

GENERALIZED DEGREE DISTANCE OF FOUR TRANSFORMATION GRAPHS

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ABSTRACT. The *generalized degree distance*, denoted by $H_\lambda(G)$, is defined as

$$H_\lambda(G) = \frac{1}{2} \sum_{u,v \in V(G)} (d_G(u) + d_G(v)) d_G^\lambda(u, v),$$

where λ is any real number. In this paper, we discuss the mathematical properties for the generalized degree distance of four edge grafting transformations of graph.

1. Introduction

A single number that can be used to characterize some property of the graph of a molecule is called a *topological index*. Topological index is a graph theoretical property that is preserved by isomorphism. The chemical information derived through *topological index* has been found useful in chemical documentation, isomer discrimination, structure property correlations. The interest in topological in-dices is mainly related to their use in non-empirical quantitative structure-property relationships and quantitative structural-activity relationships. One of the oldest and well-studied distance based graph invariants associated with a connected graph G is the *number* $W(G)$, also termed as *Wiener index* in chemical or mathematical chemistry literature, which is defined Wiener as the sum of distance over all unordered vertex pairs in G , namely,

$$W(G) = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u, v),$$

where $d_G(u, v)$ is the shortest distance between two vertices. Another distance based graph invariant, defined in a fully analogous manner to wiener index, is the *Harary index*, which

2010 *Mathematics Subject Classification*. 05C12, 05C76.

Key words and phrases. optional, but desirable.

is equal to the sum of the the reciprocal distances overall unordered vertex pairs in G that is,

$$H(G) = \frac{1}{2} \sum_{u,v \in V(G)} \frac{1}{d_G(u,v)}.$$

Dobrynin and Kochetova [3] and Gutman [4] independently proposed a vertex-degree-weighted version of Wiener index called *degree distance*, which is defined for a connected graph G as,

$$DD(G) = \frac{1}{2} \sum_{u,v \in V(G)} (d_G(u) + d_G(v))d_G(u,v).$$

The *additively weighted Harary index* (H_A) or *reciprocal degree distance* (RDD) is defined in [1] as,

$$H_A(G) = RDD(G) = \frac{1}{2} \sum_{u,v \in V(G)} \frac{(d_G(u) + d_G(v))}{d_G(u,v)}.$$

Hua and Zhang [7] have obtained lower and upper bounds for the reciprocal degree distance of graph in terms of other graph invariants. The chemical applications and mathematical properties of the reciprocal degree distance are well studied in [1, 11, 14]. The *generalized degree distance*, denoted by $H_\lambda(G)$, is defined as,

$$H_\lambda(G) = \frac{1}{2} \sum_{u,v \in V(G)} (d_G(u) + d_G(v))d_G^\lambda(u,v),$$

where λ is any real number. If $\lambda = 1$, then $H_\lambda(G) = DD(G)$ and if $\lambda = -1$, then $H_\lambda(G) = RDD(G)$. The generalized degree distance of unicyclic and bi cyclic graphs are studied by Hamzeh et. al [5, 6]. Also they are given the generalized degree distance of Cartesian product, join, symmetric difference, composition and disjunction of two graphs. The generalized degree distance of the strong and tensor product of graphs are obtained in [12, 13]. In this sequence, here we find the bounds for the generalized degree distance of four edge grafting transformations of graph. A pendent path at v in a graph G is a path in which no vertex other than v is incident with any edge of G outside the path, where the degree of v is at least three. Call v a pendent vertex (or a leaf) in G , if $d_G(v) = 1$.

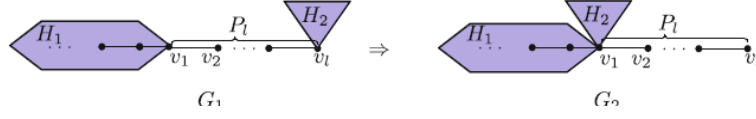
2. α -transformation

Let G_1 be a simple graph, as depicted in Fig.1, where H_1, H_2 are two connected graphs. Let

$$G_2 = G_1 - \{v_l x : x \in N_{H_2}(v_l)\} + \{v_1 x : x \in N_{H_2}(v_l)\}.$$

We call that G_2 is obtained by α - transformation on G_1 . In particular, if G_1 is a tree, Kelmas [9] used this tree-transformation to prove the some results on the number of spanning trees of graph in [9]. Recently, Bollobas and tyomkyn [2] used this tree-transformation to count the total number of walks (resp. closed walks, paths) of trees. Here we obtain the generalized degree distance of α -transformation.

THEOREM 2.1. *Let G_2 be the graph obtained from G_1 by α -transformation. Then $H_\lambda(G_1) < H_\lambda(G_2)$ if $\lambda < 0$ and $H_\lambda(G_1) > H_\lambda(G_2)$ if $\lambda > 0$.*


 FIGURE 1. α -transformation

PROOF. For each $x \in V(H_1) \setminus \{v_1\}$ and for each vertex $y \in V(H_2) \setminus \{v_l\}$, one can easily check that,

$$d_{G_1}(x) = d_{H_1}(x) = d_{G_2}(x)$$

and

$$d_{G_1}(y) = d_{H_2}(y) = d_{G_2}(y)$$

. Moreover, for each $y \in V(H_2)$,

$$d_{H_2}^\lambda(v_l, y) = t$$

in G_1 if and only if

$$d_{H_2}^\lambda(v_1, y) = t$$

in G_2 . For each vertex $x \in V(H_1) \setminus \{v_1\}$, we have

$$\begin{aligned} D_{G_1}^\lambda(x) &= \sum_{y \in V(H_1) \setminus \{v_1, x\}} d_{H_1}^\lambda(x, y) + \sum_{k=0}^{l-1} (d_{H_1}(x, v_1) + k)^\lambda \\ &\quad + \sum_{y \in V(H_2) \setminus \{v_l\}} (d_{H_1}(x, v_1) + l - 1 + d_{H_2}(v_l, y))^\lambda \end{aligned}$$

and

$$\begin{aligned} D_{G_2}^\lambda(x) &= \sum_{y \in V(H_1) \setminus \{v_1, x\}} d_{H_1}^\lambda(x, y) + \sum_{k=0}^{l-1} (d_{H_1}(x, v_1) + k)^\lambda \\ &\quad + \sum_{y \in V(H_2) \setminus \{v_1\}} (d_{H_1}(x, v_1) + d_{H_2}(v_1, y))^\lambda. \end{aligned}$$

To obtain $D_{G_1}^\lambda(x) - D_{G_2}^\lambda(x)$ we have.

$$\begin{aligned} D_{G_1}^\lambda(x) - D_{G_2}^\lambda(x) &= \sum_{y \in V(H_2) \setminus \{v_l\}} (d_{H_1}(x, v_1) + l - 1 + d_{H_2}(v_l, y))^\lambda \\ &\quad - \sum_{y \in V(H_2) \setminus \{v_1\}} (d_{H_1}(x, v_1) + d_{H_2}(v_1, y))^\lambda \\ &= \sum_{y \in V(H_2) \setminus \{v_l\}} \sum_{a=0}^{\lambda} \binom{\lambda}{a} (d_{H_1}(x, v_1) + l - 1)^{\lambda-a} (d_{H_2}(v_l, y))^a \\ &\quad - \sum_{y \in V(H_2) \setminus \{v_1\}} \sum_{a=0}^{\lambda} \binom{\lambda}{a} (d_{H_1}(x, v_1))^{\lambda-a} (d_{H_2}(v_1, y))^a. \end{aligned}$$

By direct calculation, we have $D_{G_1}^\lambda(x) < D_{G_2}^\lambda(x)$, when $\lambda < 0$, and $D_{G_1}^\lambda(x) > D_{G_2}^\lambda(x)$, when $\lambda > 0$.

Observe that $d_{G_1}(x) = d_{H_1}(x) = d_{G_2}(x)$ for all $x \in V(H_1) \setminus \{v_1\}$. If $\lambda > 0$ then

$$\sum_{x \in V(H_1) \setminus \{v_1\}} d_{G_1}(x) D_{G_1}^\lambda(x) > \sum_{x \in V(H_1) \setminus \{v_1\}} d_{G_2}(x) D_{G_2}^\lambda(x)$$

and if $\lambda < 0$, we get

$$(1) \quad \sum_{x \in V(H_1) \setminus \{v_1\}} d_{G_1}(x) D_{G_1}^\lambda(x) < \sum_{x \in V(H_1) \setminus \{v_1\}} d_{G_2}(x) D_{G_2}^\lambda(x).$$

For each vertex x in $V(H_2) \setminus \{v_l\}$, we have

$$\begin{aligned} D_{G_1}^\lambda(x) &= \sum_{y \in V(H_2) \setminus \{(v_l), x\}} d_{H_2}^\lambda(x, y) + \sum_{k=0}^{l-1} (d_{H_2}(x, v_l) + k)^\lambda \\ &+ \sum_{y \in V(H_1) \setminus \{v_1\}} (d_{H_2}(x, v_l) + l - 1 + d_{H_1}(v_1, y))^\lambda \end{aligned}$$

and

$$\begin{aligned} D_{G_2}^\lambda(x) &= \sum_{y \in V(H_2) \setminus \{(v_l), x\}} d_{H_2}^\lambda(x, y) + \sum_{k=0}^{l-1} (d_{H_2}(x, v_l) + k)^\lambda \\ &+ \sum_{y \in V(H_1) \setminus \{v_1\}} (d_{H_2}(x, v_l) + d_{H_1}(v_1, y))^\lambda. \end{aligned}$$

Next we compute

$$\begin{aligned} D_{G_1}^\lambda(x) - D_{G_2}^\lambda(x) &= \sum_{y \in V(H_1) \setminus \{v_1\}} (d_{H_2}(x, v_l) + l - 1 + d_{H_1}(v_1, y))^\lambda \\ &- \sum_{y \in V(H_1) \setminus \{v_1\}} (d_{H_2}(x, v_l) + d_{H_1}(v_1, y))^\lambda. \end{aligned}$$

once again using simple calculation we get

$$D_{G_1}^\lambda(x) < D_{G_2}^\lambda(x) \text{ when } \lambda < 0$$

and

$$D_{G_1}^\lambda(x) > D_{G_2}^\lambda(x) \text{ when } \lambda > 0$$

one can note that $d_{G_1}(x) = d_{H_2}(x) = d_{G_2}(x)$, for all x in $V(H_2) \setminus \{v_l\}$.

Hence whenever $\lambda < 0$ we get

$$\sum_{x \in V(H_2) \setminus \{v_l\}} d_{G_1}(x) D_{G_1}^\lambda(x) < \sum_{x \in V(H_2) \setminus \{v_l\}} d_{G_2}(x) D_{G_2}^\lambda(x),$$

and if $\lambda > 0$ we get

$$(2) \quad \sum_{x \in V(H_2) \setminus \{v_l\}} d_{G_1}(x) D_{G_1}^\lambda(x) > \sum_{x \in V(H_2) \setminus \{v_l\}} d_{G_2}(x) D_{G_2}^\lambda(x)$$

For each $v_j \in V(P_l) = \{v_1, v_2, \dots, v_l\}$, we have

$$\begin{aligned} (3) \quad D_{G_1}^\lambda(v_j) &= \sum_{x \in V(H_1) \setminus \{v_1\}} (d_{H_1}(x, v_1) + j - 1)^\lambda \\ &+ D_{P_l}^\lambda(v_j) + \sum_{x \in V(H_2) \setminus \{v_l\}} (d_{H_2}(x, v_l) + l - j)^\lambda \end{aligned}$$

$$(4) \quad \begin{aligned} D_{G_2}^\lambda(v_j) &= \sum_{x \in V(H_1) \setminus \{v_1\}} (d_{H_1}(x, v_1) + j - 1)^\lambda \\ &+ D_{P_l}^\lambda(v_j) + \sum_{x \in V(H_2) \setminus \{v_l\}} (d_{H_2}(x, v_l) + j - 1)^\lambda \end{aligned}$$

One can easily observe that $d_{G_1}(v_j) = d_{G_2}(v_j) = 2$, for each $v_j \in V(P_l) \setminus \{v_1, v_l\}$, and it is easily to find that $d_{G_1}(v_1) = d_{H_1}(v_1) + 1$, $d_{G_1}(v_l) = d_{H_2}(v_l) + 1$, $d_{G_2}(v_1) = d_{H_1}(v_1) + d_{H_2}(v_1) + 1$ and $d_{G_2}(v_l) = 1$. thus

$$\begin{aligned} \sum_{j=1}^l d_{G_1}(v_j) D_{G_1}^\lambda(v_j) &= (d_{H_1}(v_1) + 1) \left(\sum_{x \in V(H_1) \setminus \{v_1\}} d_{H_1}^\lambda(x, v_1) + D_{P_l}^\lambda(v_1) \right. \\ &+ \left. \sum_{x \in V(H_2) \setminus \{v_l\}} (d_{H_2}(x, v_l) + l - 1)^\lambda \right) \\ &+ (d_{H_2}(v_l) + 1) \left(\sum_{x \in V(H_2) \setminus \{v_l\}} d_{H_2}^\lambda(x, v_l) + D_{P_l}^\lambda(v_l) \right. \\ &+ \left. \sum_{x \in V(H_1) \setminus \{v_1\}} (d_{H_2}(x, v_1) + l - 1)^\lambda \right) \\ &+ 2 \sum_{j=2}^{l-1} \left(\sum_{x \in V(H_1) \setminus \{v_1\}} (d_{H_1}(x, v_1) + j - 1)^\lambda + D_{P_l}^\lambda(v_j) \right. \\ &+ \left. \sum_{x \in V(H_2) \setminus \{v_l\}} (d_{H_2}(x, v_l) + l - j)^\lambda \right), \end{aligned}$$

by Eq. 3, and

$$\begin{aligned} \sum_{j=1}^l d_{G_2}(v_j) D_{G_2}^\lambda(v_j) &= (d_{H_1}(v_1) + d_{H_2}(v_l) + 1) \left(\sum_{x \in V(H_1) \setminus \{v_1\}} d_{H_1}^\lambda(x, v_1) + D_{P_l}^\lambda(v_1) \right. \\ &+ \left. \sum_{x \in V(H_2) \setminus \{v_l\}} d_{H_2}^\lambda(x, v_l) \right) \\ &+ 2 \sum_{j=2}^{l-1} \left(\sum_{x \in V(H_1) \setminus \{v_1\}} (d_{H_1}(x, v_1) + j - 1)^\lambda + D_{P_l}^\lambda(v_j) \right. \\ &+ \sum_{x \in V(H_2) \setminus \{v_l\}} (d_{H_2}(x, v_l) + j - 1)^\lambda \\ &+ \sum_{x \in V(H_2) \setminus \{v_l\}} (d_{H_2}(x, v_1) + l - 1)^\lambda + D_{P_l}^\lambda(v_l) \\ &+ \left. \sum_{x \in V(H_1) \setminus \{v_1\}} (d_{H_1}(x, v_1) + l - 1)^\lambda \right), \end{aligned}$$

by Eq. 4. where $D_{P_l}^\lambda(v_1) = D_{P_1}^\lambda(v_l)$.

Hence

$$\begin{aligned} \sum_{j=1}^l d_{G_2}(v_j)D_{G_2}^\lambda(v_j) - \sum_{j=1}^l d_{G_1}(v_j)D_{G_1}^\lambda(v_j) &= d_{H_1}(v_1)\left(\sum_{x \in V(H_2) \setminus \{v_l\}} (d_{H_2}(x, v_l))^\lambda \right. \\ &\quad \left. - \sum_{x \in V(H_2) \setminus \{v_l\}} (d_{H_2}(x, v_l) + l - 1)^\lambda\right) \\ &\quad + d_{H_2}(v_l)\left(\sum_{x \in V(H_1) \setminus \{v_1\}} (d_{H_1}(x, v_1))^\lambda \right. \\ &\quad \left. - \sum_{x \in V(H_1) \setminus \{v_1\}} (d_{H_1}(x, v_1) + l - 1)^\lambda\right) \\ &= d_{H_1}(v_1)\left(\sum_{x \in V(H_2) \setminus \{v_l\}} (d_{H_2}^\lambda(x, v_l)) \right. \\ &\quad \left. - \sum_{x \in V(H_2) \setminus \{v_l\}} \sum_{a=0}^{\lambda} \binom{\lambda}{a} (d_{H_2}(x, v_l))^{\lambda-a} (l-1)^a\right) \\ &\quad + d_{H_2}(v_l)\left(\sum_{x \in V(H_1) \setminus \{v_1\}} (d_{H_1}^\lambda(x, v_1)) \right. \\ &\quad \left. - \sum_{x \in V(H_1) \setminus \{v_1\}} \sum_{a=0}^{\lambda} \binom{\lambda}{a} (d_{H_1}(x, v_1))^{\lambda-a} (l-1)^a\right) \end{aligned}$$

Simplify the above equation , we have;

when $\lambda > 0$, $\sum_{j=1}^l d_{G_2}(v_j)D_{G_2}^\lambda(v_j) > \sum_{j=1}^l d_{G_1}(v_j)D_{G_1}^\lambda(v_j)$ and if $\lambda < 0$, we have

$$\sum_{j=1}^l d_{G_2}(v_j)D_{G_2}^\lambda(v_j) < \sum_{j=1}^l d_{G_1}(v_j)D_{G_1}^\lambda(v_j) \dots \dots \dots (5)$$

From (1) , (2) and (5) , we have

$H_\lambda(G_1) < H_\lambda(G_2)$ if $\lambda < 0$ and $H_\lambda(G_1) > H_\lambda(G_2)$ if $\lambda > 0$. □

3. ρ -transformation

Let T be an n -vertex tree as shown in Fig. 2 , where $wv \in E(T)$ with $d_T(v) = m+1 (m \geq 2)$, T_1, T_2, \dots, T_m are subtrees under v with root vertices v_1, v_2, \dots, v_m such that the tree T_m is actually a path P_l , H_1 contains a path $P_k = us \dots$ satisfying $|V(H_1)| \geq l + 2$ and $|V(P_k)| \geq |V(P_l)|$. Let $T' = T - \{vv_1, vv_2, \dots, vv_{m-1}\} + \{wv_1, wv_2, \dots, wv_{m-1}\}$, see Fig. 2 . Clearly T' is obtained from T by ρ - transformation . Ilić [8] used ρ - transformation to study the Laplacian coefficient of trees. Here we obtain the generalized degree distance of ρ -transformation .

THEOREM 3.1. *Let T' be the n -vertex tree obtained from T by ρ -transformation. Then we have $H_\lambda(T) < H_\lambda(T')$ if $\lambda < 0$ and $H_\lambda(T) > H_\lambda(T')$ if $\lambda > 0$.*

PROOF. Let $H_2 = T_1 \cup T_2 \cup \dots \cup T_{m-1}$. Let $P_l = u_1u_2 \dots u_l$ be a path on length L . One can see that H_1 contains a path $P_k = us \dots$ whose length is no less than that of $T_m = P_l$.

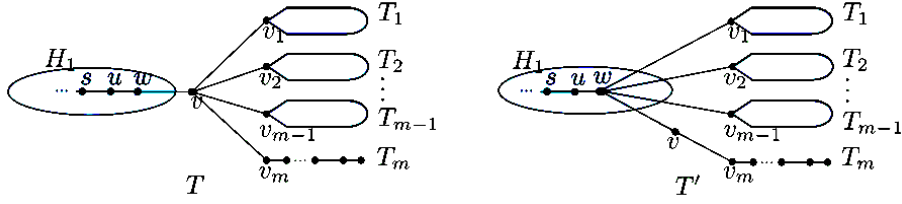


FIGURE 2. ρ -transformation

Observe that for all $x \in V(H_1) \setminus V(P_k)$, we have

$$D_T^\lambda(x) = \sum_{y \in V(H_1) \setminus \{x, w\}} d_{H_1}^\lambda(x, y) + d_{H_1}^\lambda(x, w) + (d_{H_1}(x, w) + 1)^\lambda + \sum_{j=1}^l (d_{H_1}(x, w) + 1 + j)^\lambda + \sum_{y \in V(H_2)} (d_{H_1}(x, w) + 1 + d_{T[V(H_2) \cup \{v\}]}(y, v))^\lambda$$

and

$$D_{T'}^\lambda(x) = \sum_{y \in V(H_1) \setminus \{x, w\}} d_{H_1}^\lambda(x, y) + d_{H_1}^\lambda(x, w) + (d_{H_1}(x, w) + 1)^\lambda + \sum_{j=1}^l (d_{H_1}(x, w) + j + 1)^\lambda + \sum_{y \in V(H_2)} (d_{H_1}(x, w) + d_{T'[V(H_2) \cup \{w\}]}(y, w))^\lambda, \tag{3.1}$$

where for each y in $V(H_2)$ we have $(d_{T[V(H_2) \cup \{v\}]}(y, v))^\lambda = (d_{T'[V(H_2) \cup \{w\}]}(y, w))^\lambda$ Now we obtain $D_{T'}^\lambda(x) - D_T^\lambda(x)$.

$$\begin{aligned} D_{T'}^\lambda(x) - D_T^\lambda(x) &= \sum_{y \in V(H_2)} (d_{H_1}(x, w) + d_{T'[V(H_2) \cup \{w\}]}(y, w))^\lambda - \sum_{y \in V(H_2)} (d_{H_1}(x, w) + 1 + (d_{T[V(H_2) \cup \{v\}]}(y, v))^\lambda)^\lambda \\ &= \sum_{y \in V(H_2)} \sum_{a=0}^{\lambda} \binom{\lambda}{a} (d_{H_1}^{\lambda-a}(x, w) + d_{T'[V(H_2) \cup \{w\}]}^a(y, w))^\lambda - \sum_{y \in V(H_2)} \sum_{a=0}^{\lambda} \binom{\lambda}{a} (d_{H_1}(x, w) + 1)^{\lambda-a} d_{T[V(H_2) \cup \{v\}]}^a(y, v)^\lambda \\ &= \sum_{y \in V(H_2)} \sum_{a=0}^{\lambda} \binom{\lambda}{a} d_{T'[V(H_2) \cup \{w\}]}^a(y, w) \left[d_{H_1}^{\lambda-a}(x, w) - (d_{H_1}(x, w) + 1)^{\lambda-a} \right] \end{aligned}$$

If $\lambda < 0$, then $(d_{H_1}^{\lambda-a}(x, w) < (d_{H_1}(x, w) + 1)^{\lambda-a}$ thus $D_{T'}^\lambda(x) < D_T^\lambda(x)$.

If $\lambda > 0$, then

$$(d_{H_1}^{\lambda-a}(x, w) > (d_{H_1}(x, w) + 1)^{\lambda-a} \text{ thus } D_{T'}^\lambda(x) > D_T^\lambda(x).$$

Observe that for each vertex x in $V(H_1) \setminus V(P_k)$, we have $d_T(x) = d_{H_1}(x) = d'_T(x)$. Hence

$$\sum_{x \in V(H_1) \setminus V(P_k)} d_T(x) D_T^\lambda(x) < \sum_{x \in V(H_1) \setminus V(P_k)} d_T(x) D_{T'}^\lambda(x). \text{ If } \lambda < 0$$

and

$$\sum_{x \in V(H_1) \setminus V(P_k)} d_T(x) D_T^\lambda(x) > \sum_{x \in V(H_1) \setminus V(P_k)} d_T(x) D_{T'}^\lambda(x). \text{ if } \lambda > 0. (1)$$

Similarly, for each vertex x in $V(H_2)$, we have

$$\begin{aligned} D_T^\lambda(x) &= \sum_{y \in V(H_2) \setminus \{x\}} d_{H_2}^\lambda(x, y) + \left(d_{T[V(H_2) \cup \{v\}]}(x, v) \right)^\lambda + \\ &\quad \left(d_{T[V(H_2) \cup \{v\}]}(x, v) + 1 \right)^\lambda \\ &\quad + \sum_{j=1}^l \left(d_{T[V(H_2) \cup \{v\}]}(x, v) + j \right)^\lambda \\ &\quad + \sum_{y \in V(H_1) \setminus \{V(P_k), w\}} \left(d_{T[V(H_2) \cup \{v\}]}(x, v) + 1 + d_{H_1}(y, w) \right)^\lambda \\ &\quad + \sum_{j=1}^k \left(d_{T[V(H_2) \cup \{v\}]}(x, v) + j + 1 \right)^\lambda \end{aligned}$$

and

$$\begin{aligned} D_{T'}^\lambda(x) &= \sum_{y \in V(H_2) \setminus \{x\}} d_{H_2}^\lambda(x, y) + \left(d_{T'[V(H_2) \cup \{w\}]}(x, w) \right)^\lambda \\ &\quad \left(d_{T'[V(H_2) \cup \{w\}]}(x, w) + 1 \right)^\lambda \\ &\quad + \sum_{j=1}^l \left(d_{T'[V(H_2) \cup \{w\}]}(x, w) + j + 1 \right)^\lambda, \\ &\quad + \sum_{y \in V(H_1) \setminus \{V(P_k), w\}} \left(d_{H_2 \cup \{w\}}(x, w) + d_{H_1}(y, w) \right)^\lambda \\ &\quad + \sum_{j=1}^k \left(d_{T'[V(H_2) \cup \{w\}]}(x, w) + j \right)^\lambda. \end{aligned}$$

By using similar calculation on (1) we have

$$\begin{aligned}
 D_{T'}^\lambda(x) - D_T^\lambda(x) &= \sum_{j=1}^l \left(d_{T'[V(H_2) \cup \{w\}]}(x, w) + j + 1 \right)^\lambda \\
 &\quad - \sum_{j=1}^l \left(d_{T'[V(H_2) \cup \{v\}]}(x, v) + j \right)^\lambda \\
 &\quad + \sum_{j=1}^k \left(d_{T'[V(H_2) \cup \{w\}]}(x, w) + j \right)^\lambda \\
 &\quad - \sum_{j=1}^k \left(d_{T'[V(H_2) \cup \{v\}]}(x, v) + j + 1 \right)^\lambda \\
 &\quad + \sum_{y \in V(H_1) \setminus \{V(P_k), w\}} \left(d_{H_2 \cup \{w\}}(x, w) \right. \\
 &\quad \left. + d_{H_1}(y, w) - d_{T'[V(H_2) \cup \{v\}]}(x, v) + 1 + d_{H_1}(y, w) \right)^\lambda.
 \end{aligned}$$

If H_1 is a path, that is, $k \geq l + 1$, then

$$\begin{aligned}
 D_{T'}^\lambda(x) - D_T^\lambda(x) &= \sum_{j=l+1}^k \left(d_{T'[V(H_2) \cup \{w\}]}(x, w) + j \right)^\lambda \\
 &\quad - \sum_{j=l+1}^k \left(d_{T'[V(H_2) \cup \{w\}]}(x, w) + j + 1 \right)^\lambda \\
 &= \sum_{j=l+1}^k \sum_{a=0}^{\lambda} \binom{\lambda}{a} d_{T'[V(H_2) \cup \{w\}]}^{\lambda-a}(x, w) (j)^a \\
 &\quad - \sum_{j=l+1}^k \sum_{a=0}^{\lambda} \binom{\lambda}{a} \left(d_{T'[V(H_2) \cup \{w\}]}(x, w) + 1 \right)^{\lambda-a} (j)^a \\
 &= \sum_{j=l+1}^k \sum_{a=0}^{\lambda} \binom{\lambda}{a} (j)^a \left(d_{T'[V(H_2) \cup \{w\}]}^{\lambda-a}(x, w) - \left(d_{T'[V(H_2) \cup \{w\}]}(x, w) + 1 \right)^{\lambda-a} \right).
 \end{aligned}$$

If $\lambda > 0$, then $\left(d_{T'[V(H_2) \cup \{w\}]}^{\lambda-a}(x, w) \right) > \left(d_{T'[V(H_2) \cup \{w\}]}(x, w) + 1 \right)^{\lambda-a}$

and if $\lambda < 0$, then $\left(d_{T'[V(H_2) \cup \{w\}]}^{\lambda-a}(x, w) \right) < \left(d_{T'[V(H_2) \cup \{w\}]}(x, w) + 1 \right)^{\lambda-a}$. (2)

Hence $D_{T'}^\lambda(x) - D_T^\lambda(x) < 0$ when $\lambda > 0$ and $D_{T'}^\lambda(x) - D_T^\lambda(x) > 0$ when $\lambda < 0$
 Again if H_1 is a path, that is, $k \geq l + 1$ then we have

$$\begin{aligned}
 D_{T'}^\lambda(x) - D_T^\lambda(x) &= \sum_{y \in V(H_1) \setminus \{V(P_k), w\}} \left[\left(d_{H_2 \cup \{w\}}(x, w) + d_{H_1}(y, w) \right)^\lambda \right. \\
 &\quad \left. - \left(d_{T'[V(H_2) \cup \{v\}]}(x, v) + 1 + d_{H_1}(y, w) \right)^\lambda \right]
 \end{aligned}$$

A similar argument in (2) , we have if $\lambda < 0$, then $D_{T'}^\lambda(x) - D_T^\lambda(x) < 0$ and if $\lambda > 0$, then $D_{T'}^\lambda(x) - D_T^\lambda(x) > 0$.

One can easily observe that for each $x \in V(H_2)$, $d_T(x) = d_{H_2}(x) = d_{T'}(x)$. Then

$$\sum_{x \in V(H_2)} d_T(x) D_T^\lambda(x) > \sum_{x \in V(H_2)} d_{T'}(x) D_{T'}^\lambda(x) \text{ when } \lambda > 0 \text{ and}$$

$$\sum_{x \in V(H_2)} d_T(x) D_T^\lambda(x) < \sum_{x \in V(H_2)} d_{T'}(x) D_{T'}^\lambda(x) \text{ when } \lambda < 0 \dots\dots\dots(3)$$

For each $u_j \in P_l$, $1 \leq j \leq l$, let $d(u_j) = d_T(u_j) = d_{T'}(u_j)$. Then

$$D_{T'}^\lambda(u_j) = D_{P_l}^\lambda(u_j) + j^\lambda + (j + 1)^\lambda + \sum_{x \in V(H_2)} (d_{T'[V(H_2) \cup \{v\}]}(x, v) + j)^\lambda$$

$$+ \sum_{i=1}^k (j + 1 + i)^\lambda + \sum_{x \in V(H_1) \setminus \{V(P_k), w\}} (j + 1 + d_{H_1}(x, w))^\lambda$$

and

$$D_T^\lambda(u_j) = D_{P_l}^\lambda(u_j) + j^\lambda + (j + 1)^\lambda + \sum_{x \in V(H_2)} (d_{T[V(H_2) \cup \{w\}]}(x, w) + j + 1)^\lambda$$

$$+ \sum_{i=1}^k (j + 1 + i)^\lambda + \sum_{x \in V(H_1) \setminus \{V(P_k), w\}} (j + 1 + d_{H_1}(x, w))^\lambda.$$

This implies

$$\sum_{j=1}^l d_{T'}(u_j) D_{T'}^\lambda(u_j) - \sum_{j=1}^l d_T(u_j) D_T^\lambda(u_j) = \sum_{j=1}^l d(u_j) \sum_{x \in V(H_2)} (d_{T'[V(H_2) \cup \{w\}]}(x, w) + j + 1)^\lambda$$

$$- \sum_{j=1}^l d(u_j) \sum_{x \in V(H_2)} (d_{T[V(H_2) \cup \{v\}]}(x, v) + j)^\lambda \dots\dots\dots(4)$$

For each $w_j \in P_k$, $1 \leq j \leq k$, let $d(w_j) = d_T(w_j) = d_{T'}(w_j)$, then we have

$$D_{T'}^\lambda(w_j) = D_{P_k}^\lambda(w_j) + j^\lambda + (j + 1)^\lambda + \sum_{x \in V(H_2)} (d_{T'[V(H_2) \cup \{v\}]}(x, v) + j + 1)^\lambda$$

$$+ \sum_{i=1}^l (j + 1 + i)^\lambda + \sum_{x \in V(H_1) \setminus \{V(P_k), w\}} d_{H_1}^\lambda(x, w_j).$$

and

$$D_{T'}^\lambda(w_j) = D_{P_k}^\lambda(w_j) + j^\lambda + (j + 1)^\lambda + \sum_{x \in V(H_2)} (d_{T'[V(H_2) \cup \{w\}]}(x, w) + j)^\lambda + \sum_{i=1}^l (j + 1 + i)^\lambda + \sum_{x \in V(H_1) \setminus \{V_{(P_k)}, w\}} d_{H_1}^\lambda(x, w_j).$$

This gives

$$\begin{aligned} \sum_{j=1}^k d_{T'}(w_j) D_{T'}^\lambda(w_j) - \sum_{j=1}^k d_T(w_j) D_T^\lambda(w_j) &= \sum_{j=1}^k d(w_j) \\ &\times \sum_{x \in V(H_2)} (d_{T'[V(H_2) \cup \{w\}]}(x, w) + j)^\lambda \\ &- \sum_{j=1}^k d(w_j) \times \\ &\sum_{x \in V(H_2)} (d_{T[V(H_2) \cup \{v\}]}(x, v) + j + 1)^\lambda \dots 5 \end{aligned}$$

If H_1 is not a path, then, for each $1 \leq j \leq l \leq k$, we have $d(w_j) \geq d(u_j)$ there exists $j \in \{1, 2, \dots, k\}$ such that $d(w_j) > d(u_j)$. Hence, a similar of above results we have at $\lambda > 0$

$$\begin{aligned} \sum_{j=1}^l d_{T'}(u_j) D_{T'}^\lambda(u_j) - \sum_{j=1}^l d_T(u_j) D_T^\lambda(u_j) + \sum_{j=1}^k d_{T'}(w_j) D_{T'}^\lambda(w_j) - \sum_{j=1}^k d_T(w_j) D_T^\lambda(w_j) \\ > \sum_{j=l+1}^k d(w_j) \sum_{x \in V(H_2)} (d_{T'[V(H_2) \cup \{w\}]}(x, w) + j)^\lambda \\ - \sum_{j=l+1}^k d(w_j) \sum_{x \in V(H_2)} (d_{T'[V(H_2) \cup \{w\}]}(x, w) + j + 1)^\lambda > 0. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{j=1}^l d_{T'}(u_j) D_{T'}^\lambda(u_j) + \sum_{j=1}^k d_{T'}(w_j) D_{T'}^\lambda(w_j) > \sum_{j=1}^l d_T(u_j) D_T^\lambda(u_j) \\ + \sum_{j=1}^k d_T(w_j) D_T^\lambda(w_j) \dots \dots \dots (6) \end{aligned}$$

In case of $\lambda < 0$, H_1 is not a path, then, for each $1 \leq j \leq l \leq k$, we have $d(w_j) \geq d(u_j)$ there exists $j \in \{1, 2, \dots, k\}$ such that $d(w_j) > d(u_j)$.

Hence, in view of above results, we have

$$\begin{aligned} & \sum_{j=1}^l d_{T'}(u_j)D_{T'}^\lambda(u_j) - \sum_{j=1}^l d_T(u_j)D_T^\lambda(u_j) + \sum_{j=1}^k d_{T'}(w_j)D_{T'}^\lambda(w_j) - \sum_{j=1}^k d_T(w_j)D_T^\lambda(w_j) \\ & < \sum_{j=l+1}^k d(w_j) \sum_{x \in V(H_2)} \left(d_{T'[V(H_2) \cup \{w\}]}(x, w) + j \right)^\lambda \\ & - \sum_{j=l+1}^k d(w_j) \sum_{x \in V(H_2)} \left(d_{T'[V(H_2) \cup \{w\}]}(x, w) + j + 1 \right)^\lambda < 0. \end{aligned}$$

Thus,

$$\sum_{j=1}^l d_{T'}(u_j)D_{T'}^\lambda(u_j) + \sum_{j=1}^k d_{T'}(w_j)D_{T'}^\lambda(w_j) < \sum_{j=1}^l d_{(T)}(u_j)D_{(T)}^\lambda(u_j) + \sum_{j=1}^k d_{(T)}(w_j)D_{(T)}^\lambda(w_j).$$

However, if H_1 is a path and $\lambda > 0$ we have

$$\sum_{j=1}^l d_{T'}(u_j)D_{T'}^\lambda(u_j) + \sum_{j=1}^k d_{T'}(w_j)D_{T'}^\lambda(w_j) \geq \sum_{j=1}^l d_T(u_j)D_T^\lambda(u_j) + \sum_{j=1}^k d_T(w_j)D_T^\lambda(w_j)$$

and $\lambda < 0$ we have

$$\sum_{j=1}^l d_{T'}(u_j)D_{T'}^\lambda(u_j) + \sum_{j=1}^k d_{T'}(w_j)D_{T'}^\lambda(w_j) \leq \sum_{j=1}^l d_T(u_j)D_T^\lambda(u_j) + \sum_{j=1}^k d_T(w_j)D_T^\lambda(w_j),$$

By definition of $D_G^\lambda(u)$, we have

$$\begin{aligned} D_{T'}^\lambda(w) &= \sum_{x \in V(H_1) \setminus \{V(P_k), w\}} d_{H_1}^\lambda(x, w) + \sum_{j=1}^k (j+1)^\lambda + \sum_{j=1}^l (j)^\lambda + 1 \\ &+ \sum_{x \in V(H_2)} \left(d_{T'[V(H_2) \cup \{v\}]}(x, v) + 1 \right)^\lambda \end{aligned}$$

and

$$\begin{aligned} D_{T'}^\lambda(w) &= \sum_{x \in V(H_1) \setminus \{V(P_k), w\}} d_{H_1}^\lambda(x, w) + \sum_{j=1}^l (j+1)^\lambda + \sum_{j=1}^k (j)^\lambda + 1 \\ &+ \sum_{x \in V(H_2)} d_{T'[V(H_2) \cup \{w\}]}^\lambda(x, w). \end{aligned}$$

$$\begin{aligned} D_T^\lambda(v) &= \sum_{x \in V(H_1) \setminus \{V(P_k), w\}} \left(d_{H_1}(x, w) + 1 \right)^\lambda + \sum_{j=l}^k (j+1)^\lambda + 1 \\ &+ \sum_{j=1}^l (j)^\lambda + \sum_{x \in V(H_2)} d_{T'[V(H_2) \cup \{v\}]}^\lambda(x, v) \end{aligned}$$

and

$$D_{T'}^\lambda(v) = \sum_{x \in V(H_1) \setminus \{V(P_k), w\}} (d_{H_1}(x, w) + 1)^\lambda + \sum_{j=1}^k (j + 1)^\lambda + 1$$

$$+ \sum_{j=1}^l (j)^\lambda + \sum_{x \in V(H_2)} (d_{T'[V(H_2) \cup \{w\}]}(x, w) + 1)^\lambda.$$

Substituting $d_T(w) = d_{H_1}(w) + 1$, $d_T(v) = m + 1$, $d_{T'}(w) = d_{H_1}(w) + m$ and $d_{T'}(v) = 2$. Hence

$$d_{T'}(w)D_{T'}^\lambda(w) + d_{T'}(v)D_{T'}^\lambda(v) - d_T(w)D_T^\lambda(w) - d_T(v)D_T^\lambda(v)$$

$$= (m - 1) \left(\sum_{j=l+1}^k (j)^\lambda - \sum_{j=l+1}^k (j + 1)^\lambda \right) + (d_{H_1}(w) - 1) \left(\sum_{x \in V(H_2)} d_{T'[V(H_2) \cup \{w\}]}(x, w)^\lambda \right.$$

$$\left. - \sum_{x \in V(H_2)} (d_{T'[V(H_2) \cup \{w\}]}(x, w) + 1)^\lambda \right).$$

Observe that $m > 1$, we have $d_{H_1}(w) > 1$. If $\lambda > 0$ we have $d_{T'}(w)D_{T'}^\lambda(w) + d_{T'}(v)D_{T'}^\lambda(v) > d_T(w)D_T^\lambda(w) + d_T(v)D_T^\lambda(v)$ and if $\lambda < 0$ we have

$$d_{T'}(w)D_{T'}^\lambda(w) + d_{T'}(v)D_{T'}^\lambda(v) < d_T(w)D_T^\lambda(w) - d_T(v)D_T^\lambda(v).$$

Therefore from (1),(2),(3),(4),(5) and (6), we obtain

$$\text{when } \lambda < 0 \ H_\lambda(T) < H_\lambda(T'),$$

$$\text{and if } \lambda > 0, H_\lambda(T) > H_\lambda(T').$$

Hence get desired result . □

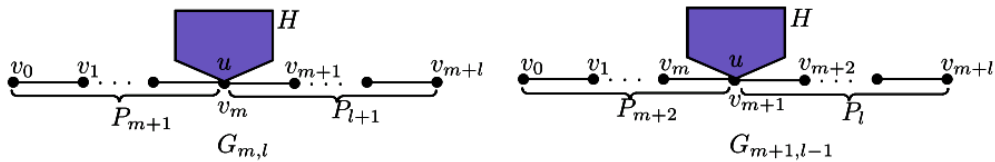


FIGURE 3. Graphs $G_{m,l}$ and $G_{m+1,l-1}$

THEOREM 3.2. *Let u be a vertex of connected graph H and for non-negative integers m and l , let $G_{m,l}$ be the graph obtained from H by adding two pendent paths of length m and l , respectively, to u of H ; see Fig. 3. If $l \geq m + 2$, we have $H_\lambda(G_{m,l}) < H_\lambda(G_{m+1,l-1})$ if $\lambda < 0$ and if $\lambda > 0$ we have $H_\lambda(G_{m,l}) > H_\lambda(G_{m+1,l-1})$*

PROOF. Let $G_1 := G_{m,l}$ and $G_2 := G_{m+1,l-1}$.
For all $x \in V(H) \setminus \{u\}$, we have

$$\begin{aligned}
D_{G_1}^\lambda(x) &= \sum_{y \in V(H) \setminus \{u,x\}} d_H^\lambda(x,y) + \sum_{j=0}^{m+l} (d_H(x,u) + d_{P_k}(v_m, v_j))^\lambda \\
&= \sum_{y \in V(H) \setminus \{u,x\}} d_H^\lambda(x,y) + \sum_{j=0}^{m-1} (d_H(x,u) + m - j)^\lambda + d_H^\lambda(x,u) \\
&\quad + \sum_{j=m+1}^{m+l} (d_H(x,u) + j - m)^\lambda \\
&= \sum_{y \in V(H) \setminus \{u,x\}} d_H^\lambda(x,y) + \sum_{j=0}^{m-1} (d_H(x,u) + m - j)^\lambda + d_H^\lambda(x,u) \\
&\quad + \sum_{j=1}^l (d_H(x,u) + j)^\lambda
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
D_{G_2}^\lambda(x) &= \sum_{y \in V(H) \setminus \{u,x\}} d_H^\lambda(x,y) + \sum_{j=0}^{m+l} (d_H(x,u) + d_{P_k}(v_{m+1}, v_j))^\lambda \\
&= \sum_{y \in V(H) \setminus \{u,x\}} d_H^\lambda(x,y) + \sum_{j=0}^m (d_H(x,u) + m + 1 - j)^\lambda + d_H^\lambda(x,u) \\
&\quad + \sum_{j=m+2}^{m+l} (d_H(x,u) + j - (m + 1))^\lambda \\
&= \sum_{y \in V(H) \setminus \{u,x\}} d_H^\lambda(x,y) + \sum_{j=0}^m (d_H(x,u) + m + 1 - j)^\lambda + d_H^\lambda(x,u) \\
&\quad + \sum_{j=1}^{l-1} (d_H(x,u) + j)^\lambda.
\end{aligned}$$

Now on subtracting $D_{G_2}^\lambda(x) - D_{G_1}^\lambda(x) = (d_H(x,u) + m + 1)^\lambda - (d_H(x,u) + l)^\lambda > 0$ whenever $\lambda < 0, l < m + 1$ and $(d_H(x,u) + m + 1)^\lambda - (d_H(x,u) + l)^\lambda < 0$ whenever $\lambda > 0$.
The above inequality are obtained from $l > m + 1$. Note that, for each x in $V(H) \setminus \{u\}$, we have

$d_{G_1}(x) = d_H(x) = d_{G_2}(x)$. Hence,

$$\sum_{x \in V(H) \setminus \{u\}} d_{G_1}(x) D_{G_1}^\lambda(x) > \sum_{x \in V(H) \setminus \{u\}} d_{G_2}(x) D_{G_2}^\lambda(x), \text{ when } \lambda > 0$$

and

$$\sum_{x \in V(H) \setminus \{u\}} d_{G_1}(x) D_{G_1}^\lambda(x) < \sum_{x \in V(H) \setminus \{u\}} d_{G_2}(x) D_{G_2}^\lambda(x), \text{ when } \lambda < 0.$$

Next we focus the vertex on the two pendent paths in G_1 (resp. G_2).

In fact, G_1 (resp. G_2) can be obtained by identifying the vertex u of H with the vertex v_m (resp. v_{m+1}) of path $P_k := P_{m+l+1} = v_0v_1 \dots v_mv_{m+1} \dots v_{m+l}$. For each $v_j \in V(P_k)$, we have

$$D_{G_1}^\lambda(v_j) = D_{P_k}^\lambda(v_j) + \sum_{x \in V(H) \setminus \{u\}} (d_H(x, u) + d_{P_k}(v_j, v_m))^\lambda$$

and

$$D_{G_2}^\lambda(v_j) = D_{P_k}^\lambda(v_j) + \sum_{x \in V(H) \setminus \{u\}} (d_H(x, u) + d_{P_k}(v_j, v_{m+1}))^\lambda.$$

Note that $d_{G_1}(v_j) = d_{G_2}(v_j)$ if $j \in \{0, 1, \dots, m+l\} \setminus \{m, m+1\}$, whereas $d_{G_1}(v_m) \leq d_H(u) + 2$, $d_{G_1}(v_{m+1}) = 2$, $d_{G_2}(v_m) = 2$ and $d_{G_2}(v_{m+1}) = d_H(u) + 2$. Hence,

$$\begin{aligned} \sum_{j=0}^{m+l} d_{G_1}(v_j) D_{G_1}^\lambda(v_j) &= \sum_{j=0}^{m+l} d_{G_1}(v_j) D_{P_k}^\lambda(v_j) \\ &+ \sum_{j=0}^{m+l} d_{G_1}(v_j) \sum_{x \in V(H) \setminus \{u\}} (d_H(x, u) + d_{P_k}(v_j, v_m))^\lambda \\ &\leq (d_H(u) + 2) D_{P_k}^\lambda(v_m) + 2 D_{P_k}^\lambda(v_{m+1}) \\ &+ \sum_{j=0, j \neq m, m+1}^{m+l} d_{G_1}(v_j) D_{P_k}^\lambda(v_j) \\ &+ \sum_{x \in V(H) \setminus \{u\}} (d_H(x, u) \\ &+ m)^\lambda + 2 \sum_{j=1}^{m-1} \sum_{x \in V(H) \setminus \{u\}} (d_H(x, u) + m - j)^\lambda \\ &+ (d_H(u) + 2) d_H^\lambda(x, u) \\ &+ 2 \sum_{j=1}^{l-1} \sum_{x \in V(H) \setminus \{u\}} (d_H(x, u) + j)^\lambda \\ &+ \sum_{x \in V(H) \setminus \{u\}} (d_H(x, u) + l)^\lambda \end{aligned}$$

and

$$\begin{aligned}
\sum_{j=0}^{m+l} d_{G_2}(v_j) D_{G_2}^\lambda(v_j) &= \sum_{j=0}^{m+l} d_{G_2}(v_j) D_{P_k}^\lambda(v_j) \\
&+ \sum_{j=0}^{m+l} d_{G_2}(v_j) \sum_{x \in V(H) \setminus \{u\}} (d_H(x, u) + d_{P_k}(v_j, v_{m+1}))^\lambda \\
&= 2D_{P_k}^\lambda(v_m) + (d_H(u) + 2)D_{P_k}^\lambda(v_{m+1}) \\
&+ \sum_{j=0, j \neq m, m+1}^{m+l} d_{G_2}(v_j) D_{P_k}^\lambda(v_j) \\
&+ \sum_{x \in V(H) \setminus \{u\}} (d_H(x, u) + m + 1)^\lambda \\
&+ 2 \sum_{j=0}^{m-1} \sum_{x \in V(H) \setminus \{u\}} (d_H(x, u) + m - j)^\lambda \\
&+ (d_H(u) + 2)d_H^\lambda(x, u) \\
&+ 2 \sum_{j=1}^{l-2} \sum_{x \in V(H) \setminus \{u\}} (d_H(x, u) + j)^\lambda \\
&+ \sum_{x \in V(H) \setminus \{u\}} (d_H(x, u) + l - 1)^\lambda.
\end{aligned}$$

Observe that one can find

$$\begin{aligned}
\sum_{j=0}^{m+l} d_{G_2}(v_j) D_{G_2}^\lambda(v_j) - \sum_{j=0}^{m+l} d_{G_1}(v_j) D_{G_1}^\lambda(v_j) &\leq d_H(u)(D_{P_k}^\lambda(v_{m+1}) - D_{P_k}^\lambda(v_m)) \\
&+ \sum_{x \in V(H) \setminus \{u\}} (d_H^\lambda(x, u) + m + 1)^\lambda \\
&- \sum_{x \in V(H) \setminus \{u\}} (d_H(x, u) + l - 1)^\lambda \\
&+ \sum_{x \in V(H) \setminus \{u\}} (d_H(x, u) + m)^\lambda \\
&- \sum_{x \in V(H) \setminus \{u\}} (d_H(x, u) + l)^\lambda.
\end{aligned}$$

By simple calculation, we have

$$\sum_{j=0}^{m+l} d_{G_2}(v_j) D_{G_2}^\lambda(v_j) > \sum_{j=0}^{m+l} d_{G_1}(v_j) D_{G_1}^\lambda(v_j), \text{ when } \lambda < 0$$

and

$$\sum_{j=0}^{m+l} d_{G_2}(v_j) D_{G_2}^\lambda(v_j) < \sum_{j=0}^{m+l} d_{G_1}(v_j) D_{G_1}^\lambda(v_j), \text{ when } \lambda > 0.$$

On above observations we find $H_\lambda(G_{m,l}) < H_\lambda(G_{m+l,l-1})$ if $\lambda < 0$ and if $\lambda > 0$ we have $H_\lambda(G_{m,l}) > H_\lambda(G_{m+l,l-1})$. Hence desired □

4. θ -transformation

Let uw be a cut edge of a bipartite graph U with $d_U(w) \geq 2$. G is obtained from U and the star S_{k+2} by identifying u with a pendent vertex of S_{k+2} whose center is v . Let $G[v \rightarrow w; 2]$ be the graph obtained from G by deleting all edges $vz, z \in w$ and adding all edges $wz, z \in w$, where $w = N_G(v) \setminus \{u\}$.

In notation, $G[v \rightarrow w; 2] = G - \{vz : z \in w\} + \{wz : z \in w\}$ and we say $G[v \rightarrow w; 2]$ is obtained from G by θ -transformation. Graphs $G, G[v \rightarrow w; 2]$ are depicted in Fig. 4. The Laplacian permanent of trees with given bi-partition using above θ -transformation is studied by Li and Zhang [10]. Here we are to use the θ -transformation as a tool to study the generalized degree distance of trees.

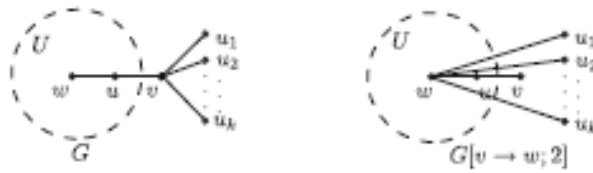


FIGURE 4. $G \Rightarrow G[v \rightarrow w; 2]$ by θ -transformation

THEOREM 4.1. *Let G and $G[v \rightarrow w; 2]$ be the bipartite graphs with some labelled vertices as above. Then*

$$H_\lambda G[v \rightarrow w; 2] > H_\lambda(G), \text{ If } \lambda > 0$$

and if $\lambda < 0$ we have $H_\lambda G[v \rightarrow w; 2] < H_\lambda(G)$.

PROOF. Let $G' = G[v \rightarrow w; 2]$ and let T_1 be the component in $G - \{wu, uv\}$ which contains u . Define $A = V(U) \setminus (V(T_1) \cup \{w\})$. One can observe that, for all $x \in A$, we have

$$(4.1) \quad \begin{aligned} D_G^\lambda(x) &= \sum_{y \in A \setminus \{x\}} (d_U(x, y))^\lambda + (d_U(x, w))^\lambda \\ &+ \sum_{y \in V(T_1)} (d_G(x, y))^\lambda + (d_U(x, w + 2))^\lambda + k(d_U(x, w) + 3)^\lambda, \end{aligned}$$

$$(4.2) \quad \begin{aligned} D_{G'}^\lambda(x) &= \sum_{y \in A \setminus \{x\}} (d_U(x, y))^\lambda + (d_U(x, w))^\lambda \\ &+ \sum_{y \in V(T_1)} (d_{G'}(x, y))^\lambda + (d_U(x, w + 2))^\lambda + k(d_U(x, w) + 1)^\lambda. \end{aligned}$$

Note that

$$\sum_{y \in V(T_1)} d_G^\lambda(x, y) = \sum_{y \in V(T_1)} d_{G'}^\lambda(x, y)$$

Hence $D_{G'}^\lambda(x) < D_G^\lambda(x)$, if $\lambda < 0$ and $D_{G'}^\lambda(x) > D_G^\lambda(x)$ when $\lambda > 0$. It is easy to check that for all x in A , one has $d_G(x) = d_U(x) = d_{G'}(x)$. So we obtain that Whenever $\lambda < 0$

$$\sum_{x \in A} d_G(x) D_G^\lambda(x) < \sum_{x \in A} d_{G'}(x) D_{G'}^\lambda(x)$$

and if $\lambda > 0$,

$$\sum_{x \in A} d_G(x) D_G^\lambda(x) > \sum_{x \in A} d_{G'}(x) D_{G'}^\lambda(x).$$

For each vertex x in $V(T_1)$, it is easily check that $d_G(x) = d_{T_1}(x) = d_{G'}(x)$ and $D_G^\lambda(x) = D_{G'}^\lambda(x)$. Hence

$$\sum_{x \in V(T_1)} d_G(x) D_G^\lambda(x) = \sum_{x \in V(T_1)} d_{G'}(x) D_{G'}^\lambda(x).$$

For each $u_j \in X' = \{u_1, u_2, \dots, u_k\}$, by direct calculation we have

$$D_G^\lambda(u_j) = 1 + (2)^\lambda(k-1) + \sum_{x \in V(T_1)} (d_G(x, v) + 1)^\lambda + (3)^\lambda + \sum_{x \in A} d_U(x, w + 3)^\lambda,$$

$$D_{G'}^\lambda(u_j) = 1 + (2)^\lambda(k-1) + \sum_{x \in V(T_1)} (d_{G'}(x, w) + 1)^\lambda + (3)^\lambda + \sum_{x \in A} (d_U(x, w) + 1)^\lambda.$$

It is routine to check that $d_G^\lambda(x, v) = d_{G'}^\lambda(x, w)$ for $x \in V(T_1)$.

Hence we have $D_{G'}^\lambda(u_j) < D_G^\lambda(u_j)$. Note that $d_G(u_j) = d_{G'}(u_j) = 1$ for $u_j \in X'$. Hence when $\lambda < 0$ we have

$$\sum_{u_j \in X'} d_G(u_j) D_G^\lambda(u_j) < \sum_{u_j \in X'} d_{G'}(u_j) D_{G'}^\lambda(u_j)$$

and if $\lambda > 0$ we have

$$\sum_{u_j \in X'} d_G(u_j) D_G^\lambda(u_j) > \sum_{u_j \in X'} d_{G'}(u_j) D_{G'}^\lambda(u_j).$$

By overall calculation, we have

$$\begin{aligned} D_G^\lambda(w) &= \sum_{x \in A} (d_U(x, w))^\lambda + \sum_{x \in V(T_1)} (d_G(x, w))^\lambda + (2)^\lambda + k(3)^\lambda \\ \text{and } D_G^\lambda(v) &= \sum_{x \in A} (d_U(x, w) + 2)^\lambda + \sum_{x \in V(T_1)} (d_G(x, v))^\lambda + (2)^\lambda + k \end{aligned}$$

Similarly, we obtain

$$D_{G'}^\lambda(w) = \sum_{x \in A} (d_U(x, w))^\lambda + \sum_{x \in V(T_1)} (d_{G'}(x, w))^\lambda + (2)^\lambda + k$$

and $D_{G'}^\lambda(v) = \sum_{x \in A} (d_U(x, w) + 2)^\lambda + \sum_{x \in V(T_1)} (d_{G'}(x, v))^\lambda + (2)^\lambda + k(3)^\lambda$.

One can see that $d_G(w) = d_U(w) + 1$, $d_G(v) = k + 1$, $d_{G'}(w) = d_U(w) + k + 1$, $d_{G'}(v) = 1$ and for each vertex x in T_1 , $d_G(x, w) = d_G(x, v) = d_{G'}(x, w) = d_{G'}(x, v)$. Observe that the difference of

$$\begin{aligned} & d_{G'}(w)D_{G'}^\lambda(w) + d_{G'}(v)D_{G'}^\lambda(v) - d_G(w)D_G^\lambda(w) - d_G(v)D_G^\lambda(v) \\ &= k \left(\sum_{x \in A} (d_U(x, w))^\lambda + \sum_{x \in V(T_1)} (d_{G'}(x, w))^\lambda + (2)^\lambda \right) + k(d_U(w) + k + 1) \\ & - k(3)^\lambda(d_U(w) + 1) - k \left(\sum_{x \in A} (d_U(x, w) + 2)^\lambda + \sum_{x \in V(T_1)} (d_{G'}(x, v))^\lambda + (2)^\lambda \right) \\ & + k(3)^\lambda - k(k - 1) \\ & > \frac{(2)^\lambda}{(3)^\lambda} k d_U(w) + \frac{(1)^\lambda}{(3)^\lambda} k^2 + \frac{(5)^\lambda}{(3)^\lambda} k > 0, \end{aligned}$$

i.e., when $\lambda > 0$ we have

$$d_{G'}(w)D_{G'}^\lambda(w) + d_{G'}(v)D_{G'}^\lambda(v) > d_G(w)D_G^\lambda(w) + d_G(v)D_G^\lambda(v)$$

but when $\lambda < 0$ we have

$$d_{G'}(w)D_{G'}^\lambda(w) + d_{G'}(v)D_{G'}^\lambda(v) < d_G(w)D_G^\lambda(w) + d_G(v)D_G^\lambda(v).$$

In view of the above results we get the following inequality

$$H_\lambda G[v \rightarrow w; 2] > H_\lambda(G), \text{ If } \lambda > 0$$

and

$$H_\lambda G[v \rightarrow w; 2] < H_\lambda(G), \text{ if } \lambda < 0 \text{ as desired}$$

□

5. Conclusion

In this paper, we determine the generalized degree distance of some graphs by using the transformation of graphs, viz. θ -transformation, ρ -transformation etc.

Acknowledgment

The authors would like to express their sincere gratitude to the referees for a very careful reading of this paper and for all their valuable commits.

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Received by editors 09.12.2018; Revised version 27.01.2019 and 06.02.2019; Available online 18.02.2019.

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