JOURNAL OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4866, ISSN (o) 2303-4947
www.imvibl.org /JOURNALS / JOURNAL
Vol. 8(2018), 35-51
DOI: 10.7251/JIMVI1801035T

Former
BULLETIN OF SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

FUZZY STRONGLY BAIRE SPACES

Ganesan Thangaraj and Raja Palani

Abstract. In this paper, the concepts of fuzzy strongly nowhere dense sets and fuzzy strongly first category sets, fuzzy strongly residual sets in fuzzy topological spaces are introduced and studied. By means of fuzzy strongly nowhere dense sets, the notion of fuzzy strongly Baire space is defined and several characterizations of fuzzy strongly Baire space are obtained.

1. Introduction

In 1965, L.A. Zadeh [**15**] introduced the concept of fuzzy sets as a new approach for modeling uncertainties. The potential of fuzzy notion was realized by the researchers and has successfully been applied in all branches of Mathematics. The concept of fuzzy topological space was introduced by C.L. Chang [**6**] in 1968. The paper of Chang paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. Several mathematicians have tried all the pivotal concepts of general topology for extension to the fuzzy setting.

In 1899, Rene Louis Baire [**2**] introduced the concepts of first and second category sets in his doctoral thesis. In classical topology, Baire space, named in honor of Rene Louis Baire, was first introduced in Bourbaki's [**4**] *Topologie generale* Chapter IX. The concepts of Baire spaces have been studied extensively in classical topology in [**6**]. The concept of Baire spaces in fuzzy setting was introduced and studied by G.Thangaraj and S.Anjalmose in [**11**]. The notion of fuzzy simply open sets by means of fuzzy nowhere denseness of fuzzy boundary sets in fuzzy topological spaces is introduced and studied by G.Thangaraj and K.Dinakaran in [**12**].

The purpose of this paper is to introduce and study a new class of fuzzy topological spaces called fuzzy strongly Baire spaces. In section 3, the notions of fuzzy

2010 *Mathematics Subject Classification.* 54A40, 03E72.

Key words and phrases. Fuzzy dense set, fuzzy nowhere dense set,fuzzy simply open set, fuzzy first category set, fuzzy Baire space, fuzzy weakly Baire space.

strongly nowhere dense sets, fuzzy strongly first category sets and fuzzy strongly second category sets in fuzzy topological spaces are introduced and studied. In section 4, by means of fuzzy strongly nowhere dense sets, the concept of fuzzy strongly Baire spaces is introduced and several characterizations of fuzzy strongly Baire spaces are established. The inter-relationships between fuzzy strongly Baire spaces, fuzzy weakly Baire spaces and fuzzy Baire spaces are investigated. Several examples are given to illustrate the concepts introduced in this paper.

2. Preliminaries

In order to make the exposition self-contained, we give some basic notions and results used in the sequel. In this work by (X, T) or simply by X, we will denote a fuzzy topological space due to Chang (1968). Let X be a non-empty set and I the unit interval $[0, 1]$. A fuzzy set λ in X is a mapping from X into I ([**15**]).

Definition 2.1. ([**6**]) Let λ and μ be any two fuzzy sets in a fuzzy topological space (X, T). Then

(i) $\lambda \vee \mu : X \to [0, 1]$ is defined as follows : $(\lambda \vee \mu)(x) = Max\{\lambda(x), \mu(x)\}$.
(ii) $\lambda \wedge \mu : X \to [0, 1]$ is defined as follows : $(\lambda \wedge \mu)(x) = Min\{\lambda(x), \mu(x)\}$.
(iii) $\mu = \lambda^c \Leftrightarrow \mu(x) = 1 - \lambda(x)$.

More generally, for a family $\{\lambda_i : i \in I\}$ of fuzzy sets in (X, T), $\psi = \vee_i \lambda_i$ and $\delta = \wedge_i \lambda_i$ are defined respectively as $\psi(x) = sup_i\{\lambda_i(x) : x \in X\}$ and $\delta(x) = inf_i\{\lambda_i(x) : x \in X\}$.

Definition 2.2. ([**6**]) Let (X, T) be a fuzzy topological space and λ be any fuzzy set in (X, T). The interior $int(\lambda)$ and the closure $cl(\lambda)$ are defined respectively as follows:
(i). $int(\lambda) = \vee\{\mu/\mu \leqslant \lambda, \mu \in T\}$.
(ii). $cl(\lambda) = \wedge\{\mu/\lambda \leqslant \mu, 1 - \mu \in T\}$.

Lemma 2.1 ([**1**]). *For a fuzzy set λ of a fuzzy topological space X,*
(i). $1 - int(\lambda) = cl(1 - \lambda)$,
(ii). $1 - cl(\lambda) = int(1 - \lambda)$.

Definition 2.3. ([**8**]) A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy dense if there exists no fuzzy closed set μ in (X, T) such that $\lambda < \mu < 1$.

Definition 2.4. ([**8**]) A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy nowhere dense if there exists no non-zero fuzzy open set μ in (X, T) such that $\mu < cl(\lambda)$. That is., $int[cl(\lambda)] = 0$, in (X, T).

Definition 2.5. ([**3**]) A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy G_δ-set in (X, T) if $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$, where $\lambda_i \in T$.

Definition 2.6. ([**3**]) A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy F_σ-set in (X, T) if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $1 - \lambda_i \in T$.

Definition 2.7. ([**8**]) A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy first category set if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T). Any other fuzzy set in (X, T) is said to be of fuzzy second category.

Definition 2.8. ([**11**]) Let λ be a fuzzy first category set in a fuzzy topological space (X, T). Then $1 - \lambda$ is called a fuzzy residual set in (X, T).

Definition 2.9. ([**8**]) A fuzzy topological space (X, T) is called fuzzy first category if the fuzzy set 1_X is a fuzzy first category set in (X, T). That is, $1_X = \vee_{i=1}^{\infty}(\lambda_i)$ where $(\lambda_i)'s$ are fuzzy nowhere dense sets in (X, T). Otherwise (X, T) will be called a fuzzy second category space.

Lemma 2.2 ([**1**]). *For a family $A = \{\lambda_\alpha\}$ of fuzzy sets of a fuzzy space X, $\vee\big(cl(\lambda_\alpha)\big) \leqslant cl\big(\vee(\lambda_\alpha)\big)$. In case A is a finite set, $\vee\big(cl(\lambda_\alpha)\big) = cl\big(\vee(\lambda_\alpha)\big)$. Also $\vee\big(int(\lambda_\alpha)\big) \leqslant int\big(\vee(\lambda_\alpha)\big)$.*

Definition 2.10. ('cite10) A fuzzy topological space (X, T) is said to be fuzzy strongly irresolvable space, if $cl[int(\lambda)] = 1$ for each fuzzy dense set λ in (X, T).

Definition 2.11. ([**7**]) Let λ be a fuzzy set in a fuzzy topological space (X, T). The fuzzy boundary of λ is defined as $Bd(\lambda) = cl(\lambda) \wedge cl(1 - \lambda)$.

Definition 2.12. ([**12**]) A fuzzy set λ in a fuzzy topological space (X, T), is called a fuzzy simply open set if $Bd(\lambda)$ is a fuzzy nowhere dense set in (X, T). That is, λ is a fuzzy simply open set in (X, T) if $intcl[Bd(\lambda)] = 0$, in (X, T).

Definition 2.13. ([**14**]) A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy σ-nowhere dense set, if λ is a fuzzy F_σ-set in (X, T) such that $int(\lambda) = 0$.

Definition 2.14. ([**3**]) A fuzzy topological space (X, T) is called a fuzzy submaximal space if for each fuzzy set λ in (X, T) such that $cl(\lambda) = 1$, then $\lambda \in T$ in (X, T).

Definition 2.15. ([**11**]) Let (X, T) be a fuzzy topological space. Then (X, T) is called a fuzzy Baire space if $int[\vee_{i=1}^{\infty}(\lambda_i)] = 0$, where $(\lambda_i)'s$ are fuzzy nowhere dense sets in (X, T).

Definition 2.16. ([**13**]) A fuzzy topological space (X, T) is called a fuzzy weakly Baire space if $int[\vee_{i=1}^{\infty}(\mu_i)] = 0$, where $\mu_i = cl(\lambda_i) \wedge (1 - \lambda_i)$ and $(\lambda_i)'s$ are fuzzy regular open sets in (X, T).

Definition 2.17. ([**5**]) A fuzzy topological space (X, T) is said to be fuzzy hyper-connected if every non-null fuzzy open subset of (X, T) is fuzzy dense set in (X, T). That is., a fuzzy topological space (X, T) is hyper-connected if $cl(\mu_i) = 1$, for all $\mu_i \in T$.

Definition 2.18. ([**9**]) A fuzzy topological space (X, T) is called a fuzzy open hereditarily irresolvable space if $intcl(\lambda) \neq 0$, then $int(\lambda) \neq 0$ for any non-zero fuzzy set λ in (X, T).

Theorem 2.1 ([**11**]). *Let (X, T) be a fuzzy topological space. Then the following are equivalent:*

(1) *(X, T) is a fuzzy Baire space.*
(2) *$int(\lambda) = 0$ for every fuzzy first category set λ in (X, T).*
(3) *$cl(\mu) = 1$ for every fuzzy residual set μ in (X, T).*

3. Fuzzy Strongly nowhere dense set

DEFINITION 3.1. Let (X, T) be a fuzzy topological space. A fuzzy set λ defined on X is called a fuzzy strongly nowhere dense set, if $\lambda \wedge (1-\lambda)$ is a fuzzy nowhere dense set in (X, T). That is., λ is a fuzzy strongly nowhere dense set in (X, T), if $intcl[\lambda \wedge (1-\lambda)] = 0$ in (X, T).

EXAMPLE 3.1. Let $X = \{a, b, c\}$. Then the fuzzy sets α, β, γ, θ and δ are defined on X as follows:
$\alpha : X \to [0, 1]$ defined as $\alpha(a) = 0.5; \alpha(b) = 0.4; \alpha(c) = 0.6$.
$\beta : X \to [0, 1]$ defined as $\beta(a) = 0.6; \beta(b) = 0.5; \beta(c) = 0.7$.
$\gamma : X \to [0, 1]$ defined as $\gamma(a) = 0.6; \gamma(b) = 0.6; \gamma(c) = 0.7$.
$\theta : X \to [0, 1]$ defined as $\theta(a) = 0.4; \theta(b) = 0.6; \theta(c) = 0.3$.
$\delta : X \to [0, 1]$ defined as $\delta(a) = 0.3; \delta(b) = 0.1; \delta(c) = 0.7$.

Then, $T = \{0, \alpha, \beta, \gamma, 1\}$ is a fuzzy topology on X. On computation, we see that $cl(\alpha) = 1$; $cl(\beta) = 1$; $cl(\gamma) = 1$; $cl(\delta) = 1$ $cl(\theta) = 1-\alpha$; $cl(1-\delta) = 1$; $int(1-\alpha) = 0 = int(1-\beta) = int(1-\gamma)$; $int(\delta) = 0$; $int(1-\delta) = 0$ in (X, T).

Now $int[cl(\theta)] = int(1-\alpha) = 0$ and hence θ is a fuzzy nowhere dense set in (X, T). Also $intcl[\theta \wedge (1-\theta)] = int[cl(1-\gamma)] = int(1-\gamma) = 0$ and hence θ is a fuzzy strongly nowhere dense set in (X, T).

Now $int[cl(\delta)] = int(1) = 1 \neq 0$ and hence δ is not a fuzzy nowhere dense set in (X, T). Also, $intcl[\delta \wedge (1-\delta)] = int(1-\gamma) = 0$ and hence δ is a fuzzy strongly nowhere dense set in (X, T).

PROPOSITION 3.1. *If λ is a fuzzy nowhere dense set in a fuzzy topological space (X, T), then λ is a fuzzy strongly nowhere dense set in (X, T).*

PROOF. Let λ be a fuzzy nowhere dense set in (X, T).Then $intcl(\lambda) = 0$, in(X, T). Since $\lambda \wedge (1-\lambda) \leqslant \lambda$ in (X, T), $intcl[\lambda \wedge (1-\lambda)] \leqslant intcl(\lambda)$ and hence $intcl[\lambda \wedge (1-\lambda)] \leqslant 0$. That is, $intcl[\lambda \wedge (1-\lambda)] = 0$. Hence λ is a fuzzy strongly nowhere dense set in (X, T). □

REMARK 3.1. A fuzzy strongly nowhere dense set in a fuzzy topological space (X, T) need not be a fuzzy nowhere dense set in (X, T). For, in example 3.1, δ is a fuzzy strongly nowhere dense set, but not a fuzzy nowhere dense set in (X, T).

PROPOSITION 3.2. *If $int(\lambda)$ is a fuzzy dense set, for a fuzzy set λ defined on X, in a fuzzy topological space (X, T), then λ is a fuzzy strongly nowhere dense set in (X, T).*

PROOF. Suppose that $int(\lambda)$ is a fuzzy dense set in(X, T). Then $cl[int(\lambda)] = 1$ in (X, T) and $1 - cl[int(\lambda)] = 0$. This implies that $intcl(1-\lambda) = 0$ in (X, T). Since $\lambda \wedge (1-\lambda) \leqslant (1-\lambda)$, $intcl[\lambda \wedge (1-\lambda)] \leqslant intcl(1-\lambda)$ and hence $intcl[\lambda \wedge (1-\lambda)] \leqslant 0$. That is., $intcl[\lambda \wedge (1-\lambda)] = 0$. Hence λ is a fuzzy strongly nowhere dense set in(X, T). □

PROPOSITION 3.3. *If $1-\lambda$ is a fuzzy nowhere dense set in a fuzzy topological space (X, T), then λ is a fuzzy strongly nowhere dense set in (X, T).*

Proof. Suppose that $1-\lambda$ is a fuzzy nowhere dense set in (X,T). Then, $int[cl(1-\lambda)]=0$ in (X,T). Since $\lambda\wedge(1-\lambda)\leqslant 1-\lambda$, $intcl[\lambda\wedge(1-\lambda)]\leqslant int[cl(1-\lambda)]$ and hence $intcl[\lambda\wedge(1-\lambda)]\leqslant 0$. That is., $intcl[\lambda\wedge(1-\lambda)]=0$ in (X,T) and hence λ is a fuzzy strongly nowhere dense set in (X,T). □

Proposition 3.4. *If $cl[int(1-\lambda)]=1$, for a fuzzy set λ defined on X in a fuzzy topological space (X,T), then λ is a fuzzy strongly nowhere dense set in (X,T).*

Proof. Suppose that $cl[int(1-\lambda)]=1$ in (X,T). Then $1-cl[int(1-\lambda)]=0$ and $1-\{1-int[cl(\lambda)]\}=0$. This implies that $int[cl(\lambda)]=0$ in (X,T). Thus λ is a fuzzy nowhere dense set in (X,T). Then, by proposition 3.1, λ is a fuzzy strongly nowhere dense set in (X,T). □

Proposition 3.5. *If λ is a fuzzy strongly nowhere dense set in a fuzzy topological space (X,T), then $1-\lambda$ is also a fuzzy strongly nowhere dense set in (X,T).*

Proof. Let λ be a fuzzy strongly nowhere dense set in (X,T). Then $intcl[\lambda\wedge(1-\lambda)]=0$ in (X,T). Now $intcl\{(1-\lambda)\wedge[1-(1-\lambda)]\}=intcl([1-\lambda)\wedge\lambda]$ and hence $intcl\{(1-\lambda)\wedge[1-(1-\lambda)]\}=0$. This implies that $1-\lambda$ is a fuzzy strongly nowhere dense set in (X,T). □

Proposition 3.6. *If λ is a fuzzy nowhere dense set in a fuzzy topological space (X,T), then $1-\lambda$ is a fuzzy strongly nowhere dense set in (X,T).*

Proof. Let λ be a fuzzy nowhere dense set in (X,T). Then, by proposition 3.1, λ is a fuzzy strongly nowhere dense set in (X,T), and by proposotion 3.5, $1-\lambda$ is a fuzzy strongly nowhere dense set in(X,T). □

Proposition 3.7. *If λ is a fuzzy strongly nowhere dense set in a fuzzy topological space (X,T), then $cl(\lambda)\vee cl(1-\lambda)=1$, in (X,T).*

Proof. Let λ be a fuzzy strongly nowhere dense set in (X,T). Then $intcl[\lambda\wedge(1-\lambda)]=0$ in (X,T). Now $1-intcl[\lambda\wedge(1-\lambda)]=1$ and hence $clint[1-\{\lambda\wedge(1-\lambda)\}]=1$, in (X,T). But, $clint[1-\{\lambda\wedge(1-\lambda)\}]\leqslant cl[1-\{\lambda\wedge(1-\lambda)\}]$ implies that $1\leqslant cl[1-\{\lambda\wedge(1-\lambda)\}]$. Thus, $cl[1-\{\lambda\wedge(1-\lambda)\}]=1$, in (X,T). This implies that $cl[(1-\lambda)\vee\lambda]=1$ in (X,T). But, by lemma 2.2, $cl[(1-\lambda)\vee\lambda]=cl(1-\lambda)\vee cl(\lambda)$. Hence $cl(\lambda)\vee cl(1-\lambda)=1$, in (X,T). □

Proposition 3.8. *If λ is a fuzzy set defined on X such that $int[bd(\lambda)]=0$ in a fuzzy topological space (X,T), then λ is a fuzzy strongly nowhere dense set in (X,T).*

Proof. Let λ be a fuzzy set defined on X such that $int[bd(\lambda)]=0$ in (X,T). Since $bd(\lambda)=cl(\lambda)\wedge cl(1-\lambda)$ and $cl(\lambda)\wedge cl(1-\lambda)\geqslant cl[\lambda\wedge(1-\lambda)]$, we have $bd(\lambda)\geqslant cl[\lambda\wedge(1-\lambda)]$ and hence $intcl[\lambda\wedge(1-\lambda)]\leqslant int[bd(\lambda)]$ in (X,T). Then $intcl[\lambda\wedge(1-\lambda)]\leqslant 0$. That is., $intcl[\lambda\wedge(1-\lambda)]=0$ and hence λ is a fuzzy strongly nowhere dense set in (X,T). □

Proposition 3.9. *If λ is a fuzzy simply open set in a fuzzy topological space (X,T), then λ is a fuzzy strongly nowhere dense set in (X,T).*

Proof. Let λ be a fuzzy simply open set in (X,T). Then $intcl[bd(\lambda)] = 0$, in (X,T). But $int[bd(\lambda)] \leqslant intcl[bd(\lambda)]$ implies that $int[bd(\lambda)] = 0$ in (X,T). Then, by proposition 3.8, λ is a fuzzy strongly nowhere dense set in (X,T). □

Remark 3.2. A fuzzy strongly nowhere dense set in a fuzzy topological space need not be a fuzzy simply open set. For, in example 3.1, δ is a fuzzy strongly nowhere dense set in (X,T), but not a fuzzy simply open set in (X,T), since $intcl[Bd(\delta)] = intcl[cl(\delta) \wedge cl(1-\delta)] = intcl[1 \wedge 1] = 1 \neq 0$, in (X,T).

Example 3.2. Let $X = \{a,b,c\}$. Then the fuzzy sets λ, μ, γ, are defined on X as follows:
$\lambda : X \to [0,1]$ defined as $\lambda(a) = 0.8; \lambda(b) = 0.5; \lambda(c) = 0.7$.
$\mu : X \to [0,1]$ defined as $\mu(a) = 0.6; \mu(b) = 0.9; \mu(c) = 0.4$.
$\gamma : X \to [0,1]$ defined as $\gamma(a) = 0.4; \gamma(b) = 0.7; \gamma(c) = 0.8$.

Then, $T = \{0, \lambda, \mu, \gamma, \lambda \vee \mu, \lambda \vee \gamma, \mu \vee \gamma, \lambda \wedge \mu, \lambda \wedge \gamma, \mu \wedge \gamma, [\lambda \wedge (\mu \vee \gamma)], [\lambda \vee (\mu \wedge \gamma)], [\mu \wedge (\lambda \vee \gamma)], [\mu \vee (\lambda \wedge \gamma)], [\gamma \wedge (\lambda \vee \mu)], [\gamma \vee (\lambda \wedge \mu)], \lambda \vee \mu \vee \gamma, \lambda \wedge \mu \wedge \gamma, 1\}$ is a fuzzy topology on X. On computation, $cl(\lambda) = 1$ and $cl(1-\lambda) = 1-\lambda$ and $intcl[Bd(\lambda)] = intcl[cl(\lambda) \wedge cl(1-\lambda)] = intcl[1 \wedge (1-\lambda)] = intcl(1-\lambda) = int(1-\lambda) = 1 - cl(\lambda) = 1 - 1 = 0$. Hence λ is a fuzzy simply open set in (X,T) but not a fuzzy nowhere dense set in (X,T), since $intcl(\lambda) = int(1) \neq 0$, in (X,T).

Remark 3.3. The inter-relations between fuzzy nowhere dense sets, fuzzy simply open sets and fuzzy strongly nowhere dense sets in a fuzzy topological space can be summarized as follows:

Fuzzy nowhere dense set $\longrightarrow$ Fuzzy simply open set

$\searrow$ $\swarrow$

Fuzzy strongly nowhere dense set

Proposition 3.10. *If λ is a fuzzy closed set with $int(\lambda) = 0$ in a fuzzy topological space (X,T), then λ is a fuzzy strongly nowhere dense set in (X,T).*

Proof. Let λ be a fuzzy closed set with $int(\lambda) = 0$ in (X,T). Then,

$$intcl[cl(\lambda) \wedge cl(1-\lambda)] = intcl[\lambda \wedge (1 - int(\lambda))] = intcl[\lambda \wedge 1] = intcl(\lambda) = int(\lambda) = 0,$$

and hence λ is a fuzzy simply open set in (X,T). By proposition 3.9, λ is a fuzzy strongly nowhere dense set in (X,T). □

Proposition 3.11. *If λ is a fuzzy open and fuzzy dense set in a fuzzy topological space (X,T), then λ is a fuzzy strongly nowhere dense set in (X,T).*

Proof. Let λ be a fuzzy open and fuzzy dense set in (X,T). Then $1-\lambda$ is a fuzzy closed set with $int(1-\lambda) = 1 - cl(\lambda) = 1 - 1 = 0$ in (X,T). Then, by proposition 3.10, $1-\lambda$ is a fuzzy strongly nowhere dense set in (X,T) and by proposition 3.5, $1-(1-\lambda)$ is a fuzzy strongly nowhere dense set in (X,T) and thus λ is a fuzzy strongly nowhere dense set in (X,T). □

THEOREM 3.1 ([**14**]). *If λ is a fuzzy nowhere dense set in a fuzzy topological space (X,T), then $cl(\lambda)$ is a fuzzy simply open set in (X,T).*

PROPOSITION 3.12. *If λ is a fuzzy nowhere dense set in a fuzzy topological space (X,T), then $cl(\lambda)$ is a fuzzy strongly nowhere dense set in (X,T).*

PROOF. Let λ be a fuzzy nowhere dense set in (X,T). Then, by theorem 3.1, $cl(\lambda)$ is a fuzzy simply open set in (X,T) and then, by proposition 3.9, $cl(\lambda)$ is a fuzzy strongly nowhere dense set in (X,T). □

THEOREM 3.2 ([**12**]). *If $int(\lambda)=0$ for a fuzzy set λ in a fuzzy strongly irresolvable space (X,T), then λ is a fuzzy simply open set in (X,T).*

PROPOSITION 3.13. *If $int(\lambda)=0$ for a fuzzy set λ in a fuzzy strongly irresolvable space (X,T), then λ is a fuzzy strongly nowhere dense set in (X,T).*

PROOF. Let λ be a fuzzy set in (X,T) such that $int(\lambda)=0$. Since (X,T) is a fuzzy strongly irresolvable space, by theorem 3.2, λ is a fuzzy simply open set in (X,T) and then, by proposition 3.9, λ is a fuzzy strongly nowhere dense set in (X,T). □

THEOREM 3.3 ([**12**]). *If λ is a fuzzy dense and fuzzy G_δ in a fuzzy strongly irresolvable space (X,T), then $1-\lambda$ is a fuzzy simply open set in (X,T).*

PROPOSITION 3.14. *If λ is a fuzzy dense and fuzzy G_δ set in a fuzzy strongly irresolvable space (X,T), then $1-\lambda$ is a fuzzy strongly nowhere dense set in (X,T).*

PROOF. Let λ be a fuzzy dense and fuzzy G_δ set in (X,T). Since (X,T) is a fuzzy strongly irresolvable space, by theorem 3.3, $1-\lambda$ is a fuzzy simply open set in (X,T). Then, by proposition 3.9, $1-\lambda$ is a fuzzy strongly nowhere dense set in (X,T). □

REMARK 3.4. In view of proposition 3.14 and proposition 3.5, we have the following result: "If λ is a fuzzy dense and fuzzy G_δ set in a fuzzy strongly irresolvable space (X,T), then λ is a fuzzy strongly nowhere dense set in (X,T)".

THEOREM 3.4 ([**14**]). *In a fuzzy topological space (X,T), a fuzzy set λ is a fuzzy σ - nowhere dense set in (X,T) if and only if $1-\lambda$ is a fuzzy dense and fuzzy G_δ set in (X,T).*

PROPOSITION 3.15. *If λ is a fuzzy σ - nowhere dense set in a fuzzy strongly irresolvable space (X,T), then λ is a fuzzy strongly nowhere dense set in (X,T).*

PROOF. Let λ be a fuzzy σ - nowhere dense set in (X,T). Then, by theorem 3.4, $1-\lambda$ is a fuzzy dense and fuzzy G_δ set in (X,T) and by proposition 3.14, $1-(1-\lambda)$ is a fuzzy strongly nowhere dense set in (X,T). Thus, λ is a fuzzy strongly nowhere dense set in (X,T). □

Remark 3.5. If λ and μ are fuzzy strongly nowhere dense sets in a fuzzy topological space (X,T), then $\lambda \wedge \mu$ need not be a fuzzy strongly nowhere dense set in (X,T). For, consider the following example:

Example 3.3. Let $X = \{a,b,c\}$. The fuzzy sets λ and μ are defined on X as follows:
$\lambda : X \to [0,1]$ defined as $\lambda(a) = 0.3; \lambda(b) = 0.2; \lambda(c) = 0.7$.
$\mu : X \to [0,1]$ defined as $\mu(a) = 0.8; \mu(b) = 0.8; \mu(c) = 0.4$.

Then, $T = \{0, \lambda, \mu, (\lambda\vee\mu), (\lambda\wedge\mu), 1\}$ is a fuzzy topology on X. On computation, we see that $int(1-\lambda) = 0$; $int(1-\mu) = 0$; $int(1-[\lambda\vee\mu]) = 0$; $int(1-[\lambda\wedge\mu]) = \lambda\wedge\mu$ and$cl(\lambda) = 1$; $cl(\mu) = 1$; $cl(\lambda\vee\mu) = 1$; $cl(\lambda\wedge\mu) = 1-[\lambda\wedge\mu]$. Now, $intcl[\lambda\wedge(1-\lambda)] = int(1-\lambda) = 0$, $intcl[\mu\wedge(1-\mu)] = int(1-\mu) = 0$ and hence λ and μ are fuzzy strongly nowhere dense sets in (X,T). But $intcl[(\lambda \wedge \mu) \wedge (1 - [\lambda \wedge \mu])] = int[1 - (\lambda \wedge \mu)] = \lambda \wedge \mu \neq 0$ and hence $\lambda \wedge \mu$ is not a fuzzy strongly nowhere dense set in (X,T).

Definition 3.2. A fuzzy set λ in a fuzzy topological space (X,T) is called a fuzzy strongly first category set if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $(\lambda_i)'s$ are fuzzy strongly nowhere dense sets in (X,T). Any other fuzzy set in (X,T) is said to be a fuzzy strongly second category set in (X,T).

Definition 3.3. If λ is a fuzzy strongly first category set in a fuzzy topological space (X,T), then $1 - \lambda$ is a fuzzy strongly residual set in (X,T).

Definition 3.4. A fuzzy topological space (X,T) is called a fuzzy strongly first category space, if the fuzzy set 1_X is a fuzzy strongly first category set in (X,T). That is, $1_X = \vee_{i=1}^{\infty}(\lambda_i)$, where $(\lambda_i)'s$ are fuzzy strongly nowhere dense sets in (X,T). Otherwise (X,T) will be called a fuzzy strongly second category space.

Example 3.4. Let $X = \{a,b,c\}$. The fuzzy sets λ, μ, γ, are defined on X as follows:
$\lambda : X \to [0,1]$ defined as $\lambda(a) = 0.3; \lambda(b) = 0.2; \lambda(c) = 0.7$.
$\mu : X \to [0,1]$ defined as $\mu(a) = 0.8; \mu(b) = 0.8; \mu(c) = 0.4$.
$\gamma : X \to [0,1]$ defined as $\gamma(a) = 0.8; \gamma(b) = 0.7; \gamma(c) = 0.6$.

Then, $T = \{0, \lambda, \mu, (\lambda\vee\mu), (\lambda\wedge\mu), 1\}$ is a fuzzy topology on X. On computation, we see that $int(1-\lambda) = 0$; $int(1-\mu) = 0$; $int(1-[\lambda\vee\mu]) = 0$; $int(1-[\lambda\wedge\mu]) = \lambda\wedge\mu$ and $cl[1-\gamma] = 1-(\lambda\wedge\mu), cl(\lambda\wedge\mu) = 1-(\lambda\wedge\mu)$ in (X,T). Also $intcl[\lambda\wedge(1-\lambda)] = int(1-\lambda) = 0$, $intcl[\mu \wedge (1-\mu)] = int(1-\mu) = 0$, $intcl[(\lambda \vee \mu) \wedge 1 - (\lambda \vee \mu)] = int[1-(\lambda\vee\mu)] = 0$, $intcl[(\lambda\vee\gamma)\wedge 1-(\lambda\vee\gamma)] = int(1-\lambda) = 0$ and $intcl[\mu\vee(\lambda\wedge\gamma)\wedge 1-\{1-[\mu\vee(\lambda\wedge\gamma])\}] = int(1-\mu) = 0$. The fuzzy strongly nowhere dense sets in (X,T) are $\lambda, \mu, \lambda\vee\mu, \lambda\vee\gamma, [\mu\vee(\lambda\wedge\gamma)], 1-\lambda, 1-\mu, 1-(\lambda\vee\mu), 1-(\lambda\vee\gamma) and 1-[\mu\vee(\lambda\wedge\gamma)]$. Now $\lambda\vee\mu = \mu\vee(\lambda\vee\gamma)\vee(1-\lambda)\vee(1-\mu)\vee[\mu\vee(\lambda\wedge\gamma)]\vee[1-(\lambda\vee\mu)]\vee[1-(\lambda\vee\gamma)]\vee[1-\mu\vee(\lambda\wedge\gamma)]$ implies that $\lambda \vee \mu$ is a fuzzy strongly first category set in (X,T) and $1 - (\lambda \vee \mu)$ is a fuzzy strongly residual set in (X,T).

Proposition 3.16. *If λ is a fuzzy first category set in a fuzzy topological space (X,T), then λ is a fuzzy strongly first category set in (X,T).*

Proof. Let λ be a fuzzy first category set in (X,T). Then $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $(\lambda_i)'s$ are fuzzy nowhere dense sets in (X,T). By proposition 3.1, the fuzzy nowhere dense sets $(\lambda_i)'s$ are fuzzy strongly nowhere dense sets in (X, T) and hence $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $(\lambda_i)'s$ are fuzzy strongly nowhere dense sets in (X, T), implies that λ is a fuzzy strongly first category set in (X, T). □

Remark 3.6. The converse of the above proposition need not be true. A fuzzy strongly first category set need not be a fuzzy first category set in a fuzzy topological space. For, in example 3.4, $\lambda \vee \mu$ is a fuzzy strongly first category set in (X, T) but not a fuzzy first category set in (X, T), since

$$\lambda \vee \mu \neq (1-\lambda) \vee (1-\mu) \vee [1-(\lambda \vee \mu)] \vee [1-(\lambda \vee \gamma)] \vee [1-\{\mu \vee (\lambda \wedge \gamma)\}],$$

where

$$1-\lambda, 1-\mu, 1-(\lambda \vee \mu), 1-(\lambda \vee \gamma), 1-[\mu \vee (\lambda \wedge \gamma)]$$

are fuzzy nowhere dense sets in (X, T).

Proposition 3.17. *If $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $(\lambda_i)'s$ are fuzzy sets defined on X such that $int(\lambda_i) = 0$ in a fuzzy strongly irresolvable space (X, T), then λ is a fuzzy strongly first category set in (X, T).*

Proof. Suppose that $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $int(\lambda_i) = 0$ in (X, T). Since (X, T) is a fuzzy strongly irresolvable space (X, T) and $int(\lambda_i) = 0$ in (X, T), by proposition 3.13, $(\lambda_i)'s$ are fuzzy strongly nowhere dense sets in (X, T) and hence λ is a fuzzy strongly first category set in (X, T). □

Proposition 3.18. *If $\mu = \wedge_{i=1}^{\infty}(\mu_i)$, where $(\mu_i)'s$ are fuzzy dense sets in a fuzzy strongly irresolvable space (X, T), then μ is a fuzzy strongly residual set in (X, T).*

Proof. Suppose that $\mu = \wedge_{i=1}^{\infty}(\mu_i)$, where $(\mu_i)'s$ are fuzzy dense sets in (X, T). Since $cl(\mu_i) = 1$ in (X, T), $1 - cl(\mu_i) = 0$ and hence $int(1-\mu_i) = 0$ in (X, T). Since (X, T) is a fuzzy strongly irresolvable space and $int(1-\mu_i) = 0$ in (X, T), by proposition 3.13, $(1-\mu_i)'s$ are fuzzy strongly nowhere dense sets in (X, T). Hence $\vee_{i=1}^{\infty}(1-\mu_i)$ is a fuzzy strongly first category set in (X, T). Now $\vee_{i=1}^{\infty}(1-\mu_i) = 1 - \wedge_{i=1}^{\infty}(\mu_i) = 1-\mu$. Then $1-\mu$ is a fuzzy strongly first category set in (X, T) and thus μ is a fuzzy strongly residual set in (X, T). □

Proposition 3.19. *If $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $(\lambda_i)'s$ are fuzzy closed sets with $int(\lambda_i) = 0$ in (X, T), then λ is a fuzzy strongly first category set in (X, T).*

Proof. Suppose that $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $1-\lambda_i \in T$ and $int(\lambda_i) = 0$ in (X, T). Now by proposition 3.10, the fuzzy closed sets $(\lambda_i)'s$ with $int(\lambda_i) = 0$, are fuzzy strongly nowhere dense sets in (X, T) and hence $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $(\lambda_i)'s$ are fuzzy strongly nowhere dense sets in (X, T), implies that λ is a fuzzy strongly first category set in (X, T). □

Proposition 3.20. *If λ is a fuzzy F_σ set such that $int(\lambda) = 0$ in (X, T), then λ is a fuzzy strongly first category set in (X, T).*

PROOF. Let λ be a fuzzy F_σ set in (X, T). Then $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $(\lambda_i)'s$ are fuzzy closed sets in (X, T). By lemma 2.2, $\vee_{i=1}^{\infty} int(\lambda_i) \leqslant int\{\vee_{i=1}^{\infty}(\lambda_i)\}$, in (X, T). This implies that $\vee_{i=1}^{\infty} int(\lambda_i) \leqslant int\{\vee_{i=1}^{\infty}(\lambda_i)\} = int(\lambda)$. Since $int(\lambda) = 0$, $\vee_{i=1}^{\infty} int(\lambda_i) = 0$ in (X, T). This implies that $int(\lambda_i) = 0$. Thus $(\lambda_i)'s$ are fuzzy closed sets with $int(\lambda_i) = 0$, in (X, T). Then, by proposition 3.10, $(\lambda_i)'s$ are fuzzy strongly nowhere dense sets in (X, T). Hence $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $(\lambda_i)'s$ are fuzzy strongly nowhere dense sets in (X, T), implies that λ is a fuzzy strongly first category set in (X, T). □

PROPOSITION 3.21. *If λ is a fuzzy σ-nowhere dense set in a fuzzy topological space (X, T) then λ is a fuzzy strongly first category set in (X, T).*

PROOF. Let λ be a fuzzy σ-nowhere dense set in (X, T). Then λ is a fuzzy F_σ set in (X, T) such that $int(\lambda) = 0$. Then, by proposition 3.20, λ is a fuzzy strongly first category set in (X, T). □

PROPOSITION 3.22. *If $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $(\lambda_i)'s$ are fuzzy simply open sets in a fuzzy topological space (X, T), then λ is a fuzzy strongly first category set in (X, T).*

PROOF. Suppose that $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $(\lambda_i)'s$ are fuzzy simply open sets in (X, T). By proposition 3.9, the fuzzy simply open sets $(\lambda_i)'s$ are fuzzy strongly nowhere dense sets in (X, T) and hence λ is a fuzzy strongly first category set in (X, T). □

PROPOSITION 3.23. *If $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $int[bd(\lambda_i)] = 0$ in (X, T), then λ is a fuzzy strongly first category set in (X, T).*

PROOF. The proof follows from proposition 3.8. □

PROPOSITION 3.24. *If $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $(\lambda_i)'s$ are fuzzy open and fuzzy dense set in a fuzzy topological space (X, T), then λ is a fuzzy strongly first category set in (X, T).*

PROOF. The proof follows from proposition 3.11. □

THEOREM 3.5 (10). *If the fuzzy topological space (X, T) is a fuzzy submaximal space, then (X, T) is a fuzzy strongly irresolvable space.*

PROPOSITION 3.25. *If $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $int(\lambda_i) = 0$ in a fuzzy submaximal space (X, T), then λ is a fuzzy strongly first category set in (X, T).*

PROOF. Suppose that $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $int(\lambda_i) = 0$ in (X, T). Since (X, T) is a fuzzy sub-maximal space, by theorem 3.5, (X, T) is a fuzzy strongly irresolvable space. Thus $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $int(\lambda_i) = 0$ in a fuzzy strongly irresolvable space (X, T). By proposition 3.17, λ is a fuzzy strongly first category set in (X, T). □

PROPOSITION 3.26. *If $\mu = \wedge_{i=1}^{\infty}(\mu_i)$ where $(\mu_i)'s$ are fuzzy dense sets in a fuzzy submaximal space (X, T), then μ is a fuzzy G_δ and fuzzy strongly residual set in (X, T).*

PROOF. Suppose that $\mu = \wedge_{i=1}^{\infty}(\mu_i)$ where $(\mu_i)'s$ are fuzzy dense sets in (X, T). Since (X, T) is a fuzzy submaximal space, by theorem 3.5, (X, T) is a fuzzy strongly irresolvable space. Thus $\mu = \wedge_{i=1}^{\infty}(\mu_i)$ where $cl(\mu_i) = 1$ in the fuzzy strongly irresolvable space (X, T). Then, by proposition 3.18, μ is a fuzzy strongly residual set. Also since (X, T) is a fuzzy submaximal space, the fuzzy dense sets $(\mu_i)'s$ are fuzzy open sets in (X, T) and hence $\mu = \wedge_{i=1}^{\infty}(\mu_i)$ is a fuzzy G_δ in (X, T). Thus μ is a fuzzy G_δ and fuzzy strongly residual set in (X, T). □

The following proposition establishes the existence of fuzzy strongly first category sets in a fuzzy topological space whenever there are fuzzy first category sets.

PROPOSITION 3.27. *If λ is a fuzzy first category set in a fuzzy topological space (X, T), then there exists a fuzzy strongly first category set μ in (X, T) such that $cl(\lambda) \geqslant \mu$.*

PROOF. Let λ be a fuzzy first category set in (X, T). Then $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $(\lambda_i)'s$ are fuzzy nowhere dense sets in (X, T). By proposition 3.12, $\{cl(\lambda_i)\}'s$ are fuzzy strongly nowhere dense sets in (X, T). Then $\vee_{i=1}^{\infty} cl(\lambda_i)$ is a fuzzy strongly first category set in (X, T). Let $\mu = \vee_{i=1}^{\infty} cl(\lambda_i)$. Now $\vee_{i=1}^{\infty} cl(\lambda_i) \leqslant cl[\vee_{i=1}^{\infty}(\lambda_i)]$ implies that $\mu \leqslant cl(\lambda)$, in (X, T). □

4. Fuzzy strongly Baire space

DEFINITION 4.1. A fuzzy topological space (X, T) is called a fuzzy strongly Baire space if $cl[\vee_{i=1}^{\infty}(\lambda_i)] = 1$, where $(\lambda_i)'s$ are fuzzy strongly nowhere dense sets in (X, T).

EXAMPLE 4.1. Let $X = \{a, b, c\}$. The fuzzy sets λ, μ, γ, are defined on X as follows:

$\lambda : X \to [0, 1]$ defined as $\lambda(a) = 0.7; \lambda(b) = 0.6; \lambda(c) = 0.2$.
$\mu : X \to [0, 1]$ defined as $\mu(a) = 0.5; \mu(b) = 0.4; \mu(c) = 0.8$.
$\gamma : X \to [0, 1]$ defined as $\gamma(a) = 0.5; \gamma(b) = 0.3; \gamma(c) = 0.6$.

Then, $T = \{0, \lambda, \mu, \gamma, \lambda \vee \mu, \lambda \vee \gamma, \lambda \wedge \mu, \lambda \wedge \gamma, \gamma \vee (\lambda \wedge \mu), 1\}$ is a fuzzy topology on X. On computation, $intcl(1 - \lambda) = 0$, $intcl[1 - (\lambda \vee \mu)] = 0$, $intcl[1 - (\lambda \vee \gamma)] = 0$. The fuzzy nowhere dense sets in (X, T) are $1 - \lambda, 1 - (\lambda \vee \mu)$ and $1 - (\lambda \vee \gamma)$. Also, $int\{cl[\lambda \wedge (1 - \lambda)]\} = 0$; $int\{cl[(\lambda \vee \mu) \wedge \{1 - (\lambda \vee \mu)\}]\} = 0$; $int\{cl[(\lambda \vee \gamma) \wedge \{1 - (\lambda \vee \gamma)\}]\} = 0$;

The fuzzy strongly nowhere dense sets in (X, T) are $\lambda, \lambda \vee \mu, \lambda \vee \gamma, 1 - \lambda, 1 - (\lambda \vee \mu), 1 - (\lambda \vee \gamma)$ and $cl\{\lambda \vee (\lambda \vee \mu) \vee (\lambda \vee \gamma) \vee (1 - \lambda) \vee [1 - (\lambda \vee \mu)] \vee [1 - (\lambda \vee \gamma)]\} = cl(\lambda \vee \mu) = 1$. Hence (X, T) is a fuzzy strongly Baire space.

PROPOSITION 4.1. *Let (X, T) be a fuzzy topological space. Then, the following are equivalent:*

(i). (X, T) is a fuzzy strongly Baire space.
(ii). $cl(\lambda) = 1$, for each fuzzy strongly first category set λ in (X, T).
(iii). $int(\mu) = 0$, for each fuzzy strongly residual set μ in (X, T).

Proof. $(i) \Rightarrow (ii)$. Let λ be a fuzzy strongly first category set in (X,T). Then $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $(\lambda_i)'s$ are fuzzy strongly nowhere dense sets in (X,T). Since (X,T) is a fuzzy strongly Baire space, $cl[\vee_{i=1}^{\infty}(\lambda_i)] = 1$ and hence $cl(\lambda) = 1$, in (X,T).

$(ii) \Rightarrow (iii)$ Let μ be a fuzzy strongly residual set in (X,T). Then $1-\mu$ is a fuzzy strongly first category set in (X,T). By hypothesis, $cl(1-\mu) = 1$ in (X,T). Then $1 - int(\mu) = cl(1-\mu) = 1$ and hence $int(\mu) = 0$ in (X,T).

$(iii) \Rightarrow (i)$ Let λ be a fuzzy strongly first category set in (X,T). Then $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $(\lambda_i)'s$ are fuzzy strongly nowhere dense sets in (X,T). Since λ is a fuzzy strongly first category set in (X,T), $1-\lambda$ is a fuzzy strongly residual set in (X,T). By hypothesis, $int(1-\lambda) = 0$ in (X,T). Now $1 - cl(\lambda) = int(1-\lambda) = 0$ implies that $cl(\lambda) = 1$ and then $cl[\vee_{i=1}^{\infty}(\lambda_i)] = 1$, where $(\lambda_i)'s$ are fuzzy strongly nowhere dense sets in (X,T), implies that (X,T) is a fuzzy strongly Baire space. □

Proposition 4.2. *If (X,T) is a fuzzy strongly Baire space, then*
(i) $cl[\vee_{i=1}^{\infty}(\lambda_i)] = 1$, where $1-\lambda_i \in T$ and $int(\lambda_i) = 0$ in (X,T).
(ii) $cl[\vee_{i=1}^{\infty}(\lambda_i)] = 1$, where $(\lambda_i)'s$ are fuzzy simply open sets in (X,T).
(iii) $cl[\vee_{i=1}^{\infty}(\lambda_i)] = 1$, where $int[Bd(\lambda_i)] = 0$ in (X,T).

Proof. (i) Let $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $(\lambda_i)'s$ are fuzzy closed sets with $int(\lambda_i) = 0$. Then, by proposition 3.19, λ is a fuzzy strongly first category set in (X,T). Since (X,T) is a fuzzy strongly Baire space, by proposition 4.1, $cl(\lambda) = 1$ in (X,T) Thus $cl[\vee_{i=1}^{\infty}(\lambda_i)] = 1$, where $1-\lambda_i \in T$ and $int(\lambda_i) = 0$.

(ii) Let $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $(\lambda_i)'s$ are fuzzy simply open sets in (X,T). Then, by proposition 3.22, λ is a fuzzy strongly first category set in (X,T). Since (X,T) is a fuzzy strongly Baire space, by proposition 4.1, $cl(\lambda) = 1$ in (X,T). Thus $cl[\vee_{i=1}^{\infty}(\lambda_i)] = 1$, where $(\lambda_i)'s$ are fuzzy simply open sets in a fuzzy strongly Baire space (X,T).

(iii) Let $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $(\lambda_i)'s$ are the fuzzy sets on (X) with $int[bd(\lambda_i)] = 0$ in (X,T). Then by proposition 3.23, λ is a fuzzy strongly first category set in (X,T). Since (X,T) is a fuzzy strongly Baire space, by proposition 4.1, $cl(\lambda) = 1$ in (X,T). Thus $cl[\vee_{i=1}^{\infty}(\lambda_i)] = 1$, where $int[bd(\lambda_i)] = 0$ in (X,T). □

Proposition 4.3. *If each fuzzy open set is a fuzzy dense set in a fuzzy topological space (X,T), then (X,T) is a fuzzy strongly Baire space.*

Proof. Let $(\lambda_i)'s$ be fuzzy open sets in (X,T). By hypothesis, $(\lambda_i)'s$ are fuzzy dense sets in (X,T). Thus $(\lambda_i)'s$ are fuzzy open and fuzzy dense sets in (X,T). Then by proposition 3.11, $(\lambda_i)'s$ are fuzzy strongly nowhere dense sets in (X,T). Let $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$. Then λ is a fuzzy strongly first category set in (X,T). Now $cl(\lambda) = cl[\vee_{i=1}^{\infty}(\lambda_i)] \geqslant \vee_{i=1}^{\infty} cl(\lambda_i) = \vee_{i=1}^{\infty}(1) = 1$. That is., $cl(\lambda) = 1$ in (X,T). Then, by proposition 4.1, (X,T) is a fuzzy strongly Baire space. □

Proposition 4.4. *If (X,T) is a fuzzy hyperconnected space, then (X,T) is a fuzzy strongly Baire space.*

Proof. Let (X,T) be a fuzzy hyperconnected space. Then each fuzzy open set is a fuzzy dense set in (X,T). Then, by proposition 4.3, (X,T) is a fuzzy strongly Baire space. □

Proposition 4.5. *If (X,T) is a fuzzy strongly Baire and fuzzy strongly irresolvable space, then*
(i). $cl[\vee_{i=1}^{\infty}(\lambda_i)] = 1$, where $int(\lambda_i) = 0$, in (X,T).
(ii). $int[\wedge_{i=1}^{\infty}(\mu_i)] = 0$, where $cl(\mu_i) = 1$, in (X,T).

Proof. Let (X,T) be a fuzzy strongly Baire and fuzzy strongly irresolvable space.

(i). Suppose that $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $int(\lambda_i) = 0$, in (X,T). Since (X,T) is a fuzzy strongly irresolvable space, by proposition 3.17, λ is a fuzzy strongly first category set in (X,T). Also since (X,T) is a fuzzy strongly Baire space, by proposition 4.1, $cl(\lambda) = 1$ in (X,T). Hence $cl[\vee_{i=1}^{\infty}(\lambda_i)] = 1$, where $int(\lambda_i) = 0$, in (X,T).

(ii). Suppose that $\mu = \wedge_{i=1}^{\infty}(\mu_i)$, where $cl(\mu_i) = 1$, in (X,T). Since (X,T) is a fuzzy strongly irresolvable space, by proposition 3.18, μ is a fuzzy strongly residual set in (X,T). Since (X,T) is a fuzzy strongly Baire space, by proposition 4.1, $int(\mu) = 0$ and hence $int[\wedge_{i=1}^{\infty}(\mu_i)] = 0$, where $cl(\mu_i) = 1$, in (X,T). □

Proposition 4.6. *If $(\lambda_i)'s(i = 1 to \infty)$ are fuzzy simply open sets in a fuzzy strongly Baire space, then $(\lambda_i)'s$ are not fuzzy dense sets in (X,T).*

Proof. Let $(\lambda_i)'s$ $(i = 1 to \infty)$ be fuzzy simply open sets in (X,T). Suppose that $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$. Then by proposition 3.22, λ is a fuzzy strongly first category set in (X,T). Since (X,T) is a fuzzy strongly Baire space, by proposition 4.1, $cl(\lambda) = 1$ in (X,T). Then $cl[\vee_{i=1}^{\infty}(\lambda_i)] = 1$ in (X,T). But $\vee_{i=1}^{\infty} cl(\lambda_i) < cl[\vee_{i=1}^{\infty}(\lambda_i)]$ implies that $\vee_{i=1}^{\infty} cl(\lambda_i) \neq 1$ in (X,T) and hence $cl(\lambda_i) \neq 1$. Thus the fuzzy simply open sets $(\lambda_i)'s$ are not fuzzy dense sets in (X,T). □

Proposition 4.7. *If λ is a fuzzy first category set in a fuzzy strongly Baire space, then λ is a fuzzy dense set in (X,T).*

Proof. Let λ be a fuzzy first category set in (X,T). By proposition 3.27, there exists a fuzzy strongly first category set μ in (X,T) such that $cl(\lambda) \geqslant \mu$. Then $cl[cl(\lambda)] \geqslant cl(\mu)$. Since (X,T) is a fuzzy strongly Baire space by proposition 4.1, for the fuzzy strongly first category set μ in (X,T), $cl(\mu) = 1$. Then $cl(\lambda) \geqslant 1$. That is, $cl(\lambda) = 1$. Hence λ is a fuzzy dense set in (X,T). □

Proposition 4.8. *If δ is a fuzzy residual set in a fuzzy strongly Baire space, then $int(\delta) = 0$ in (X,T).*

Proof. Let δ be a fuzzy residual set in (X,T). Then $1-\delta$ is a fuzzy first category set in (X,T). Since (X,T) is a fuzzy strongly Baire space, by proposition 4.7, $cl(1-\delta) = 1$ in (X,T). Then $1 - int(\delta) = 1$ and hence $int(\delta) = 0$ in (X,T). □

Proposition 4.9. *If $int[\wedge_{i=1}^{\infty}(\mu_i)] = 0$, where $(\mu_i)'s$ are fuzzy dense sets in a fuzzy submaximal space (X,T), then (X,T) is a fuzzy strongly Baire space.*

PROOF. Let $\mu = \wedge_{i=1}^{\infty}(\mu_i)$, where $cl(\mu_i) = 1$ in (X,T). Since (X,T) is a fuzzy submaximal space, by proposition 3.26, μ is a fuzzy strongly residual set in (X,T). Now $int(\mu) = int[\wedge_{i=1}^{\infty}(\mu_i)] = 0$, implies, by proposition 4.1, that (X,T) is a fuzzy strongly Baire space. □

REMARK 4.1. A fuzzy strongly Baire space need not be fuzzy Baire space. For, consider the following example.

EXAMPLE 4.2. Let $X = \{a, b, c\}$. The fuzzy sets λ, μ and γ are defined on X as follows:

$\lambda : X \to [0,1]$ defined as $\lambda(a) = 0.4; \lambda(b) = 0.5; \lambda(c) = 0.7$.

$\mu : X \to [0,1]$ defined as $\mu(a) = 0.7; \mu(b) = 0.6; \mu(c) = 0.5$.

$\gamma : X \to [0,1]$ defined as $\gamma(a) = 0.3; \gamma(b) = 0.4; \gamma(c) = 0.4$.

Then, $T = \{0, \lambda, \mu, \lambda \vee \mu, \lambda \wedge \mu, 1\}$ is a fuzzy topology on X. On computation, $int(1-\lambda) = 0, int(1-\mu) = 0, int[1-(\lambda \vee \mu)] = 0 int[1-(\lambda \wedge \mu)] = \lambda \wedge \mu$ and $int(\gamma) = 0$. Also, $cl(\lambda) = 1, cl(\mu) = 1, cl(\lambda \vee \mu) = 1$ and $cl(\lambda \wedge \mu) = 1-(\lambda \wedge \mu)$ $cl(\gamma) = 1-\mu$. The fuzzy nowhere dense sets in (X,T) are $1-\lambda, 1-\mu, 1-(\lambda \vee \mu)$ and γ and $int[(1-\lambda) \vee (1-\mu) \vee [1-(\lambda \vee \mu)] \vee \gamma] = \lambda \wedge \mu \neq 0$. Hence (X,T) is not a fuzzy Baire space.

The fuzzy strongly nowhere dense sets in (X,T) are $\lambda, 1-\lambda, \mu, 1-\mu, \lambda \vee \mu, 1-(\lambda \vee \mu), \gamma, 1-\gamma$ and $cl[\lambda \vee \mu \vee (\lambda \vee \mu) \vee \gamma \vee (1-\lambda) \vee (1-\mu) \vee \{1-(\lambda \vee \mu)\} \vee (1-\gamma)] = cl(\lambda \vee \mu) = 1$. Hence (X,T) is a fuzzy strongly Baire space.

PROPOSITION 4.10. *If λ is a fuzzy first category set in a fuzzy topological space (X,T) then there exists a fuzzy strongly first category set μ in (X,T) such that $\lambda \leqslant \mu$.*

PROOF. Let λ be a fuzzy first category set in (X,T). Then $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $(\lambda_i)'s$ are fuzzy nowhere dense sets in (X,T). By proposition 3.6, $(1-\lambda_i)'s$ are fuzzy strongly nowhere dense sets in (X,T). Also by proposition 3.1, $(\lambda_i)'s$ are fuzzy strongly nowhere dense sets in (X,T). Then $\{(\lambda_i) \vee (1-\lambda_i)\}$ are fuzzy strongly nowhere dense sets in (X,T).

Now $\lambda_i \leqslant (\lambda_i) \vee (1-\lambda_i)$ implies that

$$\vee_{i=1}^{\infty}(\lambda_i) \leqslant \vee_{i=1}^{\infty}[\lambda_i \vee (1-\lambda_i)] \text{——(A)}$$

Let $\mu = \vee_{i=1}^{\infty}[\lambda_i \vee (1-\lambda_i)]$, then μ is a fuzzy strongly first category set in (X,T). Then from (A), we have $\lambda \leqslant \mu$. □

PROPOSITION 4.11. *If δ is a fuzzy residual set in a fuzzy topological space (X,T) then there exists a fuzzy strongly residual set η in (X,T) such that $\eta \leqslant \delta$.*

PROOF. Let δ be a fuzzy residual set in (X,T). Then $1-\delta$ is a fuzzy first category set in (X,T). Then, by proposition 4.10, there exists a fuzzy strongly first category set μ in (X,T) such that $1-\delta \leqslant \mu$. Then $1-\mu \leqslant \delta$. Let $1-\mu = \eta$ and thus η is a fuzzy strongly residual set in (X,T). Hence $\eta \leqslant \delta$ in (X,T). □

PROPOSITION 4.12. *If $int(\mu) = 0$ for each fuzzy strongly first category set μ in a fuzzy topological space (X,T), then (X,T) is a fuzzy Baire space.*

Proof. Let λ be a fuzzy first category set in (X, T). Then by proposition 4.10, there exists a fuzzy strongly first category set μ in (X, T) such that $\lambda \leqslant \mu$. Then $int(\lambda) \leqslant int(\mu)$ in (X, T). By hypothesis $int(\mu) = 0$ in (X, T) and thus $int(\lambda) = 0$ in (X, T). That is, $int(\lambda) = 0$ in (X, T). Thus, by theorem 2.1, (X, T) is a fuzzy Baire space. □

Proposition 4.13. *If $cl(\eta) = 1$, for each fuzzy strongly residual set η in a fuzzy topological space (X, T), then (X, T) is a fuzzy Baire space.*

Proof. Let δ be a fuzzy residual set in (X, T). Then by proposition 4.11, there exists a fuzzy strongly residual set η in (X, T) such that $\eta \leqslant \delta$. Then $cl(\eta) \leqslant cl(\delta)$ in (X, T). By hypothesis $cl(\eta) = 1$ in (X, T) and thus $1 \leqslant cl(\delta)$. That is, $cl(\delta) = 1$ in (X, T). Hence, by theorem 2.1, (X, T) is a fuzzy Baire space. □

Remark 4.2. From propositions 3.27 and 4.10, it is clear that if λ is a fuzzy first category set in a fuzzy topological space there are fuzzy strongly first category sets $(\mu_i)'s$ in (X, T) such that $\lambda \leqslant \mu_i \leqslant cl(\lambda)$.

Proposition 4.14. *If $int(\mu) = 0$ for each fuzzy strongly first category set μ in a fuzzy strongly Baire space (X, T), then (X, T) is a fuzzy Baire space.*

Proof. The proof follows from proposition 4.12. □

Proposition 4.15. *If $cl(\eta) = 1$, for each fuzzy strongly residual set η in a fuzzy strongly Baire space (X, T), then (X, T) is a fuzzy Baire space.*

Proof. The proof follows from proposition 4.13. □

Proposition 4.16. *If $cl(\lambda) = 1$, for a fuzzy first category set in a fuzzy topological space (X, T) then there exists a fuzzy strongly first category set μ in (X, T) such that $cl(\mu) = 1$.*

Proof. Let λ be a fuzzy first category set in (X, T). Then by proposition 4.10, there exists a fuzzy strongly first category set μ in (X, T) such that $\lambda \leqslant \mu$. Then $cl(\lambda) \leqslant cl(\mu)$, in (X, T). By hypothesis, $cl(\lambda) = 1$ in (X, T) and hence $1 \leqslant cl(\mu)$. That is, $cl(\mu) = 1$ in (X, T). □

Proposition 4.17. *If each fuzzy first category set λ is a fuzzy dense set in a fuzzy Baire space (X, T) then (X, T) is a fuzzy strongly Baire space.*

Proof. Let λ be a fuzzy first category set in a fuzzy Baire space (X, T) such that $cl(\lambda) = 1$. Then by proposition 4.16, there exists a fuzzy strongly first category set μ in (X, T) such that $cl(\mu) = 1$. Hence, by proposition 4.1, (X, T) is a fuzzy strongly Baire space. □

Proposition 4.18. *If $int(\delta) = 0$ for a fuzzy residual set in a fuzzy topological space (X, T), then there exists a fuzzy strongly residual set η in (X, T) such that $int(\eta) = 0$.*

Proof. The proof follows from proposition 4.11. □

PROPOSITION 4.19. *If* $int(\delta) = 0$ *for a fuzzy residual set in a fuzzy Baire space* (X, T) *then* (X, T) *is a fuzzy strongly Baire space.*

PROOF. The proof follows from proposition 4.18. □

REMARK 4.3. A fuzzy strongly Baire space need not be a fuzzy weakly Baire space. For, consider the following example.

EXAMPLE 4.3. Let $X = \{a, b, c\}$. The fuzzy sets λ, μ and γ are defined on X as follows:

$\lambda : X \to [0, 1]$ defined as $\lambda(a) = 0.3; \lambda(b) = 0.6; \lambda(c) = 0.5$.

$\mu : X \to [0, 1]$ defined as $\mu(a) = 0.5; \mu(b) = 0.4; \mu(c) = 0.7$.

$\gamma : X \to [0, 1]$ defined as $\gamma(a) = 0.4; \gamma(b) = 0.5; \gamma(c) = 0.2$.

Then, $T = \{0, \lambda, \mu, \lambda \vee \mu, \lambda \wedge \mu, 1\}$ is a fuzzy topology on X. On computation, $int(1-\lambda) = \lambda \wedge \mu, int(1-\mu) = 0, int[1-(\lambda \vee \mu)] = 0, int[1-(\lambda \wedge \mu)] = \lambda, int(\gamma) = 0, int(1-\gamma) = \mu, cl(\lambda) = 1-(\lambda \wedge \mu), cl(\mu) = 1, cl(\lambda \wedge \mu) = 1-\lambda, cl(\lambda \vee \mu) = 1 and cl(\gamma) = 1-\mu$. The fuzzy nowhere dense sets in (X, T) are $1-\mu, 1-(\lambda \vee \mu)$ and γ. Now $int[(1-\mu) \vee [1-(\lambda \vee \mu)] \vee \gamma] = int(1-\mu) = 0$ and hence (X, T) is a fuzzy Baire space.

On computation, we have

$int\{cl[\lambda \wedge (1-\lambda)]\} = int(1-\lambda) = \lambda \wedge \mu \neq 0; int\{cl[\mu \wedge (1-\mu)]\} = int[1-(\lambda \vee \mu)] = 0$ $int\{cl[(\lambda \vee \mu) \wedge \{1-(\lambda \vee \mu)\}]\} = int[1-(\lambda \vee \mu)] = 0; int\{cl[(\lambda \wedge \mu) \wedge \{1-(\lambda \wedge \mu)\}]\} = int(1-\lambda) = \lambda \wedge \mu \neq 0; int\{cl[\gamma \wedge (1-\gamma)]\} = int(1-\mu) = 0$.

The fuzzy strongly nowhere dense sets in (X, T) are $\mu, \lambda \vee \mu, \gamma, 1-\mu, 1-(\lambda \vee \mu), 1-\gamma$ and $cl[(\mu) \vee (\lambda \vee \mu) \vee (\gamma) \vee (1-\mu) \vee \{1-(\lambda \vee \mu)\} \vee (1-\gamma)] = 1$. Hence (X, T) is a fuzzy strongly Baire space.

The fuzzy regular open sets in (X, T) are λ and $\lambda \wedge \mu$. Let $\eta_1 = cl(\lambda) \wedge (1-\lambda)$. Then $\eta_1 = [1-(\lambda \wedge \mu)] \wedge (1-\lambda)$. Let $\eta_2 = cl(\lambda \wedge \mu) \wedge [1-(\lambda \wedge \mu)]$. Then $\eta_2 = (1-\lambda) \wedge [1-(\lambda \wedge \mu)]$. Now, $int(\eta_1 \vee \eta_2) = int\{(1-\lambda) \wedge [1-(\lambda \wedge \mu)]\} = int(1-\lambda) = \lambda \wedge \mu \neq 0$. Hence, (X, T) is not a fuzzy weakly Baire space.

THEOREM 4.1 (11). *If* (X, T) *is a fuzzy weakly Baire and fuzzy open hereditarily irresolvable space, then* (X, T) *is a fuzzy Baire space.*

PROPOSITION 4.20. *If each fuzzy first category set is a fuzzy dense set in a fuzzy weakly Baire and fuzzy open hereditarily irresolvable space* (X, T), *then* (X, T) *is a fuzzy strongly Baire space.*

PROOF. Let λ be a fuzzy first category set in (X, T) such that $cl(\lambda) = 1$. Since (X, T) is a fuzzy weakly Baire and fuzzy open hereditarily irresolvable space, by theorem 4.1, (X, T) is a fuzzy Baire space. Thus $cl(\lambda) = 1$, for the fuzzy first category set λ in the fuzzy Baire space (X, T). Then, by proposition 4.17, (X, T) is a fuzzy strongly Baire space. □

PROPOSITION 4.21. *If* $int(\delta) = 0$ *for each fuzzy residual set* δ *in a fuzzy weakly Baire and fuzzy open hereditarily irresolvable space* (X, T), *then* (X, T) *is a fuzzy strongly Baire space.*

PROOF. The proof follows from theorem 4.1 and proposition 4.19. □

References

[1] K. K. Azad. On fuzzy semi-continuity, fuzzy almost continuity and fuzzy weakly continuity. *J. Math. Anal. Appl.*, **82**(1)(1981), 14–32.

[2] R. L. Baire. Sur les fonctions de variables reelles. *Annali di Matematica Pura ed Applicata*, **3**(1)(1899), 1–123.

[3] G. Balasubramanian. Maximal fuzzy topologies. *Kybernetika*, **31**(5)(1995), 459–464.

[4] N. Bourbaki. *Topologie Generale.* Act. Sci. Ind. no. 1045 Paris, (1948).

[5] M. Caldas, G. Navalgi and R. Saref. On fuzzy weakly semi-open function. *Proyecciones*, **21**(1)(2002), 51–63.

[6] C. L. Chang. Fuzzy topological spaces. *J. Math. Anal. Appl.*, **24**(1)(1968), 182–190.

[7] P. M. Pu and Y. M Liu. Fuzzy topology - I: Neighborhood structure of a fuzzy point and Moore - Smith convergence. *J. Math. Anal. Appl.*, **76**(2)(1980), 571–599.

[8] G.Thangaraj and G.Balasubramanian. On somewhat fuzzy continuous functions. *The Journal of Fuzzy Mathematics*, **11**(3)(2003), 725–736.

[9] G.Thangaraj and G.Balasubramanian. On fuzzy resolvable and fuzzy irresolvable spaces. *Fuzzy Sets, Rough sets and Multivaried Operations and Appl.*, **1**(2)(2009), 173–180.

[10] G.Thangaraj and V.Seenivasan. On fuzzy strongly irresolvable spaces. *Proc. Nat. Conf. on Fuzzy Math. and Graph Theory* (pp. 1-6), Jamal Mohamed College, Trichy, Tamilnadu, India,(2008).

[11] G.Thangaraj and S.Anjalmose. On fuzzy Baire spaces. *The Journal of Fuzzy Mathematics*, **21**(3)(2013), 667–676.

[12] G.Thangaraj and K.Dinakaran. On fuzzy simply continuous functions. *The Journal of Fuzzy Mathematics*, **25**(1)(2017), 99-124.

[13] G.Thangaraj and R.Palani. On fuzzy weakly Baire spaces. *Bull. Int. Math. Virtual Inst.*, **7**(3)(2017), 479-489.

[14] G.Thangaraj and E.Poongothai. On fuzzy σ-Baire space. *Inter. J. Fuzzy Math. Sys.*, **3**(4)(2013), 275–283.

[15] L. A. Zadeh. Fuzzy sets. *Inform. and Control*, **8**(3)(1965), 338–353.

Receibed by editors 12.06.2017; Revised version 04.09.2017; Available online 11.09.2017.

G. THANGARAJ: DEPARTMENT OF MATHEMATICS, THIRUVALLUVAR UNIVERSITY, SERKKADU, VELLORE - 632115, TAMIL NADU, INDIA

E-mail address: g.thangaraj@rediffmail.com

R. PALANI: DEPARTMENT OF MATHEMATICS, THIRUVALLUVAR UNIVERSITY, SERKKADU, VELLORE - 632115, TAMIL NADU, INDIA.

E-mail address: palani7010@gmail.com