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A COMMON FIXED POINT RESULT FOR TWO PAIRS OF WEAKLY TANGENTIAL MAPS IN B-METRIC SPACES

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Abstract. In a previous paper, published by the author in 2011, the so called property (W.T) was introduced. By this property, it is aimed a common generalization of several concepts, like the concept of noncompatible mappings due to Jungck (1986), the property (E.A) of Aamri and Moutawakil (2002) and the concept of asymptotically regular maps due to Browder and Petryshyn (1966). The purpose of this paper is to use that property (W.T) to prove a general common fixed point result for two pairs of weakly compatible maps under a contractive condition of Lipschitz type in the setting of b-metric spaces. The well-posedness of the fixed point problem for these maps is also investigated. Our main result involves a Lipschitz type condition which is is not a contractive condition of the classical type. An example applying our result is furnished.

1. Introduction

Let $(X, d; s)$ be a b-metric space with constant $s \geqslant 1$ (see Definition 2.1 below) and let A, B, S and T be selfmappings of the b-metric sapce $(X, d; s)$.

To simplify notations, for all $x, y \in X$, we set

$$\sigma(x,y) := d(Sx,Ty) + d(Sx,Ax) + d(Ty,By) + d(Sx,By) + d(Ty,Ax). \tag{1.1}$$

In this paper, we study the common fixed point problem for two weakly compatible pairs (A, S) and (B, T) of selfmappings of a b-metric space $(X, d; s)$ which are

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satisfying the following Lipshitz type condition: There exists a constant $k \in [0,1)$ such that

$$d(Ax, By) \leqslant k\,\sigma(x,y), \quad \text{for all } x, y \in X. \tag{1.2}$$

The contractive condition (1.2) is not strong enough to ensure common fixed points. It appeared in some papers (see [**5**] and the references therein). Here, we discuss conditions on k and on the mappings A, B, S and T ensuring the existence of common fixed points. Our main result (see Theorem 4.1) will make use of the new property called the property (W.T) (see [**5**]) which is weaker than the property (E.A) of Aamri and Moutawakil (see [**2**]). This work provides a natural continuation to the work [**5**], where a common fixed point result for four selfmappings satisfying the contraction (1.2) in a metric space was established. The main result of [**5**] reads as follows..

THEOREM 1.1 ([**5**]). *Let (A,S) and (B,T) be two weakly compatible pairs of selfmappings of a complete metric space (X,d) such that*
(i) : $AX \subseteq TX$ and $BX \subseteq SX$,
(ii) : one of AX, BX, SX or TX is a closed subspace of (X,d),
(iii) : $d(Ax,By) \leqslant k\,\sigma(x,y)$, for all $x,y \in X$, where k is such that $0 \leqslant k < \frac{1}{3}$.
If one of the pairs $\{A,S\}$ or $\{B,T\}$ satisfies the property (W.T), then A,B,S and T have a unique common fixed point.

In this paper, we prove an extension to Theorem 1.1 to b-metric spaces without supposing closedness of one of the ranges. We replace the assumption (ii) by a relaxed condition. (See Theorem 4.1 below).

This paper is organized as follows: In Section two, we recall some basic facts on b-metric spaces. In Section 3, we extend to the b-metric spaces some general definitions dealing with compatibility. We extend also the property (W.T) and show how it generalizes and unifies some well known properties for mappings like noncompatibility, the (E.A) property and other various concepts of asymptotic regularity. In Section 4, we establish our main result (see Theorem 4.1) and provide an illustrative example. In Section 5, we study the well-posedness of the fixed point problem for four maps A,B,S and T satisfying the conditions of Theorem 4.1 on a b-metric space.

2. A brief set-up on b-metric spaces

2.1. Definitions and examples.

DEFINITION 2.1. Let X be a nonempty set. A *b-metric* on X is a function $d : X \times X \to [0,\infty)$ satisfying the conditions

$$\begin{aligned} &\text{(i)} \quad d(x,y) = 0 \iff x = y,\\ &\text{(ii)} \quad d(x,y) = d(y,x),\\ &\text{(iii)} \quad d(x,y) \leqslant s[d(x,z) + d(z,y)], \end{aligned} \tag{2.1}$$

for all $x, y, z \in X$, and for some fixed number $s \geqslant 1$. The triple $(X, d; s)$ is called a b-*metric space* with parameter s. The inequality (iii) is called the s-relaxed triangle inequality or simply the s-triangle inequality.

Obviously, for $s = 1$ in (2.1), the function d becomes a metric on X. In this case the triple $(X, d; 1)$ is simply denoted by (X, d) which is the usual notation for a metric space X endowed with the metric d.

EXAMPLE 2.1. Let (X, d) is a metric space and choose $\beta \geqslant 1$, then $d^\beta(x, y) := [d(x, y)]^\beta$ (for all x and y in X) is a b-metric on X with parameter $2^{\beta-1}$.

Indeed, the axioms (i) and (ii) are satisfied by d^β. By using the following well known inequality:

$$(a+b)^\beta \leqslant 2^{\beta-1}(a^\beta + b^\beta), \quad \text{for all } a, b \in \mathbb{R}^+ := [0, +\infty), \tag{2.2}$$

we obtain

$$d^\beta(x, y) \leqslant [d(x, z) + d(z, y)]^\beta \leqslant 2^{\beta-1}[d^\beta(x, z) + d^\beta(z, y)], \quad \text{for all } x, y, z \in X.$$

EXAMPLE 2.2. Let $(X, d; s)$ be a b-metric space with parameter $s \geqslant 1$ and choose $p \in (0, 1]$. We set $d_p(x, y) := [d(x, y)]^{\frac{1}{p}}$, for all x and y in X. By using the inequality (2.2), it is easy to show that d_p is a b-metric on X with parameter: $s^{\frac{1}{p}}\, 2^{\frac{1}{p}-1}$.

An interesting class of b-metric spaces was introduced by Kirk and Shahzad (see [**28**]) called "strong b-metric spaces".

Let X be a non empty set. According to [**28**], a mapping $d : X \times X \to [0, \infty)$ is called a *strong b-metric* if it satisfies the conditions (i) and (ii) from (2.1) and

$$d(x, y) \leqslant d(x, z) + s d(y, z)\,, \tag{iv}$$

for some $s \geqslant 1$ and all $x, y, z \in X$.

2.2. Some basic results on b-metric spaces. Let $(X, d; s)$ be a b-metric space with parameter $s \geqslant 1$.

(1) By mathematical induction, it is easy to prove the following general s-triangle inequality.

$$d(x_0, x_{n+1}) \leqslant s d(x_0, x_1) + s^2 d(x_1, x_2) + \cdots + s^n d(x_{n-1}, x_n) + s^n d(x_n, x_{n+1})\,,$$

forall $n \in \mathbb{N}$, and for all $x_0, x_1, \ldots, x_{n+1} \in X$.

(2) One can introduce a topology on the b-metric space $(X, d; s)$ as follows.

We start by defining the ball $B(x, r)$ of center $x \in X$ and radius $r > 0$ by setting

$$B(x, r) = \{y \in X : d(x, y) < r\}\,.$$

A nonempty subset Y of X is called open if for every $x \in Y$ there exists a number $r_x > 0$ such that $B(x, r_x) \subset Y$. The empty set is open by definition.

We denote by $\mathcal{T}_d$ (or $\mathcal{T}(d)$) the family of all open subsets of X it follows that $\mathcal{T}_d$ satisfies the axioms of a topology. The topology $\mathcal{T}_d$ is metrizable (see for example [**16**] and the references therein).

A proof of this fact is given M. Paluszyński and K. Stempak in [**36**]. Indeed, for positive number $p \in (0,1]$, we set

$$\rho_p(x,y) = \inf \left\{ \sum_{k=1}^{n} d(x_{i-1}, x_i)^p \right\},$$

where the infimum is taken over all $n \in \mathbb{N}$ and all chains $x = x_0, x_1, \ldots, x_n = y$ of elements in X connecting x and y.

Then we have the folowing result extracted from [**36**].

Theorem 2.1 ([**36**]). *Let d be a b-metric on a nonempty set X satisfying the s-relaxed triangle inequality* (2.1)*.(iii), for some $s \geqslant 1$. If the number $p \in (0,1]$ is given by the equation $(2s)^p = 2$, then the mapping $\rho_p : X \times X \to [0,\infty)$ defined by (2.4) is a metric on X satisfying the inequalities*

$$\rho_p(x,y) \leqslant d^p(x,y) \leqslant 2\rho_p(x,y)\,, \tag{2.3}$$

for all $x, y \in X$.

Under the conditions of the theorem above, the following assertions are immediate consequences from the inequalities (2.3).

(2-a) $\mathcal{T}_d = \mathcal{T}_\rho$, that is, the topology of any b-metric space is metrizable.

(2-b) The convergence of sequences with respect to $\mathcal{T}_d$ is characterized by the following:

$$(x_n) \text{ converges to } x \text{ for the toplogy } \mathcal{T}_d \iff \lim_{n\to+\infty} d(x,x_n) = 0\,,$$

for any sequence (x_n) in X and $x \in X$.

(2-c) The toplogical space $(X, \mathcal{T}_d)$ is Haussdorff, so the limit point of a converging sequence is unique.

(2-d) Let $(X_1, d_1; s_1)$ and $(X_2, d_2; s_2)$ be two b-metric spaces. Let $T : X_1 \to X_2$ be a map. Let $z \in X$. Then the following assertions are equivalent:
(i) The map T is continuous at z.
(ii) For every sequence (x_n) in X , we have:
$\lim_{n\to+\infty} d_1(x_n, z) = 0 \Longrightarrow \lim_{n\to+\infty} d_2(Tx_n, Tz) = 0$.

(3) Contrary to the usual metric case, the topology $\mathcal{T}_d$ generated by a b-metric d has some peculiarities:

(3-a) A ball $B(x,r)$ need not be in $\mathcal{T}_d$. For an example, see [**16**] and [**36**].

(3-b) The b-metric d could not be continuous on $X \times X$ endowed with the product toplogy.

(3-c) The b-metric d could not be separately continuous. That is, in general, for some point $z \in X$, the map $x \mapsto d(x,z)$ could not be continuous on the topological space $(X, \mathcal{T}_d)$.

(4) When $(X, d; s)$ is a strong b-metric space, the drawbacks above disappear. Indeed,

- the openness of the balls $B(x,r)$ is ensured, and

• the continuity of the map d on $X \times X$ (with the toplogical product) is ensured by the following inequality:

$$|d(x,y) - d(u,v)| \leqslant s[d(x,u) + d(y,v)]\,, \forall x,y,u,v \in X.$$

(5) Let $(X,d;s)$ be a b-metric space. For a subset Y of X, we denote $\overline{Y}$ the closure of Y. We recall that, by definition, $\overline{Y}$ is the intersection over all closed subsets of X containing Y.

It is easy to prove the following lemma which provides two characterizations of the closure of a subset of a b-metric space.

LEMMA 2.1. *Let $(X,d;s)$ be a b-metric space and let Y be a subset of X. Let $x \in X$. Then the following assertions suivantes are equivalent:*
(a) $x \in \overline{Y}$.
(b) *There exists a sequence $(y_n)_n$ of points in Y such that* $\lim_{n\to+\infty} d(x,y_n) = 0$.
(c) $d(x,Y) = 0$, *where as usual,* $d(x,Y) := \inf\{d(x,y) : y \in Y\}$.

(6) Completeness and completion.

A *Cauchy sequence* in a b-metric space $(X,d,;s)$ is a sequence (x_n) in X which is satisfying $\lim_{m,n\to\infty} d(x_n,x_m) = 0$.

By the s-triangle inequality, it easy to show that every convergent sequence is Cauchy.

The b-metric space $(X,d;s)$ is said to be *complete* if every Cauchy sequence converges to some $x \in X$.

By a *completion* of a b-metric space $(X,d;s)$ one understands a complete b-metric space $(Y,\rho;s')$ such that there exists an isometric embedding $j : X \to Y$ such that $j(X)$ dense in Y.

By an *isometric embedding* of a b-metric space $(X_1,d_1;s_1)$ into a b-metric space $(X_2,d_2;s_2)$ one understands a mapping $f : X_1 \to X_2$ such that

$$d_2(f(x),f(y)) = d_1(x,y)\,, \quad \forall\, x,y \in X_1.$$

It seems that the problem of completeness of b-metric spaces is still open (see [**8**]).

Concerning the completeness of strong b-metric spaces, the following question raised in [**28**, p. 128] is: *Does every strong b-metric space admit a completion*?

This question was answered in the affirmative in [**8**].

2.3. Historical note. It seems that the birthday of b-metrics is the year 1970. According to [**16**], the b-metrics were known under different names since the year 1970 in the work of Coifman and de Guzman [**17**] on spaces of homogeneous type. See also the subsequent works by Macías and Segovia [**30, 31**] on the same topic.

In 1989, Bakhtin (see [**10**]) established the contraction mapping principle in quasimetric spaces. (i.e., in b-metric spaces).

In 1993, Czerwick studied contraction mappings in b-metric spaces with parameter $s = 2$ (see [**19, 20**]).

In 2010, Khamsi and Hussain [**26**] reintroduced the concept of a b-metric space under the name *metric-type space*. (See also [**27**]).

2.4. Some references. Concerning other complements and results from the general theory of b-metric spaces and fixed point theory on them, there is now a very extensive and fast growing-up literature with a lot of research papers, books (see for example [**28**]) and surveys on this interesting topic. For instance, one is invited to consult the nice survey by S. Cobzaş [**16**] and the rich list of references therein.

During the two last decades many authors were interested by fixed point theory in b-metric spaces. In [**13**], Ekeland's variational principle was investigated for a class of b-metric spaces. In [**6**] and [**7**] the author introduced some classes of implicit relations on b-metric spaces and used them to investigate fixed points of a pair of maps. In [**35**] (see also [**11**]) fixed points for ϕ-contractions were studied.

In [**46**] fixed points in b-metric spaces for four maps satisfying Ćirić and Hardy-Rogers type conditions are discussed. An extension of a fixed point theorem of Reich is done in [**33**]. In [**34**], fixed points for TAC-contractive mappings ([**15**]) are investigated in b-metric spaces. In [**9**], some fixed point theorems for C-class functions in b-metric spaces were investigated. In [**18**], some general fixed point results in compact or ordered b-metric spaces were established. In [**40**], Geraghty and Ćirić type fixed point theorems in b-metric spaces were provided. Recently, in [**32**] the authors studied (in b-metric spaces) Caristi-Kirk type and Boyd& Wong-Browder-Matkowski-Rus type fixed point results.

It is not possible to quote all works connected to fixed point theory in b-metric spaces. We have just gathered a few of them dealing with general fixed or common fixed point results close to the spirit of this work. I apologize for the non cited papers dealing with generalizations of b-metric spaces, set-valued maps, coupled fixed points, cyclic maps and many other specific topics.

3. Some definitions

In this section, we extend some definitions and concepts already known for maps on usual metric spaces to the more general setting of b-metric spaces.

We start by the so-called property (W.T). This property was introduced in [**5**] for two selfmappings of a metric space. It is easily extended to the case of b-metric spaces as follows.

Let $(X, d; s)$ be a b-metric space with constant $s \geqslant 1$ and let S and T be selfmappings of (X, d).

Definition 3.1. Let $(X, d; s)$ be a b-metric metric space and let $S, T : X \to X$ be selfmappings. S and T are said to be weakly tangential if there exists a sequence $\{x_n\}$ of points in X such that $\lim_{n\to\infty} d(Sx_n, Tx_n) = 0$.

We say also that the pair $\{S, T\}$ satisfies the property (W.T).

The compatibility property introduced by Jungck (see Jungck [**24**]) for usual metrics may also be extended to the b-metric case:

The selfmappings S and T of a b-metric space $(X, d; s)$ are called compatible if, $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that: $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some t in X.

This concept was frequently used to establish existence theorems of common fixed points in the case of metric spaces. The study on common fixed point theory for noncompatible mappings is also interesting. Work along these lines has been initiated by Pant [**37**], [**38**], [**39**].

It is clear that two selfmappings S, T of a b-metric space $(X, d; s)$ will be noncompatible if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$ but $\lim_{n\to\infty} d(STx_n, TSx_n)$ is either non-zero or not exists.

Thus, if S, T are non noncompatible selfmappings of $(X, d; s)$, then the pair $\{S, T\}$ satisfies the property (W.T).

The property (E.A) introduced in 2002 by Aamri and Moutawakil [**2**] for metric spaces can also be extended to b-metric spaces as follows.

Definition 3.2. Let S and T be two selfmappings of a b-metric space $(X, d; s)$. We say that S and T satisfy property (E.A) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$.

It is easy to see that if S and T satisfy the property (E.A) then they also satisfy the property (W.T).

The concept of asymptotically regular selfmapping introduced by Browder and Petryshyn (see [**14**]) in metric spaces can also be extended to the setting of b-metric spaces as follows.

Definition 3.3. Let T be a selfmapping of a b-metric space $(X, d; s)$. Then, T is said to be asymptotically regular at a point x in X, if

$$\lim_{n\to\infty} d(T^n x, T^n Tx) = 0,$$

where $T^n x$ denotes the n-th iterate of T at x.

We denote I the identity mapping. We observe that if T is asymptotically regular at the point x, then the mappings T and I are weakly tangential. Thus, the property (W.T) generalizes also the concept of asymptotically regular mappings.

The concept of asymptotically regular selfmaps of a metric space was extended to the case of a pair of selfmappings of a metric space as follows:

Let (X, d) be a metric space, T and S be selfmappings on X, with $T(X) \subset S(X)$. Let x_0 be a given point in X. Choose a point x_1 in X such that $Sx_1 = Tx_0$. This can be done since $T(X) \subset S(X)$. Continuing this process, having chosen $x_1, \ldots x_k$, we choose x_{k+1} in X such that

$$Sx_{k+1} = Tx_k, \quad k = 0, 1, 2, \ldots.$$

The sequence $\{Sx_n\}$ is called a T-sequence with initial point x_0.

In [**1**] the following definition is given.

Definition 3.4. Let T and S be selfmappings on a metric space X, with $T(X) \subset S(X)$ and $x_0 \in X$. A mapping T is said to be asymptotically S-regular at point x_0 if $\lim_{n\to+\infty} d(Sx_n, Sx_{n+1}) = 0$, where $\{Sx_n\}$ is a T-sequence with initial point x_0.

As before this definition may be extended to the setting of b-metric spaces. It is then evident to see that if a mapping T is asymptotically S-regular at point x_0 in a b-metric space (X, d), then the pair $\{S, T\}$ satisfies the property (W.T).

In 1984, Rhoades et al. (see [**45**]) first introduced the following definition.

Definition 3.5. Let T and S be selfmappings on a metric space (X, d).

A sequence $(x_n)_{n\geqslant 0}$ in X is said to be asymptotically S-regular with respcet to T if $\lim_{n\to+\infty} d(Tx_n, Sx_n) = 0$.

When T is the identity map, the above definition reduces to the that of Engl [**21**].

We notice that Definition 2.5 is more general than Definition 2.4.

As above, it is natural to extend Definition 2.5 to b-metric spaces.

Definition 3.6. Let T and S be selfmappings on a b-metric space $(X, d; s)$.

A sequence $(x_n)_{n\geqslant 0}$ in X is said to be asymptotically S-regular with respcet to T if $\lim_{n\to+\infty} d(Tx_n, Sx_n) = 0$.

Thus if T and S are selfmappings of a b-metric space $(X, d; s)$ have an asymptotically S-regular sequence in X with respcet to T, then the pair $\{S, T\}$ is weakly tangential.

In the case of metric spaces, it was shown (see [**5**]) through some examples that the notion of weakly tangential mappings is actually new.

As we see, the property (W.T) generalizes the property of noncompatiblity, the property (E.A) and the property of asymptotic regularity.

The property (W.T) generalizes also the property of asymptotically S-regular mappings and asymptotically S-regular sequences.

Jungck (see [**25**]) introduced the concept of weak compatibility for a pair of selfmappings of a metric space. This concept is easily extended to b-metric spaces.

Definition 3.7. Two selfmappings S and T of a b-metric space $(X, d; s)$ are said to be weakly compatible if $Tu = Su$, for some $u \in X$, then $STu = TSu$.

4. Main result

The main result of this paper reads as follows.

Theorem 4.1. *Let $(X, d; s)$ be a complete b-metric space with constant $s \geqslant 1$. Let $\{A, S\}$ and $\{B, T\}$ be two weakly compatible pairs of selfmappings of (X, d) satisfying the following conditions:*

(H.1) *$AX \subseteq TX$ and $BX \subseteq SX$,*
(H.2) *$\overline{AX} \cap \overline{BX} \subset T(X) \cup S(X)$.*
(H.3) *$d(Ax, By) \leqslant k\,\sigma(x, y)$, for all $x, y \in X$, where k is such that $0 \leqslant k < \kappa(s)$, with $\kappa(s) := \frac{1}{s(1+s+s^2)}$.*

If one of the pairs $\{A, S\}$ or $\{B, T\}$ satisfies the property (W.T), then A, B, S and T have a unique common fixed point (say) $z \in X$.

Moreover, if the map S (resp. T) is continuous at the common fixed point z, then the map A (resp. B) is continuous at this point.

PROOF. (I) Suppose that the pair $\{A, S\}$ satisfies the property (W.T). Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} d(Ax_n, Sx_n) = 0, \tag{4.1}$$

(1) First we show that the pair $\{B, T\}$ satisfies the property (W.T).

Indeed, since $AX \subseteq TX$, then for each integer n, there exists y_n in X such that

$$Ax_n = Ty_n. \tag{4.2}$$

By using (H.3), we have

$d(Ax_n, By_n) \leqslant$

$k[d(Sx_n, Ty_n) + d(Ax_n, Sx_n) + d(By_n, Ty_n) + d(Sx_n, By_n) + d(Ax_n, Ty_n)].$

By using (4.2) and the s-triangle inequality, we obtain after easy computations, the following inequality

$$d(Ax_n, By_n) \leqslant \frac{(s+2)k}{1-(1+s)k} d(Ax_n, Sx_n). \tag{4.3}$$

We observe that $0 < 1 - (1+s)k$.

By letting n to infinity in (4.3)(4.3), we obtain

$$\lim_{n\to\infty} d(Ax_n, By_n) = 0. \tag{4.4}$$

By (4.1) and (4.4), we get

$$\lim_{n\to\infty} d(Ax_n, Sx_n) = 0 = \lim_{n\to\infty} d(By_n, Ty_n). \tag{4.5}$$

(4.5) shows that both pairs $\{A, S\}$ and $\{B, T\}$ satisfy the property (W.T).

(2) Next, we prove that the sequence $\{Ax_n\}$ is a Cauchy sequence.

By using the assumption (H.3) and the s-triangle inequality, we have

$$\begin{aligned}
& d(Ax_m, Ax_n) \leqslant s[d(Ax_m, By_n) + d(By_n, Ax_n)] \\
\leqslant\ & sk[d(Sx_m, Ty_n) + d(Sx_m, Ax_m) + d(Ty_n, By_n) + d(Sx_m, By_n) \\
+\ & d(Ax_m, Ty_n)] + sd(Ax_n, By_n) \\
\leqslant\ & sk[s[d(Sx_m, Ax_m) + d(Ax_m, Ax_n)] + d(Sx_m, Ax_m) + d(Ty_n, By_n) \\
+\ & sd(Sx_m, Ax_m) + s^2 d(Ax_m, Ax_n) + s^2 d(Ax_n, By_n) + d(Ax_m, Ax_n)] \\
+\ & sd(Ax_n, By_n) \\
=\ & s(1+2s)kd(Sx_m, Ax_m) + s(1+k(1+s^2))d(Ax_n, By_n) \\
+\ & s(1+s+s^2)kd(Ax_m, Ax_n).
\end{aligned}$$

Therefore, we have

(4.6)

$$d(Ax_m, Ax_n) \leqslant \frac{s(1+2s)k}{1-s(1+s+s^2)k} d(Sx_m, Ax_m) + \frac{s(1+k(1+s^2))}{1-s(1+s+s^2)k} d(Ax_n, By_n).$$

From (4.5) and (4.6), we deduce that

$$\lim_{n,m\to\infty} d(Ax_m, Ax_n) = 0,$$

which implies that the sequence $\{Ax_n\}$ is a Cauchy sequence.

Since X is complete, then there exists a point (say) z in X such that the sequence $\{Ax_n\}$ converges to z. By virtue of (4.2) and (4.5), we conclude that we have

$$z = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n. \tag{4.7}$$

(3) Next, we show that z is a common fixed point for the selfmappings A, B, S and T.

Indeed, by virtue of (H.2) and (4.7) we deduce that $z \in T(X) \cup S(X)$.

There are two cases:

(a) Suppose that $z \in T(X)$, then there exists $u \in X$ such that $z = Tu$. By applying (H.3), we get

$$d(Ax_n, Bu) \leqslant k[d(Sx_n, Tu)+d(Ax_n, Sx_n)+d(Bu, Tu)+d(Sx_n, Bu)+d(Ax_n, Tu)],$$

which, by application of the s-triangle inequality, gives

$$d(Ax_n, Bu) \leqslant k[(1+s)d(Sx_n, z) + (1+s)d(Bu, Tu) + d(Ax_n, Sx_n) + d(Ax_n, z)],$$

which, by letting $n \to \infty$, implies that

$$\limsup_{n\to+\infty} d(Ax_n, Bu) \leqslant (1+s)kd(Tu, Bu).$$

On the other hand, we have

$$d(Tu, Bu) \leqslant s[d(z, Ax_n) + d(Ax_n, Bu)], \quad \forall n \geqslant 0. \tag{4.8}$$

which gives

$$d(Tu, Bu) \leqslant s \limsup_{n\to+\infty} d(Ax_n, Bu) \tag{4.9}$$

From (4.8) and (4.9), we infer that

$$(1 - s(1+s)k)d(Tu, Bu) \leqslant 0. \tag{4.10}$$

Since $0 \leqslant k < \frac{1}{s(1+s+s^2)k}$, then $1 - s(1+s)k > 0$. Therefore it follows from (4.10) that $d(Tu, Bu) = 0$. That is $Tu = Bu$. Thus, we have $z = Tu = Bu$.

Since $B(X) \subset S(X)$, then there exists $v \in X$ such that $Bu = Sv$. Then $z = Tu = Bu = Sv$. By applying the inequality (H.3), we get

$$\begin{aligned} d(Av, Sv) &= d(Av, Bu) \\ &\leqslant k[d(Sv, Tu) + d(Av, Sv) + d(Bu, Tu) + d(Sv, Bu) + d(Av, Tu)] \\ &= 2kd(Av, Sv), \end{aligned}$$

which implies that $Av = Sv$. Because $2k < 1$. Hence, we obtain

$$z = Tu = Bu = Sv = Av. \tag{4.11}$$

(b) Suppose that $z \in S(X)$. Then all conclusions in (4.11) will be obtained by similar arguments to those used in the case (a) above.

Since $\{A, S\}$ and $\{B, T\}$ are weakly compatible, it follows that

$$Bz = Tz \quad \text{and} \quad Az = Sz \tag{4.12}$$

Now, by using (4.12), we show that $z = Az$. To this end, we start by observing that

$$\sigma(z,u) = d(Sz,Tu) + d(Az,Sz) + d(Bu,Tu) + d(Sz,Bu) + d(Az,Tu) = 3d(Az,z).$$

So, by virtue of the assumption (H.3), we get

$$d(Az,z) = d(Az,Bu) \leqslant k\sigma(z,u) = 3kd(Az,z),$$

which (since $k \in [0, \frac{1}{3})$ implies that $d(Az,z) = 0$. Thus we get $z = Az$. Hence, we obtain $z = Az = Sz$.

Now, we show that $z = Bz$. To this end, we observe that

$$\begin{aligned}\sigma(v,z) &= d(Sv,Tz) + d(Av,Sv) + d(Bz,Tz) + d(Sv,Bz) + d(Av,Tz)\\ &= d(z,Bz) + d(z,Bz) + d(z,Bz) = 3d(z,Bz).\end{aligned}$$

By virtue of the assumption (H.3), we get

$$d(z,Bz) = d(Av,Bz) \leqslant k\sigma(v,z) = 3kd(z,Bz),$$

which implies that $d(Bz,z) = 0$. Thus we have $z = Bz = Tz$. Hence, we have

$$z = Bz = Tz = Az = Sz.$$

We conclude that z is a common fixed point for A, B, S and T.

(II) If we suppose that the pair $\{B,T\}$ satisfies the property (W.T), then by using arguments similar to ones invoked in (I), we obtain the same conclusions as in (I).

(III) It remains to prove the uniqueness of the fixed common fixed point z. Suppose that w is another common fixed point for the mappings A, B, S and T, such that $w \neq z$. Obviously we have $\sigma(w,z) = 3d(w,z) > 0$. Then, by applying the condition (H 3), we obtain

$$d(w,z) = d(Aw,Bz) \leqslant k\sigma(w,z) = 3kd(w,z),$$

which is a contradiction. So the mappings A, B, S and T have a unique common fixed point.

(IV) Suppose that S is continuous at the point z. Let (x_n) be a sequence converging to z. By using (H 3), for every integer n, we have

$$\begin{aligned}d(Ax_n,z) &\leqslant k[d(Sx_n,z) + d(Ax_n,Sx_n) + d(Sx_n,z) + d(Ax_n,z)]\\ &\leqslant k[2d(Sx_n,z) + s(d(Ax_n,z) + d(z,Sx_n)) + d(Ax_n,z)]\\ &= k(2+s)d(Sx_n,z) + k(1+s)d(Ax_n,z),\end{aligned}$$

which infers (as $k(1+s) < 1$) the following inequality:

$$d(Ax_n,z) \leqslant \frac{k(2+s)}{1-k(1+s)}d(Sx_n,z). \tag{4.13}$$

By letting $n \to +\infty$ in the inequality (4.13), we get $\lim_{n\to+\infty} d(Ax_n,z) = 0$. According to the observation (2-d), this implies the continuity of A at the point z.

If T is continuous at z, then one can prove as in (IV) above the continuity of B at z. This completes the proof. $\square$

As a consequence, we have the following.

Corollary 4.1. *Let $\{A, S\}$ and $\{B, T\}$ be two weakly compatible pairs of selfmappings of a complete b-metric space $(X, d; s)$ with constant $s \geqslant 1$ such that*
$(H1) : AX \subseteq TX$ *and* $BX \subseteq SX$,
$(H2) : \overline{AX} \cap \overline{BX} \subset T(X) \cup S(X)$.
$(H3) : d(Ax, By) \leqslant k\,\sigma(x, y)$, *for all* $x, y \in X$, *where* k *is such that* $0 \leqslant k < \kappa(s)$, *with* $\kappa(s) := \frac{1}{s(1+s+s^2)}$.

If one of the following four conditions is satisfied.
(i) A and S are noncompatible, or
(ii) the pair $\{A, S\}$ satisfies the property (E.A), or
(iii) B and T are noncompatible, or
(iv) the pair $\{B, T\}$ satisfies the property (E.A).

Then the mappings A, B, S and T have a unique common fixed point.
Moreover, if the map S (resp. T) is continuous at the common fixed point z, then the map A (resp. B) is continuous at this point.

Remark 4.1. Suppose that $\{A, S\}$ and $\{B, T\}$ are two pairs of selfmappings of a b-metric space $(X, d; s)$ satisfying:
$(i) : AX \subseteq TX$ and $BX \subseteq SX$,
(ii) : one of AX, BX, SX or TX is a closed subspace of (X, d).
Then we have
(H.2) : $\overline{AX} \cap \overline{BX} \subset T(X) \cup S(X)$.

Hence, Theorem 4.1 generalizes and extends Theorem 1.1 to b-metric spaces.
We end this section by giving an illustrative example.

Example 4.1. Let $X := [0, +\infty)$ be equipped with the b-metric d given by $d(x, y) := |x - y|^3$ or all $x, y \in X$. The parameter of d is $s = 4$. We have $\kappa(4) = \frac{1}{84}$. We define four maps A, B, S and T on X by setting

$$A(x) := \frac{1}{2}\ln(1 + x), \qquad B(x) := \frac{1}{2}\ln(1 + \frac{x}{4}),$$

$$S(x) := e^{4x} - 1, \qquad T(x) := e^{x} - 1.$$

Then we have:
(1) The b-metric space $(X, d; 4)$ is complete.
(2) $AX = BX = SX = TX = X$.
(3) From (2) and (1), we deduce that the assumptions (H 1) and (H 2) are satisfied.
(4) The pair $\{A, S\}$ is weakly compatible. Indeed, for all $x \in X$, we have

$$Ax = Sx \iff \frac{1}{2}\ln(1 + x) = e^{4x} - 1 \iff x = 0,$$

In that case, we have $AS(0) = SA(0) = 0$.
(5) Similarly, the pair $\{B, T\}$ is weakly compatible.

(6) Choose a sequence x_n in $(0,+\infty)$ such converging to zero in the b-metric space $(X,d;4)$, (for instance $x_n := \frac{1}{n+1}$ for all $n \geqslant 0$). Then we have

$$\lim_{n\to+\infty} d(A(x_n),S(x_n)) = \lim_{n\to+\infty} \left|\frac{1}{2}\ln(1+x_n) - e^{4x_n} + 1\right|^3 = 0,$$

which shows that the pair $\{A,S\}$ satisfies the property (W.T).

(7) For all $x,y \in X$, we have

$$\begin{aligned} d(Ax,By) = |Ax-By|^3 &= \frac{1}{8}\left|\ln(1+x) - \ln(1+\frac{y}{4})\right|^3 \\ &\leqslant \frac{1}{8}\left|x-\frac{y}{4}\right|^3 = \frac{1}{8\times 64}|4x-y|^3 \\ &\leqslant \frac{1}{8\times 64}\left|e^{4x}-e^{y}\right|^3 = \frac{1}{512}d(Sx,Ty) \\ &\leqslant \frac{1}{512}\left[d(Sx,Ty)+d(Sx,Ax)+d(Ty,By)+d(Sx,By)+d(Ty,Ax)\right] . \end{aligned}$$

So the assumption (H 3) is satisfied with $k = \frac{1}{512} < \kappa(4) = \frac{1}{84}$.

All conditions of Theorem 4.1 are fulfilled and the unique common fixed point of the selfmappings A,B,S and T is zero.

5. Well-posedness

After the works of F.S. De Blasi and J. Myjak [**12**] and of S. Reich and A.J. Zaslavski [**44**], many authors have been interested by the study of well-posednes of fixed point problems (see [**29**], [**42**], [**47**], [**41**], [**43**], [**3**], [**4**], [**5**] and [**1**]).

Definition 5.1. Let (X,d) be a metric space and $T:(X,d)\to(X,d)$ be a mapping. The fixed point problem of T is said to be well posed if:

(i) T has a unique fixed point z in X,

(ii) for any sequence $\{x_n\}$ of points in X such that $\lim_{n\to\infty} d(Tx_n,x_n)=0$, we have $\lim_{n\to\infty} d(x_n,z)=0$.

The following definition is a natural extension of Definition 4.1 to the case of b-metric spaces.

Definition 5.2. Let (X,d) be a b-metric space with constant $s \geqslant 1$. Let $\mathcal{A}$ be a set of selfmappings $T: X\to X$. The fixed point problem of the collection $\mathcal{A}$ is said to be well-posed if:

(i) the set $\mathcal{A}$ has a unique strict fixed point z in X,

(ii) for any sequence $\{x_n\}$ of points in X such that

$$\lim_{n\to\infty} d(Tx_n,x_n)=0, \quad \forall T\in\mathcal{A},$$

we have $\lim_{n\to\infty} d(x_n,z)=0$.

According to this definition, we investigate the well-posedness of the common fixed point problem for the set of four selfmappings A,B,S,T of a b-metric space $(X,d;s)$ satisfying the conditions of Theorem 4.1.

THEOREM 5.1. *Let $\{A, S\}$ and $\{B, T\}$ be two weakly compatible pairs of self-mappings of a complete b-metric space $(X, d; s)$ such that*
$(H1) : AX \subseteq TX$ *and* $BX \subseteq SX$,
$(H2) : \overline{AX} \cap \overline{BX} \subset T(X) \cup S(X)$.
$(H3)$: $d(Ax, By) \leqslant k\,\sigma(x, y)$, *for all* $x, y \in X$, *where* k *is such that* $0 \leqslant k < \frac{1}{s(1+s+s^2)}$.

If one of the pairs $\{A, S\}$ or $\{B, T\}$ satisfies the property (W.T), then the fixed point problem of A, B, S and T is well-posed..

PROOF. By Theorem 4.1, we know that the mappings A, S, B, T have a unique common fixed point z in X. Let $\{u_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} d(Au_n, u_n) = \lim_{n\to\infty} d(Su_n, u_n) = \lim_{n\to\infty} d(Bu_n, u_n) = \lim_{n\to\infty} d(Tu_n, u_n) = 0.$$

We have to show that $\lim_{n\to\infty} d(u_n, z) = 0$. By using the inequality (H 3) and the s-triangle inequality, we have successively

$$\begin{aligned} d(u_n, z) &\leqslant sd(u_n, Au_n) + sd(Au_n, Bz) \\ &\leqslant sd(u_n, Au_n) + sk[d(Su_n, z) + d(Su_n, Au_n) + 0 + d(Su_n, z) + d(Au_n, z)] \\ &\leqslant sd(u_n, Au_n) + sk[3sd(Su_n, u_n) + 3sd(u_n, z) + 2sd(Au_n, u_n)] \\ &= 3ks^2 d(u_n, z) + 3ks^2 d(Su_n, u_n) + s(1 + 2ks)d(Au_n, u_n), \end{aligned}$$

from which we obtain

$$d(u_n, z) \leqslant \frac{3ks^2}{1 - 3ks^2} d(Su_n, u_n) + \frac{s(1 + 2ks)}{1 - 3ks^2} d(Au_n, u_n), \quad \forall n \geqslant 0. \tag{5.1}$$

The inequality (5.1) holds true because $1 - 3ks^2 > 0$.

Letting n go to infinity in (5.1), we get

$$\lim_{n\to\infty} d(u_n, z) = 0,$$

which implies that the strict fixed point problem for the mappings A, B, S, T is well posed. □

6. An open problem

Let $(X, d; s)$ with constant $s \geqslant 1$. Let $\{A, S\}$ and $\{B, T\}$ be two weakly compatible pairs of selfmappings of X satisfying the assumptions (H 1) and (H 2). We know that all the results of Theorems 4.1 and 5.1 are true, if the maps satisfy $(H3)$: $d(Ax, By) \leqslant k\,\sigma(x, y)$, for all $x, y \in X$, where k is such that $0 \leqslant k < \frac{1}{s(1+s+s^2)}$.

We raise the question: The constant $\kappa(s) := \frac{1}{s(1+s+s^2)}$ is it the better bound ?

In another manner, what is the optimal bound for k to obtain the conculsions of Theorem 4.1 and Theorem 5.1. ?

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