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BANG-BANG PROPERTY OF TIME OPTIMAL CONTROLS FOR THE HEAT EQUATION IN THE PRESENCE OF A SCALE PARAMETER

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Abstract. In this paper we apply the classical control theory for the heat equation depending of a scale parameter. The main results establish a Pontryagyn type maximum principle and give sufficient conditions for the bang-bang property of optimal controls. In this fact, in first time we build exact solution. The dependence of this solution compared to the scale parameter thus lead to study the existence and uniqueness of the time optimal control for the heat equation. More precisely, Supposing the $L^\infty-$controllability to zero, we can establish a bang-bang type property in the presence of a scale parameter. Numerical example is given in the last section to illustrate our main result.

1. Introduction

Let m be a positive integer, let $\Omega \in \mathbb{R}^m$, $\partial\Omega$ is of class C^2 and let $\omega \subset \Omega$ be a non-empty open. The general control problem for the linear heat equation, on a compact Ω, can be stated as follows:

$$\left\{ \begin{array}{lcll} \dot{y}(x,t) & = & \Delta y(x,t) + \chi_\omega(x)\, u(x,t) & \forall\,(x,t) \in \Omega \times \mathbb{R}_+ \\ y(x,t) & = & 0 & \forall\, x \in \partial\Omega,\ \forall\, t \in \mathbb{R}_+ \\ y(x,0) & = & y_0 \in L^2(\Omega) & \end{array} \right. \tag{1.1}$$

where $T > 0$, $\omega \subset \Omega$, while χ_ω denotes the characteristic function of the set ω:

$$\forall\, x \in \Omega : \quad \chi_\omega(x) = \left\{ \begin{array}{ll} 1 & \text{if} \quad x \in \omega \\ 0 & \text{if not} \end{array} \right.$$

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where u is in $L^\infty([0,T],U)$, and where:

$$\|u\|_{L^\infty([0,T],U)} \leqslant 1 \quad \text{for almost all } t \in [0,T]$$

The objective of the time optimal control for the system (1) is to find the control $u^\star$ which is accessible, with a final time $\tau^\star$ as small as possible (We refer for instance to [4-8]).

One can obtain the bang-bang property by means of the maximum principle, if the system can be controled exactly (see for example, the article of J. Lohac and M. Tucsnak [10]).

It is now well known that for specific equations (for the general heat equation, for instance), the exact controllability is not verified. This is the reason why we choose to study new conditions, without using the maximum principle. With an assumption on the $L^\infty-$ null controllability, we can establish the bang-bang property (see also the paper of S. Micu, I. Roventa and M. Tucsnak [11], and the paper of G. Wang [12]).

The main result in this article asserts that to study the existence and uniqueness of an optimal time control problem for the linear heat equation, in the presence of a scaling parameter, and to determine whether the bang-bang property can be satisfied.

The rest of the paper is organized as follows. Section 2 is devoted to some definitions and preliminary results. In Section 3 we build the invariant solutions for the linear heat equation. In Section 4 we prove the existence and uniqueness of the time optimal control for the heat equation, in the presence of a scaling parameter. Numerical results are made in Section 5.

2. Preliminaries

We first introduce some notation. Let X and Y be Hilbert spaces. If $P \in \mathcal{L}(X;Y)$ then the null-space and the range of P are the subspaces of X and Y respectively defined by

$$Ker\ P = \{x \in X, Px = 0\}, \qquad Ran\ P = \{Px, x \in X\}.$$

Throughout this paper, X and U are complex Hilbert spaces, identified with their duals. The inner product and the norm in X are denoted by $\langle \cdot, \cdot \rangle_X$ and $\|\cdot\|_X$ respectively. We denote by $S = (S(t))_{t\geqslant 0}$ a strongly continuous semigroup on X generated by an operator $A : D(A) \to X$.

Let $B \in \mathcal{L}(U,X)$ be a control operator, let $y_0 \in X$ and let $u \in L^2([0,\infty),U)$. We consider the infinite dimensional system described by the equation

$$\dot{y}(t) = Ay(t) + Bu(t), \quad t \geqslant 0 \quad y(0) = y_0 \in X. \tag{2.1}$$

With the above notation, the solution y of (2.1) is defined by

$$y(t) = S(t)y_0 + \Phi_t u, \quad (t \geqslant 0), \tag{2.2}$$

where $\Phi_t \in \mathcal{L}(L^2([0,t],U);X)$ is given by

$$\Phi_t u = \int_0^t S(t-\sigma)Bu(\sigma)d\sigma, \quad u \in L^2([0,\infty),U). \tag{2.3}$$

The maps (Φ_t) are called *input to state maps.*

Recall the following classical definitions and preliminary results (We refer for instance to [10-16]):

Definition 2.1.

- *The pair (A,B) is said approximatively controllable in time τ if Ran Φ_τ is dense in X.*

- *The pair (A,B) is exactly controllable in time τ if Ran $\Phi_\tau = X$.*

Definition 2.2. *Let $e \subset [0,\tau]$ be a set of positive Lebesgue measure. The pair (A,B) is said approximatively controllable in time τ from e if the range of the map $\Phi_{\tau,e} \in \mathcal{L}(L^2([0,\tau],U),X)$ defined by*

$$\Phi_{\tau,e}u = \int_e S(\tau-\sigma)Bu(\sigma)d\sigma, \quad (u \in L^2([0,\tau],U))$$

is dense in X.

We are now in position to give a precise definition of time optimal controls.

Definition 2.3. *Let $y_0, y_1 \in X$ with $y_0 \neq y_1$. A function $u^* \in L^\infty([0,\infty),U)$ is said a time optimal control for the pair (A,B), associated to the initial state y_0 and the final state y_1, if there exists $\tau^* > 0$ such that*

1. $y_1 = S(\tau^*)y_0 + \Phi_{\tau*}u^*$ *and* $\|u^*\|_{L^\infty([0,\tau^*],U)} \leqslant 1$;

2. *if $\tau > 0$ is such that there exists $u \in L^\infty([0,\tau],U)$ with*
$$y_1 = S(\tau)y_0 + \Phi_\tau u, \quad \|u\|_{L^\infty([0,\tau],U)} \leqslant 1,$$
then $\tau \geqslant \tau^*$.

Proposition 2.1. *([5]). With the above notation and assumptions, assume moreover that $y_0, y_1 \in X$, $y_0 \neq y_1$ are such that there exists $t > 0$ with $y_1 - S(t)y_0 \in B_1^\infty(t)$.*

Then there exists $\tau^* > 0$ such that

$$\tau^* = \min\{t > 0 \mid y_1 - S(t)y_0 \in B_1^\infty(t)\} > 0$$

where $B_1^\infty(t) = \{\Phi_t u, \quad u \in L^\infty([0,t],u), \quad \|u\|_{L^\infty([0,t],u} \leqslant 1\}$.

In other words τ^* is the minimal time in which y_0 can be steered to y_1 by controls satisfying $\|u(t)\| \leqslant 1$ for almost every t. Therefore, any $u^* \in L^\infty([0,\tau^*],u)$ satisfying

$$y_1 - S(\tau^*)y_0 = \Phi_{\tau^*}u^*,$$

is called a time optimal control for the pair (A,B).

The below result provides a class of infinite dimensional system for which the maximum principle from the linear finite dimensional case can be extended in its classical form.

THEOREM 2.1. *[10](Pontryagyn maximum principle)*
Suppose that (A,B) is exactly controllable in any time $\tau > 0$. Then, for every $y_0, y_1 \in X$ with $y_0 \neq y_1$, there exists a time optimal control u^* steering y_0 to y_1 in time $\tau^* = \tau^*(y_0,y_1)$. Moreover, there exists $\eta \in X, \eta \neq 0$ such that

(2.4)
$$\langle B^*S^*(\tau^*-t)\eta, u(t)\rangle_U = \max_{v\in U, \|v\|_U \leqslant 1} \langle B^*S^*(\tau^*-t)\eta, v\rangle_U, (t \in (0,\tau^*) \ a.e.)$$

The following result shows that, under an extra assumption, the time optimal controls in the above theorem are bang-bang.

COROLLARY 2.1. *([6]). With the notation and the assumptions in Theorem 2.5, assume moreover that the pair (A,B) is approximatively controllable in time τ^* from any $e \subset [0,\tau^*]$ of positive measure. Then the time optimal control u^* is bang-bang, in the sense that*

$$\|u^*(t)\|_U = 1 \qquad (t \in (0,\tau^*) \quad a.e.). \tag{2.5}$$

Moreover, the time optimal control is unique.

THEOREM 2.2. *([11]). Suppose that (A,B) is L^∞ null controllability in any time $\tau > 0$ over $e \subset [0,\tau]$ of positive measure. Then, for every $y_0, y_1 \in X$ with $y_0 \neq y_1$, there exists a unique time optimal control u^* steering y_0 to y_1 in time τ^* and u^* has the bang-bang property (2.5).*

In the following results, we are interested to a method introduced by Lebeau and Robbiano [15] to study the null controllability of the heat equation. More precisely, supposing the L^∞-controllability to zero, we can establish a bang-bang type property(We refer for instance to [11,12].).

The operator $A : D(A) \to X$ is supposed to be a self-adjoint (possibly unbounded) operator on X such that

$$\langle A\psi, \psi\rangle \leqslant 0 \qquad (\ \psi \in D(A)).$$

Such an operator will be briefly called a *negative operator*. We also assume that A is diagonalizable with an orthonormal basis of eigenvectors $\{\varphi_k\}_{k\geqslant 0}$ and corresponding family of eigenvalues $\{-\lambda_k\}_{k\geqslant 0}$, where the sequence $\{\lambda_k\}$ is positive, non decreasing and satisfies $\lambda_k \to \infty$ as k tends to infinity. According to classical results, this holds, in particular, if A has compact resolvents. With the above assumptions on A, we have

$$A\psi = -\sum_{k\geqslant 0} \lambda_k \langle \psi, \varphi_k \rangle \varphi_k \quad (\psi \in D(A)), \tag{2.6}$$

so that the semigroup S generated by A is a contraction semigroup on X satisfying

$$S(t)y = \sum_{k\geqslant 0} e^{-\lambda_k t} \langle y, \varphi_k \rangle \varphi_k \quad (t \geqslant 0, y \in X). \tag{2.7}$$

We denote by Ψ_τ^d (respectively $\Psi_{\tau,e}^d$) the output maps corresponding (respectively the restriction to a set of positive measure $e \subset [0,\tau]$ of these output maps) corresponding to the pair (A^*, B^*), i.e., we set :

$$(\Psi_\tau^d y_0)(t) = B^* S^*(t) y_0 \quad (y_0 \in X, t \in [0,\tau]),$$

$$\Psi_{\tau,e}^d \in \mathcal{L}(X, L^2([0,\tau], U)), \quad \Psi_{\tau,e}^d = \chi_e \Psi_\tau^d.$$

We are now in a position to enunciate the following results :

Proposition 2.2. *[10]*
Let $\tau > 0, e \subset [0,\tau]$ be a set of positive measure and $K_{\tau,e} > 0$. The following two properties are equivalent

1. *The inequality*

$$K_{\tau,e} \|\Psi_{\tau,\acute{e}}^d \varphi\|_{L^1([0,\tau],U)} \geqslant \|S^*(\tau)\varphi\|_X, \tag{2.8}$$

holds for any $\varphi \in X$, where $\acute{e} = \{(x, \tau - t) | (x,t) \in e\}$.

2. *The pair (A,B) is L^∞-null controllable in time τ over e at cost not larger than $K_{\tau,e}$.*

We denote by μ_k the Lebesgue measure in $\mathbb{R}^k$ and
$V_\zeta = span\{\varphi_k \mid \lambda_k^{\frac{1}{2}} \leqslant \zeta, \ \forall \zeta > 0\}$.

Theorem 2.3. *([12]). Let $\tau > 0$ and $e \subset [0,\tau]$ be a set of positive measure and $K_{\tau,e} > 0$. Moreover, assume that there exist positive constants $d_0 > 0$ and $d_1 > 0$ such that for every $\zeta > 0, s, t > 0$, with $0 < s < t \leqslant \tau$ and $\varepsilon := \{\sigma \in e \mid s \leqslant \sigma \leqslant t\}$ of positive measure, we have*

$$\|S(\tau)^* \varphi\|_X \leqslant \tfrac{d_0 e^{d_1 \zeta}}{\mu(\varepsilon)} \|\Psi_{\tau,\varepsilon'}^d \varphi\|_{L^1([0,\tau];U)}, \quad \forall \varphi \in V_\zeta, \tag{2.9}$$

where $\varepsilon' := \{\tau - \sigma \mid \sigma \in \varepsilon\}$.

Then the pair (A,B) is L^∞-null controllable in time τ over e.

Proposition 2.3. *for any $\zeta > 0$ and if the paire (A,B) satisfies the inequality*

$$\exists d_0, d_1 > 0, \forall \varphi \in V_\varsigma, \|\varphi\|_X \leqslant d_0 e^{d_1 \varsigma} \|B^* \varphi\|_U, \tag{2.10}$$

Then the pair (A,B) satisfying the inequality (2.9), so imply that is L^∞-null controllable in time τ over e.

Proof. For any $\tau > 0$ and any $e \subset [0,\tau]$ of positive measure, we consider the the following adjoint system of (2.1) :

$$\begin{cases} \dot{z}(t) = -A^* z(t), \quad t \in [0,\tau], \\ \qquad z(t) \in V_\zeta. \end{cases} \tag{2.11}$$

we can write $z(\tau) = \sum\limits_{k,\varphi_k \in V_\zeta} \alpha_k \varphi_k$ where $(\alpha_k)_k \subset \mathbb{R}$ and $(\varphi_k)_k$ are the eigenvectors of the operator A corresponding to the eigenvalue $(\lambda_k)_k$

Then the solution z of (2.11) writes :

$$z(t) = \sum_{k,\varphi_k \in V_\zeta} e^{-\lambda_k(\tau - t)} \alpha_k \varphi_k. \tag{2.12}$$

From the inequality (2.10) imply that for every $(a_k)_k \subset \mathbb{R}$ we have :

$$\| \sum_{k,\varphi_k \in V_\zeta} a_k \varphi_k \|_X \leqslant d_0 e^{d_1 \zeta} \| B^* \sum_{k,\varphi_k \in V_\zeta} a_k \varphi_k \|_U. \tag{2.13}$$

Let $a_k = e^{-\lambda_k(\tau-t)} \alpha_k$ in (2.13). Then Integrating the above formula over the measurable set $\varepsilon := e \cap [s,t]$ for $0 < s < t \leqslant \tau$ we deduce that :

$$\int_\varepsilon \| \sum_{k,\varphi_k \in V_\zeta} e^{-\lambda_k(\tau-t)} \alpha_k \varphi_k \|_X dt \leqslant d_0 e^{d_1\zeta} \int_\varepsilon \| B^* \sum_{k,\varphi_k \in V_\zeta} e^{-\lambda_k(\tau-t)} \alpha_k \varphi_k \|_U dt$$

$$\Leftrightarrow \int_\varepsilon (\sum_{k,\varphi_k \in V_\zeta} e^{-2\lambda_k(\tau-t)} |\alpha_k|^2)^{1/2} dt \leqslant d_0 e^{d_1\zeta} \| \chi_\varepsilon(.) B^* \sum_{k,\varphi_k \in V_\zeta} e^{-\lambda_k(\tau-.)} \alpha_k \varphi_k \|_{L^1([0,\tau];U)}$$

$$\Rightarrow \mu(\varepsilon) \| \sum_{k,\varphi_k \in V_\zeta} e^{-\lambda_k(\tau)} \alpha_k \varphi_k \|_X \leqslant d_0 e^{d_1\zeta} \| \chi_{\varepsilon'}(.) B^* \sum_{k,\varphi_k \in V_\zeta} e^{-\lambda_k(.)} \alpha_k \varphi_k \|_{L^1([0,\tau];U)}, \tag{2.14}$$

where $\varepsilon' := \{\tau - t \mid t \in \varepsilon\}$.

Let $\varphi = \sum\limits_{k,\varphi_k \in V_\zeta} \alpha_k \varphi_k$. From (2.14) we have for every $\varphi \in V_\varepsilon$:

$$\|S(\tau)^* \varphi\|_X \leqslant \frac{d_0 e^{d_1\zeta}}{\mu(\varepsilon)} \| \chi_{\varepsilon'}(.) B^* S(.)^* \varphi \|_{L^1([0,\tau];U)} = \frac{d_0 e^{d_1\zeta}}{\mu(\varepsilon)} \| \Psi^d_{\tau,\varepsilon'} \varphi \|_{L^1([0,\tau];U)}.$$

Thus, the inequality (2.9) of the theorem (2.3) is verified, which implies that the paire (A,B) is L^∞-null controllable in time τ over e, which ends the proof. $\square$

3. A scaling invariant solutions for linear heat equation and main objective

We consider the linear heat equation of one-dimensional:

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} \quad \forall\, (x,t) \,\in\, \mathbb{R} \times [0,T]$$

where T is a positive real number, with the initial condition:

$$y(x,0) = y^0(x) \quad \forall\, x \,\in\, \mathbb{R}$$

and where y^0 denotes a given function.
The analytical solution is given, for any $(x,t) \,\in\, \mathbb{R} \times [0,T]$, by:

$$y_{classical}(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} y^0(z)\, e^{-\frac{(x-z)^2}{4t}}\, dz$$

The linear heat equation has a natural scaling invariance. Let us denote by y a solution. Then, for any strictly positive real number λ, the mapping:

$$(t,x) \mapsto y_\lambda(t,x) = \lambda y(\lambda^2\, t, \lambda\, x)$$

is also a solution.
By applying the method developed by Jean-Yves Chemin and Claire David [1], [3], one can introduce the mapping F, belonging to $L^2_{loc}(\mathbb{R}) \times \mathbb{R}_+^\star \times \mathbb{N}^\star$, by:

$$F(y^0, \lambda, N_0) = y^0 + \varepsilon \sum_{j=1}^{N_0} \lambda^{-j}\, y^0(\lambda^{-j}\cdot) \quad , \quad \varepsilon \,\in\, \{-1,+1\},\, N_0 \,\in\, \mathbb{N}^\star$$

The building of thus mapping takes its origin in the profile theory, introduced by P. Gérard et H. Bahouri [3]. It is based on the idea that two solutions of an evolution equation, of scales, sufficiently different, almost not interact.
We are thus interested, in the following, to initial data of the form:

$$y^0(x) + \varepsilon \sum_{j=1}^{N_0} y^0_{\lambda,j}(x) = y^0(x) + \varepsilon \sum_{j=1}^{N_0} \frac{1}{\lambda^j}\, y^0\left(\frac{x}{\lambda^j}\right) \quad , \quad \lambda > 0$$

The exact analytical solution y_λ, which depends on the space variable x, the time variable t, and the scaling parameter λ, is given by:

$$y_\lambda(x,t,\lambda) = y_{classical}(x,t) + \varepsilon\, \frac{1}{2\,\sqrt{\pi\, t}} \int_{-\infty}^{\infty} \sum_{j=1}^{N_0} y^0_{\lambda,j}(u)\, e^{-\frac{(x-u)^2}{4\,t}}\, du$$

It is interesting to note that:

$$y_\lambda(x,t,\lambda) = y_{classical}(x,t) + \varepsilon \sum_{j=1}^{N_0} \frac{1}{\lambda}\, y_{classical}\left(\frac{x}{\lambda^j}, \frac{t}{\lambda^{2j}}\right)$$

Hence, we have:

$$\Delta y_\lambda(x,t,\lambda) = \Delta y_{classical}(x,t) + \frac{\varepsilon}{\lambda}\sum_{j=1}^{N_0} \Delta y_{classical}\left(\frac{x}{\lambda^j}, \frac{t}{\lambda^{2j}}\right)$$

One builds thus an exact solution of the afore mentioned linear heat equation. The dependence of this solution towards the scaling parameter λ, naturally leads to an internal control problem, which can be formulated as follows:

Let m be a positive integer, let $\Omega \in \mathbb{R}^m$, $\partial\Omega$ is of class C^2 and let $\omega \subset \Omega$ be a non-empty open. we consider the following control problem for the linear heat equation with scale invariance

$$(3.1)\quad \left\{\begin{array}{lcll} \dot{y_\lambda}(x,t,\lambda) & = & \Delta y_\lambda(x,t,\lambda) + \chi_\omega(x)u(x,t) & \forall\ (x,t,\lambda) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+^\star \\ y_\lambda(x,t,\lambda) & = & 0 & \forall\ x\ \in\ \partial\Omega, \forall\ (t,\lambda) \in \mathbb{R}_+ \times \mathbb{R}_+^\star \\ y_\lambda(x,0,\lambda) & = & y_\lambda^0 \in L^2(\Omega) & \forall\ \lambda\ \in\ \mathbb{R}_+^\star \end{array}\right.$$

where χ_ω is the characteristic function of ω:

$$\forall\, x\ \in\ \Omega\ :\quad \chi_\omega(x) = \left\{\begin{array}{ccc} 1 & \text{if} & x\ \in\ \omega \\ 0 & \text{if not} & \end{array}\right.$$

In the following Section we study the existence and uniqueness of time optimal controls for (3.1) with the presence of a scale parameter. More precisely, supposing the L^∞-controllability to zero, we can establish a bang-bang type property.

4. Time optimal controls for (3.1)

We are now in a position to enunciate the our main result :

Proposition 4.1. *We consider the optimal control problem for heat equation in the presence of a scale parameter defined by (3.1).*
For every $y_\lambda^0, y_\lambda^f \in L^2(\Omega)$, with $y_\lambda^0 \neq y_\lambda^f$, there exits a unique solution u^* of the time optimal control problem (3.1) steering y_λ^0 to y_λ^f in time τ^* and this solution u^* has the bang-bang property:

$$(4.1)\qquad \|u(\sigma)\|_{L^2(\omega)} = 1, \quad (for\ all\ \sigma \in [0,\tau^*] a.e.).$$

Proof. Let $u \in L^2([0,T];U)$ with $\forall t \geqslant 0, \|u(.,t)\|_{l^2(\Omega)} \leqslant 1$.

We set $X = L^2(\Omega), U = L^2(\omega)$, the operator A is defined by

$$A := \left\{\begin{array}{l} D(A) = H^2(\Omega) \cap H_0^1(\Omega) \to X \\ \varphi \to A\varphi = \Delta\varphi, \end{array}\right.$$

and the operator $B \in \mathcal{L}(U,X)$ with $Bu = \chi_\omega u, \quad u \in L^2(\omega)$.

With the above assumptions and notation, we can write the system (3.1) as follows :

$$\dot{y_\lambda}(t) = Ay_\lambda(t) + Bu(t), \quad y_\lambda(0) = y_\lambda^0, \tag{4.2}$$

where $A = \Delta$ and $B = \chi_\omega$.

and we know that A is a negative operator and diagonalizable with an orthonormal basis of eigenvectors $\{\varphi_k\}_{k\geqslant 0}$ and corresponding family of eigenvalues $\{-\lambda_k\}_{k\geqslant 0}$, where the sequence $\{\lambda_k\}$is positive, non decreasing and satisfies $\lambda_k \to \infty$ as k tends to infinity. It is also known that the operator A has compact resolvents.

In order to prove proposition 4.1 we need to apply theorem 2.2. Then we only need to establish the inequality (3.10) and to use the proposition 2.3 for to conclude.

It is known that the inequality (2.9) is verified for the operator $A = \Delta$ and $B = \chi_\omega$ (see theorem 3 of G. Lebeau et E. Zuazua [16]).

Then, the (A, B) is L^∞ null controllability in any time $\tau > 0$ over $e \subset [0, \tau]$ of positive measure.

The existence of a time optimal control u^* is given by Proposition 2.1. And the uniqueness is verified because that the control u^* verifies the bang-bang property and by the using the strict convexity of U (see Corollary 2.1.)

Now, we prove the bang-bang property (4.1). Assume, by contradiction, we suppose that u^* is time optimal control in time τ^* and that there exist $\epsilon > 0$ and $e \subset [0, \tau^*]$ of positive measure such that :

$$\|u^*(\sigma)\|_U \leqslant 1 - \epsilon, \quad \forall \sigma \in e.$$

We denote by $y_\lambda^*(t) = y_\lambda(t, y_\lambda^0, u^*)$ the trajectory engendered by u^* from the initial point y_λ^0 in time t with the presence of a scale parameter $\lambda > 0$.

We remark that there exist $\delta_0 > 0$ such that :
$e_0 = \{t \in [\delta_0, \tau^* - \delta_0] \mid t \in e\}$ of positive measure .

It is easily seen that $y_\lambda^*(t) \xrightarrow{t \to 0} y_\lambda^0$. Then there exist $0 < \delta < \delta_0$ such that :

$$\|y_\lambda^0 - y_\lambda^*(\delta)\|_X \leqslant \tfrac{\epsilon}{M}, \quad \textit{where } M = \sup_{0<\delta<\delta_0} \{C_{\tau^*-\delta, e_0-\delta}\}. \tag{4.3}$$

note that $C_{\tau^*-\delta, e_0-\delta}$ is called the control cost in time $\tau^* - \delta$ over $e_0 - \delta$ and is defined by the smallest constant of $K_{\tau,e}$ which verifies the observability inequality (2.8) (see proposition 2.2).

From the L^∞ null controllability in time $\tau^* - \delta$ over $e_0 - \delta$, we know that there exist a control $v \in L^\infty([0, \tau^*]; U)$ such that the support of v included in $e_0 - \delta$ and we have:

$$0 = S(\tau^* - \delta)(y^0_\lambda - y^*_\lambda(\delta)) + \Phi_{\tau^*-\delta,e_0-\delta} v$$

$$= S(\tau^* - \delta)(y^0_\lambda - y^*_\lambda(\delta)) + \int_0^{\tau^*-\delta} \chi_{e_0-\delta}(\sigma) S(\tau^* - \delta - \sigma) B v(\sigma) d\sigma$$

$$= S(\tau^* - \delta)(y^0_\lambda - y^*_\lambda(\delta)) + \int_\sigma^{\tau^*} \chi_{e_0-\delta}(\sigma - \delta) S(\tau^* - \sigma) B v(\sigma - \delta) d\sigma \tag{4.4}$$

with

$$\|v(\sigma)\|_U \leqslant C_{\tau^*-\delta,e_0-\delta} \|y^0_\lambda - y^*_\lambda(\delta)\|_X \leqslant C_{\tau^*-\delta,e_0-\delta} \tfrac{\epsilon}{M} \leqslant \epsilon \quad (\sigma \in e_0 - \delta) \tag{4.5}$$

we denote $\tilde{v}(\sigma) = v(\sigma - \delta)$. Then the support of v is included in $e_0 - \delta$, and we deduce that $\tilde{v}(\sigma) = v(\sigma - \delta)$ is not null if $\sigma - \delta \in \text{supp } v \subset e_0 - \delta$. This means that the support of v is icluded in e_0.

Now, we can write (4.4) and (4.5) as follows :

$$0 = S(\tau^* - \delta)(y^0_\lambda - y^*_\lambda(\delta)) + \int_\sigma^{\tau^*} \chi_{e_0}(\sigma) S(\tau^* - \sigma) B \tilde{v}(\sigma) d\sigma, \tag{4.6}$$

with

$$\|\tilde{v}(\sigma)\|_U \leqslant \epsilon, \quad \sigma \in e_0. \tag{4.7}$$

Now we prove that u^* is not the optimal control in τ^* for obtain the contradiction.

Let $\tilde{u}(t) = u^*(t + \delta) + \tilde{v}(t + \delta)$ for every $t \in [0, \tau^* - \delta]$.

We first check that $\tilde{u} \in \mathcal{L}_1(\tau^* - \delta)$, where $\mathcal{L}_1(t)$ is a set of an admissible control which defined by :

$$\mathcal{L}_1(t) = \{u \in L^\infty([0, t]; U) \mid \|u\|_{L^\infty([0,t];U)} \leqslant 1\}$$

.

In this fact, if $t + \delta \in e_0$, therefore in view of (4.7), we obtain that

$$\|\tilde{u}(t)\|_U \leqslant \|u^*(t + \delta)\|_U + \|\tilde{v}(t + \delta)\|_U \leqslant (1 - \epsilon) + \epsilon = 1.$$

And if $t + \delta$ is not included in e_0, we have that $\|\tilde{u}(t)\|_U = \|u^*(t + \delta)\|_U \leqslant 1$.

By using (4.6) and the fact that the support of $\tilde{v}$ is included in e_0, we obtain :

$$S(\tau^*-\delta)y_\lambda^0+\Phi_{\tau^*-\delta}\tilde{u} = S(\tau^*-\delta)(y_\lambda^0-y_\lambda^*(\delta))+S(\tau^*-\delta)\left(S(\delta)y_\lambda^0 + \int_0^\delta S(\delta-\sigma)Bu^*(\sigma)d\sigma \right)$$

$$+\Phi_{\tau^*-\delta}\tilde{v}(.+\delta) + \Phi_{\tau^*-\delta}u^*(.+\delta)$$

$$= S(\tau^*-\delta)(y_\lambda^0 - y_\lambda^*(\delta)) + \Phi_{\tau^*-\delta}\tilde{v}(.+\delta) + S(\tau^*)y_\lambda^0 + \int_0^\delta S(\tau^*-\sigma)Bu^*(\sigma)d\sigma$$

$$+ \int_0^{\tau^*-\delta} S(\tau^*-\delta-\sigma)Bu^*(\sigma+\delta)d\sigma$$

$$= S(\tau^*-\delta)(y_\lambda^0 - y_\lambda^*(\delta)) + \int_0^{\tau^*-\delta} S(\tau^*-\delta-\sigma)B\tilde{v}(\sigma+\delta)d\sigma + S(\tau^*)y_\lambda^0$$

$$+ \int_0^\delta S(\tau^*-\sigma)Bu^*(\sigma)d\sigma + \int_\delta^{\tau^*} S(\tau^*-\sigma)Bu^*(\sigma)d\sigma$$

$$= S(\tau^*-\delta)(y_\lambda^0 - y_\lambda^*(\delta)) + \int_\delta^{\tau^*} \chi_{e_0}(\sigma)S(\tau^*-\sigma)B\tilde{v}(\sigma)d\sigma + S(\tau^*)y_\lambda^0 + \Phi_{\tau^*}u^*$$

$$= 0 + y_\lambda^f.$$

Hence, the control u^* steer y_λ^0 to y_λ^f in time $\tau^* - \delta$ which contradicts with the optimality of u^* in time τ^*.

Finally, u^* has the bang-bang property.

□

5. Numerical results

Our numerical application is carried out by means of a direct type method (total discretization), of the afore mentioned linear heat equation, in the presence of a scaling parameter $\Lambda > 0$, with an internal control, in a domain $\omega \subset [0,1]$. More precisely, we consider the following system:

$$\Lambda^2\dot{y}(\Lambda x, \Lambda^2 t) = \Lambda^2\Delta y(\Lambda x, \Lambda^2 t) +_{[\frac{1}{3},\frac{2}{3}]}(x)\,u(x), \forall x \in [0,1], \forall t \in [0,t_f] \tag{5.1}$$

$$y(0,t,\Lambda) = 0 \quad , \quad y(1,t,\Lambda) = 0 \quad , \quad \forall t \in [0,t_f],\ \Lambda > 0 \tag{5.2}$$

The discretization is carried out by finite differences, with an implicit Euler scheme in time. To this purpose, let us consider the time discretization:

$$0 = t_0 < t_1 < \ldots < t_i < \ldots < t_n = t_f$$

and the space discretization:

$$0 = x_0 < x_1 < \ldots < x_j < \ldots < x_{N+1} = 1$$

For any integer i belonging to $\{0, \ldots, n\}$, and for any j belonging to $\{0, \ldots, N+1\}$, let us denote by:

$$y_{i,j} = y_\Lambda(t_i, x_j)$$

the value of the solution at $t = t_i$ and $x = x_j$, for the scaling parameter Λ.
We assume:

$$\dot{y}_\lambda(t_i, x_j) \approx \frac{y_{i,j} - y_{i-1,j}}{t_h}$$

and:

$$\Delta y_\Lambda(t_i, x_j) \approx \frac{y_{i,j+1} + y_{i,j-1} - 2\, y_{i,j}}{x_h^2}$$

where:

$$t_h = \frac{t_f}{n} \quad , \quad x_h = \frac{\Lambda}{N+1} \quad , \quad t_f = \Lambda^2$$

For each integer i belonging to $\{1, \ldots, N\}$, we set:

$$Y_i = \begin{pmatrix} y_{i,1} \\ y_{i,2} \\ \vdots \\ y_{i,N} \end{pmatrix}$$

The related discrete system can be written under the following matrix form:

$$\Lambda^2 \, \frac{Y_{i+1} - Y_i}{t_h} = A_h . Y_{i+1} + B_h . U_{i+1} \tag{5.3}$$

where the $N \times N$ matrix A_h is given by:

$$A_h = \frac{\Lambda^2}{x_h^2} \begin{pmatrix} -2 & 1 & 0 & \ldots & 0 \\ 1 & -2 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & -2 & 1 \\ 0 & \ldots & 0 & 1 & -2 \end{pmatrix}$$

and where the $N \times N$ diagonal matrix B_h is given by:

$$B_h = \mathrm{diag}\ (\alpha_1, \ldots, \alpha_N)$$

while, for any integer i belonging to $\{1, \ldots, N\}$:

$$\alpha_i = \begin{cases} 1 & \text{if} \quad \dfrac{1}{3} \leqslant x_i \leqslant \dfrac{2}{3} \\ 0 & \text{otherwise} \end{cases}$$

and:

$$U_{i+1} = \begin{pmatrix} u_{i,1} \\ \vdots \\ u_{i,N} \end{pmatrix}$$

where $u_{i,j}$ is the value chosen for the control at $t = t_i$ and $x = x_j$.

Basic calculations enable one to solve, for all i in $\{1, \ldots, N\}$, the system equivalent to (5.3) :

$$C\, Y_{i+1} = Y_i + \frac{t_h}{\Lambda^2}\, B_h.U_i$$

where

$$C = I_N - \frac{1}{\Lambda^2}\, t_h.A_h$$

I_N denoting the $N \times N$ identity matrix.

At each time step, the matrix C is inverted, in order to calculate Y_{i+1}.
Let us denote by X a variable which contains the whole set of values

$$\left\{ u_{i,j} \,\middle|\, i \in [1, N],\ j \in [1, N] \right\} \cup \{t_f\}$$

One has to bear in mind that the principle of direct methods lay in minimizing a function F that yields t_f with constraints, i.e., for each integer i of $\{1, \ldots, N\}$:

$$\|U_i\|_{L^2([\frac{1}{3}, \frac{2}{3}])} \leqslant 1$$

with the final condition:

$$Y(t_f) = Y^f$$

The initial and final conditions are:

$$\forall\, x \in K : \quad y_0(x) = \sin(\pi\, x) \quad , \quad y^f(x) = 0$$

Our simulation is carried for: $n = N$
In practice, we choose the number of discretizations ($N = 10$ for instance), while changing the value of the scaling parameter.

Numerical results are given in the Table 1.

Comparison with the shooting method (The indirect method)

To transform the problem (5.1)-(5.2) into a control problem governed by ordinary differential equation, we discretize the heat equation (5.1) in the spatial direction x. By dividing the interval $[0,1]$ into N intervals of lenght $x_h = \frac{\Lambda}{N}$, we obtain the following optimal control governed by ordinary differential equations.

$$\min \int_0^{t_f} dt \tag{5.4}$$

$$\dot{Y}_i(t) = A_h Y_i(t) + B_h U_i(t), \quad Y_i(0) = Y_{i_0}, \quad \forall \in [0, t_f] \tag{5.5}$$

where A_h and B_h are defined in (5.3).

Furthermore, by using the shooting method technique [17] for the system , we get the adjoint system solution $P_i(t)$ and the solution of $Y_i(t)$ with a final condition $Y_i(tf) = 0$
The obtained results are illustrated in Table 1, and a comparison with the shooting method is done, show that the obtained results are very close to each other.

Λ	1	2	8	10	15	20
$\tau^\star$ (Total discretization method)	0.6129	2.5506	44.8033	63.9311	79.1337	109.8435
$\tau^\star$ (Shooting method)	0.6013	2.5089	43.9041	62.7097	77.8107	109.1337

TABLE 1. Final time - Optimal time

The following figures show the evolution of the L^2 norm of the time control made by the two methods with the scale parameter change. A comparison between the obtained results by the presented method with those obtained directly using the shooting method is made which shows that are the same.

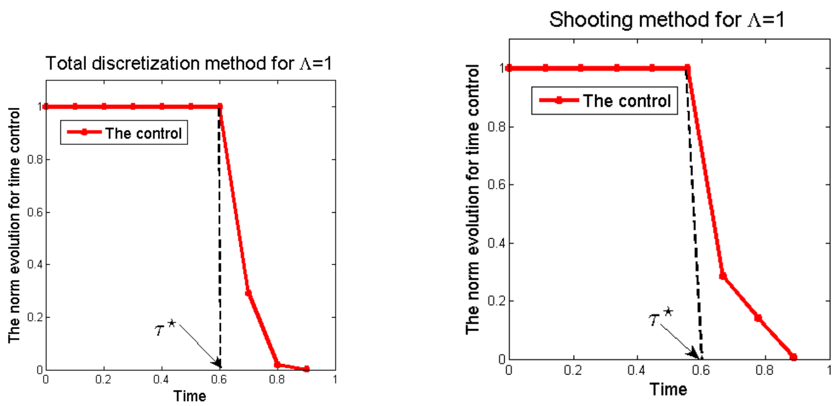

FIGURE 1. The norm evolution L^2 for time control with $\Lambda = 1$

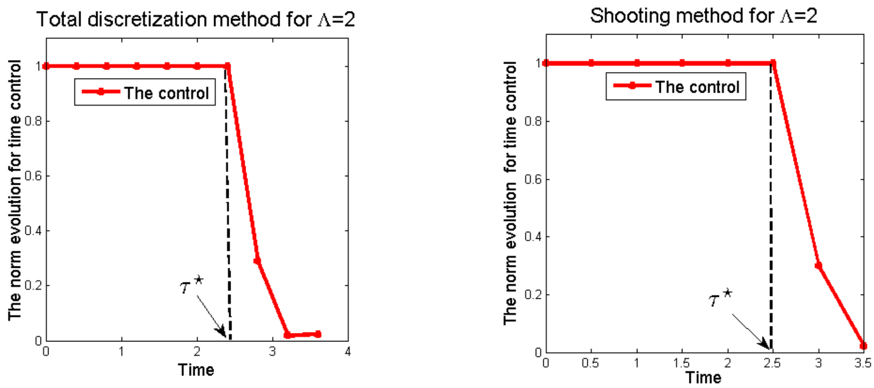

FIGURE 2. The norm evolution L^2 for time control with $\Lambda = 2$

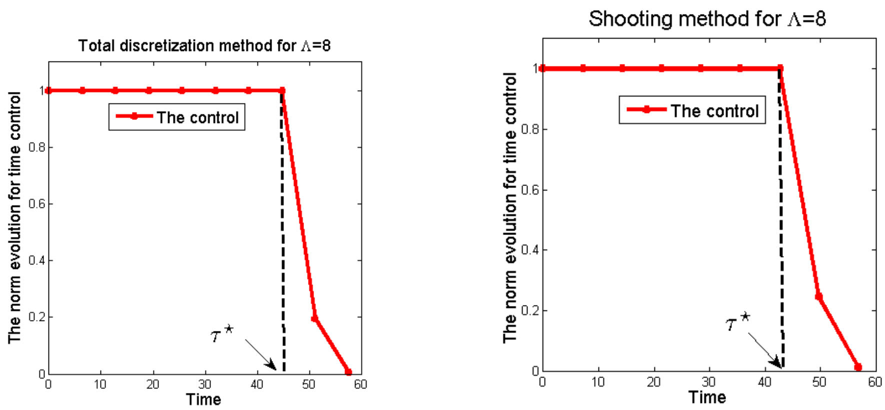

FIGURE 3. The norm evolution L^2 for time control with $\Lambda = 8$

It is interesting to note that the L^2 norm of the time control take the value 1 for any time t in $[0, \tau^\star]$. Thus, the control has the bang-bang property. Moreover, as the scaling parameter Λ increases, so does the numerical final time, with the bang-bang property still verified.

6. Perspectives

Our approach has conventionaly, consisted in studying a system with an internal control, in the presence of a scaling parameter. It is interesting to note that, due to the expression of the solution with parameter:

$$y_\lambda(x,t,\Lambda) = y_{classical}(x,t) + \varepsilon \sum_{j=1}^{N_0} \frac{1}{\Lambda^j}\, y_{classical}\left(\frac{x}{\Lambda^j}, \frac{t}{\Lambda^{2j}}\right)$$

it appears interesting to consider a control of the form:

$$\varepsilon \sum_{j=1}^{N_0} \frac{u_j}{\Lambda^j}\, y_{classical}\left(\frac{x}{\Lambda^j}, \frac{t}{\Lambda^{2j}}\right)$$

This thus leads to an affine control system. For any integer j belonging to $\{1,\ldots,N_0\}$, the control u_j corresponds to a displacement in the direction

$$f_j = \varepsilon \frac{1}{\Lambda^j}\, y_{classical}\left(\frac{x}{\Lambda^j}, \frac{t}{\Lambda^{2j}}\right)$$

It is then natural to study in the Lie algebra generated by the family

$$(f_j)_{1\leqslant i\leqslant N_0}$$

in the spirit of what is presented in [18], [19], in so far the displacements on subintervals of K, in the given directions f_i, f_j, $i \neq j$, involve their Lie bracket.

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