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CONSTRUCTIVE EFFECTS OF NOISE IN L-G PREY PREDATOR MODEL WITH S-H FUNCTIONAL RESPONSE WITH HARVESTING ON PREY

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ABSTRACT. In this paper, we consider a three-species Leslie-Gower prey-predator nutrition cable ideal with harvesting and Sokol-Howell functional response is proposed. Existences of possible steady states with stability analysis of interior equilibrium point are examined. We also analyzed the impact of noise and harvesting on the dynamics of the planned nutrition chain structure. Numerical simulations are being carried out through Mat lab.

1. Introduction

Nonlinear mathematical modelling is more attractive and popular in biological research trends.Deterministic computational strategies are extensively recycled to comprehend the dynamic forces of interrelating species. They reveal similar dynamical behaviours such as interior steady state, steadiness of local and global and limit cycle attractor. The rising basic needs, such as nutrition and resources leads to an enormous exposure of several ecological resources. Whereas another side huge need of protective measures are required to save the ecosystem from environmental constraints. A pioneer work Lotka-Volterra predator-prey model, proposed by Lotka [**19**] and Volterra [**31**] independently, is the first and simplest mathematical model. The Lotka-Volterra predator-prey model has not mapped with many real world problems and complex situations, so a number of changes in the model have been done by many researchers to improve the real world solutions. Out of those innovative research contents and techniques one of the wide range solution provider model or an improvised innovative model is proposed and analysed by Leslie and

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Gower [**14**] a predator-prey model. Which becomes more prominent and popularized as L-G (Leslie-Gower) predator prey model, in which the predator growth function is different from the predator predation function. Many Scientists assumed that the growth rate of predator is described by a function of the ratio of predators and their prey. Hsu and Huang [**10, 11**] studied this model and showed that the scheme has unique affirmative steady state and is globally stable under all biologically admissible parameters. May [**20**] improved the Leslie Gower predator-prey model by replacing the Holling type-I functional response by Holling type - II. Many Scientists contributed more innovative and inspirable modified models studied in the predator-prey structure with the Leslie–Gower scheme [**12**] - [**16**]. Mathematical revisions for a Leslie-Gower type models studied by many researchers. Many authors [**12**]-[**28**] modified and improvised more creatively the L-G model to obtain remarkable results and observations under certain environments which guarantee the global steadiness of the affirmative steady state for a predator-prey model with altered L-G and Holling type functional response scheme. Most of the inspirable studies done by many scientists [**20**] - [**2**] are more remarkable and further extendable with new techniques. The recent and informative research methodologies carried out by few researchers [**22**] - [**23**] are more creative and innovative in wide range experimental areas which inspires us to simulate and analyse the present model both theoretically and graphically.

The real world problems becomes more complex and arises the need and necessary situation for better and improved results in theoretical studies as well as in simulated graphical illustrations also which leads to an innovative concept - functional response is indulged in to the system for effective results, done by [**25**] - [**13**]. Modified L-G type model with Sokol–Howell functional response under environmental constrainable influence and harvesting influence - a new approach, which is not studied so far as per our literature collection.

This article is planned as follows: in Segment 2, the scientific model is defined. In Segment 3, the boundedness conditions are examined. Exploration of steady states of the structure is derived in Segment 4. In Segment 5, local stability is being discussed using Routh-Hurwitz criteria. Stochastic stability is discussed in Segment 6. In Segment 7, numerical study is carried out to obtain the behaviour of the model. Lastly, the article finishes with concluding remarks in Segment 8.

2. Mathematical Model

We defined the 3-species nourishment cable model at time t comprising of the prey population density denoted by $N_1(t)$ the predator population density denoted by $N_2(t)$, and the top predator whose population density denoted by $N_3(t)$. The predator $N_2(t)$ preys on its sole food N_1 at the lower level according to simplified Holling type IV functional response, while the top predator N_3 preys on N_2 at the second level according to the modified Leslie-Gower type. The dynamics of the ideal described above can be symbolized by the following set of equations:

$$\frac{dN_1}{dt} = \alpha_0 N_1 - \beta_0 N_1^2 - \frac{\gamma_0 N_1 N_2}{c_0 + N_1^2} - q_1 E_1 N_1 \tag{2.1}$$

$$\frac{dN_2}{dt} = \frac{\gamma_1 N_1 N_2}{c_1 + N_1^2} - \frac{\gamma_2 N_2 N_3}{c_2 + N_2} - \alpha_1 N_2 - \eta N_2^2 \tag{2.2}$$

$$\frac{dN_3}{dt} = c_4 N_3^2 - \frac{\gamma_3 N_3^2}{c_3 + N_2} \tag{2.3}$$

where α_0 represent intrinsic growth rate of prey, α_1 represents intrinsic death rate of predator,β_0 represents intra-specific competition among prey species, q_1 represents catch ability coefficients of prey, $\gamma_i(i = 0 - 3)$ and are the maximum values attainable by each per capita rate. $c_i(i = 0-4)$ are positive constants.E_1 represents the effort applied to harvest the prey populations.

$$N_1(0) \geqslant 0, N_2(0) \geqslant 0, N_3(0) \geqslant 0 \tag{2.4}$$

3. Positive invariance and boundedness

Feasibility or biologically positivity studies aim to objectively and rationally uncover the strength of the proposed model in the given environment. Biologically positive insures the population never become negative and population always survive. The following theorems ensure that the positivity and boundedness of the system (2.1) - (2.3).

THEOREM 3.1. *Every solution* $(N_1(t),\, N_2(t),\, N_3(t))$ *of* $(2.1) - (2.3)$ *with condition* (2.4) *is positive invariant.*

PROOF. From (2.1) it is observed that

$$\frac{dN_1}{N_1} = \left[\alpha_0 - \beta_0 N_1 - \frac{\gamma_0 N_2}{c_0 + N_1^2} - q_1 E_1\right] dt = \phi_1(N_1, N_2)dt \text{ (say)},$$

where $\phi_1(N_1, N_2) = \alpha_0 - \beta_0 N_1 - \frac{\gamma_0 N_2}{c_0+N_1^2} - q_1 E_1$.

Integrating in the region $[0, t]$ we get

$$N_1(t) = N_1(0) \exp\left(\int \phi_1(N_1, N_2)dt\right) > 0 \text{ for all t.}$$

From (2.2), it is observed that

$$\frac{dN_2}{N_2} = \left[\frac{\gamma_1 N_1}{c_1 + N_1^2} - \frac{\gamma_2 N_3}{c_2 + N_2} - \alpha_1 - \eta N_2\right] dt = \phi_2(N_1, N_2, N_3)dt \text{ (say)},$$

where $\phi_2(N_1, N_2, N_3) = \frac{\gamma_1 N_1}{c_1+N_1^2} - \frac{\gamma_2 N_3}{c_2+N_2} - \alpha_1 - \eta N_2$. Integrating in the region $[0, t]$ we get

$$N_2(t) = N_2(0) \exp\left(\int \phi_2(N_1, N_2, N_3)dt\right) > 0 \text{ for all t.}$$

From (2.3), it is observed that

$$\frac{dN_3}{N_3} = \left[c_4 N_3 - \frac{\gamma_3 N_3}{c_3 + N_2}\right] \; dt \; = \phi_3(N_2, N_3)dt \;\; \text{(say)}$$

where $\phi_3(N_2, N_3) = c_4 N_3 - \frac{\gamma_3 N_3}{c_3 + N_2}$. Integrating in the region $[0, t]$ we get

$$N_3(t) = N_3(0)\exp\left(\int \phi_3(N_2, N_3)dt\right) > 0 \text{ for all t.}$$

Hence, all solutions starting from interior of the first octant In (R_+) remain positive in it for future time. This completes the proof □

THEOREM 3.2. *Every non-negative solutions of* $(2.1)-(2.3)$ *that initiate in* $\Re^3_+$ *are uniformly bounded.*

PROOF. Let $N_1(t), N_2(t), N_3(t)$ be any solution of the system (2.1)-(2.3). Since, from (2.1) $\frac{dN_1}{dt} \leqslant N_1(\alpha_0 - \beta_0 N_1)$, we have

$$\lim_{t\to\infty} \sup N_1(t) \leqslant \frac{\alpha_0}{\beta_0}.$$

Let $w = N_1 + \frac{\gamma_0}{\gamma_1} N_2 + \alpha N_3$. Differentiate with respect to t we get

$$\frac{dw}{dt} = \frac{dN_1}{dt} + \frac{\gamma_0}{\gamma_1}\frac{dN_2}{dt} + \alpha\frac{dN_3}{dt}.$$

Substituting (2.1)-(2.3) in above equation, we obtain

$$\frac{dw}{dt} + \theta w \leqslant \frac{\alpha_0}{\beta_0} + \frac{\alpha_0}{4\alpha_1\beta_0} + \frac{N}{\alpha_1}$$

$$\frac{dw}{dt} + \theta w \leqslant \mu \quad \text{since } \frac{\alpha_0}{\beta_0} + \frac{\alpha_0}{4\alpha_1\beta_0} + \frac{N}{\alpha_1} = \mu(say))$$

Applying Lemma on differential in equalities, we obtain

$$0 \leqslant w(N_1, N_2, N_3) \leqslant (\mu/\theta)\left(1 - e^{-\theta t}\right) + \left(w(N_1(0), N_2(0), N_3(0))/e^{\theta t}\right)$$

and for $t \to \infty$ we have $0 \leqslant w(N_1, N_2, N_3) \leqslant (\mu/\theta)$. Hence every solutions of system (2.1)-(2.3)enter the region

$$\Gamma = \left\{(N_1, N_2, N_3) \in R^3_+ : 0 \leqslant N_1 \leqslant \frac{\alpha_0}{\beta_0}, 0 \leqslant w \leqslant (\mu/\theta) + \varepsilon, \forall\varepsilon > 0\right\}.$$

This completes the proof. □

4. Steady state analysis

For the system (2.1)-(2.3) the possible steady states are

(1) $E_0(0,0,0)$, (2) $E_1\left(\tilde{N}_1, 0, 0\right)$, (3) $E_2\left(\bar{N}_1, \bar{N}_2, 0\right)$, (4) $E_3(N_1^*, N_2^*, N_3^*)$.

(I) In the absence of species, i.e. $E_0(0,0,0)$, the system is always exists.

(II) In the absence of predator species only, i.e $E_1\left(\tilde{N}_1, 0, 0\right)$

$$\tilde{N}_1 = \frac{1}{\beta_0}(\alpha_0 - q_1 E_1) \tag{4.1}$$

and for $\tilde{N}_1$ to be positive if

$$\alpha_0 \geqslant (q_1 E_1) \tag{4.2}$$

(III) In the absence of top predator species, Existence Of $E_2(\bar{N}_1, \bar{N}_2, 0)$

$$\frac{dN_1}{dt} = N_1\left[\alpha_0 - \beta_0 N_1 - \frac{\gamma_0 N_2}{c_0 + N_1^2} - q_1 E_1\right] \tag{4.3}$$

$$\frac{dN_2}{dt} = N_2\left[\frac{\gamma_1 N_1}{c_1 + N_1^2} - \frac{\gamma_2 N_3}{c_2 + N_2} - \alpha_1 - \eta N_2\right] \tag{4.4}$$

For the above steady state $\frac{dN_1}{dt} = \frac{dN_2}{dt} = 0$

Then From equation (4.4), we have

$$(\alpha_1 + \eta N_2)\, N_1^2 - \gamma_1 N_1 + (c_1\alpha_1 + c_1\eta N_2) = 0.$$

This implies that

$$\bar{N}_1 = \frac{\gamma_1 \pm \sqrt{\gamma_1^2 - 4c_1\,(\alpha_1 + \eta N_2)^2}}{2\,(\alpha_1 + \eta N_2)}.$$

For $\bar{N}_1$ to be positive if $\gamma_1^2 > 4c_1\,(\alpha_1 + \eta N_2)^2$ i.e., $\gamma_1 > 2\sqrt{c_1}\,(\alpha_1 + \eta N_2)$ From equation (4.3), we have

$$\alpha_0 - q_1 E_1 - \beta_0 N_1 = \frac{\gamma_0 N_2}{c_0 + N_1^2}.$$

which implies that

$$\bar{N}_2 = \frac{1}{\gamma_0}\left[(\alpha_0 - q_1 E_1 - \beta_0 N_1)\left(c_0 + N_1^2\right)\right]$$

For $\bar{N}_2$ to be positive if $\alpha_0 - q_1 E_1 > \beta_0 N_1$ i.e $\frac{\alpha_0 - q_1 E_1}{\beta_0} > \bar{N}_1$.

Assume that $\alpha_0 - q_1 E_1 > 0$ throughout our analysis

(IV) The positive steady state $E_3\,(N_1^*, N_2^*, N_3^*)$ exists in interior of the first octant if and only if there is a positive solution of the following equations

$$g_1 = \alpha_0 - \beta_0 N_1 - \frac{\gamma_0 N_2}{c_0 + N_1^2} - q_1 E_1 = 0 \tag{4.5}$$

$$g_2 = \frac{\gamma_1 N_1}{c_1 + N_1^2} - \frac{\gamma_2 N_3}{c_2 + N_2} - \alpha_1 - \eta N_2 = 0 \tag{4.6}$$

$$g_3 = c_4 N_3 - \frac{\gamma_3 N_3}{c_3 + N_2} = 0 \tag{4.7}$$

From (4.7),

$$N_2^* = \frac{\gamma_3 - c_4 c_3}{c_4} \tag{4.8}$$

For $(N_2)^*$ to be positive if

$$\gamma_3 \geqslant c_4 c_3 \tag{4.9}$$

from (4.5), let N_1^* is the positive root of the equation

$$h(N_1) = N_1^3 + A_1 N_1^2 + B_1 N_1 + C_1 N_1 = 0 \tag{4.10}$$

where $A_1 = -\frac{(\alpha_0 - q_1 E_1)}{\beta_o}$; $B_1 = c_0$; $C_1 = \frac{1}{\beta_0}\left[\gamma_0 N_2^* - c_0(\alpha_0 - q_1 E_1)\right]$

Now since $0 \leqslant N_1^* \leqslant \frac{\alpha_0 - q_1 E_1}{\beta_0}$, then $h(0) = C_1 < 0$ if

$$N_2^* < \frac{c_0}{\gamma_0}(\alpha_0 - q_1 E_1) \tag{4.11}$$

$h\left(\frac{\alpha_0 - q_1 E_1}{\beta_0}\right) = \frac{\gamma_0 N_2^*}{\beta_0} > 0$. Thus $h(0)h\left(\frac{\alpha_0 - q_1 E_1}{\beta_0}\right) < 0$. Then there is a positive root of $h(N_1) = 0$ lies in $\left(0,\ \frac{\alpha_0 - q_1 E_1}{\beta_0}\right)$, when $N_2^* < \frac{c_0}{\gamma_0}(\alpha_0 - q_1 E_1)$ is satisfied.

From (4.6), it is observed that

$$N_3^* = \frac{c_2 + N_2^*}{\gamma_2}\left[\frac{\gamma_1 N_1^*}{(c_1 + (N_1^*)^{2)}} - \alpha_1 - \eta N_2^*\right] \tag{4.12}$$

For N_3^* is to be positive if

$$\alpha_1 < \frac{\gamma_1 N_1^*}{c_1 + (N_1^*)^2} - \eta N_2^* \tag{4.13}$$

Thus the positive steady state $E_3\,(N_1^*, N_2^*, N_3^*)$ exists, if (4.11), (4.12) and(4.13) exists.

5. Local stability

In order to examine the dynamical performance of (2.1)-(2.3) near the interior steady state, the jacobian matrix $J(E_3)$ of system (2.1)-(2.3) at $E_3\,(N_1^*, N_2^*, N_3^*)$ is calculated as follows.

$$J\left(E_3\left(N_1^*, N_2^*, N_3^*\right)\right) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{5.1}$$

where

(5.2)

$$a_{11} = -\beta_0(N_1)^* + \frac{2\gamma_0((N_1)^*)^2(N_2)^*}{((c_0 + (N_1)^*)^2)^2};\ a_{12} = \frac{-\gamma_0(N_1)^*}{((c_0 + (N_1)^*)^2)^2}\ ;\ \ a_{13} = 0;$$

$$a_{21} = \frac{\gamma_1(c_1 - ((N_1)^*)^2)((N_2)^*)}{((c_1 + (N_1)^*)^2)};\ \ a_{22} = \frac{\gamma_2(N_3)^* N_2^*}{(c_2 + (N_2)^*)^2} - \eta N_2^*;\ \ a_{23} = \frac{-\gamma_2(N_2)^*}{(c_2 + (N_2)^*)};$$

$$a_{31} = 0;\quad a_{32} = \frac{\gamma_3(N_3^*)^2}{(c_3 + N_2^*)^2};\ \ a_{33} = c_4 N_3^* - \frac{\gamma_3 N_3^*}{(c_3 + N_2^*)}.$$

The characteristic equation of the variational matrix (4.12) of the system (2.1)-(2.3) is in the form of

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0 \tag{5.3}$$

where $A = -(a_{11} + a_{22} + a_{33})$; $B = a_{11}a_{22} + a_{22}a_{33} - a_{32}a_{23} - a_{21}a_{12}$; $C = a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33} - a_{12}a_{31}a_{23}$. By Routh-Hurwitz criteria, the steady state $E_2\,(N_1^*, N_2^*, N_3^*)$ is locally asymptotically stable if $A > 0$, $C > 0$, $(AB - C) > 0$.

For proving $A > 0$, $C > 0$, the following conditions should satisfy.

$$\beta_0 > \frac{2\gamma_0 N_1^* N_2^*}{\left(c_0 + (N_1^*)^2\right)^2} \tag{5.4}$$

$$\eta > \frac{\gamma_2 N_3^*}{\left(c_2 + N_2^*\right)^2} \tag{5.5}$$

and also for proving $(AB - C) > 0$, if

$$\left[\beta_0 - \frac{2\gamma_0 N_1^* N_2^*}{\left(c_0 + (N_1^*)^2\right)^2}\right]\left[\frac{\gamma_2 N_3^*}{\left(c_2 + N_2^*\right)^2} - \eta\right] > \frac{\gamma_0\gamma_1\left(c_1 - N_1^2\right)}{\left(c_1 + N_1^2\right)^2\left(c_0 + N_1^2\right)} \tag{5.6}$$

Therefore, based on this analysis, the locally asymptotically stable in interior R_+^3 of the positive steady state $E_2\left(N_1^*, N_2^*, N_3^*\right)$ is deliberated in the subsequent proposition.

Proposition 5.1. *The positive interior steady state $E_3\left(N_1^*, N_2^*, N_3^*\right)$ is asymptotically locally stable provided the conditions (5.2)-(5.6) hold.*

6. Stochastic analysis

Deterministic models are stable with a cyclic behavior in the common period for the sizes of species population. Moreover these models may be inadequate for capturing the exact variability in nature. In predator-prey model the random fluctuations are also undeniably arising from either environmental variability or internal species. In fact, biological systems are inherently random in nature and noise play a vital role in the structure and function of such system. These random fluctuations result in changing some degree of parameter in the deterministic environment.Now we allow stochastic perturbations of the variables (N_1^*, N_2^*, N_3^*) around their values at the positive equilibrium E_2. We consider the white noise stochastic perturbations which are proportional to the distances of N_1, N_2, N_3 from N_1^*, N_2^*, N_3^*. So the stochastically perturbed system (2.1) is given by

$$\begin{aligned} dN_1 &= (\alpha_0 N_1 - \beta_0 N_1^2 - \frac{\gamma_0 N_1 N_2}{c_0 + N_1^2} - q_1 E_1 N_1)dt + \sigma_1 (N_1 - N_1^*) d\xi_t^1 \\ dN_2 &= (\frac{\gamma_1 N_1 N_2}{c_1 + N_1^2} - \frac{\gamma_2 N_2 N_3}{c_2 + N_2} - \alpha_1 N_2 - \eta N_2^2)dt + \sigma_2 (N_2 - N_2^*) d\xi_t^2 \\ dN_3 &= (c_4 N_3^2 - \frac{\gamma_3 N_3^2}{c_3 + N_2})dt + \sigma_3 (N_3 - N_3^*) d\xi_t^3 \end{aligned} \tag{6.1}$$

where $\sigma_i, i = 1, 2$ are real constants, $\xi_t^i = \xi_t(t), i = 1, 2$ are independent (see Standard Wiener processes [28]). We thoroughly studied the dynamical behavior of ideal (2.1)-(2.3) with respect to stochasticity near E_3 for (6.1) and comparing the results with those obtained for (2.1)-(2.3).

We will consider (6.1) as the Ito stochastic differential system. To analyze the stochastic stability of E_3, we consider the linear system of (6.1)around E_3 as follows:

$$du(t) = f(u(t))dt + g(u(t))d\xi(t) \tag{6.2}$$

where $u(t) = Col(u_1(t), u_2(t)) : f(u(t)) = J(u(t))$ and

$$f(u(t)) = \begin{bmatrix} -\beta_0 N_1^* + \frac{2\gamma_0 (N_1^*)^2 N_2^*}{(c_0+(N_1^*)^2)^2} & \frac{-\gamma_0 N_1^*}{(c_0+N_1^*)^2)} & 0 \\ \frac{\gamma_1 (c_1-(N_1^*)^2) N_2^*}{(c_1+(N_1^*)^2)} & \frac{\gamma_2 N_3^* N_2^*}{(c_2+N_2^*)^2} - \eta N_2^* & \frac{-\gamma_2 N_2^*}{(c_2+N_2^*)} \\ 0 & \frac{\gamma_3 (N_3^*)^2}{(c_3+N_2^*)^2} & c_4 N_3^* - \frac{\gamma_3 N_3^*}{(c_3+N_2^*)} \end{bmatrix}$$

$$g(u(t)) = \begin{bmatrix} \sigma_1 u_1 & 0 & 0 \\ 0 & \sigma_2 u_2 & 0 \\ 0 & 0 & \sigma_3 u_3 \end{bmatrix}$$

$\therefore\ d\xi(t) = col\,(\xi_1(t), \xi_2(t), \xi_3(t)); u_1 = N_1 - N_1^*, u_2 = N_2 - N_2^*, u_3 = N_3 - N_3^*$

Let $U = \{(t \geqslant t_0) \times R^n, t_0 \in R^+\}$. Hence $V_2 \in C_2^0(U)$ is a continuous function with respect to t and a twice continuously differentiable function with respect to u. With reference to [28, 29], we have

$$LV(t,u) = \frac{\partial V(t,u)}{\partial t} + f^T(u)\frac{\partial V(t,u)}{\partial u} + \frac{1}{2}Tr\left(g^T(u)\frac{\partial V^2(t,u)}{\partial u^2}g(u)\right) \tag{6.3}$$

where T means transposition

$$\frac{\partial V}{\partial u} = col\left(\frac{\partial V}{\partial u_1}, \frac{\partial V}{\partial u_2}, \frac{\partial V}{\partial u_3}\right)^T; \frac{\partial V^2(t,u)}{\partial u^2} = col(\frac{\partial V^2}{\partial u_i \partial u_j});\ i,j = 1,2,3 \tag{6.4}$$

THEOREM 6.1. *If there exists a function* $V_2 \in C_2^0(U)$ *satisfying the following*

$$M_1 |u|^p \leqslant V(t,u) \leqslant M_2 |u|^p ; LV(t,u) \leqslant -M_3 |u|^p \ \ M_i > 0, p > 0 \tag{6.5}$$

THEOREM 6.2. *Suppose that*

$$\sigma_1^2 < 2\left[(\beta_0 N_1^* - \tfrac{2\gamma_0 (N_1^*)^2 (N_2^*)}{(c_0+(N_1^*)^2)^2})\right],\ \sigma_2^2 < 2\left[\eta N_2^* - \tfrac{\gamma_2 N_3^* N_2^*}{(c_2+N_2^*)^2}\right],$$
$$\sigma_3^2 < 2\left[\tfrac{\gamma_3 N_3^*}{(c_3+N_2^*)} - c_4 N_3^*\right].$$

Then the zero solution of (6.2)is asymptotically mean square stable.

PROOF. Let us consider the Lyapunov function

$$V(u) = \frac{1}{2}\left(w_1 u_1^2 + w_2 u_2^2 + w_3 u_3^2\right), w_i > 0 \tag{6.6}$$

where w_i are real +ve constants. Obviously inequalities (6.5) hold true with p = 2.

Moreover the inequalities in (6.5)are true when p = 2 and we have

(6.7)

$$
\begin{aligned}
LV(u) = w_1 &\left((-\beta_0 N_1^* + \tfrac{2\gamma_0 (N_1^*)^2 N_2^*}{(c_0+(N_1^*)^2)^2})u_1 + (\tfrac{-\gamma_0 N_1^*}{(c_0+(N_1^*)^2)})u_2\right) u_1 \\
&+w_2 \left(\left(\tfrac{\gamma_1 (c_1-(N_1^*)^2)N_2^*}{(c_1+(N_1^*)^2)}\right) u_1 + \left(\tfrac{\gamma_2 N_3^* N_2^*}{(c_2+N_2^*)^2} - \eta N_2^*\right) u_2 + \left(\tfrac{-\gamma_2 N_2^*}{(c_2+N_2^*)}\right) u_3\right) u_2 \\
&+w_3 \left(\left(\tfrac{\gamma_3 (N_3^*)^2}{(c_3+N_2^*)^2}\right) u_2 + \left(c_4 N_3^* - \tfrac{\gamma_3 N_3^*}{(c_3+N_2^*)}\right) u_3\right) u_3 \\
&+\tfrac{1}{2} Tr\left(g^T(u) \tfrac{\partial V^2(t,u)}{\partial u^2} g(u)\right)
\end{aligned}
$$

We can easily observe that

$$
\frac{\partial^2 V}{\partial u^2} = \begin{bmatrix} w_1 & 0 & 0 \\ 0 & w_2 & 0 \\ 0 & 0 & w_3 \end{bmatrix}
$$

and hence

$$
g^T(u) \frac{\partial V^2(t,u)}{\partial u^2} g(u) = \begin{bmatrix} w_1 \sigma_1^2 u_1^2 & 0 & 0 \\ 0 & w_2 \sigma_2^2 u_2^2 & 0 \\ 0 & 0 & w_3 \sigma_3^2 u_3^2 \end{bmatrix}
$$

with

$$
\frac{1}{2} Tr\left(g^T(u) \frac{\partial V^2(t,u)}{\partial u^2} g(u)\right) = \frac{1}{2}\left(w_1 \sigma_1^2 u_1^2 + w_2 \sigma_2^2 u_2^2 + w_3 \sigma_3^2 u_3^2\right) \tag{6.8}
$$

From (6.8), we have if we choose $N_1^* w_1 = N_2^* w_2$ in (6.6)along (6.4) we get

$$
\begin{aligned}
LV(t,u) = -w_1 &\left[(\beta_0 N_1^* - \frac{2\gamma_0 (N_1^*)^2 (N_2^*)}{(c_0+(N_1^*)^2)^2}) - \frac{1}{2}\sigma_1^2\right] u_1^2 \\
-w_2 &\left[\eta N_2^* - \frac{\gamma_2 (N_3^*)(N_2^*)}{(c_2+N_2^*)^2} - \frac{1}{2}\sigma_2^2\right] u_2^2 \\
-w_3 &\left[+\frac{\gamma_3 N_3^*}{(c_3+N_2^*)} - c_4 N_3^* - \frac{1}{2}\sigma_3^2\right] u_3^2
\end{aligned}
$$

which is negative definite function. Hence the proof is completed based on theorem. □

7. Numerical simulations

In this segment, we validate and justify our mathematical findings by computer simulations with help of MATLAB software considering different sets of parameter values as follows.

EXAMPLE 7.1. For the parameters

$$
\alpha_0 = 0.47,\ \beta_0 = 0.075,\ \gamma_0 = 1,\ \gamma_1 = 2,\ \gamma_2 = 0.605,\ \gamma_3 = 1, c_0 = 15,\ c_1 = 10,
$$

$$
c_2 = 20,\ c_3 = 0.407,\ c_4 = 0.147,\ q_1 = 0.03,\ E_1 = 2,\ \alpha_1 = 0.105,\ \eta = 0.01
$$

with densities $N_1(0) = 1.2$, $N_2(0) = 1.3$, $N_3(0) = 0.65$. Figure 1 represents the variations of populations against time and Figure 2 represents phase portrait diagram among species.

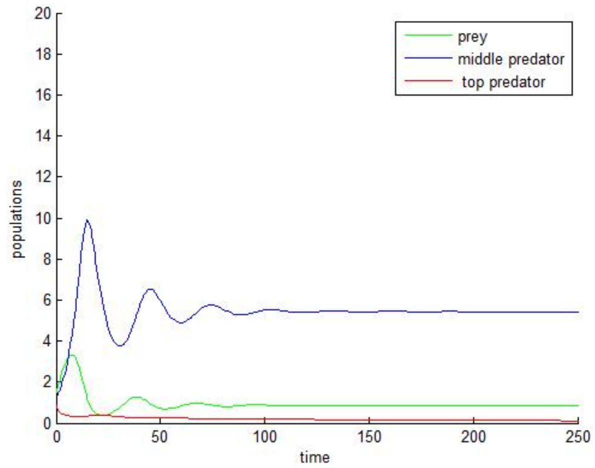

FIGURE 1. Variations of populations of the system (2.1)-(2.3) with respect to time

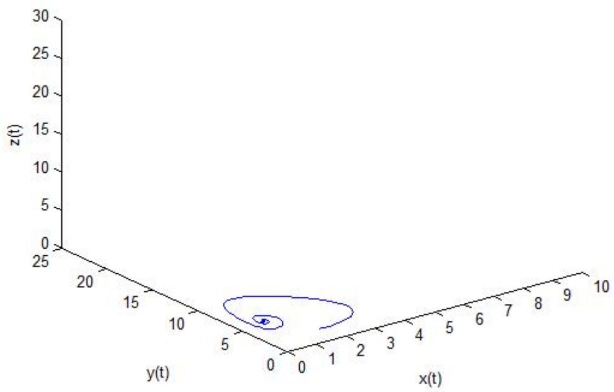

FIGURE 2. Phase portrait of the system (2.1)-(2.3) with respect to time

EXAMPLE 7.2. For the parameters

$$\alpha_0 = 0.47,\ \beta_0 = 0.075,\ \gamma_0 = 1,\ \gamma_1 = 2,\ \gamma_2 = 0.605,\ \gamma_3 = 1, c_0 = 5,\ c_1 = 10,$$
$$c_2 = 20,\ c_3 = 1,\ c_4 = 0.147,\ q_1 = 0.03,\ E_1 = 2,\ \alpha_1 = 0.105,\ \eta = 0.01$$

with densities $N_1(0) = 0.62$, $N_2(0) = 1.3$, $N_3(0) = 0.65$, $\sigma_1 = 0.02$, $\sigma_2 = 0.02$, $\sigma_3 = 0.02$. Figure 3 represents the variations of populations against time and Figure 4 represents phase portrait diagram among species.

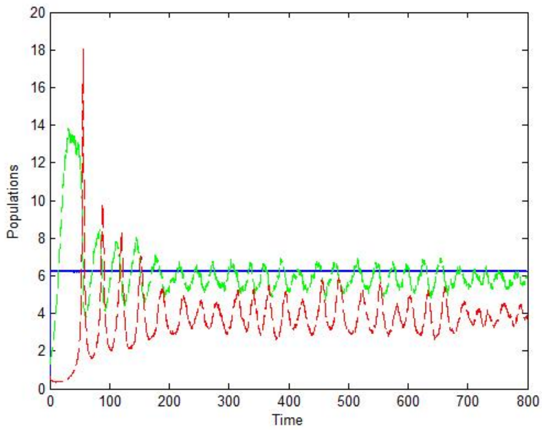

FIGURE 3. Variations of populations with low noise of strength $\sigma_1 = 0.02, \sigma_2 = 0.02, \sigma_3 = 0.02$

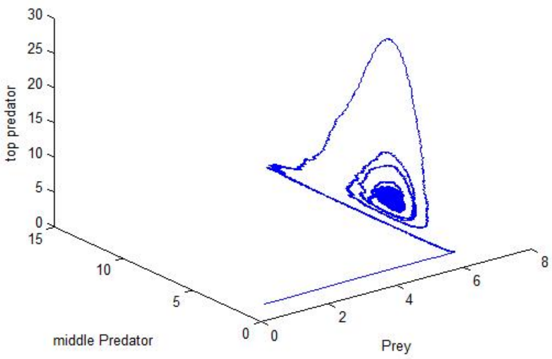

FIGURE 4. phase portrait of the system (6.1) with low noise of strength $\sigma_1 = 0.02$, $\sigma_2 = 0.02$, $\sigma_3 = 0.02$

EXAMPLE 7.3. For the parameters

$$\alpha_0 = 0.47,\ \beta_0 = 0.075,\ \gamma_0 = 1,\ \gamma_1 = 2,\ \gamma_2 = 0.605,\ \gamma_3 = 1, c_0 = 5,\ c_1 = 10,$$

$$c_2 = 20,\ c_3 = 1,\ c_4 = 0.147,\ q_1 = 0.03,\ E_1 = 2,\ \alpha_1 = 0.105,\ \eta = 0.01$$

with densities $N_1(0) = 0.62$, $N_2(0) = 1.3$, $N_3(0) = 0.65$,$\sigma_1 = 0.08$, $\sigma_2 = 0.08$, $\sigma_3 = 0.08$
Figure 5 represents the variations of populations against time and Figure 6 represents phase portrait diagram among species.

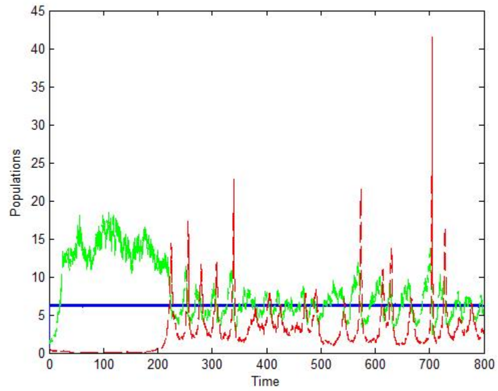

FIGURE 5. Variations of populations with medium strength of noise with oscillations $\sigma_1 = 0.08$, $\sigma_2 = 0.08$, $\sigma_3 = 0.08$

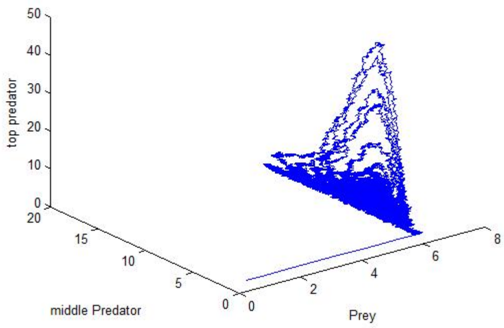

FIGURE 6. phase portrait of the system (6.1) with medium strength of noise with oscillations $\sigma_1 = 0.08$, $\sigma_2 = 0.08$, $\sigma_3 = 0.08$

EXAMPLE 7.4. For the parameters

$$\alpha_0 = 0.47,\ \beta_0 = 0.075,\ \gamma_0 = 1,\ \gamma_1 = 2,\ \gamma_2 = 0.605,\ \gamma_3 = 1, c_0 = 5,\ c_1 = 10,$$
$$c_2 = 20,\ c_3 = 1,\ c_4 = 0.147,\ q_1 = 0.03,\ E_1 = 2,\ \alpha_1 = 0.105,\ \eta = 0.01$$

with densities $N_1(0) = 0.62$, $N_2(0) = 1.3$, $N_3(0) = 0.65$,$\sigma_1 = 0.2$, $\sigma_2 = 0.2$, $\sigma_3 = 0.2$
Figure 7 represents the variations of populations against time and Figure 8 represents phase portrait diagram among species.

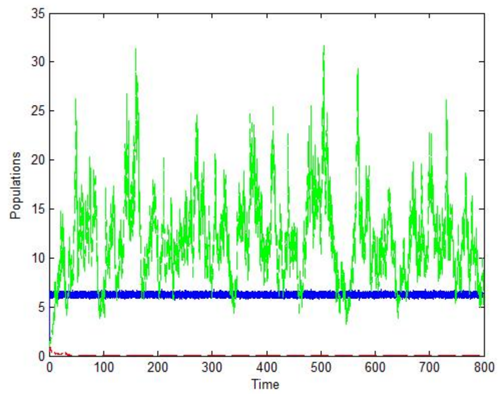

FIGURE 7. Variations of populations (6.1) with High strength of noise $\sigma_1 = 0.2$, $\sigma_2 = 0.2$, $\sigma_3 = 0.2$

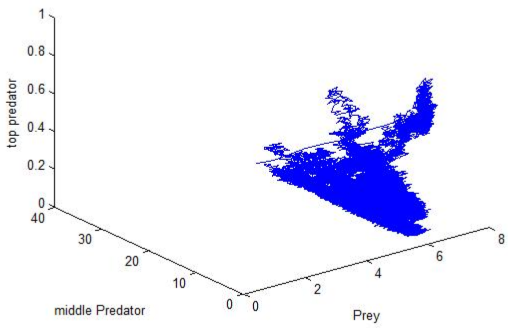

FIGURE 8. Figure: 8 phase portrait of the system (6.1) with High strength of noise $\sigma_1 = 0.2$, $\sigma_2 = 0.2$, $\sigma_3 = 0.2$

8. Concluding remarks

In this paper we have studied Leslie-Gower prey-predator model along with Sokel-Howell functional response around the interior steady state. We examined the system by introducing stochastic perturbations. By using stochastic differential equation we have showed that zero solution of this stochastic system is asymptotically mean square stable through the construction of Lyapunov function. In stochastic system, population variations have a great role for the stochastic stability. The noise in the equation results in a big variance of fluctuations around the equilibrium point which suggests that our system oscillates with respect to the noisy environment. From the Numerical simulation we conclude that the inclusion of stochastic perturbation creates a noteworthy variation in the intensity of populations due to change of responsive parameters cause's chaotic dynamics with low, medium and high variances of oscillations from Figures 3- Figure 8.

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