline{v}_{2t}\}$, respectively. Let $x = |V_1^{(1)} \cap S|$, $y = |V_1^{(2)} \cap S|$, $\overline{x} = |V_2^{(1)} \cap S|$, $\overline{y} = |V_2^{(2)} \cap S|$, where $\overline{y}$ is the number of vertices non-adjacent to one of $\overline{x}$ vertices. Then S is η_T-set if and only if following condition or its complement is satisfied:*

$$(x = 0,\ y = t,\ \overline{x} = \lceil \tfrac{t}{2} \rceil,\ \overline{y} = \lfloor \tfrac{t}{2} \rfloor).$$

Furthermore,

$$\eta_T(tK_2\overline{tK_2}) = \frac{9t^2 + 5t}{8}.$$

PROOF. By theorem (2.1), we can write following expression for $\eta_T(S)$.

$$\eta_T(S) = \eta_T(S_1, \overline{S_1}) + \eta_T(S_1, \overline{S_2}) + \eta_T(S_2, \overline{S_1}) + \eta_T(S_2, \overline{S_2})$$

By theorem (2.6), we know $\eta_T(S_2, \overline{S}_2)$ and have two options for $\overline{x}$, $\overline{y}$. These options are ($\overline{x} = \lceil \frac{t}{2} \rceil$, $\overline{y} = \lfloor \frac{t}{2} \rfloor$) or ($\overline{x} = \lfloor \frac{t}{2} \rfloor$, $\overline{y} = \lceil \frac{t}{2} \rceil$). They are complements of each other and have the same total influence number. Additionally, we must find the values of x, y for maximum value of $\eta_T(S_1, \overline{S}_1)$. For every $v_i \in V_1^{(1)}$, $v_j \in V_1^{(2)}$, since $(v_i, v_j) \in E(tK_2)$, where $j = i + 1$ and $i \in \{1, 3, ..., 2t-1\}$, $j \in \{2, 4, ..., 2t\}$, we must choose all vertices v_i or all vertices v_j for S_1. Thus we get $x = 0$, $y = t$ or $x = t$, $y = 0$.

For the set S, assume without loss of generlity that $\overline{x} = \lceil \frac{t}{2} \rceil$ and $\overline{y} = \lfloor \frac{t}{2} \rfloor$. We have two cases depending on x and y.

Case 1. Let $x = 0$ and $y = t$.

Case 1.1. Let t be even. In this case $\frac{t}{2}$ is only valid for $\overline{x}$, $\overline{y}$ and we get following value of $\eta_T(S)$ for $tK_2\overline{tK_2}$ complementary prism by using the values of x, y, $\overline{x}$, $\overline{y}$.

$$\begin{aligned}\eta_T(S) =& \frac{1}{2}t + \frac{1}{8}t(t-1) + \frac{1}{2}t^2 + \frac{1}{2}(\frac{t}{2}) + \frac{1}{4}(\frac{t}{2})(t-1) + \frac{1}{4}(\frac{t}{2})t + \frac{1}{2}(\frac{t}{2})\\ &+ \frac{1}{4}(\frac{t}{2})(t-1) + \frac{1}{4}(\frac{t}{2})t\\ =& \frac{9t^2+5t}{8}.\end{aligned}$$

Case 1.2. Let t be odd. In this case, $\overline{x} = \frac{t+1}{2}$ and $\overline{y} = \frac{t-1}{2}$ is only valid for $\overline{x}$, $\overline{y}$ and we get following value of $\eta_T(S)$ for $tK_2\overline{tK_2}$ complementary prism by using the values of x, y, $\overline{x}$, $\overline{y}$.

$$\begin{aligned}\eta_T(S) =& \frac{1}{2}t + \frac{1}{8}t(t-1) + \frac{1}{2}t^2 - \frac{1}{4} + \frac{1}{2}(\frac{t+1}{2}) + \frac{1}{4}(\frac{t+1}{2})(t-1)\\ &+ \frac{1}{4}(\frac{t-1}{2})t + \frac{1}{2}(\frac{t+1}{2}) + \frac{1}{4}(\frac{t+1}{2})(t-1) + \frac{1}{4}(\frac{t-1}{2})t\\ =& \frac{9t^2+5t}{8}.\end{aligned}$$

Case 2. Let $x = t$ and $y = 0$.

Case 2.1. Let t be even. In this case, $\frac{t}{2}$ is only valid for $\overline{x}$, $\overline{y}$ and we get following value of $\eta_T(S)$ for $tK_2\overline{tK_2}$ complementary prism by using the values of x, y, $\overline{x}$, $\overline{y}$.

$$\begin{aligned}\eta_T(S) =& \frac{1}{2}t + \frac{1}{8}t(t-1) + \frac{1}{2}t^2 + \frac{1}{2}(\frac{t}{2}) + \frac{1}{4}(\frac{t}{2})(t-1) + \frac{1}{4}(\frac{t}{2})t + \frac{1}{2}(\frac{t}{2})\\ &+ \frac{1}{4}(\frac{t}{2})(t-1) + \frac{1}{4}(\frac{t}{2})t\\ =& \frac{9t^2+5t}{8}.\end{aligned}$$

Case 2.2. Let t be odd. In this case, $\overline{x} = \frac{t+1}{2}$ and $\overline{y} = \frac{t-1}{2}$ is only valid for $\overline{x}$, $\overline{y}$ and we get following value of $\eta_T(S)$ for $tK_2\overline{tK_2}$ complementary prism by using the values of x, y, $\overline{x}$, $\overline{y}$.

$$\begin{aligned}\eta_T(S) =& \frac{1}{2}t + \frac{1}{8}t(t-1) + \frac{1}{2}t^2 - \frac{1}{4} + \frac{1}{2}(\frac{t-1}{2}) + \frac{1}{4}(\frac{t-1}{2})(t-1)\\ &+ \frac{1}{4}(\frac{t+1}{2})t + \frac{1}{2}(\frac{t-1}{2}) + \frac{1}{4}(\frac{t-1}{2})(t-1) + \frac{1}{4}(\frac{t+1}{2})t\\ =& \frac{9t^2+5t-4}{8}.\end{aligned}$$

By Case 1 and Case 2, the total influence number of $tK_2\overline{tK_2}$ is

$$\eta_T(tK_2\overline{tK_2}) = \frac{9t^2+5t}{8}.$$

□

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