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ZEROID, REGULAR AND IDEMPOTENT ELEMENTS IN Γ-SEMIRINGS

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Abstract. In this paper, we introduce the notion of zeroid, regular and idempotent elements in $\Gamma-$semiring. We study the properties of zeroid, regular and idempotent elements in $\Gamma-$semiring and we prove that,if M is a totally ordered regular $\Gamma-$semiring with unity element, in which $\Gamma-$semigroup M is negatively ordered then M is a commutative ordered $\Gamma-$semiring.

1. Introduction

In 1995, the notion of $\Gamma-$semiring was introduced by Murali Krishna rao [**7, 8, 9, 10**] not only generalizes the notion of semiring and $\Gamma-$ring but also the notion of ternary semiring. Semiring, the algebraic structure which is a common generalization of rings and distributive lattices, was first introduced by American mathematician Vandiver [**15**] in 1934, but non trivial examples of semirings had appeared in the earlier studies on the theory of commutative ideals of rings by German mathematician Richard Dedekind in 19th century. Semiring is a universal algebra with two binary operations called addition and multiplication where one of them is distributive over the other. Bounded distributive lattices are commutative semirings which are both additively and multiplicatively idempotent. A natural example of semiring which is not a ring, is the set of all natural numbers under usual addition and multiplication of numbers.

Semiring, as the basic algebraic structure, was used in the areas of theoretical computer science as well as in the solutions of graph theory and optimization theory and in particular for studying automata, coding theory and formal languages. Semiring theory has many applications in other branches of mathematics.Many

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semirings have order structure in addition to algebraic structure.The notion of $\Gamma-$ring was introduced by Nobusawa [**13**] as a generalization of ring in 1964. Sen [**14**] introduced the notion of $\Gamma-$semigroup in 1981. The notion of ternary algebraic system was introduced by Lehmer [**3**] in 1932, Lister [**4**] introduced ternary ring. Dutta & Sardar [**2**] introduced the notion of operator semirings of $\Gamma-$semiring. Murali Krishna Rao and Venkateswarlu [**11**] studied regular $\Gamma-$incline and field $\Gamma-$semiring. The set of all negative integers $\mathbb{Z}$ is not a semiring with respect to usual addition and multiplication but $\mathbb{Z}$ forms a $\Gamma-$semiring where $\Gamma = \mathbb{Z}$. The important reason for the development of $\Gamma-$semiring is a generalization of results of rings, $\Gamma-$rings, semirings, semigroups and ternary semirings. Von Neumann [**12**] introduced the concept of regular elements in a ring. Meenakshi and Anbalagan [**5**] studied regular elements in an incline and proved that regular commutative incline is a distributive lattice.Meenakshi et.al [**6**] studied ideals in incline. In this paper, we introduce the notion of zeroid, regular and idempotent elements in $\Gamma-$semiring. We study the properties of zeroid, regular and idempotent elements.

2. Preliminaries

In this section we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1. [1] A set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) will be called semiring provided

(i) addition is a commutative operation.
(ii) multiplication distributes over addition both from the left and from the right.
(iii) there exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$.

Definition 2.2. Let M and Γ be two non- empty sets. Then M is called a $\Gamma-$semigroup if it satisfies

(i) $x\alpha y \in M$,
(ii) $x\alpha(y\beta z) = (x\alpha y)\beta z$, for all $x, y, z \in M, \alpha, \beta \in \Gamma$.

Definition 2.3. Let $(M, +)$ and $(\Gamma, +)$ be semigroups. If there exists a mapping $M \times \Gamma \times M \to M$ (images to be denoted by $x\alpha y, x, y \in M, \alpha \in \Gamma$) satisfying the following axioms for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$,

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$,
(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$,
(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$
(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$,

then M is called a $\Gamma-$semiring.

Definition 2.4. A $\Gamma-$semiring M is said to have zero element if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = x\alpha 0 = 0,$ for all $x \in M$.

Example 2.1. Every semiring M is a $\Gamma-$semiring with $\Gamma = M$ and ternary operation is defined as the usual semiring multiplication

Example 2.2. Let M be the additive semigroup of all $m \times n$ matrices over the set of non negative rational numbers and Γ be the additive semigroup of all $n \times m$ matrices over the set of non negative integers, then with respect to usual matrix multiplication M is a $\Gamma-$semiring.

Definition 2.5. Let M be a $\Gamma-$semiring. An element $1 \in M$ is said to be unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.

Definition 2.6. A $\Gamma-$semiring M is said to be commutative $\Gamma-$semiring if $x\alpha y = y\alpha x$, for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 2.7. A $\Gamma-$semiring M is said to be simple $\Gamma-$semiring if it has no proper ideals other than the zero ideal.

Definition 2.8. A $\Gamma-$semigroup M is said to satisfy left (right) cancelation law if and only if $r, s, t \in M, r \neq 0, \alpha \in \Gamma$ such that $r\alpha s = r\alpha t$ ($s\alpha r = t\alpha r$) then $s = t$.

Definition 2.9. Let M be a $\Gamma-$semiring. Then $(M, +)$ is said to be band if $a + a = a$, for all $a \in M$.

Definition 2.10. A $\Gamma-$semigroup M is said to be left (right) singular if for each $a \in M$ there exists $\alpha \in \Gamma$ such that $a\alpha b = a(a\alpha b = b)$, for all $b \in M$.

Definition 2.11. A $\Gamma-$semiring M is called an ordered $\Gamma-$semiring if it admits a compatible relation $\leqslant$ i.e. $\leqslant$ is a partial ordering on M satisfies the following conditions. If $a \leqslant b$ and $c \leqslant d$ then

(i) $a + c \leqslant b + d$ (ii) $a\alpha c \leqslant b\alpha d$ (iii) $c\alpha a \leqslant d\alpha b$, for all $a, b, c, d \in M, \alpha \in \Gamma$.

Example 2.3. Let $M = [0, 1], \Gamma = N$, $+$ and ternary operation be defined as $x + y = \max\{x, y\}, x\gamma y = \min\{x, \gamma, y\}$ for all $x, y \in M, \gamma \in \Gamma$. Then M is an ordered $\Gamma-$semiring with respect to usual ordering.

Definition 2.12. An ordered $\Gamma-$semiring M is said to be totally ordered $\Gamma-$semiring M if any two elements of M are comparable.

Definition 2.13. Let M be an ordered $\Gamma-$semiring. A non-empty subset A of M is called a left (right) ideal of an ordered $\Gamma-$semiring M if A is closed under addition, $M\Gamma A \subseteq A$ ($A\Gamma M \subseteq A$) and if for any $a \in M,\ b \in I,\ a \leqslant b \Rightarrow a \in I$. A is called an ideal of M if it is both a left ideal and a right ideal of M.

Definition 2.14. A non-empty subset A of ordered $\Gamma-$semiring M is called a $k-$ideal if A is an ideal,$x \in M,\ x + y \in A$ and $y \in A$ then $x \in A$.

Definition 2.15. In an ordered $\Gamma-$semiring M

(i) $(M, +)$ is positively ordered if $a + b \geqslant a, b$ for all $a, b \in M$.
(ii) $(M, +)$ is negatively ordered if $a + b \leqslant a, b$ for all $, a, b \in M$.
(iii) $\Gamma-$semigroup M is positively ordered if $a\alpha b \geqslant a, b$ for all $\alpha \in \Gamma, a, b \in M$.
(iv) $\Gamma-$semigroup M is negatively ordered if $a\alpha b \leqslant a, b$ for all $\alpha \in \Gamma, a, b \in M$.

Definition 2.16. In an ordered $\Gamma-$semiring M with unity 1, an element $a \in M$ is said to be left invertible (right invertible) if there exists $b \in M, \alpha \in \Gamma$ such that $b\alpha a = 1(a\alpha b = 1)$.

Definition 2.17. In an ordered $\Gamma-$semiring M with unity 1, an element $a \in M$ is said to be invertible if there exist $b \in M, \alpha \in \Gamma$ such that $a\alpha b = b\alpha a = 1$.

Definition 2.18. In an ordered $\Gamma-$semiring M, an element $u \in M$ is said to be unit if there exist $a \in M$ and $\alpha \in \Gamma$ such that $a\alpha u = 1 = u\alpha a$.

Definition 2.19. Let M and N be ordered $\Gamma-$ semirings. A mapping $f : M \to N$ is called a homomorphism if

(i) $f(a + b) = f(a) + f(b)$
(ii) $f(a\alpha b) = f(a)\alpha f(b)$, for all $a, b \in M, \alpha \in \Gamma$.

Definition 2.20. Let M be an ordered $\Gamma-$ semiring. A mapping $f : M \to M$ is called an endomorphism if

(i) f is an onto ,
(ii) $f(a + b) = f(a) + f(b)$,
(iii) $f(a\alpha b) = f(a)\alpha f(b)$, for all $a, b \in M, \alpha \in \Gamma$.

Definition 2.21. Let M be an ordered $\Gamma-$ semiring. A mapping $f : M \to M$ is called an automorphism(anti-automorphism) if

(i) f is a bijection ,
(ii) $f(a + b) = f(a) + f(b)$,
(iii) $f(a\alpha b) = f(a)\alpha f(b)$,($f(a\alpha b) = f(b)\alpha f(a)$,) for all $a, b \in M, \alpha \in \Gamma$.

3. Zeroid elements in $\Gamma-$semiring

In this section, we introduce the concept of zeroid, left zeroid and right zeroid in $\Gamma-$semirings and study the properties of zeroids in $\Gamma-$semirings.

Definition 3.1. Let M be a $\Gamma-$semiring. An element $x \in M$ is said to be left (right) zeroid of $\Gamma-$semiring if there exists $y \in M$ such that $y + x = y(x + y = y)$.

Definition 3.2. Let M be a $\Gamma-$semiring. An element $x \in M$ is said to be zeroid of $\Gamma-$semiring if there exists $y \in M$ such that $x + y = y$ or $y + x = y$.

Every additive idempotent element of $\Gamma-$semiring M is a zeroid element of M.

Definition 3.3. Let M be a $\Gamma-$semiring. Zeroid of $\Gamma-$semiring M is the set of all x in M such that $x + y = y$ or $y + x = y$, for some y in M.
Zeriod of $\Gamma-$semiring M is denoted by Z.

Theorem 3.1. *Let M be a $\Gamma-$semiring with unity element, $a + a\alpha a = a$, for all $a \in M, \alpha \in \Gamma$ and $a + 1 = a$, for all $a \in M$. If $\Gamma-$semigroup M is left cancellative then every element of M is a zeroid.*

Proof. Let $a \in M$. Then $a + a\alpha a = a$, for all $\alpha \in \Gamma$ and there exists $\beta \in \Gamma$ such that $a\beta 1 = a$. We have $a + a\beta a = a$

$$\begin{aligned}
&\Rightarrow a\beta b + a + a\beta a = a\beta b + a, b \in M\\
&\Rightarrow a\beta b + a\beta 1 + a\beta a = a\beta b + a\beta 1\\
&\Rightarrow a\beta(b+1) + a\beta a = a\beta(b+1)\\
&\Rightarrow a\beta b + a\beta a = a\beta b\\
&\Rightarrow a\beta(b+a) = a\beta b\\
&\Rightarrow b + a = b.
\end{aligned}$$

Hence every element of M is a zeroid. □

Theorem 3.2. *Let M be an additively commutative $\Gamma-$semiring. Then zeroid Z is a $k-$ideal of M.*

Proof. Let $x, y \in Z$ and $\alpha \in \Gamma$. Then there exist $u, v \in M$ such that $x+u = u$ or $u + x = u$ and $y + v = v$ or $y = y = v$.
Suppose $x + u = u$ and $y + v = v$. Then

$$\begin{aligned}
x + y + u + v &= x + y + v + u\\
&= x + v + u\\
&= x + u + v\\
&= u + v.
\end{aligned}$$

Therefore $x + y \in Z$. Similarly in all cases we can prove $x + y \in Z$. Now $x + u = u$ and $y + v = v$. Then $x\alpha y + u\alpha y = u\alpha y$ and $x\alpha y + x\alpha v = x\alpha v$. Therefore $x\alpha y \in Z$. Similarly in all cases we can prove $x\alpha y \in Z$. Hence Z is a $\Gamma-$subsemiring of M.

Let $a \in Z, m \in M$ and $\alpha \in \Gamma$. Then there exists $b \in M$ such that $a + b = b \Rightarrow a\alpha m + b\alpha m = b\alpha m$. Therefore $a\alpha m \in Z$. Similarly we can prove $m\alpha a \in Z$. Let $z \in Z$ $y \in M$ and $x + y \in Z$. Then there exist $u \in M, v \in M$ such that

$$\begin{aligned}
&x + y + u = u, x + v = v\\
\Rightarrow &y + x + u = u\\
\Rightarrow &y + x + u + v = u + v\\
\Rightarrow &y + u + v = u + v.
\end{aligned}$$

Therefore $y \in Z$. Hence Z a $k-$ideal of $\Gamma-$semiring M. □

Corollary 3.1. *Let M be an additively commutative simple $\Gamma-$semiring and $Z \neq \{0\}$. Then every element of M is a zeroid.*

Theorem 3.3. *Let M be a $\Gamma-$semiring and every element of M is a zeroid. If $\Gamma-$semigroup M is a right singular then for each $a \in M$, there exists $b \in M$ such that $a + a\alpha b = b$ or $a\alpha b + a = b$, for all $\alpha \in \Gamma$.*

Proof. Let $a \in M$ and $\alpha \in \Gamma$. Then there exists $b \in M$ such that $a + b = b$ or $b + a = b$. Suppose $a + b = b$. We have $a\alpha b = b$, since $\Gamma-$semigroup M is a right singular. Therefore $a + b = b \Rightarrow a + a\alpha b = b$. Suppose $b + a = b \Rightarrow a\alpha b + a = b$. Hence the theorem. □

THEOREM 3.4. *Let M be a $\Gamma-$semiring and $(M,+)$ be right cancellative. If a is right zeroid of M then $a + a\alpha a = a$, for all $\alpha \in \Gamma$.*

PROOF. Let a be right zeroid of M and $\alpha \in \Gamma$. Then there exists $b \in M$ such that $a + b = b$.

$$\begin{aligned}&\Rightarrow a\alpha(a+b) = a\alpha b\\&\Rightarrow a\alpha a + a\alpha b = a\alpha b\\&\Rightarrow a + a\alpha a + a\alpha b = a + a\alpha b.\end{aligned}$$

Therefore $a + a\alpha a = a$. □

COROLLARY 3.2. *Let M be a $\Gamma-$semiring and $(M,+)$ be left cancellative. If a is left zeroid of M then $a\alpha a + a = a$, for all $\alpha \in \Gamma$.*

THEOREM 3.5. *Let M be a $\Gamma-$semiring with unity element. If unity element is zeroid then every element of M is zeroid.*

PROOF. Let M be a $\Gamma-$semiring with zeroid unity element and $a \in M$. Then there exist $b \in M$ and $\alpha \in \Gamma$ such that $1 + b = b$ or $b + 1 = b$ and $a\alpha 1 = a = 1\alpha a$. Suppose $b + 1 = b$. Then

$$\begin{aligned}&b + 1 = b\\&\Rightarrow a\alpha(b+1) = a\alpha b\\&\Rightarrow a\alpha b + a\alpha 1 = a\alpha b\\&\Rightarrow a\alpha b + a = a\alpha b.\end{aligned}$$

Suppose $1 + b = b$. Then

$$\begin{aligned}&1 + b = b\\&\Rightarrow a\alpha(1+b) = a\alpha b\\&\Rightarrow a\alpha 1 + a\alpha b = a\alpha b\\&\Rightarrow a + a\alpha b = a\alpha b.\end{aligned}$$

Hence every element of M is zeroid. □

DEFINITION 3.4. A $\Gamma-$semiring M is said to be semi-subtractive $\Gamma-$semiring if for every x and y in M there exists a $z \in M$ such that $z + x = y$ or $z + y = x$.

THEOREM 3.6. *If zeroid of a semi-subtractive $\Gamma-$semiring M is empty then $(M,+)$ is cancellative.*

PROOF. Suppose $x + t = y + t, x, y, t \in M$ and $x \neq y$. Then there exists $z \in M$ such that $z + x = y$ or $z + y = x$. Suppose $x = z + y$. Then

$$\begin{aligned}&x = z + y\\&\Rightarrow x + t = z + y + t\\&\Rightarrow x + t = z + x + t.\end{aligned}$$

Therefore z is the zeroid element. Which is a contradiction. Hence $(M,+)$ is a right cancellative. Thus $(M,+)$ is cancellative. □

Definition 3.5. Let M be a $\Gamma-$semiring. Then M is said to be zero square $\Gamma-$semiring if $x\alpha x = 0$, for all $x \in M, \alpha \in \Gamma$.

Theorem 3.7. *Let M be a zero square $\Gamma-$semiring. If x is a zeroid of M then there exists $y \in M$ such that $x\alpha y = 0$ or $y\alpha x = 0$, for all $\alpha \in \Gamma$.*

Proof. Let $x \in Z$ and $\alpha \in \Gamma$. Then there exists $y \in M$ such that $x + y = y$ or $y + x = y$. Suppose $x + y = y$. Then

$$\begin{aligned} & x + y = y \\ \Rightarrow & (x + y)\alpha y = y\alpha y \\ \Rightarrow & x\alpha y + y\alpha y = y\alpha y \\ \Rightarrow & x\alpha y + 0 = 0 \\ \Rightarrow & x\alpha y = 0. \end{aligned}$$

Similarly we can prove $y\alpha x = 0$. Hence the theorem. □

Theorem 3.8. *Let M be an additively right cancellative $\Gamma-$semiring and e be an additively idempotent element of M. Then e is the additive left singular of Z.*

Proof. Let $a \in Z$. Obviously $a + e \in Z$. Then there exists $b \in M$ such that

$$\begin{aligned} & a + e + b = b \\ \Rightarrow & a + (e + b) = b \\ \Rightarrow & e + a + (e + b) = e + b = e + e + b \\ \Rightarrow & e + a = e, \text{ since } (M, +) \text{ is right cancellative .} \end{aligned}$$

Hence e is the left singular of Z. □

Theorem 3.9. *Let M be a $\Gamma-$semiring and every element of M be zeroid. If $\Gamma-$semigroup M is band then for each $a \in M$, there exist $b \in M, \alpha \in \Gamma$ such that $a + a\alpha b + b = b$.*

Proof. Let $a \in M$. Then there exist $b \in M, \alpha \in \Gamma$ such that $b = a + b$ and $b\alpha b = b$.

$$\begin{aligned} b &= a + b \\ &= a + b\alpha b \\ &= a + (a + b)\alpha b \\ &= a + a\alpha b + b\alpha b \\ &= a + a\alpha b + b. \end{aligned}$$

Hence the theorem. □

Theorem 3.10. *Let M be a $\Gamma-$semiring with identity $a + b + a\alpha b = b$, for all $a, b \in M, \alpha \in \Gamma$. If every element of M is zeroid and $(M, +)$ is left cancellative then $\Gamma-$semigroup M is a band.*

PROOF. Let $a \in M$ and $\alpha \in \Gamma$. Then there exists $b \in M$ such that $b + a = b$ or $a + b = b$. Suppose $b + a = b$. We have

$$\begin{aligned} & a + b + a\alpha b = b \\ \Rightarrow & a + b + a\alpha(b + a) = b + a \\ \Rightarrow & a + b + a\alpha b + a\alpha a = b + a \\ \Rightarrow & b + a\alpha a = b + a \\ \Rightarrow & a\alpha a = a. \end{aligned}$$

Hence $\Gamma-$semigroup M is a band. □

THEOREM 3.11. *Let M be a $\Gamma-$semiring with unity in which $\Gamma-$semigroup M is left cancellative. If $a + a\alpha a = a$, for all $a \in M, \alpha \in \Gamma$ then M is a zeroid.*

PROOF. Let $a \in M$.

$$\begin{aligned} & \text{Then } a + a\alpha a = a, \text{ for all } \alpha \in \Gamma \text{ and } a\beta 1 = a,\ \beta \in \Gamma \\ \Rightarrow & a\beta 1 + a\beta a = a \\ \Rightarrow & a\beta(1 + a) = a\beta 1 \\ \Rightarrow & 1 + a = 1. \end{aligned}$$

Therefore $a \in Z$. Hence every element of M is a zeroid. □

4. Regular and idempotent elements in ordered $\Gamma-$semiring

In this section, we introduce the notion of regular and idempotent elements in ordered $\Gamma-$semiring. We study the properties of regular and idempotent elements and relation between them in ordered $\Gamma-$semirings.

DEFINITION 4.1. Let M be an ordered $\Gamma-$semiring. An element $a \in M$ is said to be idempotent of M if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$ and a is also said to be α idempotent.

DEFINITION 4.2. Let M be an ordered $\Gamma-$semiring. Every element of M is an idempotent then M is said to be idempotent ordered $\Gamma-$semiring

DEFINITION 4.3. Let M be an ordered $\Gamma-$semiring . An element $a \in M$ is said to be regular element of M if there exist $x \in M, \alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$.

DEFINITION 4.4. Let M be an ordered $\Gamma-$semiring . Every element of M is a regular element of M then M is said to be regular ordered $\Gamma-$semiring M.

THEOREM 4.1. *Let M be an ordered $\Gamma-$semiring in which $(M, +)$ is positively ordered. If $x \in M$ and y be an idempotent of $(M, +)$ and $x \leqslant y$then x is a zeroid.*

PROOF. Let $x \in M$ and y be an idempotent and $x \leqslant y$.

$$\begin{aligned} \Rightarrow & x + y \leqslant y + y \\ \Rightarrow & x + y \leqslant y. \end{aligned}$$

We have $x + y \geqslant y$. Therefore $x + y = y$. Thus $x \in Z$ Thus x is a zeroid of M. □

Corollary 4.1. *Let M be an ordered $\Gamma-$semiring in which $(M,+)$ is negatively ordered. If $x \in M$ and y be an idempotent of $(M,+)$ and $x \leqslant y$ then y is a zeroid.*

Theorem 4.2. *Let M be an ordered $\Gamma-$semiring in which $\Gamma-$semigroup is negatively ordered. If a is a regular element of $\Gamma-$semiring M if and only if a is an idempotent element of M.*

Proof. Suppose a is a regular element of M. Then there exist $\alpha, \beta \in \Gamma$ and $x \in M$ such that $a = a\alpha x\beta a \leqslant a\alpha a \leqslant a$. Then $a\alpha a = a$. Therefore a is an idempotent of M. Conversely suppose that a is an idempotent element of M. Then there exists $\alpha \in \Gamma$ such that $a = a\alpha a$. Therefore $a = a\alpha a\alpha a$. Hence a is regular. □

Theorem 4.3. *Let M be an ordered $\Gamma-$semiring with unity element. If an element a of M is left invertible then a is a regular.*

Proof. Suppose $a \in M$ is left invertible. Then there exist $b \in M, \delta \in \Gamma$ such that $b\delta a = 1$. Since 1 is unity, there exists $\alpha \in \Gamma$ such that $a\alpha 1 = a \Rightarrow a\alpha(b\delta a) = a$. Hence a is a regular element. □

Theorem 4.4. *Let M be a regular ordered $\Gamma-$semiring and $a \in M$ in which $\Gamma-$semigroup is negatively ordered. Then $a = a\alpha x = x\beta a$, for some $\alpha, \beta \in \Gamma$ and $x \in M$.*

Proof. Let $a \in M$. Then there exist $x \in M$ and $\alpha, \beta \in \Gamma$ such that

$$a = a\alpha x\beta a \leqslant a\alpha x \leqslant a$$
$$a = a\alpha x\beta a \leqslant x\beta a \leqslant a.$$

Therefore $a\alpha x = a = x\beta a$. □

Theorem 4.5. *Let M be a regular ordered $\Gamma-$semiring. If $a \in M$ then there exist $x \in M,\ \alpha, \beta \in \Gamma$ such that $a\alpha x$ and $x\beta a$ are idempotents of M.*

Proof. Let $a \in M$. Then there exist $\alpha, \beta \in \Gamma$ and $x \in M$ such that

$$a = a\alpha x\beta a$$
$$\Rightarrow a\alpha x = a\alpha x\beta a\alpha x.$$

Therefore $a\alpha x$ is $\beta-$idempotnet of M.

Similarly we can prove $x\beta a$ is $\alpha-$idempotent of M. □

Theorem 4.6. *If M is an ordered $\Gamma-$semiring with unity and $a \in M$ is an idempotent then there exist $\alpha, \beta \in \Gamma$ such that $a = a\alpha 1\beta a$.*

Proof. Let $a \in M$. Then there exists $\beta \in \Gamma$ such that $a\beta 1 = 1\beta a = a$. Suppose a is $\alpha-$idempotent. Then $a = a\alpha a = a\alpha 1\beta a$. Hence the theorem. □

Definition 4.5. An ordered $\Gamma-$semiring $M$is said to be $\Gamma-$semiring with involution, if $*$ is defined on M satisfying the following conditions

(i) $(a+b)^* = a^* + b^*$
(ii) $(a\alpha b)^* = b^*\alpha a^*$

(iii) $(a^*)^* = a$, for all $a, b \in M$ and $\alpha \in \Gamma$.

DEFINITION 4.6. Let M be an ordered $\Gamma-$semiring with involution. An element $a \in M$ is said to be symmetric if $a^* = a$.

DEFINITION 4.7. Let M be an ordered $\Gamma-$semiring with involution. An element $a \in M$ is said to be projection if $a^* = a = a\alpha a$, for all $\alpha \in \Gamma$.

DEFINITION 4.8. Let M be an ordered $\Gamma-$semiring with involution. An element $x \in M$ is said to be Moore-Penrose inverse of a if x satisfies the following

(i) $a = a\alpha x\beta a$
(ii) $x = x\beta a\alpha x$
(iii) $(a\alpha x)^* = a\alpha x$
(iv) $(x\beta a)^* = x\beta a,\ \alpha, \beta \in \Gamma$.

It is denoted as $a^\dagger$.

THEOREM 4.7. *Let M be an ordered $\Gamma-$semiring in which $\Gamma-$semigroup$M$ is negatively ordered with involution $*$. If a is a projection then $a^\dagger$ exists and equals to a.*

PROOF. Let a be a projection. Then a is a symmetric and idempotent. By Theorem 4.4, there exists $x \in M$ such that $a = a\alpha x = x\beta a$ and $a^* = a$, since a is symmetric. Then x satisfies all properties.

(i) $a = a\alpha x\beta a$ (ii) $x = x\beta a\alpha x$
(iii) $(a\alpha x)^* = a\alpha x$ (iv) $(x\beta a)^* = x\beta a,\ \alpha, \beta \in \Gamma$.

Hence $a^\dagger$ exists and equals to a. □

THEOREM 4.8. *Let M be an ordered $\Gamma-$semiring in which $\Gamma-$semigroup$M$ is negatively ordered with involution and $a \in M$. Then a is regular and $a = a\alpha a^*\beta a$, $\alpha, \beta \in \Gamma$ if and only if a is projection.*

PROOF. Suppose a is regular and $a = a\alpha a^*\beta a$. Then by Theorem 4.4,

$$
\begin{aligned}
&a = a\alpha a^* = a^*\beta a \\
&a^* = (a\alpha a^*)^* = (a^*)^*\alpha a^* = a\alpha a^* = a \\
\Rightarrow &a\alpha a^* = a\alpha a^*\beta a\alpha a^* \\
\Rightarrow &a\alpha a^* \text{ is } \beta-\text{idempotent} . \\
\Rightarrow &a \text{ is } \beta-\text{idempotent} .
\end{aligned}
$$

Therefore a is $\beta-$idempotent and symmetric. Hence a is projection.

Converse is obvious. □

THEOREM 4.9. *Let M be an ordered $\Gamma-$semiring in which $\Gamma-$semigroup$M$ is negatively ordered with involution $*$. Then $x \in M$ such that $a = a\beta x\alpha a$ and $(a\beta x)^* = a\beta x$ if and only if $a^*\alpha a\beta x = a^*$.*

Proof. Suppose $x \in M$ such that $a = a\beta x\alpha a$ and $(a\beta x)^* = a\beta x$. Then

$$\begin{aligned} a^*\alpha a\beta x &= a^*\alpha(a\beta x) \\ &= a^*\alpha(a\beta x)^* \\ &= a^*\alpha(x^*\beta a^*) \\ &= (a\beta x\alpha a)^* \\ &= a^*. \end{aligned}$$

Conversely suppose that $a^*\alpha a\beta x = a^*$.

$$\begin{aligned} &\Rightarrow a^*\alpha a\beta x \leqslant a^*\alpha a \leqslant a^* \\ &\Rightarrow a^*\alpha a = a^* \\ &\Rightarrow (a^*)^* = (a^*\alpha a)^* \\ &\Rightarrow a = a^*\alpha a = a^*. \end{aligned}$$

Therefore a is symmetric.

$$\begin{aligned} a^*\alpha a\beta x &= a^* \\ \Rightarrow a = a^* = a^*\alpha a\beta x \leqslant a\beta x \leqslant a & \\ \Rightarrow a\beta x &= a. \\ a\beta x\alpha a &= (a\beta x)\alpha a \\ &= a^*\alpha a \\ &= a \\ \Rightarrow (a\beta x)^* &= a^* = a = a\beta x. \end{aligned}$$

Hence the theorem. □

Definition 4.9. An ordered $\Gamma-$semiring M is called a sum ordered Γ -semiring, if $a \leqslant b$ then there exists $c \in M$ such that $a + c = b$.

Theorem 4.10. *Let M be an additively idempotent ordered $\Gamma-$semiring. Then the following conditions are equivalent*

(i) 0 *is the least element of* M
(ii) $a \leqslant a + b$, *for all* $a, b \in M$
(iii) $a \leqslant b$ *if and only if* $a + b = b$
(iv) M *is sum ordered* $\Gamma-$*semiring.*

Proof. Let M be an additively idempotent ordered $\Gamma-$semiring.

$(i) \Rightarrow (ii)$: Suppose 0 is the least element of M and $a, b \in M$. We have

$$\begin{aligned} &0 \leqslant a \\ &\Rightarrow 0 + b \leqslant a + b \\ &\Rightarrow b \leqslant a + b. \end{aligned}$$

Similarly $a \leqslant a + b$.

$(ii) \Rightarrow (iii)$: Suppose $a \leqslant a + b$, for all $a, b \in M$. Let $a, b \in M$ and $a \leqslant b$. Then $a + b \leqslant b + b$, $a + b \leqslant b \leqslant a + b$. Therefore $a + b = b$.

$(iii) \Rightarrow (iv):$ Obvious.
$(iv) \Rightarrow (i):$ Suppose that M is sum ordered $\Gamma-$semiringand $a \in M$.
Then there exists $c \in M$ such that $0 + c = a \Rightarrow c = a$.
Now $0 + a = a$, for all $a \in M$. Then $0 \leqslant a$.
Hence 0 is the least element of M.

□

LEMMA 4.1. *Let M be an ordered $\Gamma-$semiring with unity 1. If $1 + 1 = 1$ holds in M then $(M, +)$ is an idempotent semigroup.*

PROOF. Let $a \in M$. Then there exists $\alpha \in \Gamma$ such that $a\alpha 1 = a$.

$$\begin{aligned} a + a &= a\alpha 1 + a\alpha 1 \\ &= a\alpha(1 + 1) \\ &= a\alpha 1 \\ &= a \end{aligned}$$

Hence $(M, +)$ is an idempotent semigroup. □

$E[+]$ denotes the set $\{x \in M \mid x + x = x\}$.

THEOREM 4.11. *Let M be an ordered $\Gamma-$semiring in which $(M, +)$ is commutative, positively ordered and cancellative semigroup. If $E[+] \neq \emptyset$,$\Gamma-$semigroup M is cancellative then $E[+]$ is a k-prime ideal of an ordered $\Gamma-$semiring M.*

PROOF. Let $x \in E[+], y \in M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned} x &= x + x \\ \Rightarrow x\alpha y &= (x + x)\alpha y \\ &= x\alpha y + x\alpha y \\ \Rightarrow x\alpha y &\in E[+]. \end{aligned}$$

Similarly $y\alpha x \in E[+]$. Suppose $x, y \in E[+]$. Then

$$\begin{aligned} & x + x = x, y + y = y \\ \Rightarrow & (x + y) + (x + y) = (x + x) + (y + y) = x + y \\ \Rightarrow & x + y \in E[+]. \end{aligned}$$

Suppose $x \leqslant y, y \in E[+]$. Then

$$\begin{aligned} & y + y = y, x \leqslant y \\ & x + x = x, x + y \leqslant y + y \\ & x + x = x, x + y \leqslant y \ \text{ and } y \leqslant x + y \\ & x + y = y. \\ & x + x + y = x + y. \\ & x + x = x. \\ \Rightarrow & x \in E[+]. \end{aligned}$$

Therefore $E[+]$ is an ideal of ordered $\Gamma-$semiring. Suppose $x, x+y \in E[+]$. Then

$$\begin{aligned} & x+x=x, x+y+x+y=x+y \\ \Rightarrow & (x+y)+(x+y)=x+y \\ \Rightarrow & (x+x)+(y+y)=x+y \\ \Rightarrow & x+(y+y)=x+y \\ \Rightarrow & y+y=y \\ \Rightarrow & y \in E[+]. \end{aligned}$$

Hence $E[+]$ is a k- ideal of an ordered $\Gamma-$semiring M.
Suppose $x\alpha y \in E[+]$ and $x, y \notin E[+]$. Then

$$\begin{aligned} & x\alpha y + x\alpha y = x\alpha y \\ \Rightarrow & x\alpha(y+y) = x\alpha y \\ \Rightarrow & y+y=y, \text{ which is contradiction.} \end{aligned}$$

Hence $E[+]$ is a k-prime ideal of M. □

THEOREM 4.12. *Let M be a simple ordered $\Gamma-$semiring in which $(M,+)$ is commutative, positively ordered and cancellative semigroup and $E[+] \neq \phi$. Then every element of M is additive idempotent.*

PROOF. By Theorem 4.11, $E[+]$ is an ideal of an ordered$\Gamma-$semiring M. Therefore $M = E[+]$, since $\Gamma-$semring M is simple. Hence the theorem. □

COROLLARY 4.2. *Let M be a simple ordered $\Gamma-$semiring in which $(M,+)$ is commutative, positively ordered and cancellative semigroup . If $E[+] = 1$ then $\mid M \mid = 1$.*

The set $\{x \mid x\alpha y = y\alpha x$, for all $y \in M$ and $\alpha \in \Gamma\}$ is center of M. It is denoted by $Z(M)$.

THEOREM 4.13. *Let M be an ordered $\Gamma-$semiring in which $\Gamma-$semigroup M is left cancellative, a is in center of M, $Z(M)$ and $\alpha \in \Gamma$. Define a mapping $I_a : M \to M$ by $I_a(x) = a\alpha x$, for all $x \in M$. Then a is an $\alpha-$idempotent of M if and only if I_a is an automorphism.*

PROOF. Let a be an $\alpha-$idempotent of $\Gamma-$semiring M and $a \in Z(M)$. Define a mapping $I_a : M \to M$ by $I_a(x) = a\alpha x$, for all $x \in M$. Then

$$\begin{aligned} I_a(x\alpha y) &= a\alpha(x\alpha y) \\ &= a\alpha a\alpha x\alpha y \\ &= (a\alpha x)\alpha(a\alpha y) \\ &= I_a(x)\alpha I_a(y). \\ \text{Suppose } I_a(x) &= I_a(y) \\ \Rightarrow a\alpha x &= a\alpha y \\ \Rightarrow x &= y. \end{aligned}$$

Obviously I_a is an onto. Hence I_a is an automorphism.

Conversely suppose that I_a is an automorphism.

$$\begin{aligned}
I_a(x\alpha y) &= I_a(x)\alpha I_a(y), \text{ for all } x, y \in M \\
\Rightarrow & a\alpha x\alpha y = a\alpha x\alpha a\alpha y \\
\Rightarrow & a\alpha(x\alpha y) = a\alpha a\alpha x\alpha y \\
\Rightarrow & (x\alpha y)\alpha a = x\alpha y\alpha a\alpha a \\
\Rightarrow & a = a\alpha a.
\end{aligned}$$

Hence a is an $\alpha-$idempotent of M. □

THEOREM 4.14. *Let M be an ordered $\Gamma-$semiring with a $\alpha-$idempotent a of M and define a mapping $I_a : M \to M$ by $I_a(x) = a\alpha x$, for all $x \in M$. If I_a is an anti automorphism then a commutes with every element of M i.e., $a \in Z(M)$.*

PROOF. Let M be an ordered $\Gamma-$semiring with a $\alpha-$idempotent a of M and I_a be an anti automorphism. Then $I_a(x) = a\alpha x$, for all $x \in M$. Let $x \in M$. Then there exists $x' \in M$ such that $I_a(x') = x$. Now

$$\begin{aligned}
x\alpha a = x\alpha a\alpha a &= x\alpha I_a(a) \\
&= I_a(x')\alpha I_a(a) \\
&= I_a(a\alpha x') \\
&= a\alpha a\alpha x' \\
&= a\alpha I_a(x') \\
&= a\alpha x.
\end{aligned}$$

Hence a commutes with every element of M. □

THEOREM 4.15. *Let M be a regular ordered $\Gamma-$semiring in which $\Gamma-$semigroup M is negatively ordered with unity element. If $a, b \in M, a \leqslant b$ then there exists $\alpha \in \Gamma$ such that $a = a\alpha a = b\alpha a = a\alpha b$.*

PROOF. Let $a, b \in M$ and $a \leqslant b$. Since M is regular. There exist $\alpha, \beta \in \Gamma$ and $x \in M$ such that $a = a\alpha x\beta a$. We have

$$\begin{aligned}
& x\beta a \leqslant a \\
\Rightarrow & a\alpha x\beta a \leqslant a\alpha a \\
\Rightarrow & a \leqslant a\alpha a \leqslant a.
\end{aligned}$$

Therefore $a = a\alpha a$.. Now

$$\begin{aligned}
a \leqslant b \Rightarrow & a\alpha a \leqslant b\alpha a \\
\Rightarrow & a \leqslant b\alpha a \leqslant a \\
\Rightarrow & a = b\alpha a.
\end{aligned}$$

Therefore $a = a\alpha a = b\alpha a$. Now

$$\begin{aligned} \leqslant b &\Rightarrow a\alpha a \leqslant a\alpha b \\ &\Rightarrow a \leqslant a\alpha b \leqslant a. \end{aligned}$$

Therefore $a = a\alpha b$. Hence $a = a\alpha a = a\alpha b = b\alpha a$. □

THEOREM 4.16. *Let M be a regular ordered $\Gamma-$semiring with unity element in which $\Gamma-$semigroup$M$ is negatively ordered. If $a, b \in M, a \leqslant b$ then there exists $\beta \in \Gamma$ such that $a\beta b = b\beta a = a$.*

PROOF. Let $a, b \in M, a \leqslant b$ and b be $\beta-$ idempotent. Then by Theorem 4.15, there exists $\alpha \in \Gamma$ such that $a = a\alpha a = b\alpha a = a\alpha b$ and $b\beta b = b$. Now

$$\begin{aligned} &a = a\alpha b = b\alpha a \\ &a = a\alpha b = a\alpha b\beta b \leqslant a\beta b \leqslant a. \end{aligned}$$

Therefore $a\beta b = a$. Now $a = b\alpha a = b\beta b\alpha a \leqslant b\beta a \leqslant a$. Hence $a = a\beta b = b\beta a$. □

THEOREM 4.17. *Let M be a totally ordered regular $\Gamma-$semiring with unity element in which $\Gamma-$semigroup$M$ is negatively ordered. Then M is a commutative ordered $\Gamma-$semiring.*

PROOF. Let $a, b \in M, \gamma \in \Gamma$ and $a \leqslant b$. Suppose $a\gamma b \leqslant b\gamma a$. Then $b\gamma a \leqslant a$. By Theorems 4.15 and 4.16, there exists $\alpha \in \Gamma$ such that $a\alpha b = a\alpha a = a$ and

$$\begin{aligned} &(b\gamma a)\alpha a = a\alpha(b\gamma a) \\ &\Rightarrow b\gamma(a\alpha a) = (a\alpha b)\gamma a, \\ &\Rightarrow b\gamma a = a\gamma a, \cdots (1). \\ a \leqslant b &\Rightarrow a\gamma a \leqslant a\gamma b \\ &\Rightarrow b\gamma a \leqslant a\gamma b, \text{ from } (1). \end{aligned}$$

Therefore, $b\gamma a = a\gamma b$ Hence M is a commutative idempotent ordered $\Gamma-$semiring. □

5. Conclusion

In this paper, we introduced the notion of zeroid in $\Gamma-$semiring, regular and idempotent elements in ordered $\Gamma-$semiring. We studied the properties of zeroid, regular and idempotent elements and relations between them. We proved that,if M is a totally ordered regular $\Gamma-$semiring in which $\Gamma-$semigroup$M$ is negatively ordered with unity element then M is a commutative ordered $\Gamma-$semiring.

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