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SYMMETRIC POSITIVE SOLUTION FOR EVEN ORDER BVPs WITH INTEGRAL BOUNDARY CONDITIONS ON TIME SCALES

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Abstract. In this paper, we study a certain even order two-point boundary value problem with mixed derivatives and integral boundary conditions on time scales and establish the existence of a symmetric positive solutions. The proofs of our main results are based on the Hölder's inequality and theory of fixed point index in cones.

1. Introduction

There are many results from differential equations that carry over naturally and easily to difference equations, while others have a completely different structure from their continuous counterparts. The study of dynamic equations on time scales sheds new light on the discrepancies between continuous differential equations and discrete difference equations. By choosing the time scale to be the set of real numbers, the general result yields a result concerning an ordinary differential equation, and by choosing the time scale to be the set of integers, the same result yields a result for difference equations. However, since there are many other time scales than just the set of real numbers or the set of integers, one has a much more general result. Thus, the two main features of the time scales calculus are unification and extension.

Recently, researchers shown a great deal of interest to establish existence of positive solutions of boundary value problems with integral boundary conditions on time scales, for details, see [**5–8**, **12–16**, **20**, **21**] and reference therein. However, there are very few papers for the existence of symmetric positive solutions

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of boundary value problems with integral boundary conditions on time scales. In 2010, Hamal and Yoruk [**11**] established the existence of a symmetric positive solution of the following fourth order boundary value problem with integral boundary conditions,

$$
\begin{aligned}
&\big(q(t)\phi\big(p(t)x^{\Delta\nabla}\big)\big)^{\Delta\nabla}(t) = \lambda f\big(t, x(t)\big),\ t \in (0,1)_{\mathbb{T}},\\
&x(0) = x(1) = \int_0^1 g(s)x(s)\nabla s,\\
&q(0)\phi(p(0)x^{\Delta\nabla}(0)) = q(1)\phi(p(1)x^{\Delta\nabla}(1)) = \int_0^1 h(s)q(s)\phi(p(s)x^{\Delta\nabla}(s))\nabla s,
\end{aligned}
\tag{1.1}
$$

by using a fixed point index theorem. In 2016, Oguz and Topal [**17**] studied the existence of symmetric positive solutions of the boundary value problem on time scales,

$$
\begin{aligned}
&u^{\Delta\nabla}(t) + f\big(t, u(t), u^{\Delta}(t)\big) = 0,\ t \in (a,b)_{\mathbb{T}},\\
&\alpha u(a) - \beta \lim_{t\to a^+} u^{\Delta}(t) = \int_a^b h_1(s)u(s)\nabla s,\\
&\alpha u(b) + \beta \lim_{t\to b^-} u^{\Delta}(t) = \int_a^b h_2(s)u(s)\nabla s,
\end{aligned}
\tag{1.2}
$$

by using monotone iterative technique. In the same year, Topal and Denk [**22**] investigated the existence of symmetric positive solutions of the following second order boundary value problem on time scales,

$$
\begin{aligned}
&\big(g(t)u^{\Delta}(t)\big)^{\nabla} + \lambda f\big(t, u(t)\big) = 0,\ t \in (a,b)_{\mathbb{T}},\\
&\alpha u(a) - \beta \lim_{t\to a^+} g(t)u^{\Delta}(t) = \int_a^b h_1(s)u(s)\nabla s,\\
&\alpha u(b) + \beta \lim_{t\to b^-} g(t)u^{\Delta}(t) = \int_a^b h_2(s)u(s)\nabla s,
\end{aligned}
\tag{1.3}
$$

by applying the Krasnoselskii fixed point theorem in cones. However, to the best of our knowledge, there is no paper on the existence of symmetric positive solutions for higher order boundary value problems with integral boundary conditions on time scales. This paper attempts to fill this gap in literature.

In this paper, we investigate the existence of symmetric positive solutions for the even order boundary value problem with mixed derivatives on time scales given by,

$$
(-1)^n u^{(\Delta\nabla)^n}(t) + w(t)f\big(t, u(t)\big) = 0,\ t \in (0,1)_{\mathbb{T}},
\tag{1.4}
$$

satisfying the boundary conditions,

$$
\begin{aligned}
&u^{(\Delta\nabla)^i}(0) = \int_0^1 a_{i+1}(s)u^{(\Delta\nabla)^i}(s)\nabla s,\ 0 \leqslant i \leqslant n-1,\\
&u^{(\Delta\nabla)^i}(1) = \int_0^1 a_{i+1}(s)u^{(\Delta\nabla)^i}(s)\nabla s,\ 0 \leqslant i \leqslant n-1,
\end{aligned}
\tag{1.5}
$$

where $n \geqslant 1$ and $\mathbb{T}$ is a symmetric time scale, by applying fixed point index theory in cones. The rest of the paper is organized in the following fashion. In Section 2, we provide some definitions and lemmas which are essential to establish our results. In Section 3, we construct the Green's function for the homogeneous problem corresponding to (1.4)-(1.5), estimate bounds for the Green's function, and some lemmas are provided which are needed in proving our main results. In Section 4, we establish a criteria for the existence of at least one symmetric positive solution for the boundary value problem (1.4)-(1.5) by using fixed point index theory in cones. Finally as an application, we give an example to illustrate the main results.

2. Preliminaries

In this section, we provide some definitions and lemmas the details can be found in [**1**–**4**, **10**, **19**, **23**].

Definition 2.1. A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$,

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\}, \quad \rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\}$$

(supplemented by $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$) are well defined. The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) < t$, respectively.

Definition 2.2. An interval time scale $\mathbb{T} = [a, b]_{\mathbb{T}}$ is said to be symmetric if for any given $t \in \mathbb{T}$, we have $b + a - t \in \mathbb{T}$ and a function $u : \mathbb{T} \to \mathbb{R}$ is said to be symmetric on $\mathbb{T}$ if for any given $t \in \mathbb{T},\ u(t) = u(b + a - t)$.

By an interval time scale, we mean the intersection of a real interval with a given time scale. i.e., $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$. Similarly other intervals can be defined.

Definition 2.3. A function $u : \mathbb{T} \to \mathbb{R}$ is said to be concave if for any $t_1, t_2 \in \mathbb{T}$ and $c \in [0, 1],\ u(ct_1 + (1 - c)t_2) \geqslant cu(t_1) + (1 - c)u(t_2)$.

Definition 2.4. Let μ_Δ and μ_∇ be the Lebesgue $\Delta-$measure and the Lebesgue $\nabla-$measure on $\mathbb{T}$, respectively. If $A \subset \mathbb{T}$ satisfies $\mu_\Delta(A)$=$\mu_\nabla(A)$, then we call A is measurable on $\mathbb{T}$, denoted $\mu(A)$ and this value is called the Lebesgue measure of A. Let P denote a proposition with respect to $t \in \mathbb{T}$.

- (i) If there exists $E_1 \subset A$ with $\mu_\Delta(E_1) = 0$ such that P holds on $A\backslash E_1$, then P is said to hold Δ–a.e. on A.
- (ii) If there exists $E_2 \subset A$ with $\mu_\nabla(E_2) = 0$ such that P holds on $A\backslash E_2$, then P is said to hold ∇–a.e. on A.

Definition 2.5. Let $E \subset \mathbb{T}$ be a ∇–measurable set and $p \in \bar{\mathbb{R}} \equiv \mathbb{R} \cup \{-\infty, +\infty\}$ be such that $p \geqslant 1$ and let $f : E \to \bar{\mathbb{R}}$ be ∇–measurable function. Say that f belongs to $L^p_\nabla(E)$ provided that either

$$\int_E |f|^p(s)\nabla s < \infty \quad \text{if} \quad p \in \mathbb{R},$$

or there exists a constant $C \in \mathbb{R}$ such that

$$|f| \leqslant C,\ \nabla - a.e.\ on\ E \text{ if } p = +\infty.$$

LEMMA 2.1. *Let $E \subset \mathbb{T}$ be a ∇–measurable set. If $f : \mathbb{T} \to \mathbb{R}$ is a ∇–integrable on E, then*

$$\int_E f(s)\nabla s = \int_E f(s)ds + \sum_{i \in I_E} \big(t_i - \rho(t_i)\big) f(t_i),$$

where $I_E := \{i \in I : t_i \in E\}$ and $\{t_i\}_{i \in I}, I \subset \mathbb{N}$, is the set of all left-scattered points of $\mathbb{T}$.

For convenience, let $J_0 := (0,1)_{\mathbb{T}}$ and we make the following assumptions throughout the paper:

(H1) $f : J_0 \times [0,\infty) \to [0,\infty)$ is continuous and $f(\cdot,u)$ is symmetric on J_0 for all $u \in [0,\infty)$,

(H2) $w \in L^p_\nabla(J_0)$ for some $1 \leqslant p \leqslant +\infty$, is symmetric on J_0 and there exists $m > 0$ such that $w(t) \geqslant m$ a.e. on J_0,

(H3) $a_i \in L^1_\nabla(J_0)$ for all $1 \leqslant i \leqslant n$, are nonnegative on J_0 and $\alpha_i \in (0,1)$ for all $1 \leqslant i \leqslant n$, where $\alpha_i = \int_0^1 a_i(t)\nabla t$ for all $1 \leqslant i \leqslant n$.

3. Green's Function and Bounds

In this section, we construct the Green's function for the homogeneous problem corresponding to (1.4)-(1.5) and estimate bounds for the Green's function.

LEMMA 3.1. *Suppose that $\alpha_j \in (0,1)$ for all $1 \leqslant j \leqslant n$. Then for any $h(t) \in C(J_0)$, boundary value problem,*

$$-u^{\Delta\nabla}(t) = h(t),\ t \in J_0, \tag{3.1}$$

$$u(0) = u(1) = \int_0^1 a_j(x)u(x)\nabla x,\ for\ 1 \leqslant j \leqslant n, \tag{3.2}$$

has a unique solution

$$u(t) = \int_0^1 G_j(t,s)h(s)\nabla s,\ for\ 1 \leqslant j \leqslant n, \tag{3.3}$$

where

$$G_j(t,s) = G(t,s) + \frac{1}{1-\alpha_j}\int_0^1 G(s,x)a_j(x)\nabla x,\ for\ 1 \leqslant j \leqslant n, \tag{3.4}$$

and

$$G(t,s) = \begin{cases} t(1-s), & t \leqslant s, \\ s(1-t), & s \leqslant t. \end{cases} \tag{3.5}$$

PROOF. Suppose that u is a solution of the problem (3.1), thereafter integrating twice, we get

$$\begin{aligned}
u(t) &= -\int_0^t \int_0^s h(r)\nabla r\Delta s + At + B \\
&= -\int_0^t (t-s)h(s)\nabla s + At + B
\end{aligned}$$

where $A = \lim_{t\to 0^+} u^{\Delta}(t)$ and $B = u(0)$. Using the boundary conditions (3.2) we have $A = \int_0^1 (1-s)h(s)\nabla s$ and

$$\begin{aligned}
B &= \int_0^1 a_j(x)u(x)\nabla x \\
&= \int_0^1 a_j(x)\Big[-\int_0^x (x-s)h(s)\nabla s + Ax + B\Big]\nabla x \\
&= \int_0^1 a_j(x)\Big[-\int_0^x (x-s)h(s)\nabla s + x\int_0^1 (1-s)h(s)\nabla s\Big]\nabla x + B\alpha_j \\
&= \int_0^1 a_j(x)\Big[\int_0^x s(1-x)h(s)\nabla s + \int_x^1 x(1-s)h(s)\nabla s\Big]\nabla x + B\alpha_j \\
&= \int_0^1 a_j(x)\Big[\int_0^1 G(s,x)h(s)\nabla s\Big]\nabla x + B\alpha_j \\
&= \int_0^1 \Big[\int_0^1 G(s,x)a_j(x)\nabla x\Big]h(s)\nabla s + B\alpha_j \\
&= \frac{1}{1-\alpha_j}\int_0^1 \Big[\int_0^1 G(s,x)a_j(x)\nabla x\Big]h(s)\nabla s.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
u(t) &= -\int_0^t (t-s)h(s)\nabla s + \int_0^1 t(1-s)h(s)\nabla s \\
&\qquad\qquad + \frac{1}{1-\alpha_j}\int_0^1 \Big[\int_0^1 G(s,x)a_j(x)\nabla x\Big]h(s)\nabla s \\
&= \int_0^t s(1-t)h(s)\nabla s + \int_t^1 t(1-s)h(s)\nabla s \\
&\qquad\qquad + \frac{1}{1-\alpha_j}\int_0^1 \Big[\int_0^1 G(s,x)a_j(x)\nabla x\Big]h(s)\nabla s \\
&= \int_0^1 G(t,s)h(s)\nabla s + \frac{1}{1-\alpha_j}\int_0^1 \Big[\int_0^1 G(s,x)a_j(x)\nabla x\Big]h(s)\nabla s \\
&= \int_0^1 \Big[G(t,s) + \frac{1}{1-\alpha_j}\int_0^1 G(s,x)a_j(x)\nabla x\Big]h(s)\nabla s \\
&= \int_0^1 G_j(t,s)h(s)\nabla s,
\end{aligned}$$

where $G_j(t,s)$ is given in (3.4). This completes the proof. $\square$

LEMMA 3.2. *Assume that* (H_3) *holds. Then* $G(t,s)$ *and* $G_j(t,s)$ *for* $1 \leqslant j \leqslant n$, *have the following properties:*

(i) $G(t,s) > 0$ *and* $G_j(t,s) > 0$ *for all* $t, s \in J_0$,

(ii) $G(s,s)G(t,t) \leqslant G(t,s) \leqslant G(s,s)$ *for all* $t, s \in J_0$,

(iii) $G(1-t, 1-s) = G(t,s)$ *for all* $t, s \in J_0$,

(iv) $\lambda_j G_j(s,s) \leqslant G_j(t,s) \leqslant G_j(s,s)$ *for all* $t, s \in J_0$, *where* $\lambda_j = \dfrac{\delta_j}{1-\alpha_j+\delta_j}$, $\delta_j = \displaystyle\int_0^1 G(x,x)a_j(x)\nabla x$,

(v) *For each* $s \in J_0$, *the function* $G(.,s)$ *and* $G_j(.,s)$ *for* $1 \leqslant j \leqslant n$, *are concave in the first argument on* J_0.

PROOF. We can easily establish the inequalities (i), (ii) and (iii). To prove the inequality (iv), let

$$g_j(s) = \frac{1}{1-\alpha_j}\int_0^1 G(s,x)a_j(x)\nabla x, \text{ for } 1 \leqslant j \leqslant n.$$

Then,

$$G_j(t,s) = G(t,s) + g_j(s). \tag{3.6}$$

Now, from (ii) we have

$$\begin{aligned} g_j(s) &\geqslant \frac{1}{1-\alpha_j}\int_0^1 G(x,x)G(s,s)a_j(x)\nabla x \\ &\geqslant \frac{\delta_j}{1-\alpha_j}G(s,s), \end{aligned}$$

which implies from (H3) that, $(1-\alpha_j)g_j(s) \geqslant \delta_j G(s,s)$. Using (3.4) and (3.6), we have $(1-\alpha_j+\delta_j)g_j(s) \geqslant \delta_j G_j(s,s)$. Since $\delta_j > 0$ and $1-\alpha_j+\delta_j > \delta_j$, it follows that $g_j(s) \geqslant \lambda_j G_j(s,s)$. Therefore, from (3.6), $G_j(t,s) \geqslant g_j(s) \geqslant \lambda_j G_j(s,s)$. The other inequality is obvious, which proves the inequality (iv). Let $c \in [0,1]$ and $t, r, s \in J_0$ with $t \leqslant r$. In order to prove concavity of $G(t,s)$ in the first argument, we consider the following cases:

Case 1. Suppose that $s \leqslant t$. Then $ct + (1-c)r \geqslant s$ and

$$\begin{aligned} G(ct+(1-c)r, s) - &cG(t,s) - (1-c)G(r,s) \\ &= s(1-ct-(1-c)r) - cs(1-t) - (1-c)s(1-r) \\ &= 0. \end{aligned}$$

Case 2. Suppose that $r \leqslant s$. Then $ct + (1-c)r \leqslant s$ and

$$\begin{aligned} G(ct+(1-c)r, s) - &cG(t,s) - (1-c)G(r,s) \\ &= s(1-ct-(1-c)r) - ct(1-s) - (1-c)s(1-r) \\ &= 2scr > 0. \end{aligned}$$

Case 3. Suppose that $t \leqslant s \leqslant r$. Then $ct + (1-c)r \geqslant s$ and

$$\begin{aligned}G(ct + (1-c)r, s) - cG(t,s) - (1-c)G(r,s) &= s(1 - ct - (1-c)r) - ct(1-s) - (1-c)s(1-r)\\ &= c(s-t) > 0.\end{aligned}$$

Now, we prove concavity of $G_j(t,s)$ for $1 \leqslant j \leqslant n$, in the first argument. For $1 \leqslant j \leqslant n$,

$$\begin{aligned}G_j(ct + (1-c)r, s) &= G(ct + (1-c)r, s) + \frac{1}{1-\alpha_j}\int_0^1 G(s,x)a_j(x)\nabla x\\ &\geqslant cG(t,s) + (1-c)G(r,s) + \frac{1}{1-\alpha_j}\int_0^1 G(s,x)a_j(x)\nabla x\\ &\geqslant c\Big[G(t,s) + \frac{1}{1-\alpha_j}\int_0^1 G(s,x)a_j(x)\nabla x\Big]\\ &\qquad + (1-c)\Big[G(r,s) + \frac{1}{1-\alpha_j}\int_0^1 G(s,x)a_j(x)\nabla x\Big]\\ &\geqslant cG_j(t,s) + (1-c)G_j(r,s).\end{aligned}$$

This completes the proof. $\square$

Lemma 3.3. *Assume that the condition (H3) is satisfied and $G_j(t,s)$ for $1 \leqslant j \leqslant n$, is given in* (3.4). *Let $H_1(t,s) = G_1(t,s)$ and recursively define*

$$H_j(t,s) = \int_0^1 H_{j-1}(t,r)G_j(r,s)\nabla r, \quad for \quad 2 \leqslant j \leqslant n. \tag{3.7}$$

Then $H_n(t,s)$ is the Green's function for the homogeneous boundary value problem

$$(-1)^n u^{(\Delta\nabla)^n}(t) = 0,\ t \in J_0,$$

$$u^{(\Delta\nabla)^i}(0) = u^{(\Delta\nabla)^i}(1) = \int_0^1 a_{i+1}(s)u^{(\Delta\nabla)^i}(s)\nabla s,\ 0 \leqslant i \leqslant n-1.$$

Lemma 3.4. *Assume that the condition (H3) is satisfied. Define $K = \prod_{j=1}^{n-1} k_j$, $L = \prod_{j=1}^{n-1} \lambda_j k_j$, then the Green's function $H_n(t,s)$ satisfies the following inequalities:*

(i) $0 \leqslant H_n(t,s) \leqslant KG_n(s,s)$, *for all $t, s \in J_0$ and*

(ii) $H_n(t,s) \geqslant \lambda_n LG_n(s,s)$, *for all $t, s \in J_0$, where $k_j = \int_0^1 G_j(s,s)\nabla s$, for $1 \leqslant j \leqslant n$.*

Proof. We can easily establish the result by induction on n. $\square$

Lemma 3.5. *The Green's function $H_j(t,s)$ for $1 \leqslant j \leqslant n$, satisfies the following conditions*

$$H_j(t,s) = H_j(1-t, 1-s)\,\forall\, t, s \in J_0 \tag{3.8}$$

and for each $s \in J_0$, $H_j(\cdot, s)\,(1 \leqslant j \leqslant n)$ *is concave in the first argument on* J_0.

Proof. The proof is by induction. For $j = 1$, the equation (3.8) is clear and assume that the equation (3.7) is true for fixed $j \geqslant 2$. Then from (3.7) and using transformation $r_1 = 1 - r$, we have

$$\begin{aligned} H_{j+1}(t,s) &= \int_0^1 H_j(t,r)G_{j+1}(r,s)\nabla r \\ &= \int_0^1 H_j(1-t,1-r)G_{j+1}(1-r,1-s)\nabla r \\ &= \int_0^1 H_j(1-t,r_1)G_{j+1}(r_1,1-s)\nabla r_1 \\ &= H_{j+1}(1-t,1-s). \end{aligned}$$

Now, to prove concavity of $H_n(.,s)$, let $c \in [0,1]$ and $t,r,s \in J_0$ with $t \leqslant r$ and using Lemma 3.2. For $n = 1$,

$$\begin{aligned} H_1(ct+(1-c)r,s) &= G_1(ct+(1-c)r,s) \\ &\geqslant cG_1(t,s)+(1-c)G_1(r,s) \\ &\geqslant cH_1t,s)+(1-c)H_1(r,s). \end{aligned}$$

Next, we assume that $H_j(ct+(1-c)r,s) \geqslant cH_j(t,s)+(1-c)H_j(r,s)$ for fixed $j \geqslant 2$.
Then

$$\begin{aligned} H_{j+1}(ct+(1-c)r,s) &= \int_0^1 H_j(ct+(1-c)r,s_1)G_{j+1}(s_1,s)\nabla s \\ &\geqslant \int_0^1 \Big[cH_j(t,s_1)+(1-c)H_j(r,s_1)\Big]G_{j+1}(s_1,s)\nabla s \\ &\geqslant c\int_0^1 H_j(t,s_1)G_{j+1}(s_1,s)\nabla s \\ &\qquad + (1-c)\int_0^1 H_j(r,s_1)G_{j+1}(s_1,s)\nabla s \\ &\geqslant cH_{j+1}(t,s)+(1-c)H_{j+1}(r,s). \end{aligned}$$

This completes the proof. □

Let X denotes the Banach space $C_{ld}(J_0,\mathbb{R})$ with norm $\|x\| = \max_{t\in J_0}|x(t)|$. Define the cone

$$\mathcal{K} = \Big\{x \in X \;:\; x(t) \geqslant 0,\; x(t) \text{ is symmetric, concave on } J_0 \text{ and } \min_{t\in J_0} \geqslant \lambda_*\|x(t)\|\Big\},$$

where $\lambda_* = \frac{\lambda_n L}{K}$. For any $x \in \mathcal{K}$, define an operator $T : \mathcal{K} \to X$ by

$$(3.9) \qquad (Tx)(t) = \int_0^1 H_n(t,s)w(s)f\big(s,x(s)\big)\nabla s.$$

LEMMA 3.6. *Assume that (H1)-(H3) hold. Then $T(\mathcal{K}) \subset \mathcal{K}$ and $T : \mathcal{K} \to \mathcal{K}$ is completely continuous.*

PROOF. It is clear that $(Tx)(t) \geqslant 0 \,\forall\, t \in J_0$. For every $x \in \mathcal{K}$ and from Lemma 3.5, we have

$$\begin{aligned}(Tx)(ct+(1-c)r) &= \int_0^1 H_n(ct+(1-c)r, s)w(s)f\big(s, x(s)\big)\nabla s\\ &\geqslant c\int_0^1 H_n(t,s)w(s)f\big(s,x(s)\big)\nabla s\\ &\qquad + (1-c)\int_0^1 H_n(r,s)w(s)f\big(s,x(s)\big)\nabla s\\ &\geqslant c(Tx)(t) + (1-c)(Tx)(r).\end{aligned}$$

Thus, it proves that T is concave on J_0. Noticing w, x are symmetric on J_0 and $f(.,x)$ is symmetric on J_0, we have

$$\begin{aligned}(Tx)(1-t) &= \int_0^1 H_n(1-t,s)w(s)f\big(s,x(s)\big)\nabla s\\ &= \int_0^1 H_n(1-t,1-s)w(1-s)f\big(1-s,x(1-s)\big)\nabla(1-s)\\ &= \int_0^1 H_n(t,s)w(s)f\big(s,x(s)\big)\nabla s\\ &= (Tx)(t).\end{aligned}$$

Therefore, Tx is symmetric on J_0. On the other hand, by Lemma 3.4 we obtain

$$\begin{aligned}(Tx)(t) &= \int_0^1 H_n(t,s)w(s)f\big(s,x(s)\big)\nabla s\\ &\leqslant K\int_0^1 G_n(s,s)w(s)f\big(s,x(s)\big)\nabla s.\end{aligned}$$

Similarly, by Lemma 3.4 we obtain

$$\begin{aligned}(Tx)(t) &= \int_0^1 H_n(t,s)w(s)f\big(s,x(s)\big)\nabla s\\ &\geqslant \lambda_n L\int_0^1 G_n(s,s)w(s)f\big(s,x(s)\big)\nabla s\\ &\geqslant \frac{\lambda_n L}{K}\|Tx\|\\ &= \lambda_*\|Tx\|\end{aligned}$$

So, $Tx \in \mathcal{K}$ and then $T(\mathcal{K}) \subset \mathcal{K}$. Next, by standard methods and the Arzela-Ascoli theorem, one can easily prove that the operator T is completely continuous. The proof is complete. □

4. Main Result

In this section, we establish the existence of at least one positive solution for the boundary value problem (1.4)-(1.5) by applying theory of fixed point index in cones. Define the sets

$$\mathcal{K}_r = \{\, x \in \mathcal{K} \,:\, \|x\| \leqslant r\}$$

and

$$\partial\mathcal{K}_r = \{\, x \in \mathcal{K} \,:\, \|x\| = r\}$$

where r is a positive real number. The following lemmas will play an important role in the proofs of our main results.

LEMMA 4.1. *([**9**]) Let P be a cone in real Banach space E and Ω be a bounded open subset of E and $\theta \in \Omega$. Suppose $A : P \cap \bar{\Omega} \to P$ is a completely continuous operator, and satisfies $Ax = \mu x$, $x \in P \cap \partial\Omega, \mu < 1$. Then $i(A, P \cap \Omega, P) = 1$.*

LEMMA 4.2. *([**19**]) Suppose $A : P \cap \bar{\Omega} \to P$ is a completely continuous operator, and satisfies*

(i) $\inf\limits_{x \in P \cap \partial\Omega} \|Ax\| > 0$; *and*

(ii) $Ax = \mu x$, $x \in P \cap \partial\Omega$, $\mu \notin (0, 1]$.

Then $i(A, P \cap \Omega, P) = 0$.

LEMMA 4.3. *([**2, 18**]) Let $J = (a, b]$ and $f \in L^p_\nabla(J)$ with $p > 1, g \in L^q_\nabla(J)$ with $q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then $fg \in L^1_\nabla(J)$ and $\|fg\|_{L^1_\nabla} \leqslant \|f\|_{L^p_\nabla}\|g\|_{L^q_\nabla}$, where*

$$\|f\|_{L^p_\nabla} := \begin{cases} \Big[\displaystyle\int_J |f|^p(s)\nabla s\Big]^{\frac{1}{p}}, & p \in \mathbb{R}, \\ inf\Big\{C \in \mathbb{R} \,/\, |f| \leqslant C \ \nabla - a.e., \ on\ J\Big\}, & p = \infty. \end{cases}$$

Moreover, if $f \in L^1_\nabla(J)$ and $g \in L^\infty_\nabla(J)$. Then $fg \in L^1_\nabla(J)$ and $\|fg\|_{L^1_\nabla} \leqslant \|f\|_{L^1_\nabla}\|g\|_{L^\infty_\nabla}$.

For convenience, we introduce the following notation:

$$F^\beta = \limsup_{x\to\beta} \max_{t\in J_0} \frac{f(t,x)}{x}, \ F_\beta = \liminf_{x\to\beta} \min_{t\in J_0} \frac{f(t,x)}{x}, \ \text{where } \beta \text{ denotes 0 or } \infty,$$

$$M_1^{-1} = \max\Big\{K\|G_n\|_{L^q_\nabla}\|w\|_{L^p_\nabla},\ K\|G_n\|_{L^1_\nabla}\|w\|_{L^\infty_\nabla},\ K\|G_n\|_{L^\infty_\nabla}\|w\|_{L^1_\nabla}\Big\},$$

$$m_1^{-1} = K\lambda_* k_n m.$$

We consider the following three cases for $w \in L^p_\nabla(J_0) : p > 1, p = 1, p = \infty$. Case $p > 1$ is treated in the following theorem.

THEOREM 4.1. *Assume that $(H1) - (H3)$ hold. In addition, suppose that*

(H_4) $0 < F^0 < M_1$ *and* $m_1 < F_\infty < \infty$, *or*

(H_5) $0 < F^\infty < M_1$, *and* $m_1 < F_0 < \infty$.

Then the boundary value problem (1.4)-(1.5) *has at least one symmetric positive* *solution.*

PROOF. We consider the case (H_4). Considering $0 < F^0 < M_1$, there exist $r > 0, \epsilon_0 > 0$ such that $M_1 - \epsilon_0 > 0$ and for any $0 < x \leqslant r$ we have

$$f(t,x) \leqslant (M_1 - \epsilon_0)x \leqslant (M_1 - \epsilon_0)r,\ t \in J_0. \tag{4.1}$$

Therefore, from Lemma 3.4 and (4.1), we have for all $x \in \partial\mathcal{K}_r$,

$$\begin{aligned}
(Tx)(t) &= \int_0^1 H_n(t,s)w(s)f\big(s,x(s)\big)\nabla s \\
&\leqslant K\int_0^1 G_n(s,s)w(s)\nabla s(M_1 - \epsilon_0)r \\
&\leqslant K\|G_n\|_{L^q_\nabla}\|w\|_{L^p_\nabla}(M_1 - \epsilon_0)r \\
&\leqslant K\|G_n\|_{L^q_\nabla}\|w\|_{L^p_\nabla}M_1 r - K\|G_n\|_{L^q_\nabla}\|w\|_{L^p_\nabla}\epsilon_0 r \\
&\leqslant r - K\|G_n\|_{L^q_\nabla}\|w\|_{L^p_\nabla}\epsilon_0 r \\
&< r.
\end{aligned}$$

So, $Tx \neq \mu x, \forall x \in \partial\mathcal{K}_r$, and $\mu \geqslant 1$. This and Lemma 4.1 imply

$$i(T,\mathcal{K}_r,\mathcal{K}) = 1. \tag{4.2}$$

Next, we show that the conditions of Lemma 4.2 hold. In fact, from $m_1 < F_\infty < \infty$, we know that there exist $R > \lambda_* r > 0, \epsilon_1 > 0$ such that

$$f(t,x) \geqslant (m_1 + \epsilon_1)x,\ \forall\, x \geqslant R,\ t \in J_0. \tag{4.3}$$

Letting $r^* = \lambda_*^{-1}R$, then $r^* > r$, and $\min_{t\in J_0} x(t) \geqslant \lambda_*\|x\| = R,\ \forall x \in \partial\mathcal{K}_{r^*}$. Now we prove that $Tx \neq \mu x,\ \forall\, x \in \partial\mathcal{K}_{r^*},\ 0 < \mu \leqslant 1$. If not, then there exist $x_0 \in \partial\mathcal{K}_{r^*}$ and $0 < \mu_0 \leqslant 1$ such that $Tx_0 = \mu_0 x_0$. So, from Lemma 3.4 and (4.3), we have

$$\begin{aligned}
x_0(t) = \mu_0^{-1}(Tx)(t) &= \mu_0^{-1}\int_0^1 H_n(t,s)w(s)f\big(s,x_0(s)\big)\nabla s \\
&\geqslant \lambda_n L\int_0^1 G_n(s,s)w(s)x_0(s)\nabla s(m_1 + \epsilon_1) \\
&\geqslant m\lambda_n L\int_0^1 G_n(s,s)\nabla s(m_1 + \epsilon_1)r^* \\
&\geqslant Km\lambda_* k_n(m_1 + \epsilon_1)r^* \\
&= r^*\Big(1 + \frac{\epsilon_1}{m_1}\Big) > r^*,
\end{aligned}$$

i.e., $r^* > r^*$, which is a contradiction. On the other hand, from the above calculations, we can also obtain $(Tx)(t) \geqslant \lambda_* r^*\Big(1+\frac{\epsilon_1}{m_1}\Big) > \lambda_* r^*$. Hence, $\inf_{x\in\partial\mathcal{K}_{r^*}}\|Tx\| > \lambda_* r^* > 0$. So, the conditions of Lemma 4.2 hold and then from Lemma 4.2, we have

$$i(T,\mathcal{K}_{r^*},\mathcal{K}) = 0. \tag{4.4}$$

On the other hand, from (4.3) and (4.4) together with the additivity of the fixed point index, we get

$$i(T,K_{r^*}\backslash\bar{\mathcal{K}_r},\mathcal{K}) = i(T,\mathcal{K}_{r^*},\mathcal{K}) - i(T,\mathcal{K}_r,\mathcal{K}) = 0 - 1 = -1. \tag{4.5}$$

Applying the solution property of the fixed point index to (4.5) yields that T has a fixed point $x^* \in \mathcal{K}_{r^*} \backslash \bar{\mathcal{K}}_r$. Thus it follows that the boundary value problem (1.4)-(1.5) has a symmetric positive solution x^*. The proof is complete. □

The following corollary deals with the case of $p = \infty$.

Corollary 4.1. *Assume that* (H_1)-(H_3) *hold. Then the boundary value problem* (1.4)-(1.5) *has at least one symmetric positive solution.*

Proof. Let $\|G_n\|_{L^1_\nabla}\|w\|_{L^\infty_\nabla}$ replace $\|G_n\|_{L^q_\nabla}\|w\|_{L^p_\nabla}$ and repeat the argument above. □

The following corollary deals with the case of $p = 1$.

Corollary 4.2. *Assume that* (H_1)-(H_3) *hold. Then the boundary value problem* (1.4)-(1.5) *has at least one symmetric positive solution.*

Proof. Similar to the proof of Theorem 4.4. For $x \in \partial K_r$, we have

$$\begin{aligned}
(Tx)(t) &= \int_0^1 H_n(t,s)w(s)f\big(s,x(s)\big)\nabla s \\
&\leqslant K\int_0^1 G_n(s,s)w(s)\nabla s(M_1-\epsilon_0)r \\
&\leqslant K\|G_n\|_{L^\infty_\nabla}\|w\|_{L^1_\nabla}(M_1-\epsilon_0)r \\
&\leqslant K\|G_n\|_{L^\infty_\nabla}\|w\|_{L^1_\nabla}M_1 r - K\|G_n\|_\infty\|w\|_{L^1_\nabla}\epsilon_0 r \\
&\leqslant r - K\|G_n\|_{L^\infty_\nabla}\|w\|_{L^1_\nabla}\epsilon_0 r \\
&< r.
\end{aligned}$$

Therefore, $Tx \neq \mu x\ \forall x \in \partial\mathcal{K}_r$ and $\mu \geqslant 1$. This and Lemma 4.1 imply that (4.2) holds. This together with (4.3) completes the proof. □

5. Example

Example 5.1. In order to illustrate our result, we present an example as follows: Let

$$\mathbb{T} = [0, \tfrac{1}{4}] \cup \{\tfrac{3}{10}, \tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{7}{10}\} \cup [\tfrac{3}{4}, 1]$$

be a bounded symmetric time scale and consider the following boundary value problem,

(5.1)
$$u^{(\Delta\nabla)^4}(t) + \Big[\frac{1}{2} + |sin(\pi t)|\Big]\times \Big[\Big(\Big(t-\frac{1}{2}\Big)^2 + 500\Big)\Big(5000u - 10\sqrt{249978}\tanh u\Big)\Big] = 0,\ t \in J_0,$$

$$u^{(\Delta\nabla)^i}(0) = u^{(\Delta\nabla)^i}(1) = \int_0^1 \frac{1}{2}u^{(\Delta\nabla)^i}(s)\nabla s,\ 0 \leqslant i \leqslant 3, \tag{5.2}$$

where $w(t) = \frac{1}{2} + |sin(\pi t)|,\ a_i = \frac{1}{2},\ 1 \leqslant i \leqslant 4$ and

$$f(t,u) = \Big(\Big(t - \frac{1}{2}\Big)^2 + 500\Big)\Big(5000u - 10\sqrt{249978}\tanh u\Big).$$

Then by computation, we have

$$k_j = \frac{39919}{160000} \approx 0.2494937500,\ \delta_j = \frac{3983}{48000} \approx 0.0829791667,$$

$$\lambda_j = \frac{3983}{27983} \approx 0.1423364185 \text{ for } 1 \leqslant j \leqslant 4 \text{ and } m = \frac{1}{2}.$$

By these values, we get $K = 1.553027021 \times 10^{-2}$, $L = 4.478443648 \times 10^{-5}$, $\lambda_* = 2.392859344 \times 10^{-4}$, $m_1 = 2.157119188 \times 10^6$, $\|w\|_{L^\infty_\nabla} = \dfrac{3}{2}$, $\|G_4\|_{L^\infty_\nabla} = \dfrac{3}{8}$,

$$\|w\|_{L^1_\nabla} = \int_0^1 |w(t)|\nabla t \approx 1.133380806,$$

$$\|G_4\|_{L^1_\nabla} = \int_0^1 G_4(s,s)\nabla s \approx 0.2494937500.$$

Choosing $p = 2$, then $q = 2$, and we have

$$\|w\|_{L^2_\nabla} = \Big[\int_0^1 |w(t)|^2\nabla t\Big]^{\frac{1}{2}} \approx 1.174011855,$$

$$\|G_4\|_{L^2_\nabla} = \Big[\int_0^1 |G_4(s,s)|^2\nabla s\Big]^{\frac{1}{2}} \approx 0.2733002570.$$

Therefore, $M_1 = 151.5004304$. Now we check the conditions of Theorem 4.1,

$$\begin{aligned} 0 < F^0 &= \limsup_{u\to 0}\max_{t\in J_0} \frac{\Big[\Big(t - \frac{1}{2}\Big)^2 + 500\Big]\Big[5000u - 10\sqrt{249978}\tanh u\Big]}{u} \\ &= \frac{2001}{4}(5000 - 10\sqrt{249978}) \\ &= 110.0550000 < M_1. \end{aligned}$$

$$\begin{aligned} m_1 < F_\infty &= \liminf_{u\to 0}\max_{t\in J_0} \frac{\Big[\Big(t - \frac{1}{2}\Big)^2 + 500\Big]\Big[5000u - 10\sqrt{249978}\tanh u\Big]}{u} \\ &= 500\liminf_{u\to 0} \frac{\Big[5000u - 10\sqrt{249978}\tanh u\Big]}{u} \\ &= 2.5 \times 10^6. \end{aligned}$$

Thus, the conditions of the Theorem 4.1 are fulfilled. Consequently, the problem (5.1) − (5.2) has a symmetric positive solution.

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