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A STUDY OF BI-QUASI-INTERIOR IDEAL AS A NEW GENERALIZATION OF IDEAL OF GENERALIZATION OF SEMIRING

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Abstract. In this paper, as a further generalization of ideals, we introduce the notion of bi- quasi-interior ideals as a generalization of bi-ideal, quasi ideal, interior ideal, bi-interior ideal and bi-quasi ideal of $\Gamma-$semiring and study the properties of bi-quasi-interior ideals of $\Gamma-$semiring.

1. Introduction

In 1995, M.Murali Krishna Rao [15, 16, 17, 18, 19] introduced the notion of $\Gamma-$semiring as a generalization of $\Gamma-$ ring, ternary semiring and semiring.As a generalization of ring, the notion of a $\Gamma-$ring was introduced by Nobusawa [28] in 1964. In 1981, Sen [29] introduced the notion of a $\Gamma-$semigroup as a generalization of semigroup. The notion of a ternary algebraic system was introduced by Lehmer [13] in 1932. Murali Krishna Rao and Venkateswarlu [24, 25, 26] studied regular $\Gamma-$incline, field $\Gamma-$semiring and derivations. Lister [14] introduced ternary ring. Dutta and Sardar [**2**] introduced the notion of operator semirings of $\Gamma-$semiring. The set of all negative integers $\mathbb{Z}$ is not a semiring with respect to usual addition and multiplication but $\mathbb{Z}$ forms a $\Gamma-$semiring where $\Gamma = \mathbb{Z}$. The important reason for the development of $\Gamma-$semiring is a generalization of results of rings, $\Gamma-$rings, semirings, semigroups and ternary semirings. The notion of a semiring was introduced by Vandiver [32] in 1934, but semirings had appeared in earlier studies on the theory of ideals of rings. A universal algebra $(S, +, \cdot)$ is called a semiring if and only if $(S, +), (S, \cdot)$ are semigroups which are connected by distributive laws, *i.e.*, $a(b + c) = ab + ac$, $(a + b)c = ac + bc$, for all $a, b, c \in S$. The theory

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of rings and theory of semigroups have considerable impact on the development of theory of semirings. Semirings play an important role in studying matrices and determinants. Semirings are useful in the areas of theoretical computer science as well as in the solution of graph theory, optimization theory, in particular for studying automata,coding theory and formal languages. Semiring theory has many applications in other branches.

Ideals play an important role in advance studies and uses of algebraic structures. Generalization of ideals in algebraic structures is necessary for further study algebraic structures. Many mathematicians proved important results and charecterization of algebraic structures by using the concept and the properties of generalization of ideals in algebraic structures. Henriksen [4] and Shabir et al. [30] studied ideals in semirings We know that the notion of an one sided ideal of any algebraic structure is a generalization of notion of an ideal. The quasi ideals are generalization of left ideal and right ideal whereas the bi-ideals are generalization of quasi ideals. In 1952, the concept of bi-ideals was introduced by Good and Hughes [3] for semigroups. The notion of bi-ideals in rings and semirings were introduced by Lajos and Szasz [11, 12]. Bi-ideal is a special case of $(m-n)$-ideal. Steinfeld [31] first introduced the notion of quasi ideals for semigroups and then for rings. Iseki [5, 6, 7, 8] introduced the concept of quasi ideal for a semiring. Quasi ideals bi-ideals in $\Gamma-$semirings studied by Jagtap and Pawar [9, 10]. Murali Krishna Rao [22, 23, 24] introduced the notion of left (right) bi-quasi ideal of semiring, $\Gamma-$semiring,$\Gamma-$semigroup and study the properties of left bi-quasi ideals. We characterize the left bi-quasi simple $\Gamma-$semiring and regular $\Gamma-$semiring using left bi-quasi ideals of $\Gamma-$semirings.Murali Krishna Rao [18] studied ideals in ordered $\Gamma-$semiring. Murali Krishna Rao [20] introduced the notion of bi-interior ideal as a generalization of quasi ideal, bi-ideal and interior ideal of semigroup and study the properties of bi-interior ideals of semigroup, simple semigroup and regular semigroup.

In this paper, as a further generalization of ideals, we introduce the notion of bi- quasi-interior ideal as a generalization of bi-ideal,quasi ideal, interior ideal, bi-interior ideal and bi-quasi ideal of $\Gamma-$semiring and study the properties of bi-quasi-interior ideals of $\Gamma-$semiring. Some characterization of bi-quasi-interior ideals of $\Gamma-$semiring,regular $\Gamma-$semiring and simple $\Gamma-$semiring. A necessary sufficient condition for a $\Gamma-$semiring to be regular and simple is proved using bi- quasi-interior ideals $\Gamma-$semiring.

2. Preliminaries

In this section we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1. [1] A set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$, respectively) will be called semiring provided

(i) Addition is a commutative operation.
(ii) Multiplication distributes over addition both from the left and from the right.

(iii) There exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$.

DEFINITION 2.2. Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. Then we call M a $\Gamma-$semiring, if there exists a mapping $M \times \Gamma \times M \to M$ (images of (x, α, y) will be denoted by $x\alpha y$, for any $x, y \in M$ and $\alpha \in \Gamma$) such that it satisfies the following axioms for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$
(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$
(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$
(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Every semiring R is a $\Gamma-$semiring with $\Gamma = R$ and ternary operation $x\gamma y$ defined as the usual semiring multiplication.

EXAMPLE 2.1. Let S be a semiring and $M_{p,q}(S)$ denote the additive abelian semigroup of all $p \times q$ matrices with identity element whose entries are from S. Then $M_{p,q}(S)$ is a $\Gamma-$ semiring with $\Gamma = M_{p,q}(S)$ ternary operation is defined by $x\alpha z = x(\alpha^t)z$ as the usual matrix multiplication, where α^t denote the transpose of the matrix α; for all x, y and $\alpha \in M_{p,q}(S)$.

EXAMPLE 2.2. Let M be the set of all natural numbers. Then $(M,\ max,\ min)$ is a semiring. If $\Gamma = M$, then M is a $\Gamma-$semiring.

EXAMPLE 2.3. Let M be the additive semigroup of all $m \times n$ matrices over the set of non negative rational numbers and Γ be the additive semigroup of all $n \times m$ matrices over the set of non negative integers.Then with respect to usual matrix multiplication M is a $\Gamma-$semiring.

DEFINITION 2.3. A $\Gamma-$semiring M is said to be commutative $\Gamma-$semiring if $x\alpha y = y\alpha x$, for all $x, y \in M$ and $\alpha \in \Gamma$.

DEFINITION 2.4. Let M be a $\Gamma-$semiring. An element $1 \in M$ is said to be unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.

DEFINITION 2.5. In a $\Gamma-$semiring M with unity 1, an element $a \in M$ is said to be left invertible (right invertible) if there exist $b \in M, \alpha \in \Gamma$ such that $b\alpha a = 1(a\alpha b = 1)$.

DEFINITION 2.6. In a $\Gamma-$semiring M with unity 1, an element $a \in M$ is said to be invertible if there exist $b \in M, \alpha \in \Gamma$ such that $a\alpha b = b\alpha a = 1$.

DEFINITION 2.7. A $\Gamma-$semiring M is said to have zero element if there exists an element $0 \in M$ such that $0+x = x = x+0$ and $0\alpha x = x\alpha 0 = 0$, for all $x \in M, \alpha \in \Gamma$.

DEFINITION 2.8. An element $a \in \Gamma-$semiring M is said to be idempotent if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$.

DEFINITION 2.9. Every element of M, is an idempotent of M then M is said to be idempotent $\Gamma-$semiring M.

DEFINITION 2.10. A non-zero element a in a $\Gamma-$semiring M is said to be zero divisor if there exits non-zero element $b \in M, \alpha \in \Gamma$ such that $a\alpha b = b\alpha a = 0$.

Definition 2.11. A $\Gamma-$semigroup M is said to be left (right) singular if for each $a \in M$ there exists $\alpha \in \Gamma$ such that $a\alpha b = a(a\alpha b = b)$, for all $b \in M$.

Definition 2.12. A $\Gamma-$semiring M is called a division $\Gamma-$semiring if for each non-zero element of M has multiplication inverse.

Definition 2.13. A non-empty subset A of $\Gamma-$semiring M is called

(i) a $\Gamma-$subsemiring of M if $(A, +)$ is a subsemigroup of $(M, +)$ and $A\Gamma A \subseteq A$.
(ii) a quasi ideal of M if A is a $\Gamma-$subsemiring of M and $A\Gamma M \cap M\Gamma A \subseteq A$.
(iii) a bi-ideal of M if A is a $\Gamma-$subsemiring of M and $A\Gamma M\Gamma A \subseteq A$.
(iv) an interior ideal of M if A is a $\Gamma-$subsemiring of M and $M\Gamma A\Gamma M \subseteq A$.
(v) a left (right) ideal of M if A is a $\Gamma-$subsemiring of M and $M\Gamma A \subseteq A(A\Gamma M \subseteq A)$.
(vi) an ideal if A is a $\Gamma-$subsemiring of $M, A\Gamma M \subseteq A$ and $M\Gamma A \subseteq A$.
(vii) a $k-$ideal if A is a $\Gamma-$subsemiring of $M, A\Gamma M \subseteq A, M\Gamma A \subseteq A$ and $x \in M,\ x + y \in A, y \in A$ then $x \in A$.

Definition 2.14. A non-empty subset B of $\Gamma-$semiring M is said to be bi-interior ideal of M if B is a $\Gamma-$subsemiring of M and $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B$.

Definition 2.15. Let M be a $\Gamma-$semiring. A non-empty subset L of M is said to be left bi-quasi ideal (right bi-quasi ideal) of M if L is a subsemigroup of $(M, +)$ and $M\Gamma L \cap L\Gamma M\Gamma L \subseteq L$ $(L\Gamma M \cap L\Gamma M\Gamma L \subseteq L)$.

Definition 2.16. Let M be a $\Gamma-$semiring. L is said to be bi-quasi ideal of M if it is both a left bi-quasi and a right bi-quasi ideal of M.

Example 2.4.
(i) Let Q be the set of all rational numbers, $M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in Q \right\}$ be the additive semigroup of M matrices and $\Gamma = M$. Ternary operation $A\alpha B$ is defined as usual matrix multiplication of A, α, B, for all $A, \alpha, B \in M$. Then M is a $\Gamma-$semiring.

(a) If $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid 0 \neq a, 0 \neq b \in Q \right\}$ then R is a quasi ideal of $\Gamma-$semiring M and R is neither a left ideal nor a right ideal.
(b) If $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid 0 \neq a \in Q \right\}$ then S is a bi-ideal of $\Gamma-$semiring M.

(ii) If $M = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in Q \right\}$ and $\Gamma = M$ then M is a $\Gamma-$semiring with respect to usual addition of matrices and ternary operation is defined as usual matrix multiplication and $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid 0 \neq a, 0 \neq b \in Q \right\}$. Then A is not a bi-ideal of $\Gamma-$semiring M.

Definition 2.17. A $\Gamma-$semiring M is called a left bi-quasi simple $\Gamma-$semiring if M has no left bi-quasi ideal other than M itself.

3. Bi-quasi-interior ideals of $\Gamma-$semirings

In this section we introduce the notion of bi-quasi-interior ideal as a generalization of bi-ideal, quasi-ideal and interior ideal of $\Gamma-$semiring and study the properties of bi-quasi-interior ideal of $\Gamma-$semiring.

Definition 3.1. A non-empty subset B of a $\Gamma-$semiring M is said to be bi-quasi-interior ideal of M if B is a $\Gamma-$subsemiring of M and $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B$.

Every bi-quasi-interior ideal of $\Gamma-$semiring M need not be bi-ideal,quasi-ideal, interior ideal bi-interior ideal. and bi-quasi ideals of $\Gamma-$semiring M.

Example 3.1. Let $\mathbb{N}$ be a the set of all natural numbers and $\Gamma = 4\mathbb{N}$ be additive Abelian semigroups. Ternary operation is defined as $(x, \alpha, y) \rightarrow x + \alpha + y$, where $+$ is the usual addition of integers. Then $\mathbb{N}$ is an $\Gamma-$semiring. A subset $I = 2\mathbb{N}$ of $\mathbb{N}$ is a bi-quasi-interior ideal of $\mathbb{N}$ but not bi-ideal, quasi-ideal, interior ideal, bi-interior ideal and bi-quasi ideals of $\Gamma-$semiring $\mathbb{N}$.

In the following theorem, we mention some important properties and we omit the proofs since proofs are straight forward.

Theorem 3.1. *Let M be a $\Gamma-$semring. Then the following are hold.*

(1) *Every left ideal is a bi-quasi-interior of M.*
(2) *Every right ideal is a bi-quasi-interior of M.*
(3) *Every quasi ideal is bi-quasi-interior of M.*
(4) *Every ideal is a bi-quasi-interior ideal of M.*
(5) *Intersection of a right ideal and a left ideal of M is a bi-quasi-interior ideal of M.*
(6) *If L is a left ideal and R is a right ideal of $\Gamma-$semiring M then $B = R\Gamma L$ is a bi-quasi-interior ideal of M.*
(7) *If B is a bi-quasi-interior ideal and T is a $\Gamma-$subsemiring of M then $B \cap T$ is a bi-quasi-interior ideals of a $\Gamma-$semiring M.*
(8) *Let M be a $\Gamma-$semiring and B be a $\Gamma-$subsemiring of M. If $B\Gamma M\Gamma M\Gamma M\Gamma B \subseteq B$ then B is a bi-quasi-interior ideal of M.*
(9) *Let M be a $\Gamma-$semiring and B be a $\Gamma-$subsemiring of M. If $M\Gamma M\Gamma M\Gamma M\Gamma B \cap B\Gamma M\Gamma M\Gamma M\Gamma M \subseteq B$ then B is a bi-quasi-interior ideal of M.*

Theorem 3.2. *Every bi-ideal of a $\Gamma-$semiring M is a bi-quasi-interior ideal of a $\Gamma-$semiring M.*

Proof. Let B be a bi-ideal of the $\Gamma-$semiring M. Then $B\Gamma M\Gamma B \subseteq B$. Therefore $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B\Gamma M\Gamma B \subseteq B$. Hence every bi-ideal of the $\Gamma-$semring M is a bi-quasi-interior ideal of M. □

Theorem 3.3. *Every interior ideal of a $\Gamma-$semring M is a bi-quasi-interior ideal of M.*

Proof. Let I be an interior ideal of a $\Gamma-$semring M. Then $I\Gamma M\Gamma I\Gamma M\Gamma I \subseteq M\Gamma I\Gamma M \subseteq I$. Hence I is a bi-quasi-interior ideal of the $\Gamma-$semiring M. □

Theorem 3.4. *Let M be a $\Gamma-$semiring and B be a $\Gamma-$subsemiring of M. B is a bi-quasi-interior ideal of M if and only if there exist a left ideal L and a right idealR such that $R\Gamma L \subseteq B \subseteq R \cap L$.*

Proof. Suppose B is a bi-quasi-interior ideal of the $\Gamma-$semiring$M$. Then $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B$. Let $R = B\Gamma M\Gamma B\Gamma M$ and $L = M\Gamma B$. Then L and R are left and right ideals of M respectively. Therefore $R\Gamma L \subseteq B \subseteq R \cap L$. Conversely suppose that there exist L and R are left and right ideals of M respectively such that $R\Gamma L \subseteq B \subseteq R \cap L$. Then

$$\begin{aligned} B\Gamma M\Gamma B\Gamma M\Gamma B &\subseteq (R \cap L)\Gamma M\Gamma(R \cap L)\Gamma M\Gamma(R \cap L) \\ &\subseteq (R)\Gamma M\Gamma(R)\Gamma M\Gamma(L) \\ &\subseteq R\Gamma L \subseteq B\ . \end{aligned}$$

Hence B is a bi-quasi-interior ideal of the $\Gamma-$semiring M. □

Theorem 3.5. *The intersection of a bi-quasi-interior ideal B of a $\Gamma-$semiring M and a right ideal A of M is always bi-quasi-interior ideal of M.*

Proof. Suppose $C = B \cap A$. Then

$$C\Gamma M\Gamma C\Gamma M\Gamma C \subseteq B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B$$
$$C\Gamma M\Gamma C\Gamma M\Gamma C \subseteq A\Gamma M\Gamma A\Gamma M\Gamma A \subseteq A.$$

since A is a left ideal of M. Therefore

$$C\Gamma A\Gamma C\Gamma M\Gamma C\Gamma M \subseteq B \cap A = C.$$

Hence the intersection of a bi-quasi-interior ideal of B of the $\Gamma-$semiring M and a $\Gamma-$subsemiring A of M is always bi-quasi-interior ideal of M. □

Theorem 3.6. *Let A and C be bi-quasi-interior ideals of a $\Gamma-$semiring M and $B = A\Gamma C$. If $C\Gamma C = C$ then B is a bi-quasi-interior ideal of M.*

Proof. Let A and C be bi-quasi-interior ideals of the$\Gamma-$semiring M and $B = A\Gamma C$. Then

$$\begin{aligned} B\Gamma B = A\Gamma C\Gamma A\Gamma C &= A\Gamma C\Gamma C\Gamma C\Gamma M\Gamma C\Gamma A\Gamma C \\ &\subseteq A\Gamma C\Gamma M\Gamma C\Gamma M\Gamma C \subseteq A\Gamma C = B. \end{aligned}$$

Obviously $B = A\Gamma C$ is a $\Gamma-$subsemiring of M

$$\begin{aligned} B\Gamma M\Gamma B\Gamma M\Gamma B &= A\Gamma C\Gamma M\Gamma A\Gamma C\Gamma M\Gamma A\Gamma C \\ &\subseteq A\Gamma M\Gamma A\Gamma M\Gamma A\Gamma C \subseteq A\Gamma C = B. \end{aligned}$$

Hence B is a bi-quasi-interior ideal of M. □

Corollary 3.1. *Let A and C be bi-quasi-interior ideals of the $\Gamma-$semiring M and $B = C\Gamma A$. If $C\Gamma C = C$ then B is a bi-quasi-interior ideal of M.*

Theorem 3.7. *the Let A and C be $\Gamma-$subsemirings of M and $B = A\Gamma C$. If A is the left ideal then B is a bi-quasi-interior ideal of M.*

PROOF. Let A and C be $\Gamma-$subsemirings of M and $B = A\Gamma C$ Suppose A is the left ideal of M. $B\Gamma B = A\Gamma C\Gamma A\Gamma C \subseteq A\Gamma C = B$.

$$B\Gamma M\Gamma B\Gamma M\Gamma B = A\Gamma C\Gamma M\Gamma A\Gamma C\Gamma M\Gamma A\Gamma C$$
$$\subseteq A\Gamma C = B.$$

Hence B is a bi-quasi-interior ideal of M. □

COROLLARY 3.2. *Let A and C be $\Gamma-$subsemirings of a$\Gamma-$semiring M and $B = A\Gamma C$. If C is a right ideal then B is a bi-quasi-interior ideal of M.*

THEOREM 3.8. *Let M be a $\Gamma-$semiring and T be a non-empty subset of M. Then every subsemiring of T containing $T\Gamma M\Gamma T\Gamma M\Gamma T$ is a bi-quasi-interior ideal of $\Gamma-$semiring M.*

PROOF. Let B be a subsemiring of T containing $T\Gamma M\Gamma T\Gamma M\Gamma T$. Then

$$B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq T\Gamma M\Gamma T\Gamma M\Gamma T \subseteq B.$$

Therefore $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B$. Hence B is a bi-quasi-interior ideal of M. □

THEOREM 3.9. *B is a bi-quasi-interior ideal of a $\Gamma-$semiring M if and only if B is a left ideal of some right ideal of $\Gamma-$semiring M.*

PROOF. Suppose B is a left ideal of some right ideal R of the $\Gamma-$semiring M. Then $R\Gamma B \subseteq B, R\Gamma M \subseteq B$. Hence

$$B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B\Gamma M\Gamma B \subseteq R\Gamma M\Gamma B \subseteq R\Gamma B \subseteq B.$$

Therefore B is a bi-quasi-interior ideal of the $\Gamma-$semiring M. Conversely suppose that B is a bi-quasi-interior ideal of the $\Gamma-$semiring M .Then $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B$. Therefore B is a left ideal of right ideal $B\Gamma M\Gamma B\Gamma M$ of the $\Gamma-$semiring$M$. □

COROLLARY 3.3. *B is a bi-quasi-interior ideal of a $\Gamma-$semiring M if and only if B is a right ideal of some left ideal of a $\Gamma-$semiring$M$.*

THEOREM 3.10. *If B is a bi-quasi-interior ideal of a $\Gamma-$semiring M, T is a $\Gamma-$subsemiring of M and $T \subseteq B$ then $B\Gamma T$ is a bi-quasi-interior ideal of M.*

PROOF. Obviously, $B\Gamma T$ is a $\Gamma-$subsemigroup of $(M, +)$. Then $B\Gamma T\Gamma B\Gamma T \subseteq B\Gamma T$. Hence $B\Gamma T$ is a $\Gamma-$subsemiring of M. We have

$$M\Gamma B\Gamma T\Gamma M \subseteq M\Gamma B\Gamma M$$
$$B\Gamma T\Gamma M\Gamma B\Gamma T \subseteq B\Gamma M\Gamma B$$

From this follows

$$B\Gamma T\Gamma M\Gamma B\Gamma T\Gamma M\Gamma B\Gamma T \subseteq B\Gamma M\Gamma B\Gamma M\Gamma B\Gamma T \subseteq B\Gamma T.$$

Hence $B\Gamma T$ is a bi-quasi-interior ideal of the $\Gamma-$semiring M. □

THEOREM 3.11. *Let B be bi-ideal of a $\Gamma-$semiring M and I be interior ideal of M. Then $B \cap I$ is bi-quasi-interior idea of M.*

Proof. Obviously $B\cap I$ is $\Gamma-$subsemiring of M. Suppose B is a bi-ideal of M and I is an interior ideal of M. Then

$$(B\cap I)\Gamma M\Gamma(B\cap I)\Gamma M\Gamma(B\Gamma I)\subseteq B(B\cap I)\Gamma M\Gamma(B\cap I)\Gamma M\Gamma(B\Gamma I)\Gamma M\subseteq I$$

Therefore $(B\cap I)\Gamma M\Gamma(B\cap I)\Gamma M\Gamma(B\Gamma I)\Gamma M\subseteq B\cap I$. Hence $B\cap I$ is a bi-quasi-interior ideal of the $\Gamma-$semiring M. □

Theorem 3.12. *Let M be a $\Gamma-$semiring and T be an additive subsemigroup of M. Then every additive subsemigroup of T containing $T\Gamma M\Gamma T\Gamma M\Gamma T$ is a bi-quasi-interior ideal of M.*

Proof. Let C be an additive subsemigroup of T containing $T\Gamma M\Gamma T\Gamma M\Gamma T$. Then

$$\begin{aligned}C\Gamma M\Gamma C\Gamma M\Gamma C&\subseteq T\Gamma M\cap T\Gamma M\Gamma T\\&\subseteq C.\end{aligned}$$

Hence C is a bi-quasi-interior ideal of the $\Gamma-$semiring M. □

Theorem 3.13. *Let M be a $\Gamma-$semiring. If $M=M\Gamma a$, for all $a\in M$. Then every bi-quasi-interior ideal of M is a quasi ideal of M.*

Proof. Let B be a bi-quasi-interior ideal of the $\Gamma-$semiring M and $a\in B$. Then

$$\begin{aligned}&B\Gamma M\Gamma B\Gamma M\Gamma B\subseteq B\\ \Rightarrow&M\Gamma a\subseteq M\Gamma B,\ (B\Gamma M=M)\\ \Rightarrow&M\subseteq M\Gamma B\subseteq M\\ \Rightarrow&M\Gamma B=M\\ \Rightarrow&B\Gamma M=B\Gamma M\Gamma B\subseteq B\Gamma M\Gamma B\Gamma M\Gamma B\subseteq B\\ \Rightarrow&M\Gamma B\cap B\Gamma M\subseteq M\Gamma M\cap B\Gamma M\subseteq B.\end{aligned}$$

Therefore B is a quasi ideal of M. Hence the theorem. □

Theorem 3.14. *The intersection of $\{B_\lambda\mid\lambda\in A\}$ bi-quasi-interior ideal of a $\Gamma-$semiring M is a bi-quasi-interior ideal of M.*

Proof. Let $B=\bigcap\limits_{\lambda\in A}B_\lambda$. Then B is a $\Gamma-$subsemiring of M.
Since B_λ is a bi-quasi-interior ideal of M, we have

$$\begin{aligned}&B_\lambda\Gamma M\Gamma B_\lambda\Gamma M\Gamma B_\lambda\subseteq B_\lambda,\ \text{for all }\lambda\in A\\ \Rightarrow&\cap B_\lambda\Gamma M\Gamma\cap B_\lambda\Gamma M\cap B_\lambda)\subseteq\cap B_\lambda\\ \Rightarrow&B\Gamma M\Gamma B\Gamma M\Gamma B\subseteq B.\end{aligned}$$

Hence B is a bi-quasi-interior ideal of M. □

Theorem 3.15. *Let B be a bi-quasi-interior ideal of a $\Gamma-$semiring M,$e\in B$ and e be $\beta-$idempotent. Then $e\Gamma B$ is a bi-quasi-interior ideal of M.*

PROOF. Let B be a bi-quasi-interior ideal of the $\Gamma-$semiring M. Suppose $x \in B \cap e\Gamma M$. Then $x \in B$ and $x = e\alpha y, \alpha \in \Gamma, y \in M$.

$$\begin{aligned} x &= e\alpha y \\ &= e\beta e\alpha y \\ &= e\beta(e\alpha y) \\ &= e\beta x \in e\Gamma B. \end{aligned}$$

Therefore

$$\begin{aligned} B \cap e\Gamma M &\subseteq e\Gamma B \\ e\Gamma B &\subseteq B \text{ and } e\Gamma B \subseteq e\Gamma M \\ \Rightarrow e\Gamma B &\subseteq B \cap e\Gamma M \\ \Rightarrow e\Gamma B &= B \cap e\Gamma M. \end{aligned}$$

Hence $e\Gamma B$ is the bi-quasi-interior ideal of M. □

COROLLARY 3.4. *Let M be a $\Gamma-$semiring M and e be $\alpha-$idempotent. Then $e\Gamma M$ and $M\Gamma e$ are bi-quasi-interior ideals of M respectively.*

THEOREM 3.16. *If B be a left bi-quasi ideal of a $\Gamma-$semiring M, then B is a bi-quasi-interior ideal of M.*

PROOF. Suppose B is a left bi-quasi ideal of the $\Gamma-$semiring M. Then

$$B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq M\Gamma B \text{ and } B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B\Gamma M\Gamma B.$$

Therefore

$$B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq M\Gamma B \cap M\Gamma B\Gamma M \subseteq B.$$

Hence B is a bi-quasi-interior ideal of M. □

COROLLARY 3.5. *If B be a right bi-quasi ideal of a $\Gamma-$semiring M, then B is a bi-quasi-interior ideal of M.*

COROLLARY 3.6. *If B be a bi-quasi ideal of a $\Gamma-$semiring M, then B is a bi-quasi-interior ideal of M.*

THEOREM 3.17. *If B be a bi–interior ideal of $\Gamma-$ semiring of M, then B is a bi-quasi-interior ideal of M.*

PROOF. Suppose B is a bi–interior ideal of $\Gamma-$semiring M. Then

$$B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq M\Gamma B\Gamma M \text{ and } B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq M\Gamma B\Gamma M.$$

Therefore $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq M\Gamma B\Gamma M \cap M\Gamma B\Gamma M \subseteq B$. Hence B is a bi-quasi-interior ideal of M. □

4. Bi-quasi-interior simple $\Gamma-$semiring

In this section,we introduce the notion of bi-quasi-interior simple $\Gamma-$semiring and characterize the bi-quasi-interior simple $\Gamma-$semiring using bi-quasi-interior ideals of $\Gamma-$semiring and study the properties of minimal bi-quasi-interior ideals of $\Gamma-$semiring.

Definition 4.1. A $\Gamma-$semiring M is said to be bi-quasi-interior simple Γ -semiring if M has no bi-quasi-interior ideals other than M itself.

Theorem 4.1. *If M is a division $\Gamma-$semiring then M is a bi-quasi-interior simple $\Gamma-$semiring.*

Proof. Let B be a proper bi-quasi-interior ideal of the division $\Gamma-$semring and $0 \neq a \in B$. Since M is a division $\Gamma-$semiring, there exist $b \in M, \alpha \in \Gamma$ such that $a\alpha b = 1$. Then there exist $\beta \in \Gamma, x \in M$ such that $a\alpha b\beta x = x = x\beta a\alpha b$. Then $x \in B\Gamma M$. Therefore $M \subseteq B\Gamma M$. We have $B\Gamma M \subseteq M$. Hence $M = B\Gamma M$. Similarly we can prove $M\Gamma B = M$.

$$\begin{aligned} M &= M\Gamma B \\ &= B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B \\ M &\subseteq B. \end{aligned}$$

Therefore $M = B$. Hence division $\Gamma-$semiring M has no properbi-quasi-interior ideals. □

Theorem 4.2. *Let M be a simple $\Gamma-$semiring. Every bi-quasi-interior ideal is bi-ideal of M.*

Proof. Let M be a simple $\Gamma-$semring and B be a bi-quasi-interior ideal of M. Then $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B$ and $M\Gamma B\Gamma M$ is an ideal of M. Since M is a simple $\Gamma-$semiring, we have $M\Gamma B\Gamma M = M$. Hence

$$\begin{aligned} &B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B \\ \Rightarrow\ &B\Gamma M\Gamma B \subseteq B. \end{aligned}$$

Hence the theorem. □

Theorem 4.3. *Let M be a $\Gamma-$semiring. Then M is a bi-quasi-interior simple $\Gamma-$semiring if and only if $(a)_{bqi} = M$, for all $a \in M$, where $(a)_{bqi}$ is the bi-quasi-interior ideal generated by a.*

Proof. Let M be a $\Gamma-$semiring. Suppose that $(a)_{bqi}$ is a bi-quasi-interior ideal generated by a and M is a bi-quasi -interior simple $\Gamma-$semiring. Then $(a)_{bqi} = M$, for all $a \in M$.

Conversely suppose that B is a bi-quasi-interior ideal of $\Gamma-$semiring M and $(a)_{bqi} = M$, for all $a \in M$. Let $b \in B$. Then $(b)_{bqi} \subseteq B \ \Rightarrow\ M = (b)_{bqi} \subseteq B \subseteq M$. Therefore M is a bi-quasi-interior simple $\Gamma-$semiring. □

Theorem 4.4. *Let M be a $\Gamma-$semiring. Then M is a bi-quasi-interior simple $\Gamma-$semiring if and only if $< a >= M$, for all $a \in M$ and where $< a >$ is the smallest bi-quasi-interior ideal generated by a.*

PROOF. Let M be a $\Gamma-$semiring. Suppose M is a bi-quasi-interior simple $\Gamma-$semiring, $a \in M$ and $B = M\Gamma a$. Then B is a left ideal of M. Therefore, by Theorem 3.4, B is a bi-quasi-interior ideal of M. Therefore $B = M$. Hence $M\Gamma a = M$, for all $a \in M$.

$$\begin{aligned} &M\Gamma a \subseteq < a > \subseteq M \\ \Rightarrow &M \subseteq < a > \subseteq M. \end{aligned}$$

Therefore $M =< a >$.

Suppose $< a >$ is the smallest bi-quasi-interior ideal of M generated by a and $< a >= M$ and A is the bi-quasi-interior ideal and $a \in A$. Then

$$\begin{aligned} &< a > \subseteq A \subseteq M \\ \Rightarrow &M \subseteq A \subseteq M. \end{aligned}$$

Therefore $A = M$. Hence M is a bi-quasi-interior simple $\Gamma-$semiring. □

THEOREM 4.5. *Let M be a $\Gamma-$semiring. Then M is a bi-quasi-interior simple $\Gamma-$semriring if and only if $a\Gamma M\Gamma a\Gamma M\Gamma a = M$, for all $a \in M$.*

PROOF. Suppose M is left bi-quasi simple $\Gamma-$semiring and $a \in M$. Therefore $a\Gamma M\Gamma a\Gamma M\Gamma a$ is a bi-quasi-interior ideal of M. Hence $a\Gamma M\Gamma a\Gamma M\Gamma a = M$, for all $a \in M$.

Conversely suppose that $a\Gamma M\Gamma a\Gamma M\Gamma a = M$, for all $a \in M$. Let B be a bi-quasi-interior ideal of the $\Gamma-$semiring M and $a \in B$.

$$\begin{aligned} M &= a\Gamma M\Gamma a\Gamma M\Gamma a \\ &\subseteq B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B. \end{aligned}$$

Therefore $M = B$. Hence M is a bi-quasi-interior simple $\Gamma-$semiring. □

THEOREM 4.6. *If B is a minimal bi-quasi-interior ideal of a $\Gamma-$semiring M then any two non-zero elements of B generated the same right ideal of M.*

PROOF. Let B be a minimal bi-quasi-interior ideal of M and $x \in B$. Then $(x)_R \cap B$ is a bi-quasi-interior ideal of M. Therefore $(x)_R \cap B \subseteq B$. Since B is a minimal bi-quasi-interior ideal of M, we have $(x)_R \cap B = B \Rightarrow B \subseteq (x)_R$. Suppose $y \in B$. Then $y \in (x)_R$, $(y)_R \subseteq (x)_R$. Therefore $(x)_R = (y)_R$. Hence the theorem. □

COROLLARY 4.1. *If B is a minimal bi-quasi-interior ideal of a $\Gamma-$semiring M, then any two non-zero elements of B generates the same left ideal of M.*

DEFINITION 4.2. A $\Gamma-$semiring M is a left (right) simple $\Gamma-$semiring if M has no proper left (right) ideal of M.

A $\Gamma-$semiring M is said to be simple $\Gamma-$semring if M has no proper ideals.

THEOREM 4.7. *If $\Gamma-$semring M is left simple $\Gamma-$semiring, then every bi-quasi-interior ideal of M is a right ideal of M.*

PROOF. Let B be a bi-quasi-interior of the left simple $\Gamma-$semiring M. Then $M\Gamma B$ is a left ideal of M and $M\Gamma B \subseteq M$. Therefore $M\Gamma B = M$. Then

$$\begin{aligned}
&M\Gamma B\Gamma M = M\Gamma M = M\\
&\Rightarrow B\Gamma M = B\Gamma M\Gamma B\\
&\Rightarrow B\Gamma M = B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B\\
&\Rightarrow B\Gamma M \subseteq B.
\end{aligned}$$

Hence every bi-quasi-interior ideal is a right ideal of M. □

COROLLARY 4.2. *If $\Gamma-$semring M is a right simple $\Gamma-$semiring, then every bi-quasi-interior ideal of M is a left ideal of M.*

COROLLARY 4.3. *Every bi-quasi-interior ideal of a left and right simple Γ - semiring M is an ideal of M.*

THEOREM 4.8. *Let M be a $\Gamma-$semiring and B be bi-quasi-interior ideal of M. Then B is minimal bi-quasi-interior ideal of M if and only if B is a bi-quasi-interior simple $\Gamma-$susemiring.*

PROOF. Let B be a minimal bi-quasi-interior ideal of a $\Gamma-$semiring M and C be a bi-quasi-interior ideal of B. Then

$$C\Gamma B\Gamma C\Gamma B\Gamma C \subseteq C.$$

Therefore $C\Gamma B\Gamma C\Gamma B\Gamma C\Gamma B$ is a bi-quasi-interior ideal of M. Since C is a bi-quasi-interior ideal of B, we have

$$\begin{aligned}
&C\Gamma B\Gamma C\Gamma B\Gamma C = B\\
&\Rightarrow B = C\Gamma B\Gamma C\Gamma B\Gamma C \subseteq C\\
&\Rightarrow B = C.
\end{aligned}$$

Conversely suppose that B is a bi-quasi-interior simple $\Gamma-$subsemiring of M. Let C be a bi-quasi-interior ideal of M and $C \subseteq B$. Since B is a bi-quasi-interior simple $\Gamma-$semiring, we have

$$\begin{aligned}
&C\Gamma B\Gamma C\Gamma B\Gamma C = C\\
&\Rightarrow C\Gamma B\Gamma C\Gamma B\Gamma C \subseteq C\Gamma M\Gamma C\Gamma M\Gamma C \subseteq B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B,\\
&\Rightarrow B = C.
\end{aligned}$$

Hence B is a minimal bi-quasi-interior ideal of M. □

THEOREM 4.9. *Let M be a $\Gamma-$semiring and $B = R\Gamma L$, where L and R are minimal left and right ideals of M repectively. Then B is a minimal bi-quasi-interior ideal of M.*

PROOF. Obviously $B = R\Gamma L$ is bi-quasi-interior ideal of M. Let A be bi-quasi-interior ideal of M such that $A \subseteq B$.

We have $M\Gamma A$ is a right ideal. Then

$$\begin{aligned} M\Gamma A &\subseteq M\Gamma B \\ &= M\Gamma R\Gamma L \\ &\subseteq L, \end{aligned}$$

since L is a left ideal of M. Similarly, we can prove $A\Gamma M \subseteq R$. Therefore $M\Gamma A = L$ and $A\Gamma M = R$. Hence

$$\begin{aligned} B &= A\Gamma M\Gamma M\Gamma A \\ &\subseteq A\Gamma M\Gamma A\Gamma M\Gamma A. \\ &\subseteq A. \end{aligned}$$

Therefore $A = B$. Hence B is a minimal bi-quasi-interior ideal of M.

□

5. Regular $\Gamma-$semiring

In this section,we characterize regular $\Gamma-$semiring using bi-quasi-interior ideals of $\Gamma-$semiring.

THEOREM 5.1. *Let M be a regular $\Gamma-$semiring. Then every bi-quasi-interior ideal of M is an ideal of M.*

PROOF. Let B be a bi-quasi-interior ideal of M. Then

$$\begin{aligned} &B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B \\ \Rightarrow &B\Gamma M \subseteq B\Gamma M\Gamma B, \text{ since } M \text{ is regular} \\ \Rightarrow &B\Gamma M \subseteq B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B. \end{aligned}$$

Similarly, we can show that $M\Gamma B \subseteq B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B$. Hence the theorem. □

THEOREM 5.2. *Let M be a regular $\Gamma-$semiring. Then every bi-quasi-interior ideal is an ideal of M.*

PROOF. Let I be a bi-quasi-interior ideal of M. By Theorem 5.1, I is an ideal of M. Hence the theorem. □

THEOREM 5.3. *Let M be a regular $\Gamma-$semiring and I be an interior ideal of M. Then $M\Gamma I\Gamma M = I$.*

PROOF. Let $a \in M$. Then there exist $x \in M, \alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. Thus there exist $\delta, \gamma \in \Gamma$ and $y \in M$ such that $x\beta a = x\beta a\delta y\gamma x\beta a$. Therefore

$$a = a\alpha x\beta a = a\alpha x\beta a\delta y\gamma x\beta a \in M\Gamma I\Gamma M.$$

Hence $I \subseteq M\Gamma I\Gamma M$. We have $M\Gamma I\Gamma M \subseteq I$. Hence $M\Gamma I\Gamma M = I$. □

THEOREM 5.4. *Let M be a regular $\Gamma-$semiring. Then B is a bi-quasi-interior ideal of M if and only if $B\Gamma M\Gamma B\Gamma M\Gamma B = B$, for all bi-quasi-interior ideals B of M.*

Proof. Suppose M is the regular $\Gamma-$semiring M , B is a bi-interior ideal of M and $x \in B$. Then $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B$ and there exist $y \in M,\ \alpha, \beta \in \Gamma$ such that $x = x\alpha y\beta x\alpha y\beta x \in B\Gamma M\Gamma B\Gamma M\Gamma B$. Therefore $x \in B\Gamma M\Gamma B\Gamma M\Gamma B$. Hence $B\Gamma M\Gamma B\Gamma M\Gamma B = B$.

Conversely suppose that $B\Gamma M\Gamma B\Gamma M\Gamma B = B$, for all bi-quasi-interior ideals B of M. Let $B = R \cap L$, where R is a right ideal and L is a left ideal of M. Then B is a bi-interior ideal of M. Therefore $(R \cap L)\Gamma M\Gamma(R \cap L)\Gamma M\Gamma(R \cap L) = R \cap L$

$$\begin{aligned} R \cap L &= (R \cap L)\Gamma M\Gamma(R \cap L)\Gamma M\Gamma(R \cap L) \\ &\subseteq R\Gamma M\Gamma L\Gamma M\Gamma L \\ &\subseteq R\Gamma L \\ &\subseteq R \cap L \text{ (since } R\Gamma L \subseteq L \text{ and } R\Gamma L \subseteq R). \end{aligned}$$

Therefore $R \cap L = R\Gamma L$. Hence M is a regular $\Gamma-$semiring. $\square$

Theorem 5.5. *Let B be the bi-quasi-interior ideal of regular $\Gamma-$semiring M. If B is a bi-quasi-interior ideal of M and B is regular $\Gamma-$subsemiring of M then any bi-quasi-interior ideal of B is a bi-quasi-interior ideal of M.*

Proof. Let A be a bi-quasi-interior ideal of the regular $\Gamma-$subsemiring B of M. Then by Theorem 5.4, $A\Gamma B\Gamma A\Gamma B\Gamma A = A$. We have $B\Gamma M\Gamma B\Gamma M\Gamma B = B$ and $A \subseteq A\Gamma B$. Now

$$\begin{aligned} & A\Gamma M\Gamma A\Gamma M\Gamma A \subseteq B\Gamma M\Gamma B\Gamma M\Gamma B = B \\ \Rightarrow\ & A\Gamma B\Gamma A\Gamma B\Gamma A = A \subseteq B \\ \Rightarrow\ & A = A\Gamma B\Gamma A\Gamma B\Gamma A \subseteq A\Gamma M\Gamma A\Gamma M\Gamma A \\ \Rightarrow\ & A\Gamma M\Gamma A\Gamma M\Gamma A = A. \end{aligned}$$

Hence A is a bi-quasi-interior ideal of M. $\square$

Theorem 5.6. *M is a regular $\Gamma-$semiring if and only if $A\Gamma B = A \cap B$ for any right ideal A and left ideal B of a $\Gamma-$semiring M.*

Proof. Let A, B be a right ideal and a left ideal of the regular ordered $\Gamma-$semiring M respectively. Obviously $A\Gamma B \subseteq A \cap B$. Let $x \in A \cap B$. Since M is a regular, there exist $\alpha, \beta \in \Gamma$ and $y \in M$ such that $x = x\alpha y\beta x$. Since $x\alpha y \in A$ and $x \in B$, $x\alpha y\beta x \in A\Gamma B$. Thus $x \in A\Gamma B$. Hence $A\Gamma B = A \cap B$.

Conversely, suppose that $A\Gamma B = A \cap B$ for any right ideal A and left ideal B of M. Let $x \in M$ and I be the right ideal generated by x and J be the left ideal generated by x. We have $x \in I \cap J = I\Gamma J$. Therefore $x = x\alpha y = z\beta x, \alpha, \beta \in \Gamma, y, z \in M$ which implies that $x = x\alpha y\gamma z\beta x$, for some $\gamma \in \Gamma$. Hence M is a regular ordered $\Gamma-$semiring. $\square$

Theorem 5.7. *Let B be $\Gamma-$subsemiring of a regular $\Gamma-$semiring M. If B can be represented as $B = R\Gamma L$, where R is a right ideal and L is a left ideal of M, then B is a bi-quasi-interior ideal of M.*

PROOF. Suppose $B = R\Gamma L$, where R is right ideal of M and L is a left ideal of M. Then

$$\begin{aligned} B\Gamma M\Gamma B\Gamma M\Gamma B &= R\Gamma L\Gamma M\Gamma R\Gamma LM\Gamma R\Gamma L \\ &\subseteq R\Gamma L = B. \end{aligned}$$

Hence B is a bi-quasi-interior ideal of the $\Gamma-$semiring M.

Conversely suppose that B is a bi-quasi-interior ideal of the regular $\Gamma-$semiring M. We have $B\Gamma M\Gamma B\Gamma M\Gamma B = B$. Let $R = B\Gamma M$ and $L = M\Gamma B$. Then $R = B\Gamma M$ is a right ideal of M and $L = M\Gamma B$ is a left ideal of M.

$$\begin{aligned} &B\Gamma M \cap M\Gamma B \subseteq B\Gamma M\Gamma B\Gamma M\Gamma B = B \\ \Rightarrow &B\Gamma M \cap M\Gamma B \subseteq B \\ \Rightarrow &R \cap L \subseteq B. \end{aligned}$$

We have

$$\begin{aligned} &B \subseteq B\Gamma M = R \text{ and } B \subseteq M\Gamma B = L \\ \Rightarrow &B \subseteq R \cap L \\ \Rightarrow &B = R \cap L = R\Gamma L, \text{ since } M \text{ is a regular } \Gamma-\text{semiring}. \end{aligned}$$

Hence B can be represented as $R\Gamma L$, where R is the right ideal and L is the left ideal of M. Hence the theorem. □

The following theorem is a necessary and sufficient condition for $\Gamma-$semiring M to be regular using bi-quasi-interior ideal.

THEOREM 5.8. *M is a regular $\Gamma-$semriring if and only if $B \cap I \cap L \subseteq B\Gamma I\Gamma L$, for any bi-quasi-interior ideal B, ideal I and left idealL of M.*

PROOF. Suppose M be a regular $\Gamma-$semiring, B, I and L are bi-quasi-interior ideal, ideal and left ideal of M respectively. Let $a \in B \cap I \cap L$. Then $a \in a\Gamma M\Gamma a$, since M is regular.

$$\begin{aligned} a \in a\Gamma M\Gamma a &\subseteq a\Gamma M\Gamma a\Gamma M\Gamma a \\ &\subseteq B\Gamma I\Gamma B \end{aligned}$$

Hence $B \cap I \cap L \subseteq B\Gamma I\Gamma L$.

Conversely suppose that $B \cap I \cap L \subseteq B\Gamma I\Gamma L$, for any bi-quasi-interior ideal B, ideal I and left ideal L of M. Let R be a right ideal and L be left ideal of M. Then by assumption

$$R \cap L = R \cap M \cap L \subseteq R\Gamma M\Gamma LR\Gamma L.$$

We have $R\Gamma L \subseteq R$ and $R\Gamma L \subseteq L$. Therefore $R\Gamma L \subseteq R \cap L$. Hence $R \cap L = R\Gamma L$. Thus M is a regular $\Gamma-$semiring. □

6. Conclusion

As a further generalization of ideals ,we introduced the notion of bi-quasi-interior ideal of $\Gamma-$semiring as a generalization of ideal ,left ideal, right ideal, bi-ideal,quasi ideal and interior ideal of semigroup and studied some of their properties. We introduced the notion of bi-quasi-interior simple $\Gamma-$semiring and characterized the bi-quasi-interior simple $\Gamma-$semiring ,regular $\Gamma-$semiring using bi-quasi-interior ideals of $\Gamma-$semiring.We proved every bi-quasi ideal of $\Gamma-$semiring and bi-interior ideal of $\Gamma-$semiring are bi-quasi-interior ideals and studied some of the properties of bi-interior ideals of$\Gamma-$semiring. In continuity of this paper, we study prime bi-quasi-interior ideals,prime, maximal and minimal bi-quasi-interior ideals of $\Gamma-$semiring.

References

[1] P. J. Allen. A fundamental theorem of homomorphism for semirings. *Proc. Amer. Math. Sco.,* **21**(2)(1969), 412–416.

[2] T. K. Dutta and S.K. Sardar. On the operator semirings of a $\Gamma-$semiring. *Southeast Asian Bull. Math.,* **26**(2)(2002), 203–213.

[3] R. A. Good and D. R. Hughes. Associated groups for a semigroup. *Bull. Amer. Math. Soc.,* **58**(6)(1952), 624–625.

[4] M. Henriksen. Ideals in semirings with commutative addition. *Amer. Math. Soc. Notices,* **6** (1958), pp. 321.

[5] K. Iseki. Quasi-ideals in semirings without zero. *Proc. Japan Acad.,* **34**(1958), 79–84.

[6] K. Iseki. Ideal theory of semiring. *Proc. Japan Acad.,* **32**(8) (1956), 554-559.

[7] K. Iseki. Ideal in semirings. *Proc. Japan Acad.,* **34**(1)(1958), 29–31.

[8] K. Izuka. On the Jacobson radical of a semiring. *Tohoku Math. J.,* **11**(2)(1959), 409–421.

[9] R. D. Jagatap and Y.S. Pawar. Quasi-ideals and minimal quasi-ideals in $\Gamma-$ semirings. *Novi Sad J. Math.,* **39**(2)(2009), 79–87.

[10] J. D. Jagatap and Y. S. Pawar. Bi-ideals in $\Gamma-$ semirings. *Bull. Int. Math. Virtual Inst.,* **6**(2)(2016), 169–179.

[11] S. Lajos. On the bi-ideals in semigroups. *Proc. Japan Acad.,* **45**(8)(1969), 710–712.

[12] S. Lajos and F. A. Szasz. On the bi-ideals in associative ring. *Proc. Japan Acad.,* **46**(6)(1970), 505–507.

[13] H. Lehmer. A ternary analogue of abelian groups. *Amer. J. Math.,* **54**(2)(1932), 329–338.

[14] G. Lister. Ternary rings. *Trans. of Amer. Math. Soc.,* **154** (1)(1971), 37–55.

[15] M. Murali Krishna Rao. $\Gamma-$semirings, I. *Southeast Asian Bull. Math.,* **19**(1)(1995), 49–54.

[16] M. Murali Krishna Rao, Γ-semirings, II. *Southeast Asian Bull. of Math.,* **21**(3)(1997), 281–287

[17] M.Murali Krishna Rao. The Jacobson radical of Γ-semiring. *Southeast Asian Bull. of Math.,* **23**(1)(1999), 127–134.

[18] M. Murali Krishna Rao. Ideals in ordered $\Gamma-$ semirings. *Discuss. Math. Gen. Algebra Appl.,* **38**(1)(2018), 47–68.

[19] M. Murali Krishna Rao. $\Gamma-$ semiring with identity. *Discuss. Math. Gen. Algebra Appl.,* **37**(2)(2017), 189–207.

[20] M. Murali Krishna Rao. Bi-interior Ideals in semigroups. (Accepted for publication in *Discuss. Math. Gen. Algebra Appl.*)

[21] M. Murali Krishna Rao. T-fuzzy ideals in ordered $\Gamma-$semirings. *Ann. Fuzzy Math. Info.,* **13**(2)(2017), 253–276.

[22] M. Murali Krishna Rao. Left bi-quasi ideals of semirings. *Bull. Int. Math. Virtual Inst. ,* **8**(1)(2018), 45–53 .

[23] M.Murali Krishna Rao. Bi-quasi-ideals and fuzzy bi-quasi-ideals of $\Gamma-$ semigroups, *Bull. Int. Math. Virtual Inst.,* **7**(2)(2017), 231–242.

[24] M. Murali Krishna Rao, B. Venkateswarlu and N.Rafi. Left Bi-quasi-ideals of $\Gamma-$semirings. *Asia Pacific J. of Math.,* **4**(2)(2017), 144–153.
[25] M. Murali Krishna Rao and B. Venkateswarlu. On generalized right derivations of $\Gamma-$ incline. *J. Int. Math. Virtual Inst.,* **6**(2016), 31–47.
[26] M. Murali Krishna Rao and B. Venkateswarlu. Right derivation of ordered $\Gamma-$semirings. *Discuss. Math. Gen. Algebra Appl.* **36**(2)(2016), 209–221.
[27] M. Murali Krishna Rao and B. Venkateswarlu. Regular $\Gamma-$incline and field $\Gamma-$semiring. *Novi Sad J. of Math.,* **45**(2)(2015), 155–171.
[28] N. Nobusawa. On a generalization of the ring theory. *Osaka. J.Math.,* **1**(1)(1964), 81 – 89.
[29] M. K. Sen. On $\Gamma-$semigroup, In *Proceeding of International Conference on Algebra and its Applications*,(New Delhi, 1981) (pp. 301308). Lecture Notes in Pure and Appl. Math. 9, New York: Decker Publication, 1984.
[30] A. M. Shabir and A Batod. A note on quasi ideal in semirings. *Southeast Asian Bull. Math.,* **27**(5)(2004), 923–928.
[31] O. Steinfeld. Uher die quasi ideals. *Von halbgruppend Publ. Math.,* Debrecen, **4** (1956), 262 - 275.
[32] H. S. Vandiver. Note on a simple type of algebra in which cancellation law of addition does not hold. *Bull. Amer. Math.,* **40** (1934), 914–920.

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