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GAMMA SEMIGROUPS ON WEAK NEARNESS APPROXIMATION SPACES

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Abstract. In this paper, we consider the problem of how to define Γ-nearness semigroup theory which extends the notion of a nearness semigroup and roughness of Γ-semigroups ([**6**] and [**11**]) to include the algebraic structures of near sets and rough sets, respectively. Also, we introduce some properties of aproximations and these algebraic structures.

1. Introduction

In 1982, the concept of a rough set was originally proposed by Pawlak [**20**] as a formal tool for modeling incompleteness and imprecision in information systems. The theory of rough sets is an extension of The set theory, in which a subset of a universe is described by a pair of ordinary sets called the lower and upper approximations. A basic notion in the Pawlak rough set model is an equivalence relation. The lower approximation of a given set is the union of all the equivalence classes which are subsets of the set, and the upper approximation is the union of all the equivalence classes which have a non-empty intersection with the set. An algebraic approach of rough sets has been given by Iwinski [**10**]. Afterwards, rough subgroups were introduced by Biswas and Nanda [**1**]. Kuroki in [**12**], introduced the notion of a rough ideal in a semigroup. Since then the subject has been investigated in many papers ([**13**], [**3**], [**4**], [**14**], [**5**], [**11**]).

In 2002, Peters introduced near set theory as an generalization of rough set theory. In this theory, Peters defined a indiscernibility relation that depends on

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the features of the objects in order to define the nearness of the objects [**23**]. More recent work considers generalized approach theory in the study of the nearness of non-empty sets that resemble each other [**21**], [**22**], [**24**], [**25**], [**26**], [**27**], [**28**].

In 2012, İnan and Öztürk investigated the concept of nearness groups [**6, 7**]. Also, in 2015, Öztürk and İnan established nearness semigroups and nearness rings [**8, 9**] (and other algebraic approaches of near sets in [**15**], [**16**], [**17**], [**18**]).

In 1986, Sen and Saha studied on Γ-semigroup for the first time in [**31**]. After this research, many mathematicians made good works on Γ-semigroups, which are parallel to the results in the semigroup theory ([**29**], [**30**], [**11**], [**2**], [**32**]).

The aim of this paper is to the concept of gamma nearness semigroup theory which extends the notion of a nearness semigroup and roughness of Γ-semigroups ([**6**] and [**11**]) to include the algebraic structures of near sets and rough sets, respectively. Also, we introduce some properties of aproximations and these algebraic structures.

2. Preliminaries

An object description is defined by means of a tuple of function values $\Phi(x)$ associated with an object $x \in X$. Assume that $B \subseteq \mathcal{F}$ is a given set of functions representing features of sample objects $X \subseteq \mathcal{O}$. Let $\varphi_i \in B$, where $\varphi_i : \mathcal{O} \to \mathbb{R}$. In combination, the functions representing object features provide a basis for an object description $\Phi : \mathcal{O} \to \mathbb{R}^L, \Phi(x) = (\varphi_1(x), \varphi_2(x), ..., \varphi_L(x))$ a vector containing measurements (returned values) associated with each functional value $\varphi_i(x)$, where the description length $\mid \Phi \mid = L$ ([**21**]).

Sample objects $X \subseteq \mathcal{O}$ are near each if and only if the objects have similar descriptions. The important thing to notice is the choice of functions $\varphi_i \in B$ used to describe an object of interest. Recall that each φ defines a description of an object. Then let $\triangle_{\varphi_i}$ denote $\triangle_{\varphi_i} = \mid \varphi_i(x') - \varphi_i(x) \mid$, where $x', x \in \mathcal{O}$. The difference φ leads to a description of the indiscernibility relation"$\sim_B$" introduced by Peters in [**21**].

Definition 2.1. ([**21**]) Let $x, x' \in \mathcal{O}$ and $B \subseteq \mathcal{F}$

$$\sim_B = \{(x, x') \in \mathcal{O} \times \mathcal{O} \mid \triangle_{\varphi_i} = 0 \text{ for all } \varphi_i \in B\}$$

is called the indiscernibility relation on $\mathcal{O}$, where description length $i \leqslant \mid \Phi \mid$.

The basic idea in the near set approach to object recognition is to compare object descriptions. Sets of objects X, X' are considered near each other if the sets contain objects with at least partial matching descriptions.

Definition 2.2. ([**21**]) Let $X, X' \subseteq \mathcal{O}$ and $B \subseteq \mathcal{F}$. Set X is called near X' if there exists $x \in X, x' \in X', \varphi_i \in B$ such that $x \sim_{\varphi_i} x'$.

Symbol	*Interpretation*
B	$B \subseteq \mathcal{F}$, set of probe functions,
r	$\binom{\lvert B\rvert}{r}$, i.e. , $\lvert B\rvert$ probe functions $\varphi_i \in B$ taken r at a time,
B_r	$r \leqslant \lvert B\rvert$ probe functions in B,
$\sim_{B_r}$	indiscernibility relation defined using B_r,
$[x]_{B_r}$	$[x]_{B_r} = \{x' \in \mathcal{O} \mid x \sim_{B_r} x'\}$, near equivalence class,
$\mathcal{O}/\sim_{B_r}$	$\mathcal{O}/\sim_{B_r} = \{[x]_{B_r} \mid x \in \mathcal{O}\} = \xi_{\mathcal{O},B_r}$, quotient set,
$N_r(B)$	$N_r(B) = \{\xi_{\mathcal{O},B_r} \mid B_r \subseteq B\}$, set of partitions,
ν_{N_r}	$\nu_{N_r} : \wp(\mathcal{O}) \times \wp(\mathcal{O}) \to [0,1]$, overlap function,
$N_r(B)_* X$	$N_r(B)_* X = \bigcup\limits_{[x]_{B_r} \subseteq X} [x]_{B_r}$, lower approximation,
$N_r(B)^* X$	$N_r(B)^* X = \bigcup\limits_{[x]_{B_r} \cap X \neq \varnothing} [x]_{B_r}$, upper approximation,
$Bnd_{N_r(B)}(X)$	$N_r(B)^* X \diagdown N_r(B)_* X = \left\{x \in N_r(B)^* X \mid x \notin N_r(B)_* X\right\}$.

Table 1 : Symbols of Nearness Approximation Space

A nearness approximation space is a tuple $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$ where the approximation space is defined with a set of perceived objects $\mathcal{O}$, set of probe functions $\mathcal{F}$ representing object features, $\sim_{B_r}$indiscernibility relation B_r defined relative to $B_r \subseteq B \subseteq \mathcal{F}$, collection of partitions (families of neighbour-hoods) $N_r(B)$, and overlap function ν_{N_r} ([**21**]).

Definition 2.3. ([**8**]) Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$ be a nearness approximation space and let "$\cdot$" be a binary operation defined on $\mathcal{O}$. Let $X \subseteq \mathcal{O}$ and $B_r \subseteq \mathcal{F}$, $r \leqslant\mid B \mid$. A indiscernibility relation $\sim_{B_r}$ on $\mathcal{O}$ is called a complete indiscernibility relation $\sim_{B_r}$on perceptual objects $\mathcal{O}$, if $[x]_{B_r} \cdot [y]Br = [x \cdot y]_{B_r}$ for all $x, y \in X$.

Theorem 2.1 ([**8**]). *Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$ be a nearness approximation space and X,Y $\subset \mathcal{O}$, then the following statements hold;*

1) $N_r(B)_*(X) \subseteq X \subseteq N_r(B)^*(X)$,
2) $N_r(B)^*(X \cup Y) = N_r(B)^*(X) \cup N_r(B)^*(Y)$,
3) $N_r(B)_*(X \cap Y) = N_r(B)_*(X) \cap N_r(B)_*(Y)$,
4) $X \subseteq Y$ *implies* $N_r(B)_*(X) \subseteq N_r(B)_*(Y)$,
5) $X \subseteq Y$ *implies* $N_r(B)^*(X) \subseteq N_r(B)^*(Y)$,
6) $N_r(B)_*(X \cup Y) \supseteq N_r(B)_*(X) \cup N_r(B)_*(Y)$,
7) $N_r(B)^*(X \cap Y) \subseteq N_r(B)^*(X) \cap N_r(B)^*(Y)$.

Definition 2.4. ([**8**]) Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$ be a nearness approximation space and "$\cdot$" be a binary operation defined on $\mathcal{O}$. A subset S of perceptual objects $\mathcal{O}$ is called a semigroup on nearness approximation space or shortly nearness semigroup if the following properties are satisfied.

1) $x \cdot y \in N_r(B)^* S$ for all $x, y \in S$;
2) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property holds in $N_r(B)^* S$ for all $x, y \in S$.

Definition 2.5. ([**8**]) Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$ be a nearness approximation space, S a nearness semigroup and I a non-empty subset of S. If $N_r(B)^* I$ is a left (right, two sided) ideal of S, then I is called a nearness left (right, two sided) ideal of S.

Definition 2.6. ([**8**]) Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$ be a nearness approximation space, S a nearness semigroup and I a non-empty subset of S. If $N_r(B)^* I$ is a bi-ideal of S, then I is called a nearness bi-ideal of S.

Definition 2.7. ([**31**]) Let $S = \{a, b, c, ...\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$ be two non-empty sets. S is called a Γ-semigroup if (i) $a\alpha b \in S$, (ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

Definition 2.8. ([**31**]) A non-empty subset B of a Γ-semigroup S is said to be a sub Γ-semigroup of S if $B\Gamma B \subseteq B$.

Definition 2.9. ([**31**]) A sub Γ-semigroup B of a Γ-semigroup S is said to be a Γ-left (resp. right) ideal of S if $S\Gamma B \subseteq B$ (resp. $B\Gamma S \subseteq B$). B is said to be a Γ-ideal of S if it is both a Γ-left ideal and a Γ-right ideal of S.

Definition 2.10. ([**2**]) Let S be a Γ-semigroup. A sub Γ-semigroup B of S is called a bi-Γ-ideal of S if $B\Gamma S\Gamma B \subseteq B$.

3. $\Gamma-$Nearness Semigroups

In this section, $\nu_{N_r} : \wp(\mathcal{O}) \times \wp(\mathcal{O}) \to [0, 1]$ is not needed which is overlap function when algebraic structures are studied on the nearness approximation space $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$. Therefore, let's start with the following definition.

Definition 3.1. Let $\mathcal{O}$ be a set of perceived objects, $\mathcal{F}$ a set of the probe functions, $\sim_{B_r}$an indiscernibility relation, and N_r a collection of partitions. Then, $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r)$ is called a weak nearness approximation space.

We will give the following theorem, which is the same proof as the proof of Theorem 2.1.

Theorem 3.1. *Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r)$ be a weak nearness approximation space and $A, B \subset \mathcal{O}$, then the following statements hold;*

i) $N_r(B)_* A \subseteq A \subseteq N_r(B)^* A$,
ii) $N_r(B)^* (A \cup B) = N_r(B)^* A \cup N_r(B)^* B$,
iii) $N_r(B)_* (A \cap B) = N_r(B)_* A \cap N_r(B)_* B$,
iv) $A \subseteq B$ *implies* $N_r(B)_* A \subseteq N_r(B)_* B$,
v) $A \subseteq B$ *implies* $N_r(B)^* A \subseteq N_r(B)^* B$,
vi) $N_r(B)_* (A \cup B) \supseteq N_r(B)_* A \cup N_r(B)_* B$,
vii) $N_r(B)^* (A \cap B) \subseteq N_r(B)^* A \cap N_r(B)^* B$.

Definition 3.2. Let $S = \{x, y, z, \ldots\} \subseteq \mathcal{O}$, and $\Gamma = \{\alpha, \beta, \gamma, \ldots\} \subseteq \acute{\mathcal{O}}$ where $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r)$ and $(\acute{\mathcal{O}}, \mathcal{F}, \sim_{B_r}, N_r)$ are two different weak near approximation spaces. If the following properties are satisfied, then S is called a Γ-semigroup on weakly approximate approximation spaces $\mathcal{O} - \acute{\mathcal{O}}$, or, in short, a Γ-nearness semigroup.

i) $x\gamma y \in N_r(B)^* S$ for all $x, y \in S$ and $\gamma \in \Gamma$;

ii) $(x\beta y)\gamma z = x\beta(y\gamma z)$ property holds in $N_r(B)^* S$ for all $x, y \in S$ and $\beta, \gamma \in \Gamma$.

Let S be a Γ-semigroup on weakly approximate approximation spaces $\mathcal{O} - \acute{\mathcal{O}}$ where $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r)$ and $(\acute{\mathcal{O}}, \mathcal{F}, \sim_{B_r}, N_r)$ are two different weak near approximation spaces. If $\mathcal{O} = \acute{\mathcal{O}}$, then S is a Γ-semigroup on weakly approximate approximation spaces $\mathcal{O}$.

Example 3.1. Let $\mathcal{O} = \{a, \beta, \gamma, b, c, d, e, f, g, h, i, j\}$ be a set of perceptual objects where

$$a = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \beta = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \gamma = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, c = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$
$$d = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, a = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, g = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$
$$h = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, i = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, j = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

for $U = \{ [a_{ij}]_{2x2} \mid a_{ij} \in \mathbb{Z}_2\}$, $r = 1$, $B = \{\varphi_1, \varphi_2, \varphi_3\} \subseteq \mathcal{F}$ be a set of probe functions, and $S = \{d, e\} \subset \mathcal{O}, \Gamma = \{\beta, \gamma\} \subset \mathcal{O}$. Values of the probe functions

$$\varphi_1 : \mathcal{O} \to V_1 = \{\alpha_1, \alpha_2, \alpha_3\},$$
$$\varphi_2 : \mathcal{O} \to V_2 = \{\alpha_1, \alpha_3, \alpha_4\},$$
$$\varphi_3 : \mathcal{O} \to V_3 = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5\}$$

are given in *Table* 2.

	a	β	γ	b	c	d	e	f	g	h	i	j
φ_1	α_1	α_2	α_3	α_2	α_1	α_3	α_2	α_1	α_1	α_1	α_1	α_1
φ_2	α_1	α_3	α_3	α_4	α_1	α_1	α_4	α_3	α_4	α_3	α_3	α_3
φ_3	α_3	α_3	α_1	α_1	α_4	α_4	α_5	α_1	α_3	α_3	α_4	α_3

Table 2

Let "·" be a binary operation of perceptual objects on $\mathcal{O}$ with the following table:

	a	β	γ	b	c	d	e	f	g	h	i	j
a	a	a	a	a	a	a	a	a	a	a	a	a
β	a	a	a	β	β	β	γ	γ	i	i	a	β
γ	a	β	γ	a	β	i	a	γ	γ	i	i	γ
b	a	a	a	b	b	b	e	e	g	g	a	b
c	a	a	a	c	c	c	f	f	h	h	a	c
d	a	b	e	a	b	g	a	e	a	g	i	d
e	a	b	e	a	b	g	a	e	a	g	g	e
f	a	c	f	a	c	h	a	f	a	h	h	f
g	a	b	e	b	a	e	e	a	g	a	g	g
h	a	c	f	c	a	f	f	a	h	a	h	h
i	a	β	γ	β	a	γ	γ	a	i	a	i	i
j	a	β	γ	b	c	d	e	f	g	h	i	j

Table 3

Let us now determine the near equivalence classes according to the indiscernibility relation of $\sim_{B_r}$ of elements of $\mathcal{O}$:

$$\begin{aligned}
[a]_{\varphi_1} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_1(a) = \alpha_1\} = \{a, c, f, g, h, i, j\} \\
&= [c]_{\varphi_1} = [f]_{\varphi_1} = [g]_{\varphi_1} = [h]_{\varphi_1} = [i]_{\varphi_1} = [j]_{\varphi_1}, \\
[\beta]_{\varphi_1} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_1(\beta) = \alpha_2\} = \{\beta, b, e\} \\
&= [b]_{\varphi_1} = [e]_{\varphi_1}, \\
[\gamma]_{\varphi_1} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_1(\gamma) = \alpha_3\} = \{\gamma, d\} \\
&= [b]_{\varphi_1}.
\end{aligned}$$

Then, we get that $\xi_{\varphi_1} = \left\{[a]_{\varphi_1}, [\beta]_{\varphi_1}, [\gamma]_{\varphi_1}\right\}$.

$$\begin{aligned}
[a]_{\varphi_2} &= \{x \in \mathcal{O} \mid \varphi_2(x) = \varphi_2(a) = \alpha_1\} = \{a, c, d\} \\
&= [c]_{\varphi_2} = [d]_{\varphi_2}, \\
[\beta]_{\varphi_2} &= \{x \in \mathcal{O} \mid \varphi_2(x) = \varphi_2(\beta) = \alpha_3\} = \{\beta, \gamma, f, h, i, j\} \\
&= [\gamma]_{\varphi_2} = [f]_{\varphi_2} = [h]_{\varphi_2} = [i]_{\varphi_2} = [j]_{\varphi_2}, \\
[b]_{\varphi_2} &= \{x \in \mathcal{O} \mid \varphi_2(\acute{x}) = \varphi_2(\gamma) = \alpha_4\} = \{b, e, g\} \\
&= [e]_{\varphi_2} = [g]_{\varphi_2}.
\end{aligned}$$

Thus, we have that $\xi_{\varphi_2} = \left\{[a]_{\varphi_2}, [\beta]_{\varphi_2}, [b]_{\varphi_2}\right\}$.

$$
\begin{aligned}
[a]_{\varphi_3} &= \{x \in \mathcal{O} \mid \varphi_3(x) = \varphi_3(a) = \alpha_3\} = \{a, \beta, g, h, j\} \\
&= [\beta]_{\varphi_3} = [g]_{\varphi_3} = [h]_{\varphi_3} = [j]_{\varphi_3}, \\
[\gamma]_{\varphi_3} &= \{x \in \mathcal{O} \mid \varphi_3(x) = \varphi_3(\gamma) = \alpha_1\} = \{\gamma, b, f\} \\
&= [b]_{\varphi_3} = [f]_{\varphi_3}, \\
[c]_{\varphi_3} &= \{x \in \mathcal{O} \mid \varphi_3(x) = \varphi_3(c) = \alpha_4\} = \{c, d, i\} \\
&= [d]_{\varphi_3} = [i]_{\varphi_3}, \\
[e]_{\varphi_3} &= \{x \in \mathcal{O} \mid \varphi_3(x) = \varphi_3(e) = \alpha_5\} = \{e\}.
\end{aligned}
$$

From hence, we obtain that $\xi_{\varphi_3} = \left\{[a]_{\varphi_3}, [\gamma]_{\varphi_3}, [c]_{\varphi_3}, [e]_{\varphi_3}\right\}$. Therefore, for $r = 1$, a set of partitions of $\mathcal{O}$ is $N_r(B) = \{\xi_{\varphi_1}, \xi_{\varphi_2}, \xi_{\varphi_3}\}$. Then, we can write

$$
\begin{aligned}
N_1(B)^* S &= \bigcup_{[x]_{\varphi_i} \cap S \neq \varnothing} [x]_{\varphi_i} \\
&= [\beta]_{\varphi_1} \cup [\gamma]_{\varphi_1} \cup [a]_{\varphi_2} \cup [b]_{\varphi_2} \cup [c]_{\varphi_3} \cup [e]_{\varphi_3} \\
&= \{\beta, b, e\} \cup \{\gamma, d\} \cup \{a, c, d\} \cup \{b, e, g\} \cup \{c, d, i\} \cup \{e\} \\
&= \{a, \beta, \gamma, b, c, d, e, g, i\}.
\end{aligned}
$$

In that case; S is a Γ-semigroup on the weak near approximation space $\mathcal{O}$ by Definition 3.2.

Now, let's give a Γ-semigroup example defined on weakly approximate approximation spaces $\mathcal{O} - \acute{\mathcal{O}}$.

EXAMPLE 3.2. Let $\mathcal{O} = \{a, b, c, d, e, f, g, h\}$ be a set of perceptual objects where

$$
\begin{aligned}
a &= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, c = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, d = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}, \\
e &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, g = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}
\end{aligned}
$$

for $U = \{\ [a_{ij}]_{1x3} \mid a_{ij} \in \mathbb{Z}_2\}$, $\acute{\mathcal{O}} = \{\alpha, \beta, \gamma, \theta, \lambda, \mu, \delta, \sigma\}$ be a set of perceptual objects where

$$
\begin{aligned}
\alpha &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \beta = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \gamma = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \theta = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \\
\lambda &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \mu = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \delta = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \sigma = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

for $U = \{ [a_{ij}]_{3x1} \mid a_{ij} \in \mathbb{Z}_2\}, r = 1,\ B = \{\varphi_1, \varphi_2, \varphi_3\} \subseteq \mathcal{F}$ be a set of probe functions, and $S = \{b, c\} \subset \mathcal{O}, \Gamma = \{\alpha, \beta\} \subset \acute{\mathcal{O}}$. Values of the probe functions

$$\varphi_1 : \mathcal{O} \to V_1 = \{\alpha_1, \alpha_2, \alpha_3\},$$
$$\varphi_2 : \mathcal{O} \to V_2 = \{\alpha_1, \alpha_3\},$$
$$\varphi_3 : \mathcal{O} \to V_3 = \{\alpha_2, \alpha_3\}$$

are given in *Table* 4.

	a	b	c	d	e	f	g	h
φ_1	α_1	α_1	α_1	α_2	α_3	α_3	α_3	α_2
φ_2	α_1	α_1	α_1	α_3	α_3	α_3	α_3	α_3
φ_3	α_2	α_2	α_2	α_2	α_3	α_3	α_3	α_3

Table 4

In this case,

$$\begin{aligned}[a]_{\varphi_1} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_1(a) = \alpha_1\} = \{a, b, c\} \\ &= [b]_{\varphi_1} = [c]_{\varphi_1}, \\ [d]_{\varphi_1} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_1(d) = \alpha_2\} = \{d, h\} \\ &= [h]_{\varphi_1}, \\ [e]_{\varphi_1} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_1(e) = \alpha_3\} = \{e, f, g\} \\ &= [f]_{\varphi_1} = [g]_{\varphi_1}.\end{aligned}$$

Then, we get that $\xi_{\varphi_1} = \left\{ [a]_{\varphi_1}, [d]_{\varphi_1}, [e]_{\varphi_1} \right\}$.

$$\begin{aligned}[a]_{\varphi_2} &= \{x \in \mathcal{O} \mid \varphi_2(x) = \varphi_2(a) = \alpha_1\} = \{a, b, c\} \\ &= [b]_{\varphi_2} = [c]_{\varphi_2}, \\ [d]_{\varphi_2} &= \{x \in \mathcal{O} \mid \varphi_2(x) = \varphi_2(\beta) = \alpha_2\} = \{d, e, f, g, h\} \\ &= [e]_{\varphi_2} = [f]_{\varphi_2} = [g]_{\varphi_2} = [h]_{\varphi_2}.\end{aligned}$$

We have that $\xi_{\varphi_2} = \left\{ [a]_{\varphi_2}, [d]_{\varphi_2} \right\}$.

$$\begin{aligned}[a]_{\varphi_3} &= \{x \in \mathcal{O} \mid \varphi_3(x) = \varphi_3(a) = \alpha_2\} = \{a, b, c, d\} \\ &= [b]_{\varphi_3} = [c]_{\varphi_3} = [d]_{\varphi_3}, \\ [e]_{\varphi_3} &= \{x \in \mathcal{O} \mid \varphi_3(x) = \varphi_3(e) = \alpha_3\} = \{e, f, g, h\} \\ &= [f]_{\varphi_3} = [g]_{\varphi_3} = [h]_{\varphi_3}.\end{aligned}$$

From hence, we obtain that $\xi_{\varphi_3} = \left\{ [a]_{\varphi_3}, [e]_{\varphi_3} \right\}$. Therefore, for $r = 1$, a set of partitions of $\mathcal{O}$ is $N_r(B) = \{\xi_{\varphi_1}, \xi_{\varphi_2}, \xi_{\varphi_3}\}$. Then, we can write

$$N_1(B)^* S = \bigcup_{[x]_{\varphi_i} \cap S \neq \varnothing} [x]_{\varphi_i} = \{a, b, c, d\}.$$

Considering the following tables of operations:

α	b	c
b	a	a
c	a	a

β	b	c
b	a	a
c	b	c

S is a Γ-semigroup on the weak near approximation space $\mathcal{O} - \acute{\mathcal{O}}$ by Definition 3.2.

DEFINITION 3.3. Let $S \subseteq \mathcal{O}$ and $\Gamma \subseteq \acute{\mathcal{O}}$ where $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r)$ and $(\acute{\mathcal{O}}, \mathcal{F}, \sim_{B_r}, N_r)$ are two different weak near approximation spaces, $B_r \subseteq \mathcal{F}$where $r \leqslant \mid B \mid$and $B \subseteq \mathcal{F}$, $\sim_{B_r}$be a indiscernibility relation on $\mathcal{O} - \acute{\mathcal{O}}$. Then, $\sim_{B_r}$ is called a congruence indiscernibility relation on Γ-nearness semigroup S, if $x \sim_{B_r} y$, where $x, y \in S$ implies $x\gamma a \sim_{B_r} y\gamma a$ and $a\gamma x \sim_{B_r} a\gamma y$ for all $a \in S$ and $\gamma \in \Gamma$.

PROPOSITION 3.1. *Let $S \subseteq \mathcal{O}$ and $\Gamma \subseteq \acute{\mathcal{O}}$ where*

$$(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r) \text{ and } (\acute{\mathcal{O}}, \mathcal{F}, \sim_{B_r}, N_r)$$

are two different weak near approximation spaces, S be a Γ-nearness semigroup. If $\sim_{B_r}$ is a congruence indiscernibility relation on S, then $[x]_{B_r}\gamma[y]_{B_r} \subseteq [x\gamma y]_{B_r}$ for all $x, y \in S$ and $\gamma \in \Gamma$.

PROOF. Let $z \in [x]_{B_r}\gamma[y]_{B_r}$. In his case, $z = a\gamma b; a \in [x]_{B_r}, \gamma \in \Gamma, b \in [y]_{B_r}$. From here $x \sim_{B_r} a$,and $y \sim_{B_r} b$, and so, we have $x\gamma y \sim_{B_r} a\gamma y$,and $a\gamma y \sim_{B_r} a\gamma b$ by hypothesis. Thus, $x\gamma y \sim_{B_r} a\gamma b \Rightarrow z = a\gamma b \in [x\gamma y]_{B_r}$. □

DEFINITION 3.4. Let $S \subseteq \mathcal{O}$ and $\Gamma \subseteq \acute{\mathcal{O}}$ where $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r)$ and $(\acute{\mathcal{O}}, \mathcal{F}, \sim_{B_r}, N_r)$ are two different weak near approximation spaces, $B_r \subseteq \mathcal{F}$ where $r \leqslant \mid B \mid$ and $B \subseteq \mathcal{F}$, $\sim_{B_r}$ be a indiscernibility relation on $\mathcal{O} - \acute{\mathcal{O}}$. Then, $\sim_{B_r}$ is called a complete congruence indiscernibility relation on Γ-nearness semigroup S, if $[x]_{B_r}\gamma[y]_{B_r} = [x\gamma y]_{B_r}$ for all $x, y \in S$ and $\gamma \in \Gamma$.

S be a Γ-nearness semigroup. Let $X\Gamma Y = \{x\gamma y \mid x \in X, \gamma \in \Gamma, \text{ and } y \in Y\}$, where subsets X and Y of S.

LEMMA 3.1. *Let $S \subseteq \mathcal{O}$ and $\Gamma \subseteq \acute{\mathcal{O}}$ where $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r)$ and $(\acute{\mathcal{O}}, \mathcal{F}, \sim_{B_r}, N_r)$ are two different weak near approximation spaces, S be a Γ-nearness semigroup. The following properties hold:*

i) If $X, Y \subseteq S$, then $(N_r(B)^* X)\Gamma(N_r(B)^* Y) \subseteq N_r(B)^* (X\Gamma Y)$.

ii) If $X, Y \subseteq S$, and $\sim_{B_r}$ is a complete congruence indiscernibility relation on S, then $(N_r(B)_* X)\Gamma(N_r(B)_* Y) \subseteq N_r(B)_* (X\Gamma Y)$.

PROOF. *i)* Let $x \in (N_r(B)^* X)\Gamma(N_r(B)^* Y)$. We have

$$x = a\gamma b;\ a \in N_r(B)^* X, b \in N_r(B)^* Y,$$

and $\gamma \in \Gamma$. $a \in N_r(B)^* X \Rightarrow [a]_{B_r} \cap X \neq \varnothing \Rightarrow \exists y \in [a]_{B_r} \cap X \Rightarrow y \in [a]_{B_r}$ and $y \in X$. Likewise, $b \in N_r(B)^* Y \Rightarrow [b]_{B_r} \cap Y \neq \varnothing \Rightarrow \exists z \in [b]_{B_r} \cap Y \Rightarrow z \in [b]_{B_r}$ and $z \in Y$. Since $w = y\gamma z \in [a]_{B_r}\gamma[b]_{B_r} \subseteq [a\gamma b]_{B_r}$, we get $w \in [a\gamma b]_{B_r}$ and $w \in X\Gamma Y$. Thus, $w \in [a\gamma b]_{B_r} \cap X\Gamma Y \Rightarrow [a\gamma b]_{B_r} \cap (X\Gamma Y) \neq \varnothing$, and so $a\gamma b = x \in N_r(B)^*(X\Gamma Y)$.

ii) Let $x \in (N_r(B)_* X)\Gamma(N_r(B)_* Y)$. We have $x = a\gamma b$; $a \in N_r(B)_* X, b \in N_r(B)_* Y$, and $\gamma \in \Gamma$. In this case, $a \in N_r(B)_* X \Rightarrow [a]_{B_r} \subseteq X$ and $b \in N_r(B)_* Y \Rightarrow [b]_{B_r} \subseteq Y$, so, we obtain $[a]_{B_r}\gamma[a]_{B_r} \subseteq X\Gamma Y$. On the other hand, since $[a\gamma b]_{B_r} = [a]_{B_r}\gamma[b]_{B_r} \subseteq X\Gamma Y$. Thus, $[a\gamma b]_{B_r} \subseteq X\Gamma Y$, and so $a\gamma b = x \in N_r(B)_*(X\Gamma Y)$. □

Definition 3.5. Let S be a Γ-semigroup on $\mathcal{O} - \acute{\mathcal{O}}$ where $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r)$ and $(\acute{\mathcal{O}}, \mathcal{F}, \sim_{B_r}, N_r)$ are two different weak near approximation spaces, and A a non-empty subset of S.

i) A is called a sub Γ-semigroup of S if $A\Gamma A \subseteq N_r(B)^* A$.

ii) A is called a upper-near sub Γ-semigroup of S if $(N_r(B)^* A)\Gamma(N_r(B)^* A) \subseteq N_r(B)^* A$.

Now, let's give an example to the sub Γ-nearness semigroup and the upper-near sub Γ-nearness semigroup.

Example 3.3. Let $\mathcal{O} = \{a, b, c, d, e, f, g, h\}$ be a set of perceptual objects where

$$\begin{aligned} a &= \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, c = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}, d = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \\ e &= \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, g = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, h = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \end{aligned}$$

for $U = \{ [a_{ij}]_{1x3} \mid a_{ij} \in \mathbb{Z}_2\}$, $\acute{\mathcal{O}} = \{\alpha, \beta, \gamma, \theta, \lambda, \mu, \delta, \sigma\}$ be a set of perceptual objects where

$$\alpha = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \beta = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \gamma = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \theta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$
$$\lambda = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \mu = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \delta = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \sigma = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

for $U = \{ [a_{ij}]_{3x1} \mid a_{ij} \in \mathbb{Z}_2\}, r = 1$, $B = \{\varphi_1, \varphi_2, \varphi_3\} \subseteq \mathcal{F}$ be a set of probe functions, and $S = \{a, g\} \subset \mathcal{O}$, $A = \{g\} \subseteq S$ $\Gamma = \{\alpha, \beta\} \subset \acute{\mathcal{O}}$. Values of the probe functions

$$\begin{aligned} \varphi_1 &: \mathcal{O} \to V_1 = \{\alpha_1, \alpha_2, \alpha_3\}, \\ \varphi_2 &: \mathcal{O} \to V_2 = \{\alpha_1, \alpha_3, \alpha_4\}, \\ \varphi_3 &: \mathcal{O} \to V_3 = \{\alpha_2, \alpha_3, \alpha_4\} \end{aligned}$$

are given in *Table* 5.

	a	b	c	d	e	f	g	h
φ_1	α_1	α_1	α_1	α_2	α_3	α_3	α_3	α_2
φ_2	α_4	α_1	α_1	α_3	α_3	α_3	α_4	α_3
φ_3	α_4	α_2	α_4	α_2	α_3	α_3	α_4	α_3

Table 5

In this case,

$$\begin{aligned} [a]_{\varphi_1} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_1(a) = \alpha_1\} = \{a, b, c\} \\ &= [b]_{\varphi_1} = [c]_{\varphi_1}, \\ [d]_{\varphi_1} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_1(d) = \alpha_2\} = \{d, h\} \\ &= [h]_{\varphi_1}, \\ [e]_{\varphi_1} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_1(e) = \alpha_3\} = \{e, f, g\} \\ &= [f]_{\varphi_1} = [g]_{\varphi_1}. \end{aligned}$$

Then, we get that $\xi_{\varphi_1} = \left\{ [a]_{\varphi_1}, [d]_{\varphi_1}, [e]_{\varphi_1} \right\}$.

$$\begin{aligned} [a]_{\varphi_2} &= \{x \in \mathcal{O} \mid \varphi_2(x) = \varphi_2(a) = \alpha_4\} = \{a, g\} \\ &= [g]_{\varphi_2}, \\ [b]_{\varphi_2} &= \{x \in \mathcal{O} \mid \varphi_2(x) = \varphi_2(b) = \alpha_1\} = \{b, c\} \\ &= [c]_{\varphi_2}, \\ [d]_{\varphi_2} &= \{x \in \mathcal{O} \mid \varphi_2(x) = \varphi_2(d) = \alpha_3\} = \{d, e, f, h\} \\ &= [e]_{\varphi_2} = [f]_{\varphi_2} = [h]_{\varphi_2}, \end{aligned}$$

We have $\xi_{\varphi_2} = \left\{ [a]_{\varphi_2}, [b]_{\varphi_2}, [d]_{\varphi_2} \right\}$.

$$\begin{aligned} [a]_{\varphi_3} &= \{x \in \mathcal{O} \mid \varphi_3(x) = \varphi_3(a) = \alpha_4\} = \{a, c, g\} \\ &= [c]_{\varphi_3} = [g]_{\varphi_3}, \\ [b]_{\varphi_3} &= \{x \in \mathcal{O} \mid \varphi_3(x) = \varphi_3(b) = \alpha_1\} = \{b, d\} \\ &= [d]_{\varphi_3}, \\ [e]_{\varphi_3} &= \{x \in \mathcal{O} \mid \varphi_3(x) = \varphi_3(e) = \alpha_3\} = \{e, f, h\} \\ &= [f]_{\varphi_3} = [h]_{\varphi_3}. \end{aligned}$$

From hence, we obtain that $\xi_{\varphi_3} = \left\{ [a]_{\varphi_3}, [b]_{\varphi_3}, [e]_{\varphi_3} \right\}$. Therefore, for $r = 1$, a set of partitions of $\mathcal{O}$ is $N_r(B) = \{\xi_{\varphi_1}, \xi_{\varphi_2}, \xi_{\varphi_3}\}$. Then, we can write

$$\begin{aligned} N_1(B)^* S &= \bigcup_{[x]_{\varphi_i} \cap S \neq \varnothing} [x]_{\varphi_i} \\ &= \{a, b, c, e, f, h\}. \end{aligned}$$

Considering the following tables of operations:

α	a	g
a	f	f
g	a	g

β	a	g
a	a	g
g	f	f

S is a Γ-semigroup on the weak near approximation space $\mathcal{O} - \acute{\mathcal{O}}$. Furthermore,

$$N_1(B)^* A = \bigcup_{[x]_{\varphi_i} \cap A \neq \varnothing} [x]_{\varphi_i} = \{a, c, e, f, g\}.$$

Since $A\Gamma A \subseteq N_r(B)^* A$, A is a sub Γ-semigroup of S. In addition to A is a upper-near sub Γ-semigroup of S, for $(N_r(B)^* A)\Gamma(N_r(B)^* A) \subseteq N_r(B)^* A$.

Example 3.4. Let $\mathcal{O} = \{a, \beta, \gamma, b, c, d, e, f, g, h, i, j\}$ be a set of perceptual objects, $r = 2$, and $B = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\} \subseteq \mathcal{F}$ be a set of probe functions. Values of the probe functions

$$\varphi_1 : \mathcal{O} \to V_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\},$$
$$\varphi_2 : \mathcal{O} \to V_2 = \{\alpha_3, \alpha_4, \alpha_5\},$$
$$\varphi_3 : \mathcal{O} \to V_3 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5\},$$
$$\varphi_4 : \mathcal{O} \to V_4 = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5\}$$

are given in *Table* 6.

	a	b	c	d	e	f	g	h	i	j
φ_1	α_3	α_1	α_2	α_2	α_5	α_3	α_4	α_4	α_4	α_3
φ_2	α_3	α_3	α_4	α_3	α_5	α_4	α_5	α_3	α_5	α_5
φ_3	α_2	α_1	α_3	α_5	α_1	α_2	α_3	α_5	α_1	α_1
φ_4	α_2	α_1	α_4	α_5	α_5	α_2	α_4	α_4	α_4	α_4

Table 6

In this case,

$$[a]_{\{\varphi_1,\varphi_2\}} = \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_2(x) = \varphi_1(a) = \varphi_2(a) = \alpha_3\} = \{a\}$$
$$[e]_{\{\varphi_1,\varphi_2\}} = \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_2(x) = \varphi_1(e) = \varphi_2(e) = \alpha_5\} = \{e\}.$$

Then, we have that $\xi_{\{\varphi_1,\varphi_2\}} = \left\{[a]_{\{\varphi_1,\varphi_2\}}, [e]_{\{\varphi_1,\varphi_2\}}\right\}$.

$$[b]_{\{\varphi_1,\varphi_3\}} = \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_3(x) = \varphi_1(b) = \varphi_2(b) = \alpha_1\} = \{b\}.$$

We get $\xi_{\{\varphi_1,\varphi_3\}} = \left\{[b]_{\{\varphi_1,\varphi_3\}}\right\}$.

$$\begin{aligned}
[b]_{\{\varphi_1,\varphi_4\}} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_4(x) = \varphi_1(b) = \varphi_4(b) = \alpha_1\} = \{b\},\\
[e]_{\{\varphi_1,\varphi_4\}} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_4(x) = \varphi_1(e) = \varphi_4(e) = \alpha_5\} = \{e\},\\
[g]_{\{\varphi_1,\varphi_4\}} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_4(x) = \varphi_1(g) = \varphi_4(g) = \alpha_4\} = \{g, h, i\}\\
&= [h]_{\{\varphi_1,\varphi_4\}} = [i]_{\{\varphi_1,\varphi_4\}}.
\end{aligned}$$

Thus, $\xi_{\{\varphi_1,\varphi_4\}} = \left\{[b]_{\{\varphi_1,\varphi_3\}}, [e]_{\{\varphi_1,\varphi_4\}}, [g]_{\{\varphi_1,\varphi_4\}}\right\}$.

$$\begin{aligned}
[c]_{\{\varphi_2,\varphi_4\}} &= \{x \in \mathcal{O} \mid \varphi_2(x) = \varphi_4(x) = \varphi_2(c) = \varphi_4(c) = \alpha_4\} = \{c\},\\
[e]_{\{\varphi_2,\varphi_4\}} &= \{x \in \mathcal{O} \mid \varphi_2(x) = \varphi_4(x) = \varphi_2(e) = \varphi_4(e) = \alpha_5\} = \{e\}.
\end{aligned}$$

We get that $\xi_{\{\varphi_2,\varphi_4\}} = \left\{[c]_{\{\varphi_2,\varphi_4\}}, [e]_{\{\varphi_2,\varphi_4\}}\right\}$.

$$\begin{aligned}
[a]_{\{\varphi_3,\varphi_4\}} &= \{x \in \mathcal{O} \mid \varphi_3(x) = \varphi_4(x) = \varphi_3(a) = \varphi_4(a) = \alpha_2\} = \{a, f\}\\
&= [f]_{\{\varphi_3,\varphi_4\}},\\
[b]_{\{\varphi_2,\varphi_4\}} &= \{x \in \mathcal{O} \mid \varphi_3(x) = \varphi_4(x) = \varphi_3(b) = \varphi_4(b) = \alpha_1\} = \{b\},\\
[d]_{\{\varphi_2,\varphi_4\}} &= \{x \in \mathcal{O} \mid \varphi_3(x) = \varphi_4(x) = \varphi_3(d) = \varphi_4(d) = \alpha_5\} = \{d\}.
\end{aligned}$$

From hence, we obtain that $\xi_{\{\varphi_3,\varphi_4\}} = \left\{[a]_{\{\varphi_3,\varphi_4\}}, [b]_{\{\varphi_2,\varphi_4\}}, [d]_{\{\varphi_2,\varphi_4\}}\right\}$. Therefore, for $r = 2$, a set of partitions of $\mathcal{O}$ is

$$N_r(B) = \left\{\xi_{\{\varphi_1,\varphi_2\}}, \xi_{\{\varphi_1,\varphi_3\}}, \xi_{\{\varphi_1,\varphi_4\}}, \xi_{\{\varphi_2,\varphi_4\}}, \xi_{\{\varphi_3,\varphi_4\}}\right\}.$$

If $S = \{e, f, g\}$, then we can write

$$\begin{aligned}
N_2(B)^* S &= \bigcup_{[x]_{\{\varphi_i,\varphi_j\}} \cap S \neq \varnothing} [x]_{\{\varphi_i,\varphi_j\}}\\
&= [e]_{\{\varphi_1,\varphi_2\}} \cup [e]_{\{\varphi_1,\varphi_4\}} \cup [g]_{\{\varphi_1,\varphi_4\}} \cup [e]_{\{\varphi_2,\varphi_4\}} \cup [a]_{\{\varphi_3,\varphi_4\}}\\
&= \{e\} \cup \{e\} \cup \{g, h, i\} \cup \{e\} \cup \{a, f\}\\
&= \{a, e, f, g, h, i\}
\end{aligned}$$

and also

$$\begin{aligned}
N_2(B)^* \left(N_2(B)^* S\right) &= \bigcup_{[x]_{\{\varphi_i,\varphi_j\}} \cap N_2(B)^* S \neq \varnothing} [x]_{\{\varphi_i,\varphi_j\}}\\
&= \{a\} \cup \{e\} \cup \{e\} \cup \{g, h, i\} \cup \{e\} \cup \{a, f\}\\
&= \{a, e, f, g, h, i\}.
\end{aligned}$$

Thus, $N_2(B)^* \left(N_2(B)^* S\right) = N_2(B)^* S$ is obtained.

Theorem 3.2. *Let S be a Γ-nearness semigroup where $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r)$ and $(\acute{\mathcal{O}}, \mathcal{F}, \sim_{B_r}, N_r)$ are two different weak near approximation spaces The following properties hold:*

i) If $\varnothing \neq A \subseteq S$, and $A\Gamma A \subseteq A$, then A is a upper-near sub Γ-semigroup of S.

ii) If A is a sub Γ-semigroup of S, and $N_r(B)^*(N_r(B)^* A) = N_r(B)^* A$, then A is a upper-near sub Γ-semigroup of S.

PROOF. *i)* Let $\varnothing \neq A \subseteq S$, and $A\Gamma A \subseteq A$. From Lemma 3.1.(i), we have

$$(N_r(B)^* A)\Gamma(N_r(B)^* A) \subseteq N_r(B)^*(A\Gamma A).$$

On the other hand, from Theorem 3.1.(v), we have that

$$N_r(B)^*(A\Gamma A) \subseteq N_r(B)^* A.$$

In this case,

$$(N_r(B)^* A)\Gamma(N_r(B)^* A) \subseteq N_r(B)^* A$$

is obtained. Hence, A is a upper-near sub Γ-semigroup of S.

ii) Since A is a sub Γ-semigroup of S, $A\Gamma A \subseteq N_r(B)^* A$. Thus, we have $N_r(B)^*(A\Gamma A) \subseteq N_r(B)^*(N_r(B)^* A) = N_r(B)^* A$ by Theorem 3.1.(v). and hypothesis. Combining this and Lemma 3.1.(i), we conclude that

$$(N_r(B)^* A)\Gamma(N_r(B)^* A) \subseteq N_r(B)^* A.$$

Hence, A is a upper-near sub $\Gamma-$semigroup of S. □

DEFINITION 3.6. Let S be a Γ-semigroup on $\mathcal{O} - \acute{\mathcal{O}}$ where $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r)$ and $(\acute{\mathcal{O}}, \mathcal{F}, \sim_{B_r}, N_r)$ are two different weak near approximation spaces, and A a sub Γ-semigroup of S.

i) A is called a Γ-right(left) ideals of S if $A\Gamma S \subseteq N_r(B)^* A$ $(S\Gamma A \subseteq N_r(B)^* A)$.

ii) A is called a upper-near Γ-right(left) ideals of S if $(N_r(B)^* A)\Gamma S \subseteq N_r(B)^* A$ $(S\Gamma(N_r(B)^* A) \subseteq N_r(B)^* A)$.

EXAMPLE 3.5. Let $\mathcal{O} = \{a, b, c, d, e, f, g, h\}$ be a set of perceptual objects where

$$a = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, c = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}, d = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix},$$
$$e = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, g = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, h = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

for $U = \{\ [a_{ij}]_{1x3} \mid a_{ij} \in \mathbb{Z}_2\}$, $\acute{\mathcal{O}} = \{\alpha, \beta, \gamma, \theta, \lambda, \mu, \delta, \sigma\}$ be a set of perceptual objects where

$$\alpha = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \beta = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \gamma = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \theta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$
$$\lambda = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \mu = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \delta = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \sigma = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

for $U = \{\ [a_{ij}]_{3x1} \mid a_{ij} \in \mathbb{Z}_2\}, r = 1$, $B = \{\varphi_1, \varphi_2, \varphi_3\} \subseteq \mathcal{F}$ be a set of probe functions, and $S = \{b, h\} \subset \mathcal{O}$, $A = \{b\} \subseteq S$ $\Gamma = \{\gamma, \delta\} \subset \acute{\mathcal{O}}$. Values of the probe functions

$$\varphi_1 : \mathcal{O} \to V_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\},$$
$$\varphi_2 : \mathcal{O} \to V_2 = \{\alpha_1, \alpha_3, \alpha_4\},$$
$$\varphi_3 : \mathcal{O} \to V_3 = \{\alpha_2, \alpha_3, \alpha_4\}$$

are given in *Table* 7.

	a	b	c	d	e	f	g	h
φ_1	α_1	α_4	α_1	α_2	α_3	α_4	α_3	α_4
φ_2	α_4	α_4	α_1	α_1	α_3	α_4	α_3	α_1
φ_3	α_4	α_4	α_2	α_2	α_3	α_3	α_2	α_3

Table 7

Then,

$$
\begin{aligned}
[a]_{\varphi_1} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_1(a) = \alpha_1\} = \{a, c\} \\
&= [c]_{\varphi_1}, \\
[b]_{\varphi_1} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_1(b) = \alpha_4\} = \{b, f, h\} \\
&= [f]_{\varphi_1} = [h]_{\varphi_1}, \\
[d]_{\varphi_1} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_1(d) = \alpha_2\} = \{d\} \\
[e]_{\varphi_1} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_1(e) = \alpha_3\} = \{e, g\} \\
&= [g]_{\varphi_1}.
\end{aligned}
$$

Then, we get that $\xi_{\varphi_1} = \left\{[a]_{\varphi_1}, [b]_{\varphi_1}, [d]_{\varphi_1}, [e]_{\varphi_1}\right\}$.

$$
\begin{aligned}
[a]_{\varphi_2} &= \{x \in \mathcal{O} \mid \varphi_2(x) = \varphi_2(a) = \alpha_4\} = \{a, b, f\} \\
&= [b]_{\varphi_2} = [f]_{\varphi_2}, \\
[c]_{\varphi_2} &= \{x \in \mathcal{O} \mid \varphi_2(x) = \varphi_2(c) = \alpha_1\} = \{c, d, h\} \\
&= [c]_{\varphi_2}, \\
[e]_{\varphi_2} &= \{x \in \mathcal{O} \mid \varphi_2(x) = \varphi_2(e) = \alpha_3\} = \{e, g\} \\
&= [g]_{\varphi_2},
\end{aligned}
$$

We have $\xi_{\varphi_2} = \left\{[a]_{\varphi_2}, [c]_{\varphi_2}, [e]_{\varphi_2}\right\}$.

$$
\begin{aligned}
[a]_{\varphi_3} &= \{x \in \mathcal{O} \mid \varphi_3(x) = \varphi_3(a) = \alpha_4\} = \{a, b\} \\
&= [b]_{\varphi_3}, \\
[c]_{\varphi_3} &= \{x \in \mathcal{O} \mid \varphi_3(x) = \varphi_3(c) = \alpha_2\} = \{c, d, g\} \\
&= [d]_{\varphi_3} = [g]_{\varphi_3}, \\
[e]_{\varphi_3} &= \{x \in \mathcal{O} \mid \varphi_3(x) = \varphi_3(e) = \alpha_3\} = \{e, f, h\} \\
&= [f]_{\varphi_3} = [h]_{\varphi_3}.
\end{aligned}
$$

From hence, we obtain that $\xi_{\varphi_3} = \left\{[a]_{\varphi_3}, [c]_{\varphi_3}, [e]_{\varphi_3}\right\}$. Therefore, for $r = 1$, a set of partitions of $\mathcal{O}$ is $N_r(B) = \{\xi_{\varphi_1}, \xi_{\varphi_2}, \xi_{\varphi_3}\}$. Then, we can write

$$N_1(B)^* S = \bigcup_{[x]_{\varphi_i} \cap S \neq \varnothing} [x]_{\varphi_i} = \{a, b, c, d, e, f, h\}.$$

Considering the following tables of operations:

γ	b	h
b	f	f
h	b	h

δ	b	h
b	b	h
h	b	h

S is a Γ-semigroup on the weak near approximation space $\mathcal{O} - \acute{\mathcal{O}}$. Furthermore,

$$N_1(B)^* A = \bigcup_{[x]_{\varphi_i} \cap A \neq \varnothing} [x]_{\varphi_i} = \{a, b, f, h\}.$$

Since $A\Gamma A \subseteq N_r(B)^* A$, A is a sub Γ-semigroup of S. In addition to A is a Γ-ideals of S. Because of $(N_r(B)^* A)\Gamma(N_r(B)^* A) \subseteq N_r(B)^* A$, A is a upper-near sub Γ-semigroup of S. Additionally, $(N_r(B)^* A)\Gamma S \subseteq N_r(B)^* A$ $(S\Gamma(N_r(B)^* A) \subseteq N_r(B)^* A)$ is obtained, and so A is a upper-near Γ-right (left) ideal of S, i.e. , A is a upper-near Γ-ideal of S.

Theorem 3.3. *Let S be a Γ-nearness semigroup where $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r)$ and $(\acute{\mathcal{O}}, \mathcal{F}, \sim_{B_r}, N_r)$ are two different weak near approximation spaces The following properties hold:*

i) If $\varnothing \neq A \subseteq S$, and $A\Gamma S \subseteq A$ $(S\Gamma A \subseteq A)$, then A is a upper-near Γ-right (left) ideal of S.

ii) If A is a Γ-right (left) ideal of S, and $N_r(B)^*(N_r(B)^* A) = N_r(B)^* A$, then A is a upper-near Γ-right (left) ideal of S.

Proof. It is done similar to the proof of Theorem 3.2. □

Definition 3.7. Let S be a Γ-semigroup on $\mathcal{O} - \acute{\mathcal{O}}$ where $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r)$ and $(\acute{\mathcal{O}}, \mathcal{F}, \sim_{B_r}, N_r)$ are two different weak near approximation spaces, and A a sub Γ-semigroup of S.

i) A is called a Γ-bi-ideals of S if $A\Gamma S\Gamma A \subseteq N_r(B)^* A$.

ii) A is called a upper-near Γ-bi-ideals of S if $(N_r(B)^* A)\Gamma S\Gamma(N_r(B)^* A) \subseteq N_r(B)^* A$.

Example 3.6. Let $\mathcal{O} = \{a, b, c, d, e, f, g, h\}$ be a set of perceptual objects where

$$a = [\,0 \;\; 1 \;\; 1\,], b = [\,1 \;\; 0 \;\; 0\,], c = [\,1 \;\; 1 \;\; 0\,], d = [\,0 \;\; 1 \;\; 0\,],$$
$$e = [\,1 \;\; 0 \;\; 1\,], f = [\,0 \;\; 0 \;\; 0\,], g = [\,0 \;\; 0 \;\; 1\,], h = [\,1 \;\; 1 \;\; 1\,]$$

for $U = \{\ [a_{ij}]_{1x3} \mid a_{ij} \in \mathbb{Z}_2\}$, $\acute{\mathcal{O}} = \{\alpha, \beta, \gamma, \theta, \lambda, \mu, \delta, \sigma\}$ be a set of perceptual objects where

$$\alpha = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \beta = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \gamma = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \theta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$
$$\lambda = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \mu = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \delta = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \sigma = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

for $U = \{\ [a_{ij}]_{3x1} \mid a_{ij} \in \mathbb{Z}_2\}, r = 1$, $B = \{\varphi_1, \varphi_2, \varphi_3\} \subseteq \mathcal{F}$ be a set of probe functions, and $S = \{b, h\} \subset \mathcal{O}$, $A = \{b\} \subseteq S$ $\Gamma = \{\gamma, \delta\} \subset \acute{\mathcal{O}}$. Values of the probe functions

$$\varphi_1 : \mathcal{O} \to V_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\},$$
$$\varphi_2 : \mathcal{O} \to V_2 = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5\},$$
$$\varphi_3 : \mathcal{O} \to V_3 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$$

are given in *Table* 8.

	a	b	c	d	e	f	g	h
φ_1	α_1	α_4	α_1	α_2	α_3	α_4	α_3	α_4
φ_2	α_5	α_4	α_1	α_1	α_3	α_4	α_5	α_1
φ_3	α_4	α_5	α_2	α_2	α_3	α_3	α_2	α_3

Table 8

Then,

$$\begin{aligned} [a]_{\varphi_1} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_1(a) = \alpha_1\} = \{a, c\} \\ &= [c]_{\varphi_1}, \\ [b]_{\varphi_1} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_1(b) = \alpha_4\} = \{b, f, h\} \\ &= [f]_{\varphi_1} = [h]_{\varphi_1}, \\ [d]_{\varphi_1} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_1(d) = \alpha_2\} = \{d\} \\ [e]_{\varphi_1} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_1(e) = \alpha_3\} = \{e, g\} \\ &= [g]_{\varphi_1}. \end{aligned}$$

Then, we get that $\xi_{\varphi_1} = \left\{[a]_{\varphi_1}, [b]_{\varphi_1}, [d]_{\varphi_1}, [e]_{\varphi_1}\right\}$.

$$\begin{aligned}
[a]_{\varphi_2} &= \{x \in \mathcal{O} \mid \varphi_2(x) = \varphi_2(a) = \alpha_5\} = \{a, g\} \\
&= [g]_{\varphi_2}, \\
[b]_{\varphi_2} &= \{x \in \mathcal{O} \mid \varphi_2(x) = \varphi_2(b) = \alpha_4\} = \{b, f\} \\
&= [f]_{\varphi_2}, \\
[c]_{\varphi_2} &= \{x \in \mathcal{O} \mid \varphi_2(x) = \varphi_2(c) = \alpha_1\} = \{c, d, h\} \\
&= [d]_{\varphi_2} = [h]_{\varphi_2}, \\
[e]_{\varphi_2} &= \{x \in \mathcal{O} \mid \varphi_2(x) = \varphi_2(e) = \alpha_3\} = \{e\}.
\end{aligned}$$

We have $\xi_{\varphi_2} = \left\{ [a]_{\varphi_2}, [a]_{\varphi_2}, [c]_{\varphi_2}, [e]_{\varphi_2} \right\}$.

$$\begin{aligned}
[a]_{\varphi_3} &= \{x \in \mathcal{O} \mid \varphi_3(x) = \varphi_3(a) = \alpha_4\} = \{a\} \\
[b]_{\varphi_3} &= \{x \in \mathcal{O} \mid \varphi_3(x) = \varphi_3(b) = \alpha_5\} = \{b\} \\
[c]_{\varphi_3} &= \{x \in \mathcal{O} \mid \varphi_3(x) = \varphi_3(c) = \alpha_2\} = \{c, d, g\} \\
&= [d]_{\varphi_3} = [d]_{\varphi_3}, \\
[e]_{\varphi_3} &= \{x \in \mathcal{O} \mid \varphi_3(x) = \varphi_3(e) = \alpha_3\} = \{e, f, h\} \\
&= [f]_{\varphi_3} = [h]_{\varphi_3}.
\end{aligned}$$

From hence, we obtain that $\xi_{\varphi_3} = \left\{ [a]_{\varphi_3}, [b]_{\varphi_3}, [c]_{\varphi_3}, [e]_{\varphi_3} \right\}$. Therefore, for $r = 1$, a set of partitions of $\mathcal{O}$ is $N_r(B) = \{\xi_{\varphi_1}, \xi_{\varphi_2}, \xi_{\varphi_3}\}$. Then, we can write

$$\begin{aligned}
N_1(B)^* S &= \bigcup_{[x]_{\varphi_i} \cap S \neq \varnothing} [x]_{\varphi_i} \\
&= \{b, c, d, e, f, h\}.
\end{aligned}$$

Considering the following tables of operations:

γ	b	h
b	f	f
h	b	h

δ	b	h
b	b	h
h	b	h

S is a Γ-semigroup on the weak near approximation space $\mathcal{O} - \acute{\mathcal{O}}$. Furthermore,

$$\begin{aligned}
N_1(B)^* A &= \bigcup_{[x]_{\varphi_i} \cap A \neq \varnothing} [x]_{\varphi_i} \\
&= \{b, f, h\}.
\end{aligned}$$

Since $A\Gamma S\Gamma A \subseteq N_r(B)^* A$, A is a sub Γ-semigroup of S.In addition, A is a Γ-bi-ideals of S. Because of $(N_r(B)^* A)\Gamma(N_r(B)^* A) \subseteq N_r(B)^* A$, A is a upper-near sub Γ-semigroup of S. Additionally, since $(N_r(B)^* A)\Gamma S\Gamma(N_r(B)^* A) \subseteq N_r(B)^* A$, A is a upper-near Γ-bi-ideal of S.

THEOREM 3.4. *Let S be a Γ-semigroup on $\mathcal{O} - \acute{\mathcal{O}}$ where $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r)$ and $(\acute{\mathcal{O}}, \mathcal{F}, \sim_{B_r}, N_r)$ are two different weak near approximation spaces The following properties hold:*

i) If $\varnothing \neq A \subseteq S$, and $A\Gamma S\Gamma A \subseteq A$, then A is a upper-near Γ-bi-ideal of S.

ii) If A is a Γ-bi-ideal of S, and $N_r(B)^*(N_r(B)^* S) = N_r(B)^* S$, then A is a upper-near Γ-bi-ideal of S.

PROOF. *i)* Let $\varnothing \neq A \subseteq S$, and $A\Gamma S\Gamma A \subseteq A$. From Theorem 3.1.(i), we have

$$(N_r(B)^* A)\Gamma S\Gamma(N_r(B)^* A) \subseteq (N_r(B)^* A)\Gamma(N_r(B)^* S)\Gamma(N_r(B)^* S).$$

From Theorem 3.1.(v) we have $N_r(B)^*(A\Gamma S\Gamma A) \subseteq N_r(B)^* A$ by $A\Gamma S\Gamma A \subseteq A$. On the other hand,

$$(N_r(B)^* A)\Gamma(N_r(B)^* S)\Gamma(N_r(B)^* S) \subseteq N_r(B)^*(A\Gamma S\Gamma A)$$

is obtained by Lemma 3.1.(i). Hence, we get that $(N_r(B)^* A)\Gamma S\Gamma(N_r(B)^* A) \subseteq N_r(B)^* A$, i.e. , A is a upper-near Γ-bi-ideal of S.

ii) Since A is a Γ-bi-ideal of S, $A\Gamma S\Gamma A \subseteq N_r(B)^* A$. Thus, we have

$$N_r(B)^*(A\Gamma S\Gamma A) \subseteq N_r(B)^*(N_r(B)^* A) = N_r(B)^* A$$

by Theorem 3.1.(v) and hypothesis. Also,

$$(N_r(B)^* A)\Gamma S\Gamma(N_r(B)^* A) \subseteq (N_r(B)^* A)\Gamma(N_r(B)^* S)\Gamma(N_r(B)^* S)$$

by Theorem 3.1.(i). Combining this and Lemma 3.1.(i), we conclude that

$$(N_r(B)^* A)\Gamma S\Gamma(N_r(B)^* A) \subseteq N_r(B)^* A.$$

Hence, A is a upper-near Γ-bi-ideal of S. □

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