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ALMOST LATTICES

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Abstract. The concept of an Almost Lattice (AL) is introduced and given certain examples of an AL which are not lattices. Also, some basic properties of an AL are proved and set of equivalent conditions are established for an AL to become a lattice. Further, the concept of AL with 0 is introduced and some basic properties of an AL with 0 are proved.

1. Introduction

It was Garett Birkhoff's (1911 - 1996) work in the mid thirties that started the general development of the lattice theory. In a brilliant series of papers, he demonstrated the importance of the lattice theory and showed that it provides a unified frame work for unrelated developments in many mathematical disciplines. V. Glivenko, Karl Menger, John Van Neumann, Oystein Ore, George Gratzer, P. R. Halmos, E. T. Schmidt, G. Szasz, M. H. Stone , R. P. Dilworth and many others have developed enough of this field for making it attractive to the mathematicians and for its further progress. The traditional approach to lattice theory proceeds from partially ordered sets to general lattices, semimodular lattices, modular lattices and finally to distributive lattices.

In this paper, we introduced the concept of an Almost lattice AL which is a generalization of a lattice and we gave certain examples of ALs which are not lattices. Also, we proved some basic properties in the class of ALs and we defined a partial ordering " $\leqslant$ " on an AL. We proved that an AL L is directed above under $\leqslant$ if and only if L is a lattice. In addition, we established sets of identities for an AL to become a lattice. The concept of zero element in an AL is introduced and we observed that an ALL can have at most one zero element and it will be the least

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element of the poset $(L, \leqslant)$. We also gave few examples of an AL with 0 and we proved some basic properties of an AL with 0. A necessary and sufficient condition for an AL with 0 to become a lattice with 0 is established. Finally, we introduced the concept of a simple AL and gave a set of identities for an AL with 0 to become a simple AL.

2. Preliminaries

In this section, we collect a few important definitions and results which are already known and which will be used more frequently in the paper.

Definition 2.1. Let A and B be non empty sets. A relation R from A to B is a subset of $A \times B$. Relations from A to A are called relations on A.

Note that if R is a relation on a non empty set A, then for any $a, b \in A$ we write aRb instead of $(a, b) \in R$ and say that "a is in relation R to b". A relation R on the set A may have some of the following properties.

(1) R is reflexive if for all a in A, we have aRa.
(2) R is symmetric, if for all a, b in A, aRb implies bRa.
(3) R is antisymmetric, if for all a, b in A, aRb and bRa implies $a = b$.
(4) R is transitive if for all a, b, c in A, aRb and bRc imply aRc.

Definition 2.2. A relation R on a non empty set A is called an equivalence relation if R is reflexive, symmetric and transitive.

Definition 2.3. A relation R on a non empty set A is called a partial order relation if R is reflexive, antisymmetric and transitive. In this case, (A, R) is called a partially ordered set or poset.

Note that if $(P, \leqslant)$ is a poset and $x, y \in P$ such that $x \leqslant y$ and $x \neq y$, then we write $x < y$.

Definition 2.4. Let $(P, \leqslant)$ be a poset. For any $x, y \in P$, we say that y covers x or x is covered by y, (denoted by $x \prec y$), if $x < y$, and there is no $z \in P$ such that $x < z < y$.

Definition 2.5. A partial order $\leqslant$ on a set P is called a total order, if for each $a, b \in R$, either $a \leqslant b$ or $b \leqslant a$. In this case, the poset $(P, \leqslant)$ is called a totally ordered set or a chain.

Definition 2.6. Let $(P, \leqslant)$ be a poset and $a \in P$. Then

(1) a is called the least element of P if $a \leqslant x$ for all $x \in P$.
(2) a is called the greatest element of P if $x \leqslant a$ for all $x \in P$.

It can be easily observed that, if least (greatest) element exists in a poset, then it is unique.

Definition 2.7. Let $(P, \leqslant)$ be a poset and $a \in P$. Then

(1) a is called a minimal element, if $x \leqslant a$, implies $x = a$ for all $x \in P$.
(2) a is called maximal element, if $a \leqslant x$, implies $a = x$ for all $x \in P$.

It can be easily verified that least (greatest) element (if exists), then it is minimal (maximal) but, converse need not be true.

Definition 2.8. Let $(P, \leqslant)$ be a poset and $S \subseteq P$. Then

(1) $a \in P$ is called a lower bound of S $\Longleftrightarrow$ for all x in S; $a \leqslant x$.
(2) $a \in P$ is called an upper bound of $S \Longleftrightarrow$ for all x in S; $x \leqslant a$.
(3) The greatest amongst the lower bounds, whenever if it exists is the infimum of S and is denoted by inf S or $\wedge S$.
(4) The least amongst the upper bound of S whenever if it exists is called supremum of S and is denoted by Sup S or $\vee S$.

Definition 2.9. (Zorn's Lemma) If $(P, \leqslant)$ is a poset such that every chain of elements in P has an upper bound in P, then P has at least one maximal element.

Definition 2.10. Let $(P, \leqslant)$ be a poset. If P has least element 0 and greatest element 1, then P is said to be a bounded poset.

Note that if $(P, \leqslant)$ is a bounded poset with bounds $0, 1$, then for any $x \in P$, we have $0 \leqslant x \leqslant 1$.

Definition 2.11. Let $(P, \leqslant)$ be a poset. Then P is called lattice ordered set if for every pair x, y of elements of P the sup(x,y) and inf(x,y) exist.

Definition 2.12. An algebra $(L, \vee, \wedge)$ of type $(2, 2)$ is called a lattice if it satisfies the following axioms. For any $x, y, z \in L$,

(1) $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$. (Commutative Law)
(2) $(x \vee y) \vee z = x \vee (y \vee z)$ and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$. (Associative Law)
(3) $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$. (Absorption Law)

It can be easily seen that in any lattice $(L, \vee, \wedge)$, $x \vee x = x$ and $x \wedge x = x$ (Idempotent Law).

Theorem 2.1. *Let $(L, \leqslant)$ be a lattice ordered set. If we define $x \wedge y$ is the inf (x, y) and $x \vee y$ is the sup(x, y), where $x, y \in L$, then $(L, \vee, \wedge)$ is a lattice.*

Theorem 2.2. *Let $(L, \vee, \wedge)$ be a lattice. If we define a relation $\leqslant$ on L, by $x \leqslant y$ if and only if $x = x \wedge y$, or equivalently $x \vee y = y$. Then $(L, \leqslant)$ is a lattice ordered set.*

Important Note: Theorem 2.1 and theorem 2.2 together imply that the concepts of lattice and lattice ordered set are equivalent. We refer to it as a lattice in future.

Definition 2.13. A lattice L with 0 is said to be pseudocomplemented if there exists a unary operation $x \mapsto x^\star$ on L such that $x \wedge y = 0$ if and only if $x \leqslant y^\star$ for all $x, y \in L$.

Theorem 2.3. *In any lattice $(L, \vee . \wedge)$, the following are equivalent:*

(1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
(2) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
(3) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
(4) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$.

Definition 2.14. A lattice $(L, \vee, \wedge)$ is called a distributive lattice if it satisfies any one of the above four conditions.

Definition 2.15. Let $(L, \vee, \wedge)$ be a lattice. Then L is said to be bounded lattice if L is bounded as a poset. That is, there exists $0, 1 \in L$ such that $0 \wedge a = 0$ and $1 \vee a = 1$ for all $a \in L$.

Definition 2.16. A bounded lattice $(L, \vee, \wedge)$ with bounds 0 and 1 is said to be complemented if to each $x \in L$, there exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$.

Definition 2.17. A bounded lattice $(L, \vee, \wedge)$ with bounds 0 and 1 is said to be complemented if to each $x \in L$, there exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$.

Definition 2.18. A complemented distributive lattice is called a Boolean algebra.

Definition 2.19. Let $(R, +, .)$ be a ring with identity element 1. Then R is called a Boolean ring if every element in R is idempotent.

Definition 2.20. A semigroup is a groupoid $(G, .)$ in which $x.(y.z) = (x.y).z$ for all $x, y, z \in G$.

Definition 2.21. A semilattice is a semigroup $(S, .)$ which satisfies the commutative and idempotent law.

Definition 2.22. Let $(S, \wedge)$ be a semi lattice with least element 0. An element $a^\star$ is a pseudocomplement of $a \in S$ if and only if $a \wedge a^\star = 0$ and $a \wedge x = 0$ implies $x \leqslant a^\star$ $\forall x \in S$.

Definition 2.23. A semilattice $(S, \wedge)$ with a least element 0 is said to be pseudocomplemented semilattice if every element in S has a pseudocomplement.

Definition 2.24. A ring R is called a p_1- ring if, to each $x \in R$, there exists a central idempotent $x^o \in R$ such that:

(1) $xx^o = x$
(2) For any idempotent e of R, $xe = x$ implies that $x^o e = x^o$.

Here, x^o is known as minimal idempotent duplicator of x in the center of R.

Definition 2.25. A semigroup S with 0 is called a Bear-Stone semigroup if, to each $x \in S$, there exists a central idempotent $x^\star \in S$ such that:

(1) $x^\star S = \{y \in S |\ xy = 0 = yx\}$
(2) The map $s \mapsto (x^\star s, x^{\star\star} s)$ is an isomorphism of S onto $x^\star S \times x^{\star\star} S$.

Definition 2.26. A pseudocomplemented distributive lattice with 0 is called a Stone lattice if, for any $x \in L$, $x^\star \vee x^{\star\star} = 1$.

Definition 2.27. A pseudocomplemented semilattice S is called strongly admissible if

(1) For each $x \in S$, there exists a dense element $d \in S$ (that is, $d^\star = 0$) such that $x = x^{\star\star} \wedge d$
(2) There is a mapping $f : S^{\star\star} \times D \longrightarrow D$, where $S^{\star\star}$ is the set of all closed elements of S and D, the set of all dense elements of S, such that, for any $x \in S$, $x \leqslant f(a,d)$ if and only if $x \wedge a \leqslant d$ for all $a \in S^{\star\star}$ and $d \in D$
(3) $f(a \vee b, d) = f(a,d) \wedge f(b,d)$ for all $a, b \in S^{\star\star}$ and $d \in D$.

Definition 2.28. A $\star-$ ring is a structure $(R, \star)$ where R is a ring and $\star$ is a map of R into R satisfying, for any $a \in R$:
i. $aa^\star = a$ and
ii. $x \in R, ax = a \Longrightarrow a^\star x = a^\star$.

Definition 2.29. A ring R is called a regular ring if, to each $a \in R$, there exists $x \in R$ such that $axa = a$.

Definition 2.30. A ring R is called a p- ring (p is prime) if, for any $x \in R$, $x^p = x$ and $px = 0$.

Definition 2.31. A ring R is called bi-regular if every principal ideal is generated by a central idempotent .

Definition 2.32. A ring R is a Bear ring if, to each $x \in R$, there exists a central idempotent $e \in R$ such that $eR = \{y \in R| \ xy = 0 = yx\}$.

Definition 2.33. Let S be a semigroup with 0 in which, to each $x \in S$, there exists a central idempotent e of S such that $ex = x$ and E_S, the semilattice of all central idempotents of S, is directed above. An element $a \in S$ is said to be $B-$central if there exists semigroups S_1 and S_2 with 0 such that S_1 has 1 also and an isomorphism of S onto $S_1 \times S_2$ which maps the element a onto the element $(1, 0)$ of $S_1 \times S_2$. The set $B(S)$ of all $B-$ central elements of S is called the Birkhoff center of S.

Definition 2.34. Let S be a semigroup with 0 satisfying the hypothesis of the above definition. Then S is called a p_1- semigroup if:
(1) For each $x \in S$, there exist $x^o \in B(S)$ such that $xx^o = x$
(2) For any $a \in B(S)$ such that $ax = x$, must $ax^o = x^o$.

3. Almost Lattices (AL)

In this section, we introduce the concept of Almost Lattice(AL) and we give some examples of an AL which are not a lattices.

Definition 3.1. An algebra $(L, \vee, \wedge)$ of type (2,2) is called an Almost Lattice if it satisfies the following axioms.
A_1. $(a \wedge b) \wedge c = (b \wedge a) \wedge c$
A_2. $(a \vee b) \wedge c = (b \vee a) \wedge c$
A_3. $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
A_4. $(a \vee b) \vee c = a \vee (b \vee c)$
A_5. $a \wedge (a \vee b) = a$

A_6. $a \vee (a \wedge b) = a$
A_7 $(a \wedge b) \vee b = b$

For brevity, in future, we will refer to this Almost Lattices as AL unless otherwise specified. Now we give some examples of an AL.

EXAMPLE 3.1. Every lattice is an AL.

In the following, we give some examples of ALs which are not Lattices.

EXAMPLE 3.2. Let $L = \{a, b, c\}$. Define $\vee$ and $\wedge$ on L as follows:

$\vee$	a	b	c
a	a	b	c
b	b	b	b
c	c	c	c

and

$\wedge$	a	b	c
a	a	a	a
b	a	b	c
c	a	b	c

Then clearly $(L, \vee, \wedge)$ is an AL, but not a lattice because $c \vee b \neq b \vee c$.

EXAMPLE 3.3. Let $L = \{a, b, c\}$. Define $\vee$ and $\wedge$ on L as follows.

$\vee$	a	b	c
a	a	a	a
b	b	b	b
c	a	b	c

and

$\wedge$	a	b	c
a	a	b	c
b	a	b	c
c	c	c	c

Then clearly $(L, \vee, \wedge)$ is an AL, but not a lattice since $a \vee b \neq b \vee a$.

EXAMPLE 3.4. Let $L = \{1, 2, 3, 4\}$. Define $\vee$ and $\wedge$ on L as follows:

$\vee$	1	2	3	4
1	1	1	3	3
2	2	2	4	4
3	3	3	3	3
4	4	4	4	4

and

$\wedge$	1	2	3	4
1	1	2	1	2
2	1	2	1	2
3	1	2	3	4
4	1	2	3	4

Then clearly $(L, \vee, \wedge)$ is an AL, but not a lattice since $3 \vee 4 \neq 4 \vee 3$.

EXAMPLE 3.5. Let L be a non empty set. Define, for any $a, b \in L, a \vee b = a = b \wedge a$. Then clearly $(L, \vee, \wedge)$ is an AL and this AL is called discrete AL.

We conclude this section by exhibiting the structure of an AL in some known algebras.

G. Suryanarayan Murti (in his doctoral thesis entitled "Boolean Center of Universal Algebra" (1980), Andhra University, Walltier, India); introduced the notion of P_1- semigroup as a common abstraction of P_1- rings and Baer- Stone semigroups. Thus the class of P_1- semigroups include the classes of Boolean rings, regular rings, P-rings, bi-regular rings, Bear rings, Stone lattices, strongly admissible semilattices etc. In the following example, we define two binary operations $\vee$ and $\wedge$ in a P_1- semigroup $(S, .)$ and with this operation, $(S, \vee, \wedge)$ becomes an AL. Thus we have an AL structure in each of the algebras mentioned above.

Example 3.6. Let $(S, .)$ be a P_1- semigroup. Let us recall that, to each $x \in S$, there exists x^0 in the Birkhoff center $B(S)$ of S which is least among the elements of $B(S)$ with the property $x^0 x = x$. Since $x^0 \in B(S)$, there exists $x^{0'} \in B(S)$ such that the mapping $y \mapsto (x^0 y, x^{0'} y)$ of S onto $x^0 S \times x^{0'} S$ is an isomorphism. Now, define for any $x, y \in S,\ x \wedge y = x^0 y$ and $x \vee y$ to be the unique element of S such that $x^0(x \vee y) = x$ and $x^{0'}(x \vee y) = x^{0'} y$. Then it can be easily verified that $(S, \vee, \wedge)$ is an AL.

4. Lattice Theory of ALs

In this section, we prove some results in an AL L. Further, we define a partial ordering $\leqslant$ on L and prove that with this partial ordering, the poset $(L, \leqslant)$ is directed above if and only if L is a Lattice. Also, we establish some sets of equivalent conditions for an AL to become a Lattice. First we prove the following.

Lemma 4.1. *Let L be an AL and $a \in L$. Then;*

$I_\vee$. $a \vee a = a$
$I_\wedge$. $a \wedge a = a$

Proof. $(I_\vee)$:- By conditions A_5 *and* A_6, in the definition of an AL, we have $a \vee a = a \vee \{a \wedge (a \vee a)\} = a$.
$(I_\wedge)$:- Again by A_6 *and* A_5, we obtain $a \wedge a = a \wedge \{a \vee (a \wedge a)\} = a$. □

Lemma 4.2. *Let L be an AL. Then for any $a, b \in L,\ a \wedge b = a$ if and only if $a \vee b = b$.*

Proof. Suppose $a \wedge b = a$. Then by A_7, we get $a \vee b = (a \wedge b) \vee b = b$. Conversely, suppose $a \vee b = b$. Then by A_5, we get $a \wedge b = a \wedge (a \vee b) = a$. □

In the following we define a partial ordering on an AL L.

Definition 4.1. Let L be an AL and $a, b \in L$. Then we define a is less than or equal to b and write as $a \leqslant b$ if and only if $a \wedge b = a$ or, equivalently $a \vee b = b$.

Theorem 4.1. *The relation $\leqslant$ is a partial ordering on an AL L and hence $(L, \leqslant)$ is a poset.*

Proof. The reflexivity of $\leqslant$ follows from $(I_\wedge)$. Let $a, b \in L$ such that $a \leqslant b$ and $b \leqslant a$. Then $a \wedge b = a$ and $b \wedge a = b$. Now, $a = a \wedge b = (a \wedge b) \wedge b = (b \wedge a) \wedge b = b \wedge (a \wedge b) = b \wedge a = b$. Therefore $\leqslant$ is antisymmetric relation on L. Suppose $a, b, c \in L$ such that $a \leqslant b$ and $b \leqslant c$. Then $a \wedge b = a$ and $b \wedge c = b$. Now, $a \wedge c = (a \wedge b) \wedge c = a \wedge (b \wedge c) = a \wedge b = a$. It follows that, $a \leqslant c$ and hence $\leqslant$ is transitive relation on L. Therefore $(L, \leqslant)$ is a poset. □

notation 4.1. Every relation θ on an AL L may not be a partial order relation in general. For, if we define a relation θ on an AL L by $a\ \theta\ b$ if and only if $a \wedge b = b$, then clearly θ is reflexive and transitive but not antisymmetric in general. For, suppose L is an AL with at least two elements. Then $(I_\wedge)$ we have $a \wedge a = a$ and hence $a\ \theta\ a\ \forall a\ \in L$. Therefore θ is reflexive relation on L. Also, for any

$a, b, c \in L$, suppose $a\ \theta\ b$ and $b\ \theta\ c$. Then $a \wedge b = b$ and $b \wedge c = c$. Now, $a \wedge c = a \wedge (b \wedge c) = (a \wedge b) \wedge c = b \wedge c = c$. Hence $a\ \theta\ c$. Therefore θ is transitive relation on L. Now, choose $a, b \in L$ *with* $a \neq b$ such that $a\ \theta\ b$ and $b\ \theta\ a$. Then $a \wedge b = b$ and $b \wedge a = a$. But, $a \neq b$ and hence θ is not antisymmetric relation on L.

In the following, we prove some basic properties of an AL.

Theorem 4.2. *Let L be an AL. Then for any $a, b, c \in L$, we have the following.*

(1) $a \leqslant b \Longrightarrow a \wedge b = b \wedge a$
(2) $a \leqslant a \vee b$
(3) $a \wedge b \leqslant b$
(4) $(a \vee b) \wedge a = a$
(5) $(a \vee b) \wedge b = b$
(6) $a \wedge b = b \Longleftrightarrow a \vee b = a$
(7) $a \leqslant b \Longrightarrow a \vee b = b \vee a$
(8) $b \vee (a \wedge b) = b$
(9) $a \vee b = b \vee a \Longrightarrow a \wedge b = b \wedge a$
(10) *If $a \leqslant c$ and $b \leqslant c$, then $a \wedge b \leqslant c$ and $a \vee b \leqslant c$*

Proof. (1) Suppose $a \leqslant b$. Then $a \wedge b = a$. Now, we have $a \wedge b = (a \wedge b) \wedge b = (b \wedge a) \wedge b = b \wedge (a \wedge b) = b \wedge a$. Hence $a \wedge b = b \wedge a$.
(2) From A_5, we have $a \wedge (a \vee b) = a$. It follows that, $a \leqslant a \vee b$.
(3) By A_3 and $I_\wedge$, we have $(a \wedge b) \wedge b = a \wedge (b \wedge b) = a \wedge b$. Therefore, $a \wedge b \leqslant b$.
(4) By (1) and (2) we get, $a = a \wedge (a \vee b) = (a \vee b) \wedge a$, since $a \leqslant a \vee b$.
(5) By (4) and A_2, $b = (b \vee a) \wedge b = (a \vee b) \wedge b$.
(6) Suppose $a \wedge b = b$. Then $a \vee b = a \vee (a \wedge b) = a$. Conversely, suppose $a \vee b = a$. Then $a \wedge b = (a \vee b) \wedge b = b$, since by (5).
(7) Suppose $a \leqslant b$. Then $b = a \vee b$. Now, by(4) we get $(a \vee b) \wedge a = a$. Hence by (6), we get $(a \vee b) \vee a = a \vee b$. It follows that, $b \vee a = (a \vee b) \vee a = a \vee b$. Therefore $a \vee b = b \vee a$.
(8) We have $a \wedge b \leqslant b$. Therefore, by (7) we get $b \vee (a \wedge b) = (a \wedge b) \vee b = b$.
(9) Suppose $a \vee b = b \vee a$. Then $b \wedge a = b \wedge \{a \wedge (a \vee b)\} = (b \wedge a) \wedge (a \vee b) = (a \wedge b) \wedge (a \vee b) = a \wedge \{b \wedge (a \vee b)\} = a \wedge \{b \wedge (b \vee a)\} = a \wedge b$.
(10) Suppose $a \leqslant c$ *and* $b \leqslant c$. Then $(a \wedge b) \wedge c = a \wedge (b \wedge c) = a \wedge b$. Hence $a \wedge b \leqslant c$. Also, $(a \vee b) \vee c = a \vee (b \vee c) = a \vee c = c$. Therefore $a \vee b \leqslant c$. □

Corollary 4.1. *Let L be an AL. Then for any $a, b \in L$, we have the following.*

(1) $(a \vee b) \vee b = a \vee b$
(2) $(a \vee b) \vee a = a \vee b$
(3) $a \vee (a \vee b) = a \vee b$
(4) $a \wedge (a \wedge b) = a \wedge b$
(5) $(a \wedge b) \wedge b = a \wedge b$
(6) $b \wedge (a \wedge b) = a \wedge b$

Definition 4.2. An AL L is said to be directed above if for any $a, b \in L$ there exists $c \in L$ such that $a, b \leqslant c$.

In the following we give a set of equivalent conditions that an AL to become a lattice.

THEOREM 4.3. *Let L be an AL. Then the following are equivalent:*

(1) *L is directed above.*
(2) *$\wedge$ is commutative.*
(3) *$\vee$ is commutative.*
(4) *L is a lattice.*

PROOF. (1) $\Longrightarrow$ (2):- Suppose L is directed above. Let $a, b \in L$. Then there exists $c \in L$ such that $a \leqslant c$ and $b \leqslant c$. This implies $a = a \wedge c$ and $b = b \wedge c$. Now, $a \wedge b = a \wedge (b \wedge c) = (a \wedge b) \wedge c = (b \wedge a) \wedge c = b \wedge (a \wedge c) = b \wedge a$.
(2) $\Longrightarrow$ (1):- Suppose $\wedge$ is commutative. Let $a, b \in L$. Then by (A_5), we get $a \wedge (a \vee b) = a$. Therefore $a \leqslant a \vee b$. Also, we have $b \wedge (a \vee b) = (a \vee b) \wedge b = b$. Hence we get $b \leqslant a \vee b$. Therefore L is directed above.
(1) $\Longrightarrow$ (3):- Suppose L is directed above. Let $a, b \in L$. Then there exists $c \in L$ such that $a \leqslant c$ and $b \leqslant c$. Therefore $a = a \wedge c$ and $b = b \wedge c$ and hence $a \vee c = c$ and $b \vee c = c$. Also, since $(a \vee b) \vee c = a \vee (b \vee c) = a \vee c = c$ and $(b \vee a) \vee c = b \vee (a \vee c) = b \vee c = c$. Now, consider $a \vee b = (a \vee b) \wedge c = (b \vee a) \wedge c = b \vee a$. Therefore $\vee$ is commutative.
(3) $\Longrightarrow$ (4):- Suppose $\vee$ is commutative. Let $a, b \in L$. Then by theorem 4.2 (9), we have $a \wedge b = b \wedge a$. Therefore L is a lattice.
(4) $\Longrightarrow$ (1):- Suppose L is a lattice. Then for any $a, b \in L$, $a \leqslant a \vee b$ *and* $b \leqslant a \vee b$. Therefore L is directed above. $\square$

THEOREM 4.4. *The following conditions are equivalent for all $a, b \in L$ of an AL L.*

(1) $(a \wedge b) \vee a = a$
(2) $a \wedge (b \vee a) = a$
(3) $(b \wedge a) \vee b = b$
(4) $b \wedge (a \vee b) = b$
(5) $a \wedge b = b \wedge a$
(6) $a \vee b = b \vee a$
(7) *L is directed above.*

PROOF. (1) $\Longrightarrow$ (5):- Suppose $(a \wedge b) \vee a = a$. Then by lemma 4.2, we get $(a \wedge b) \wedge a = a \wedge b$. Now consider, $a \wedge b = (a \wedge b) \wedge a = (b \wedge a) \wedge a = b \wedge (a \wedge a) = b \wedge a$. Therefore $a \wedge b = b \wedge a$.
(5) $\Longrightarrow$ (1):- Suppose that $a \wedge b = b \wedge a$. Then $(a \wedge b) \vee a = (b \wedge a) \vee a = a$.
(3) $\Longrightarrow$ (5):- Suppose $(b \wedge a) \vee b = b$. Then by lemma 4.2, we get $(b \wedge a) \wedge b = b \wedge a$. Now, $b \wedge a = (b \wedge a) \wedge b = (a \wedge b) \wedge b = a \wedge b$. Thus $a \wedge b = b \wedge a$.
(5) $\Longrightarrow$ (3):- Suppose $a \wedge b = b \wedge a$. Then $(b \wedge a) \vee b = (a \wedge b) \vee b = b$.
(2) $\Longrightarrow$ (6):- Suppose $a \wedge (b \vee a) = a$. Then by lemma 4.2, we have $a \vee (b \vee a) = b \vee a$. But, by corollary 4.1 we get $a \vee (b \vee a) = a \vee b$. Hence $a \vee b = b \vee a$.
(6) $\Longrightarrow$ (2):- Suppose $a \vee b = b \vee a$. Then $a \wedge (b \vee a) = a \wedge (a \vee b) = a$.
(4) $\Rightarrow$ (6):- Suppose $b \wedge (a \vee b) = b$. Then by lemma 4.2, we get $b \vee (a \vee b) = a \vee b$.

On the other hand, we have $(b\vee a)\wedge b = b$. Therefore $b\vee(a\vee b) = (b\vee a)\vee b = b\vee a$. Therefore $a\vee b = b\vee a$.
(6) $\Longrightarrow$ (4):- Suppose that $a\vee b = b\vee a$. Then $b\wedge(a\vee b) = b\wedge(b\vee a) = b$.
(2) $\Longrightarrow$ (7):- Suppose $a\wedge(b\vee a) = a$. Then we get $a \leqslant b\vee a$. But, we have $b \leqslant b\vee a$. Therefore L is directed above.
(7) $\Longrightarrow$ (2):- Suppose L is directed above. By theorem 4.3, $\vee$ is commutative. Now, consider $a\wedge(b\vee a) = a\wedge(a\vee b) = a$. By theorem 4.3, we get (7) and (5) are equivalent. □

THEOREM 4.5. *Let L be an AL. Then for all $a, b \in L$, the following are equivalent:*

(1) $a\vee b = b\vee a$
(2) $b \leqslant a\vee b$
(3) *L is directed above*
(4) *$a\vee b$ is the lub of a and b in the poset $(L, \leqslant)$.*
(5) $a \leqslant b\vee a$
(6) *$b\vee a$ is the lub of a and b in the poset $(L, \leqslant)$.*

PROOF. (1) $\Longrightarrow$ (2):- Suppose $a\vee b = b\vee a$. Then $b\wedge(a\vee b) = b\wedge(b\vee a) = b$. Therefore $b \leqslant a\vee b$.
(2) $\Longrightarrow$ (3):- Suppose $b \leqslant a\vee b$. But, we have $a\wedge(a\vee b) = a$ and hence $a \leqslant a\vee b$. Therefore L is directed above. Proof of (3) $\Longrightarrow$ (1) follows from theorem 4.3.
(2) $\Longrightarrow$ (4):- Suppose $b \leqslant a\vee b$. Now, since $a\wedge(a\vee b) = a,\ a \leqslant a\vee b$. Hence $a\vee b$ is an upper bound of a and b in the poset $(L, \leqslant)$. Suppose $x \in L$ such that x is an upper bound of a and b in the poset $(L, \leqslant)$. Then $a \leqslant x$ and $b \leqslant x$. It follows that, $a\vee x = x$ and $b\vee x = x$. Now, $(a\vee b)\vee x = a\vee(b\vee x) = a\vee x = x$. Hence, $a\vee b \leqslant x$. Therefore $a\vee b$ is the lub of a and b in the poset $(L, \leqslant)$.
(4) $\Rightarrow$ (2):- Suppose $a\vee b$ is the lub of a and b in the poset $(L, \leqslant)$. Then we have $b \leqslant a\vee b$.
(1) $\Rightarrow$ (5):- Suppose $a\vee b = b\vee a$. Then $a\wedge(b\vee a) = a\wedge(a\vee b) = a$. Hence $a \leqslant b\vee a$.
(5) $\Rightarrow$ (6):- Assume (5). Then we get $a \leqslant b\vee a$. But, we have $b \leqslant b\vee a$. Therefore $b\vee a$ is an upper bound of a and b in the poset $(L, \leqslant)$. Suppose $x \in L$ such that x is an upper bound of a and b in the poset $(L, \leqslant)$. Then $a \leqslant x$ and $b \leqslant x$. Hence $a\vee x = x$ *and* $b\vee x = x$. Now consider, $(b\vee a)\vee x = b\vee(a\vee x) = b\vee x = x$. It follows that, $b\vee a \leqslant x$. Therefore $b\vee a$ is the lub of a and b in the poset $(L, \leqslant)$. Proof of (6) $\Rightarrow$ (3) is clear. □

Though the duality principle is, in general, not valid in an AL, we have the following theorem:

THEOREM 4.6. *Let L be an AL. Then for all $a, b \in L$, the following are equivalent:*

(1) $a\wedge b = b\wedge a$
(2) $a\wedge b \leqslant a$
(3) *$a\wedge b$ is the glb of a and b in the poset $(L, \leqslant)$.*
(4) $b\wedge a \leqslant b$

(5) *$b \wedge a$ is the glb of a and b in the poset $(L, \leqslant)$.*

Proof. (1) $\Rightarrow$ (2):- Suppose $a \wedge b = b \wedge a$. Then $(a \wedge b) \vee a = (b \wedge a) \vee a = a$. It follows that, $a \wedge b \leqslant a$.
(2) $\Rightarrow$ (3):- Suppose $a \wedge b \leqslant a$. But, we have $(a \wedge b) \vee b = b$ and hence $a \wedge b \leqslant b$. Therefore $a \wedge b$ is a lower bound of a and b in the poset $(L, \leqslant)$. Suppose $c \in L$ such that c is a lower bound of a and b. Then $c \leqslant a$ and $c \leqslant b$. Now, $c \wedge (a \wedge b) = (c \wedge a) \wedge b = c \wedge b = c$. Hence $c \leqslant a \wedge b$. Therefore $a \wedge b$ is the glb of a and b in the poset $(L, \leqslant)$.
(3) $\Rightarrow$ (1):- Suppose $a \wedge b$ is the glb of a and b in the poset $(L, \leqslant)$. Then $a \wedge b \leqslant a$. Now, $a \wedge b = (a \wedge a) \wedge b = a \wedge (a \wedge b) = (a \wedge b) \wedge a = (b \wedge a) \wedge a = b \wedge (a \wedge a) = b \wedge a$. Therefore $a \wedge b = b \wedge a$.
(1) $\Rightarrow$ (4):- Suppose $a \wedge b = b \wedge a$. Now, consider $(b \wedge a) \wedge b = (a \wedge b) \wedge b = a \wedge b = b \wedge a$. Therefore $b \wedge a \leqslant b$.
(4) $\Rightarrow$ (5):- Suppose $b \wedge a \leqslant b$. But we have $b \wedge a \leqslant a$. Therefore $b \wedge a$ is a lower bound of a and b in the poset $(L, \leqslant)$. Suppose $c \in L$ such that c is a lower bound of a and b. Then $c \leqslant a$ and $c \leqslant b$. Hence $c = c \wedge a$ and $c = c \wedge b$. Now, consider $c \wedge (b \wedge a) = (c \wedge b) \wedge a = c \wedge a = c$. Hence $c \leqslant b \wedge a$. Therefore $b \wedge a$ is the glb of a and b in the poset $(L, \leqslant)$.
(5) $\Rightarrow$ (1) Suppose $b \wedge a$ is the glb of a and b in the poset $(L, \leqslant)$. Then $b \wedge a \leqslant b$. Now, $a \wedge b = a \wedge (b \wedge b) = (a \wedge b) \wedge b = (b \wedge a) \wedge b = b \wedge (b \wedge a) = (b \wedge b) \wedge a = b \wedge a$. Therefore $a \wedge b = b \wedge a$. $\square$

In the following we give an other set of equivalent condition that an AL to become a lattice.

Theorem 4.7. *Let L be an AL. Then the following are equivalent:*

(1) *L is a lattice.*
(2) *The poset $(L, \leqslant)$ is directed above.*
(3) *$a \wedge (b \vee a) = a$ for all $a, b \in L$.*
(4) *The operation $\vee$ is commutative.*
(5) *The operation $\wedge$ is commutative.*
(6) *The relation $\theta = \{(a, b) \in L \times L | a \wedge b = b\}$ is anti symmetric.*
(7) *The relation θ defined in (6) is a partial ordering on L.*

Proof. From theorem 4.3 and 4.4, conditions $(1), (2), (3), (4)$ *and* (5) are equivalent.
(5) $\Rightarrow$ (6):- Suppose the operation $\wedge$ is commutative. Let $(a, b), (b, a) \in \theta$. Then $a \wedge b = b$ *and* $b \wedge a = a$. Now, $b = a \wedge b = b \wedge a = a$. Therefore θ is antisymmetric relation on L.
(6) $\Rightarrow$ (7):- Suppose θ is antisymmetric. Then from notation 4.1, we get θ is a partial ordering on L.
(7) $\Rightarrow$ (1):- Suppose θ is a partial ordering on L. Let $a, b \in L$. Now, consider $(a \wedge b) \wedge (b \wedge a) = a \wedge (b \wedge (b \wedge a)) = a \wedge (b \wedge a) = (a \wedge b) \wedge a = (b \wedge a) \wedge a = b \wedge a$ and also $(b \wedge a) \wedge (a \wedge b) = b \wedge (a \wedge (a \wedge b)) = b \wedge (a \wedge b) = (b \wedge a) \wedge b = (a \wedge b) \wedge b = a \wedge b$. Hence $(a \wedge b, b \wedge a) \in \theta$ *and* $(b \wedge a, a \wedge b) \in \theta$. Therefore by antisymmetric we get $a \wedge b = b \wedge a$. Also, by theorem 4.3, L is a lattice. $\square$

Corollary 4.2. *Let L be an AL be and $a \in L$. Then the set $L_a = \{x \wedge a | x \in L\} = \{x \in L | x \leqslant a\}$ is a lattice under the induced operations $\vee$ and $\wedge$ on L with a as its greatest element.*

Proof. Let $x \wedge a,\ y \wedge a \in L_a$. Then $(x \wedge a) \wedge (y \wedge a) = x \wedge (a \wedge (y \wedge a)) = x \wedge ((a \wedge y) \wedge a) = x \wedge ((y \wedge a) \wedge a) = x \wedge (y \wedge a) = (x \wedge y) \wedge a \in L_a$. Also, we have $x \wedge a \leqslant a$ and $y \wedge a \leqslant a$ and hence $(x \wedge a) \vee (y \wedge a) \leqslant a$. Therefore $(x \wedge a) \vee (y \wedge a) \in L_a$. Therefore L_a is closed under $\vee$ and $\wedge$. Clearly, L_a is directed above by a. Thus $(L_a, \vee, \wedge)$ is a lattice. $\square$

Remark 4.1. From what we have done so far, it is evident that an AL L satisfies the axioms of a lattice except possibly;

(1) $(C_\wedge)$, commutativity of the operation $\wedge$.
(2) $(C_\vee)$, commutativity of the operation $\vee$.

We have observed that $(C_\wedge)$ and $(C_\vee)$ are equivalent in L (see theorem 4.10) and if we add either of these two axioms to L, then $(L, \vee, \wedge)$ will become a lattice.

Proposition 4.1. *The operations $\vee$ and $\wedge$ in an AL L are isotone. That is if $a, b \in L$ and $a \leqslant b$, then for any $c \in L$, $a \wedge c \leqslant b \wedge c$, $c \wedge a \leqslant c \wedge b$ and $c \vee a \leqslant c \vee b$.*

Proof. Suppose $a, b \in L$ such that $a \leqslant b$. Then $a = a \wedge b$ and hence $a \vee b = b$. Now, $(a \wedge c) \wedge (b \wedge c) = a \wedge (c \wedge (b \wedge c)) = a \wedge (b \wedge c) = (a \wedge b) \wedge c = a \wedge c$. Therefore $a \wedge c \leqslant b \wedge c$. Also, $(c \wedge a) \wedge (c \wedge b) = (a \wedge c) \wedge (c \wedge b) = a \wedge (c \wedge (c \wedge b)) = a \wedge (c \wedge b) = (a \wedge c) \wedge b = (c \wedge a) \wedge b = c \wedge (a \wedge b) = c \wedge a$. Therefore $c \wedge a \leqslant c \wedge b$. Further, $(c \vee a) \vee (c \vee b) = ((c \vee a) \vee c) \vee b = (c \vee a) \vee b = c \vee (a \vee b) = c \vee b$. Therefore $c \vee a \leqslant c \vee b$. $\square$

It can be easily seen that for any $a, b \in L$, $a \leqslant b$ need not generally imply that $a \vee c \leqslant b \vee c$. For, in example 3.2 we have $a \leqslant b$. But, $a \vee c = c \nleq b = b \vee c$.

Definition 4.3. Let L be an AL. Then for any $a, b \in L$, we say that a is compatible with b, written as $a \sim b$ if $a \wedge b = b \wedge a$ or, equivalently, $a \vee b = b \vee a$.

We can easily observe that the above relation $\sim$ is reflexive and symmetric but not transitive. For, in example 3.2, we have $b \sim a$ and $a \sim c$. But, $b \nsim c$.

Proposition 4.2. *Let L be an AL. Then for any $a, b \in L$, $a \sim b$ if and only if $a \wedge b \sim b \wedge a$.*

Proof. Suppose $a \sim b$. Then $a \wedge b = b \wedge a$. Now, $(a \wedge b) \wedge (b \wedge a) = (b \wedge a) \wedge (a \wedge b)$. Therefore $a \wedge b \sim b \wedge a$. Conversely, suppose $a \wedge b \sim b \wedge a$. Then $(a \wedge b) \wedge (b \wedge a) = (b \wedge a) \wedge (a \wedge b)$. Now, $(a \wedge b) \wedge (b \wedge a) = ((a \wedge b) \wedge b) \wedge a = (a \wedge b) \wedge a = (b \wedge a) \wedge a = b \wedge a$. Similarly, we can prove that $(b \wedge a) \wedge (a \wedge b) = a \wedge b$. Hence $a \wedge b = b \wedge a$. Therefore $a \sim b$. $\square$

Corollary 4.3. *Let L be an AL. Then for any $a, b \in L$, $a \sim b$ if and only if $a \vee b \sim b \vee a$.*

Proposition 4.3. *Let L be an AL. Then for any $a, b, c \in L, a \sim b$ and $a \sim c$ implies $a \sim b \wedge c$.*

Proof. Suppose $a \sim b$ *and* $a \sim c$. Then $a \wedge b = b \wedge a$ and $a \wedge c = c \wedge a$. Now, $a \wedge (b \wedge c) = (a \wedge b) \wedge c = (b \wedge a) \wedge c = b \wedge (a \wedge c) = b \wedge (c \wedge a) = (b \wedge c) \wedge a$. Thus $a \sim b \wedge c$. $\square$

Corollary 4.4. *Let L be an AL. Then for any $a, b, c \in L$, $a \sim b$ and $a \sim c$ implies $a \sim b \vee c$.*

5. Almost lattice with Zero

In this section, we introduce the concept of zero element in an AL analogous to that of the least element in a lattice. Further we give some examples of an AL with 0. We observe that an AL can have at most one zero element and it will be the least element of the poset $(L, \leqslant)$. Also, we prove some basic properties of an AL with 0. Moreover, we prove that an AL with 0 is a lattice with 0 if and only if $\sim$ is a transitive relation on L. First, we define the zero element in L.

Definition 5.1. Let L be an AL. Then an element $0 \in L$ is called a zero element of L if
(0_1) $0 \wedge a = 0$ for all $a \in L$.

Observe that, not every AL has zero element. For, in example 3.4, L does not have zero element.

Lemma 5.1. *Let L be an AL. Then L has at most one zero element.*

Proof. Suppose L has two zero elements say $0_1, 0_2$. Then we have $0_1 = 0_1 \wedge (0_2 \wedge 0_1) = (0_1 \wedge 0_2) \wedge 0_1 = (0_2 \wedge 0_1) \wedge 0_1 = 0_2 \wedge (0_1 \wedge 0_1) = 0_2 \wedge 0_1 = 0_2$. Thus $0_1 = 0_2$. $\square$

Note that we always denote the zero element of L, if it exists, by '0'. If L has 0, then the algebra $(L, \vee, \wedge, 0)$ is called an AL with '0'. In the following we give some examples of an AL with 0. First we need the following:

Lemma 5.2. *Let L be an AL with 0. Then for any $a \in L$, we have the following:*
(0_2) $a \wedge 0 = 0$.
(0_3) $a \vee 0 = a$.
(0_4) $0 \vee a = a$

Proof. Suppose L is an AL with 0. Then,
(0_2) $a \wedge 0 = a \wedge (0 \wedge a) = (a \wedge 0) \wedge a = (0 \wedge a) \wedge a = 0 \wedge a = 0$. Therefore $a \wedge 0 = 0$.
(0_3) By A_6 and (0_2), we have $a \vee 0 = a \vee (a \wedge 0) = a$.
(0_4) By A_7 and 0_1 we have, $0 \vee a = (0 \wedge a) \vee a = a$. $\square$

Now, we have the following theorem whose proof is straight forward.

Theorem 5.1. *Let L be an AL and 0 be any external element of L. For any $x, y \in L \cup \{0\}$, define:*

$$x \vee y = \begin{cases} x & \text{if } y = 0. \\ y & \text{if } x = 0 \\ x \vee y \ (\text{in } L) & \text{if } x,\ y \in L \end{cases} \quad \text{and } x \wedge y = \begin{cases} x \wedge y \ (\text{in } L) & \text{if } x, y \in L \\ 0 & \text{otherwise} \end{cases}$$

Then $(L \cup \{0\}, \vee, \wedge, 0)$ *is an AL with* 0. *We denote this AL by* L^o.

According to definition 3.1, an AL with 0 is an algebra $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ satisfying the following axioms:

(A_1) $(a \wedge b) \wedge c = (b \wedge a) \wedge c$
(A_2) $(a \vee b) \wedge c = (b \vee a) \wedge c$
(A_3) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
(A_4) $(a \vee b) \vee c = a \vee (b \vee c)$
(A_5) $a \wedge (a \vee b) = a$
(A_6) $a \vee (a \wedge b) = a$
(A_7) $(a \wedge b) \vee b = b$
(0_1) $0 \wedge a = 0$

Example 5.1. Let L be a non empty set and fix $x_o \in L$. Define binary operations $\vee$ *and* $\wedge$ on L by:

$$x \vee y = \begin{cases} y & \text{if } x = x_o \\ x & \text{if } x \neq x_o \end{cases} \quad \textit{and } x \wedge y = \begin{cases} y & \text{if } x \neq x_o \\ x_o & \text{if } x = x_o \end{cases}$$

Then $(L, \vee, \wedge, x_o)$ is an AL with x_o as its zero element and we call this AL by discrete AL with 0.

Example 5.2. Let $L = \{0, a, b, c\}$. Define $\vee$ and $\wedge$ on L as follows.

∨	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	c	c	c

and

∧	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
c	0	a	b	c

Then clearly $(L, \vee, \wedge, 0)$ is an AL with 0.

Example 5.3. Let $L = \{0, a, b, c\}$. Define $\vee$ and $\wedge$ on L as follows.

∨	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	b	b	b
c	c	a	b	c

and

∧	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	c	c	c

Then clearly $(L, \vee, \wedge, 0)$ is an AL with 0.

Example 5.4. Let $A = \{0, a\}$ and $B = \{0, b_1, b_2\}$ be two discrete ALs. Then $L = A \times B = \{(0,\ 0), (0,\ b_1), (0,\ b_2), (a,\ 0), (a,\ b_1), (a,\ b_2)\}$. Now, define $\vee$ and $\wedge$ on L by point wise operations as follows.

$\vee$	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(a,0)$	(a,b_1)	(a,b_2)
$(0,0)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(a,0)$	(a,b_1)	(a,b_2)
$(0,b_1)$	$(0,b_1)$	$(0,b_1)$	$(0,b_1)$	(a,b_1)	(a,b_1)	(a,b_1)
$(0,b_2)$	$(0,b_2)$	$(0,b_2)$	$(0,b_2)$	(a,b_2)	(a,b_2)	(a,b_2)
$(a,0)$	$(a,0)$	(a,b_1)	(a,b_2)	$(a,0)$	(a,b_1)	(a,b_2)
(a,b_1)	(a,b_1)	(a,b_1)	(a,b_1)	(a,b_1)	(a,b_1)	(a,b_1)
(a,b_2)	(a,b_2)	(a,b_2)	(a,b_2)	(a,b_2)	(a,b_2)	(a,b_2)

and

$\wedge$	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(a,0)$	(a,b_1)	(a,b_2)
$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
$(0,b_1)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$
$(0,b_2)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$
$(a,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(a,0)$	$(a,0)$	$(a,0)$
(a,b_1)	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(a,0)$	(a,b_1)	(a,b_2)
(a,b_2)	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(a,0)$	(a,b_1)	(a,b_2)

Then clearly, $(L,\vee,\wedge,(0,0))$ is an AL with $(0,0)$ as its zero element.

EXAMPLE 5.5. Let $(R,+,.,0)$ be a commutative regular ring. To each $a\in R$, *let* a_0 be the unique idempotent element in R such that $aR=a_0R$. Define for any $a,b\in R$,

(1) $a\wedge b=a_0b$
(2) $a\vee b=a+(1-a_0)b$.

Then $(R,\vee,\wedge,0)$ is an AL with 0.

EXAMPLE 5.6. Let X be a non empty set with at least two elements and let Y be any set. Let $f_0\in X^Y$. Now for any $f,g\in X^Y$ *and* $y\in Y$, define;

$$(f\vee g)(y)=\begin{cases}f(y) & \text{if } f(y)\neq f_o(y)\\ g(y) & \text{if } f(y)=f_o(y)\end{cases} \textit{ and } (f\wedge g)(y)=\begin{cases}g(y) & \text{if } f(y)\neq f_o(y)\\ f_o(y) & \text{if } f(y)=f_o(y)\end{cases}$$

Then $(X^Y,\vee,\wedge,f_o)$ is an AL with f_o as its zero element.

In the following we prove some results in an AL with zero.

LEMMA 5.3. *Let L be an AL with 0. Then for any* $a,b\in L$, $a\wedge b=0$ *if and only if* $b\wedge a=0$.

PROOF. Suppose $a\wedge b=0$. Then, $b\wedge a=b\wedge(a\wedge a)=(b\wedge a)\wedge a=(a\wedge b)\wedge a=0\wedge a=0$. Converse is clear. □

COROLLARY 5.1. *Let L be an AL. Then for any* $a,b\in L$, $a\wedge b=b\wedge a$ *whenever* $a\wedge b=0$

In the following we give a necessary and sufficient condition for an AL with zero to become a lattice with zero.

THEOREM 5.2. *Let L be an AL with* 0. *Then L is a lattice with* 0 *if and only if for any* $a,b\in L$, $a\leqslant b$ *implies* $a\vee x\leqslant b\vee x$ *for all* x *in* L.

Proof. Suppose L is a lattice with 0. Let $a, b \in L$, such that $a \leqslant b$. Then $b = a \vee b$. Let $x \in L$. Then $(a \vee x) \vee (b \vee x) = ((a \vee x) \vee b) \vee x = (a \vee (x \vee b)) \vee x = (a \vee (b \vee x)) \vee x = ((a \vee b) \vee x) \vee x = (a \vee b) \vee (x \vee x) = (a \vee b) \vee x = b \vee x$. Therefore $a \vee x \leqslant b \vee x$. Conversely, assume the condition. Let $a, b \in L$. Then we have $0 \leqslant a$. Therefore by our assumption, $0 \vee b \leqslant a \vee b$. This implies $b \leqslant a \vee b$. But, we have $a \leqslant a \vee b$. Therefore L is directed above. Thus L is a lattice with 0. □

Next we give another characterization for an AL with zero to become a lattice with zero.

Theorem 5.3. *Let L be an AL with 0. Then L is a lattice with 0 if and only if $\sim$ is a transitive relation on L.*

Proof. Suppose $\sim$ is a transitive relation on L. Let $a, b \in L$. Then we have $a \sim 0$ and $0 \sim b$. Therefore by transitive $a \sim b$. Hence $a \wedge b = b \wedge a$. Thus L is a lattice with 0. The converse is trivial since L is a lattice. □

Definition 5.2. Let L be an AL. Then an element $a \in L$ is maximal (minimal) if for any $x \in L,\ a \leqslant x$ $(x \leqslant a)$ implies $a = x$ $(x = a)$.

Now, we give a set of equivalent conditions for an element $m \in L$ to become a maximal element.

Proposition 5.1. *Let L be an AL and $m \in L$. Then the following are equivalent:*

(1) *m is maximal.*
(2) *$m \vee x = m$ for all $x \in L$.*
(3) *$m \wedge x = x$ for all $x \in L$.*

Proof. (1) $\Rightarrow$ (2):- Suppose m is maximal element. But, we have $m \leqslant m \vee x$ for all $x \in L$. It follows that, $m = m \vee x$ for all $x \in L$. Proof of (2) $\Rightarrow$ (3) is clear by theorem 4.2.
(3) $\Rightarrow$ (1):- Assume (3). Let $x \in L$ such that $m \leqslant x$. Then we have $m = m \wedge x$. Hence by (3), $m = x$. Therefore m is maximal element. □

In the following, we give a set of equivalent conditions for an element $m \in L$ to become a minimal element.

Proposition 5.2. *Let L be an AL. Then for any $m \in L$, the following are equivalent:*

(1) *m is minimal.*
(2) *$x \wedge m = m$ for all $x \in L$.*
(3) *$x \vee m = x$ for all $x \in L$.*

Proof. (1) $\Rightarrow$ (2):- Suppose m is minimal element. But, we have $x \wedge m \leqslant m$ for all $x \in L$. It follows that, $x \wedge m = m$ for all $x \in L$. Proof of (2) $\Rightarrow$ (3) follows by theorem 4.2.
(3) $\Rightarrow$ (1):- Assume (3). Let $x \in L$ such that $x \leqslant m$. Then we have $m = x \vee m$. Hence by (3), $x = m$. Therefore m is minimal element. □

COROLLARY 5.2. *L is a discrete AL if and only if every element of L is maximal.*

PROOF. Suppose L is a discrete AL. Then we have $x \wedge y = y$ for all $x, y \in L$. Thus every element in L is maximal. The converse is trivial by proposition 5.1. $\square$

DEFINITION 5.3. If L is a discrete AL, then the AL $(L^o, \vee, \wedge, 0)$ where $L^o = L \cup \{0\}$ is called a simple AL.

We immediately have, from corollary 5.2, the following theorem.

THEOREM 5.4. *Let L be an AL with 0. Then the following are equivalent:*
(1) *L is simple AL.*
(2) *Every non zero element of L is maximal.*
(3) *$a \vee b = a$ for all $a \neq 0$.*
(4) *$a \wedge b = b$ for all $a \neq 0$.*

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