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NEUTROSOPHIC IDEALS IN BCK/BCI-ALGEBRAS BASED ON NEUTROSOPHIC POINTS

Mehmet Ali Öztürk and Young Bae Jun

ABSTRACT. Given $\Phi, \Psi \in \{\in, q, \in\vee q\}$, the notions of ($\Phi$, Ψ)-neutrosophic ideals of a BCK/BCI-algebra are introduced, and related properties are investigated. Characterizations of an $(\in, \in\vee q)$-neutrosophic ideal are provided. Given special sets, so called neutrosophic $\in$-subsets, conditions for the neutrosophic $\in$-subsets to be ideals are discussed. Conditions for a neutrosophic set to be a $(q, \in\vee q)$-neutrosophic ideal are considered.

1. Introduction

The concept of neutrosophic set (NS) developed by Smarandache [5, 6] is a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. Neutrosophic set theory is applied to various part which is refered to the site

http://fs.gallup.unm.edu/neutrosophy.htm.

Jun [3] introduced the notion of neutrosophic subalgebras in BCK/BCI-algebras with several types. He provided characterizations of an $(\in, \in)$-neutrosophic subalgebra and an $(\in, \in\vee q)$-neutrosophic subalgebra. Given special sets, so called neutrosophic $\in$-subsets, neutrosophic q-subsets and neutrosophic $\in\vee q$-subsets, he considered conditions for the neutrosophic $\in$-subsets, neutrosophic q-subsets and neutrosophic $\in\vee q$-subsets to be subalgebras. He discussed conditions for a neutrosophic set to be a $(q, \in\vee q)$-neutrosophic subalgebra. In [1], Borumand Saeid and Jun provided relations between an $(\in, \in\vee q)$-neutrosophic subalgebra and a $(q, \in\vee q)$-neutrosophic subalgebra. They discussed characterization of an $(\in,$

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$\in \vee q$)-neutrosophic subalgebra by using neutrosophic $\in$-subsets, and considered conditions for an $(\in, \in \vee q)$-neutrosophic subalgebra to be a $(q, \in \vee q)$-neutrosophic subalgebra.

In this paper, we introduce the notion of (Φ, Ψ)-neutrosophic ideal of a BCK/BCI-algebra X for $\Phi, \Psi \in \{\in, q, \in \vee q\}$, and investigate related properties. We consider characterizations of an $(\in, \in \vee q)$-neutrosophic ideal. We provide conditions for an $(\in, \in \vee q)$-neutrosophic ideal to be an $(\in, \in)$-neutrosophic ideal. We consider conditions for a neutrosophic set to be a $(q, \in \vee q)$-neutrosophic ideal and an $(\in, \in \vee q)$-neutrosophic ideal. We show that every $(\in \vee q, \in \vee q)$-neutrosophic ideal is an $(\in, \in \vee q)$-neutrosophic ideal, and every $(q, \in \vee q)$-neutrosophic ideal is an $(\in, \in \vee q)$-neutrosophic ideal.

2. Preliminaries

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the axioms:

(a1) $((x * y) * (x * z)) * (z * y) = 0$,
(a2) $(x * (x * y)) * y = 0$,
(a3) $x * x = 0$,
(a4) $x * y = y * x = 0 \;\Rightarrow\; x = y$,

for all $x, y, z \in X$. If a BCI-algebra X satisfies the axiom

(a5) $0 * x = 0$ for all $x \in X$,

then we say that X is a *BCK-algebra*. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

$$0 \in I, \tag{2.1}$$

$$(\forall x, y \in X)\,(x * y \in I,\; y \in I \;\Rightarrow\; x \in I)\,. \tag{2.2}$$

We refer the reader to the books [2] and [4] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

If $\Lambda = \{1, 2\}$, we will also use $a_1 \vee a_2$ and $a_1 \wedge a_2$ instead of $\bigvee\{a_i \mid i \in \Lambda\}$ and $\bigwedge\{a_i \mid i \in \Lambda\}$, respectively.

Let X be a non-empty set. A neutrosophic set (NS) in X (see [5]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, we consider the following sets:

$$\begin{aligned}
&T_\in(A;\alpha) := \{x \in X \mid A_T(x) \geqslant \alpha\},\\
&I_\in(A;\beta) := \{x \in X \mid A_I(x) \geqslant \beta\},\\
&F_\in(A;\gamma) := \{x \in X \mid A_F(x) \leqslant \gamma\},\\
&T_q(A;\alpha) := \{x \in X \mid A_T(x) + \alpha > 1\},\\
&I_q(A;\beta) := \{x \in X \mid A_I(x) + \beta > 1\},\\
&F_q(A;\gamma) := \{x \in X \mid A_F(x) + \gamma < 1\},\\
&T_{\in\vee\, q}(A;\alpha) := \{x \in X \mid A_T(x) \geqslant \alpha \text{ or } A_T(x) + \alpha > 1\},\\
&I_{\in\vee\, q}(A;\beta) := \{x \in X \mid A_I(x) \geqslant \beta \text{ or } A_I(x) + \beta > 1\},\\
&F_{\in\vee\, q}(A;\gamma) := \{x \in X \mid A_F(x) \leqslant \gamma \text{ or } A_F(x) + \gamma < 1\}.
\end{aligned}$$

We say $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are *neutrosophic $\in$-subsets*; $T_q(A;\alpha)$, $I_q(A;\beta)$ and $F_q(A;\gamma)$ are *neutrosophic q-subsets*; and $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are *neutrosophic $\in \vee\, q$-subsets*. For $\Phi \in \{\in, q, \in \vee\, q\}$, the element of $T_\Phi(A;\alpha)$ (resp., $I_\Phi(A;\beta)$ and $F_\Phi(A;\gamma)$) is called a *neutrosophic T_Φ-point* (resp., *neutrosophic I_Φ-point* and *neutrosophic F_Φ-point*) with value α (resp., β and γ) (see [3]).

It is clear that

$$T_{\in\vee\, q}(A;\alpha) = T_\in(A;\alpha) \cup T_q(A;\alpha), \tag{2.3}$$

$$I_{\in\vee\, q}(A;\beta) = I_\in(A;\beta) \cup I_q(A;\beta), \tag{2.4}$$

$$F_{\in\vee\, q}(A;\gamma) = F_\in(A;\gamma) \cup F_q(A;\gamma). \tag{2.5}$$

3. Neutrosophic ideals

In what follows, let X be a BCK/BCI-algebra unless otherwise specified.

THEOREM 3.1. *For a neutrosophic set $A = (A_T, A_I, A_F)$ in X, the following are equivalent.*

(1) *The nonempty neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are ideals of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$.*

(2) *$A = (A_T, A_I, A_F)$ satisfies the following assertions:*

$$(\forall x \in X)\begin{pmatrix} A_T(0) \vee 0.5 \geqslant A_T(x) \\ A_I(0) \vee 0.5 \geqslant A_I(x) \\ A_F(0) \wedge 0.5 \leqslant A_F(x) \end{pmatrix} \tag{3.1}$$

and

$$(\forall x, y \in X) \left(\begin{array}{l} A_T(x) \vee 0.5 \geqslant A_T(x * y) \wedge A_T(y) \\ A_I(x) \vee 0.5 \geqslant A_I(x * y) \wedge A_I(y) \\ A_F(x) \wedge 0.5 \leqslant A_F(x * y) \vee A_F(y) \end{array} \right). \tag{3.2}$$

Proof. Assume that the nonempty neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are ideals of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$. If there exist $a, b \in X$ such that

$$A_T(0) \vee 0.5 < A_T(a) \text{ and } A_I(0) \vee 0.5 < A_I(b), \tag{3.3}$$

respectively, then $\alpha_a := A_T(a) \in (0.5, 1]$ and $\beta_b := A_I(b) \in (0.5, 1]$, and thus $a \in T_\in(A;\alpha_a)$ and $b \in I_\in(A;\beta_b)$. But (3.3) induces $A_T(0) < A_T(a)$ and $A_I(0) < A_I(b)$, which imply that $0 \notin T_\in(A;\alpha_a)$ and $0 \notin I_\in(A;\beta_b)$. This is a contradiction, and so we get $A_T(0) \vee 0.5 \geqslant A_T(x)$ and $A_I(0) \vee 0.5 \geqslant A_I(x)$ for all $x \in X$. If $A_F(0) \wedge 0.5 > A_F(x)$ for some $x \in X$, then $A_F(x) \in [0, 0.5)$. Since $F_\in(A; A_F(x))$ is an ideal of X, we have $0 \in F_\in(A; A_F(x))$ and so $A_F(0) \leqslant A_F(x)$. This is a contradiction, and so $A_F(0) \wedge 0.5 \leqslant A_F(x)$ for all $x \in X$. Suppose that $A_T(x) \vee 0.5 < A_T(x*y) \wedge A_T(y)$ for some $x, y \in X$ and take $\alpha = A_T(x*y) \wedge A_T(y)$. Then $\alpha \in (0.5, 1]$ and $y, x*y \in T_\in(A;\alpha)$. But $x \notin T_\in(A;\alpha)$ since $A_T(x) < \alpha$, a contradiction. If $A_I(a) \vee 0.5 < A_I(a*b) \wedge A_I(b)$ for some $a, b \in X$, then $a*b, b \in I_\in(A;\beta)$ and $a \notin I_\in(A;\beta)$ where $\beta = A_I(a*b) \wedge A_I(b)$. This is a contradiction. Assume that there exist $x, y \in X$ such that $A_F(x) \wedge 0.5 > A_F(x*y) \vee A_F(y) := \gamma$. Then $\gamma \in [0, 0.5)$, $x*y \in F_\in(A;\gamma)$ and $y \in F_\in(A;\gamma)$, but $x \notin F_\in(A;\gamma)$. This is a contradiction. Consequently, $A = (A_T, A_I, A_F)$ satisfies the assertions (3.1) and (3.2).

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X satisfying the conditions (3.1) and (3.2). Let $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$ be such that $T_\in(A;\alpha) \neq \emptyset$, $I_\in(A;\beta) \neq \emptyset$ and $F_\in(A;\gamma) \neq \emptyset$. For any $x \in T_\in(A;\alpha)$, $y \in I_\in(A;\beta)$ and $z \in F_\in(A;\gamma)$, we have

$$\begin{aligned} &A_T(0) \vee 0.5 \geqslant A_T(x) \geqslant \alpha > 0.5, \\ &A_I(0) \vee 0.5 \geqslant A_I(y) \geqslant \beta > 0.5, \\ &A_F(0) \wedge 0.5 \leqslant A_F(z) \leqslant \gamma < 0.5, \end{aligned}$$

and thus $A_T(0) \geqslant \alpha$, $A_I(0) \geqslant \beta$ and $A_F(0) \leqslant \gamma$. Therefore $0 \in T_\in(A;\alpha)$, $0 \in I_\in(A;\beta)$ and $0 \in F_\in(A;\gamma)$. Let $x, y, a, b, u, v \in X$ be such that $x*y \in T_\in(A;\alpha)$, $y \in T_\in(A;\alpha)$, $a*b \in I_\in(A;\beta)$, $b \in I_\in(A;\beta)$, $u*v \in F_\in(A;\gamma)$ and $v \in F_\in(A;\gamma)$. It follows from (3.2) that

$$\begin{aligned} &A_T(x) \vee 0.5 \geqslant A_T(x*y) \wedge A_T(y) \geqslant \alpha > 0.5, \\ &A_I(a) \vee 0.5 \geqslant A_I(a*b) \wedge A_I(b) \geqslant \beta > 0.5, \\ &A_F(u) \wedge 0.5 \leqslant A_F(u*v) \vee A_F(v) \leqslant \gamma < 0.5. \end{aligned}$$

and so that $A_T(x) \geqslant \alpha$, $A_I(a) \geqslant \beta$ and $A_F(u) \leqslant \gamma$, that is, $x \in T_\in(A;\alpha)$, $a \in I_\in(A;\beta)$ and $u \in F_\in(A;\gamma)$. Therefore the nonempty neutrosophic $\in$-subsets

$T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are ideals of X for all $\alpha,\beta \in (0.5,1]$ and $\gamma \in [0,0.5)$. □

Definition 3.2. Given $\Phi,\Psi \in \{\in, q, \in\vee q\}$, a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called a (Φ,Ψ)-*neutrosophic ideal* of X if the following assertions are valid.

$$(\forall x \in X)\left(\begin{array}{l} x \in T_\Phi(A;\alpha_x) \;\Rightarrow\; 0 \in T_\Psi(A;\alpha_x) \\ x \in I_\Phi(A;\beta_x) \;\Rightarrow\; 0 \in I_\Psi(A;\beta_x) \\ x \in F_\Phi(A;\gamma_x) \;\Rightarrow\; 0 \in F_\Psi(A;\gamma_x) \end{array}\right), \tag{3.4}$$

and

$$(\forall x,y \in X)\left(\begin{array}{l} x*y \in T_\Phi(A;\alpha_x),\; y \in T_\Phi(A;\alpha_y) \;\Rightarrow\; x \in T_\Psi(A;\alpha_x \wedge \alpha_y), \\ x*y \in I_\Phi(A;\beta_x),\; y \in I_\Phi(A;\beta_y) \;\Rightarrow\; x \in I_\Psi(A;\beta_x \wedge \beta_y), \\ x*y \in F_\Phi(A;\gamma_x),\; y \in F_\Phi(A;\gamma_y) \;\Rightarrow\; x \in F_\Psi(A;\gamma_x \vee \gamma_y) \end{array}\right) \tag{3.5}$$

for all $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

Theorem 3.3. *For a neutrosophic set $A = (A_T, A_I, A_F)$ in X, the following are equivalent.*

(1) *$A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q)$-neutrosophic ideal of X.*
(2) *$A = (A_T, A_I, A_F)$ satisfies the following assertions:*

$$(\forall x \in X)\left(\begin{array}{l} A_T(0) \geqslant A_T(x) \wedge 0.5 \\ A_I(0) \geqslant A_I(x) \wedge 0.5 \\ A_F(0) \leqslant A_F(x) \vee 0.5 \end{array}\right) \tag{3.6}$$

and

$$(\forall x,y \in X)\left(\begin{array}{l} A_T(x) \geqslant \bigwedge\{A_T(x*y), A_T(y), 0.5\} \\ A_I(x) \geqslant \bigwedge\{A_I(x*y), A_I(y), 0.5\} \\ A_F(x) \leqslant \bigvee\{A_F(x*y), A_F(y), 0.5\} \end{array}\right). \tag{3.7}$$

Proof. Suppose that $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q)$-neutrosophic ideal of X. Let $x \in X$ and assume that $A_T(x) < 0.5$. If $A_T(0) < A_T(x)$, then $A_T(0) < \alpha_x < A_T(x)$ for some $\alpha_x \in (0,0.5)$. It follows that $x \in T_\in(A;\alpha_x)$ and $0 \notin T_\in(A;\alpha_x)$. Also, $A_T(0) + \alpha_x < 1$, that is, $0 \notin T_q(A;\alpha_x)$. Hence $0 \notin T_{\in\vee q}(A;\alpha_x)$ which is a contradiction, and so $A_T(0) \geqslant A_T(x)$ for all $x \in X$. Now if $A_T(x) \geqslant 0.5$, then $x \in T_\in(A;0.5)$ and thus $0 \in T_{\in\vee q}(A;0.5)$. If $A_T(0) < 0.5$, then $A_T(0)+0.5 < 1$, that is, $0 \notin T_q(A;0.5)$. This is a contradiction, and thus $A_T(0) \geqslant 0.5$. Consequently, $A_T(0) \geqslant A_T(x) \wedge 0.5$ for all $x \in X$. Similarly, we know that $A_I(0) \geqslant A_I(x) \wedge 0.5$ for all $x \in X$. Assume that there exists $z \in X$ such that $A_F(0) > A_F(z) \vee 0.5$. Then $A_F(0) > \gamma_z \geqslant A_F(z) \vee 0.5$ for some $\gamma_z \in (0,1)$, which implies that $\gamma_z \geqslant 0.5$, $z \in F_\in(A;\gamma_z)$ and $0 \notin F_\in(A;\gamma_z)$. Since $A_F(0) + \gamma_z \geqslant 1$, we have $0 \notin F_q(A;\gamma_z)$. This is impossible, and so $A_F(0) \leqslant A_F(x) \vee 0.5$ for all $x \in X$. Suppose that there exist $a, b \in X$ such that $A_T(a) < \bigwedge\{A_T(a*b), A_T(b), 0.5\}$. Then

$$A_T(a) < \alpha \leqslant \bigwedge\{A_T(a*b), A_T(b), 0.5\}$$

for some $\alpha \in (0,1)$. It follows that $a*b \in T_{\in}(A;\alpha)$, $b \in T_{\in}(A;\alpha)$ and $a \notin T_{\in}(A;\alpha)$. Since $\alpha \leqslant 0.5$, we have $A_T(a)+\alpha < 2\alpha \leqslant 1$ and so $a \notin T_q(A;\alpha)$. This is a contradiction, and therefore $A_T(x) \geqslant \bigwedge\{A_T(x*y), A_T(y), 0.5\}$ for all $x, y \in X$. Let $x, y \in X$ and suppose that $A_I(x*y)\wedge A_I(y) < 0.5$. Then $A_I(x) \geqslant A_I(x*y)\wedge A_I(y)$. If not, then $A_I(x) < \beta < A_I(x*y)\wedge A_I(y)$ for some $\beta \in (0, 0.5)$. It follows that $x*y \in I_{\in}(A;\beta)$, $y \in I_{\in}(A;\beta)$ but $x \notin I_{\in\vee q}(A;\beta)$, a contradiction. Hence $A_I(x) \geqslant A_I(x*y)\wedge A_I(y)$ whenever $A_I(x*y)\wedge A_I(y) < 0.5$. If $A_I(x*y)\wedge A_I(y) \geqslant 0.5$, then $x*y \in I_{\in}(A;0.5)$ and $y \in I_{\in}(A;0.5)$, which implies that $x \in I_{\in\vee q}(A;0.5)$. Therefore $A_I(x) \geqslant 0.5$ because if $A_I(x) < 0.5$ then $A_I(x)+0.5 < 0.5+0.5 = 1$, a contradiction. Hence $A_I(x) \geqslant \bigwedge\{A_I(x*y), A_I(y), 0.5\}$ for all $x, y \in X$. Now suppose that $A_F(x) > \bigvee\{A_F(x*y), A_F(y), 0.5\}$ for some $x, y \in X$. Then there exists $\gamma \in (0,1)$ such that $A_F(x) > \gamma > \bigvee\{A_F(x*y), A_F(y), 0.5\}$. Thus $\gamma > 0.5$, $x*y \in F_{\in}(A;\gamma)$ and $y \in F_{\in}(A;\gamma)$. It follows from (3.5) that $x \in F_{\in\vee q}(A;\gamma)$. Since $A_F(x) > \gamma$ and $A_F(x)+\gamma > 2\gamma > 1$, we have $x \notin F_{\in\vee q}(A;\gamma)$ a contradiction. Therefore $A_F(x) \leqslant \bigvee\{A_F(x*y), A_F(y), 0.5\}$ for all $x, y \in X$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X satisfying the conditions (3.6) and (3.7). For any $x, y, z \in X$, let $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$ be such that $x \in T_{\in}(A;\alpha)$, $y \in I_{\in}(A;\beta)$ and $z \in F_{\in}(A;\gamma)$. Then $A_T(x) \geqslant \alpha$, $A_I(y) \geqslant \beta$ and $A_F(z) \leqslant \gamma$. Suppose that $A_T(0) < \alpha$, $A_I(0) < \beta$ and $A_F(0) > \gamma$. If $A_T(x) < 0.5$, then $A_T(0) \geqslant A_T(x)\wedge 0.5 = A_T(x) \geqslant \alpha$, a contradiction. Hence we know that $A_T(x) \geqslant 0.5$ and so

$$A_T(0)+\alpha > 2A_T(0) \geqslant 2\,(A_T(x)\wedge 0.5) = 1.$$

Hence $0 \in T_q(A;\alpha) \subseteq T_{\in\vee q}(A;\alpha)$. We can verify that $0 \in I_{\in\vee q}(A;\beta)$ by the similar way. If $A_F(x) > 0.5$, then $A_F(0) \leqslant A_F(x)\vee 0.5 = A_F(x) \leqslant \gamma$ which is a contradiction. Thus $A_F(x) \leqslant 0.5$ and so $A_F(0)+\gamma < 2A_F(0) \leqslant 2\,(A_F(x)\vee 0.5) = 1$. Hence $0 \in F_q(A;\gamma) \subseteq F_{\in\vee q}(A;\gamma)$. For any $x, y, a, b, u, v \in X$, let $\alpha_x, \alpha_y, \beta_a, \beta_b \in (0,1]$ and $\gamma_u, \gamma_v \in [0,1)$ be such that $x*y \in T_{\in}(A;\alpha_x)$, $y \in T_{\in}(A;\alpha_y)$, $a*b \in I_{\in}(A;\beta_a)$, $b \in I_{\in}(A;\beta_b)$, $u*v \in F_{\in}(A;\gamma_u)$ and $v \in F_{\in}(A;\gamma_v)$. Then $A_T(x*y) \geqslant \alpha_x$, $A_T(y) \geqslant \alpha_y$, $A_I(a*b) \geqslant \beta_a$, $A_I(b) \geqslant \beta_b$, $A_F(u*v) \leqslant \gamma_u$ and $A_F(v) \leqslant \gamma_v$. Suppose that $A_T(x) < \alpha_x \wedge \alpha_y$. If $A_T(x*y)\wedge A_T(y) < 0.5$, then

$$A_T(x) \geqslant \bigwedge\{A_T(x*y), A_T(y), 0.5\} = A_T(x*y)\wedge A_T(y) \geqslant \alpha_x \wedge \alpha_y$$

which is a contradiction. Hence $A_T(x*y)\wedge A_T(y) \geqslant 0.5$, and so

$$A_T(x)+(\alpha_x\wedge\alpha_y) > 2A_T(x) \geqslant 2\left(\bigwedge\{A_T(x*y), A_T(y), 0.5\}\right) = 1.$$

This induces $x \in T_q(A;\alpha_x\wedge\alpha_y) \subseteq T_{\in\vee q}(A;\alpha_x\wedge\alpha_y)$. Similarly, we have $a \in I_{\in\vee q}(A;\beta_a\wedge\beta_b)$. Assume that $A_F(u) > \gamma_u\vee\gamma_v$, that is, $u \notin F_{\in}(A;\gamma_u\vee\gamma_v)$. If $A_F(u*v)\vee A_F(v) > 0.5$, then

$$A_F(u) \leqslant \bigvee\{A_F(u*v), A_F(v), 0.5\} = A_F(u*v)\vee A_F(v) \leqslant \gamma_u\vee\gamma_v$$

which is a contradiction. Hence $A_F(u*v)\vee A_F(v) \leqslant 0.5$, and so

$$A_F(u)+(\gamma_u\vee\gamma_v) < 2A_F(u) \leqslant 2\left(\bigvee\{A_F(u*v), A_F(v), 0.5\}\right) = 1.$$

This induces $u \in F_q(A; \gamma_u \vee \gamma_v) \subseteq F_{\in\vee q}(A; \gamma_u \vee \gamma_v)$. Consequently, $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q)$-neutrosophic ideal of X. □

Proposition 3.4. *Every $(\in, \in\vee q)$-neutrosophic ideal $A = (A_T, A_I, A_F)$ of X satisfies:*

$$(3.8) \qquad (\forall x, y, z \in X) \left(x * y \leqslant z \;\Rightarrow \begin{cases} A_T(x) \geqslant \bigwedge\{A_T(y), A_T(z), 0.5\} \\ A_I(x) \geqslant \bigwedge\{A_I(y), A_I(z), 0.5\} \\ A_F(x) \leqslant \bigvee\{A_F(y), A_F(z), 0.5\} \end{cases} \right).$$

Proof. Let $x, y, z \in X$ be such that $x * y \leqslant z$. Then $(x * y) * z = 0$, which implies from (3.6) and (3.7) that

$$A_T(x * y) \geqslant \bigwedge\{A_T((x * y) * z), A_T(z), 0.5\} = \bigwedge\{A_T(0), A_T(z), 0.5\} \geqslant A_T(z) \wedge 0.5,$$
$$A_I(x * y) \geqslant \bigwedge\{A_I((x * y) * z), A_I(z), 0.5\} = \bigwedge\{A_I(0), A_I(z), 0.5\} \geqslant A_I(z) \wedge 0.5,$$
$$A_F(x * y) \leqslant \bigvee\{A_F((x * y) * z), A_F(z), 0.5\} = \bigvee\{A_F(0), A_F(z), 0.5\} \leqslant A_F(z) \vee 0.5.$$

It follows from (3.7) that

$$A_T(x) \geqslant \bigwedge\{A_T(x * y), A_T(y), 0.5\} \geqslant \bigwedge\{A_T(y), A_T(z), 0.5\},$$
$$A_I(x) \geqslant \bigwedge\{A_I(x * y), A_I(y), 0.5\} \geqslant \bigwedge\{A_I(y), A_I(z), 0.5\},$$
$$A_F(x) \leqslant \bigvee\{A_F(x * y), A_F(y), 0.5\} \leqslant \bigvee\{A_F(y), A_F(z), 0.5\}.$$

This completes the proof. □

Theorem 3.5. *A neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is an $(\in, \in\vee q)$-neutrosophic ideal of X if and only if the nonempty neutrosophic $\in$-subsets $T_\in(A; \alpha)$, $I_\in(A; \beta)$ and $F_\in(A; \gamma)$ are ideals of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$.*

Proof. Assume that $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q)$-neutrosophic ideal of X and let $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$ be such that $T_\in(A; \alpha) \neq \emptyset$, $I_\in(A; \beta) \neq \emptyset$ and $F_\in(A; \gamma) \neq \emptyset$. Using (3.6), we have $A_T(0) \geqslant A_T(x) \wedge 0.5$, $A_I(0) \geqslant A_I(y) \wedge 0.5$ and $A_F(0) \leqslant A_F(z) \vee 0.5$ for all $x \in T_\in(A; \alpha)$, $y \in I_\in(A; \beta)$ and $z \in F_\in(A; \gamma)$. It follows that $A_T(0) \geqslant \alpha \wedge 0.5 = \alpha$, $A_I(0) \geqslant \beta \wedge 0.5 = \beta$ and $A_F(0) \leqslant \gamma \vee 0.5 = \gamma$, that is, $0 \in T_\in(A; \alpha)$, $0 \in I_\in(A; \beta)$ and $0 \in F_\in(A; \gamma)$. Now let $x, y, a, b, u, v \in X$ be such that $x * y \in T_\in(A; \alpha)$, $y \in T_\in(A; \alpha)$, $a * b \in I_\in(A; \beta)$, $b \in I_\in(A; \beta)$, $u * v \in F_\in(A; \gamma)$ and $v \in F_\in(A; \gamma)$ for $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$. Then $A_T(x*y) \geqslant \alpha$, $A_T(y) \geqslant \alpha$, $A_I(a*b) \geqslant \beta$, $A_I(b) \geqslant \beta$, $A_F(u*v) \leqslant \gamma$ and $A_F(v) \leqslant \gamma$. It follows from (3.7) that

$$A_T(x) \geqslant \bigwedge\{A_T(x * y), A_T(y), 0.5\} \geqslant \alpha \wedge 0.5 = \alpha,$$
$$A_I(a) \geqslant \bigwedge\{A_I(a * b), A_I(b), 0.5\} \geqslant \beta \wedge 0.5 = \beta,$$
$$A_F(u) \leqslant \bigvee\{A_F(u * v), A_F(v), 0.5\} \leqslant \gamma \vee 0.5 = \gamma$$

and so that $x \in T_\in(A;\alpha)$, $a \in I_\in(A;\beta)$ and $u \in F_\in(A;\gamma)$. Hence $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are ideals of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X such that the nonempty neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are ideals of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$. If there are $x, y, z \in X$ such that $A_T(0) < A_T(x) \wedge 0.5$, $A_I(0) < A_I(y) \wedge 0.5$ and $A_F(0) > A_F(z) \vee 0.5$, then $A_T(0) < \alpha_x \leqslant A_T(x) \wedge 0.5$, $A_I(0) < \beta_y \leqslant A_I(y) \wedge 0.5$ and $A_F(0) > \gamma_z \geqslant A_F(z) \vee 0.5$ for some $\alpha_x, \beta_y \in (0, 0.5]$ and $\gamma_z \in [0, 0.5)$. Hence $0 \notin T_\in(A;\alpha_x)$, $0 \notin I_\in(A;\beta_y)$ and $0 \notin F_\in(A;\gamma_z)$, which is a contradiction. Therefore $A_T(0) \geqslant A_T(x) \wedge 0.5$, $A_I(0) \geqslant A_I(x) \wedge 0.5$ and $A_F(0) \leqslant A_F(x) \vee 0.5$ for all $x \in X$. Assume that there exist $x, y, a, b, u, v \in X$ such that $A_T(x) < \bigwedge\{A_T(x * y), A_T(y), 0.5\}$, $A_I(a) < \bigwedge\{A_I(a * b), A_I(b), 0.5\}$, and $A_F(u) > \bigvee\{A_F(u * v), A_F(v), 0.5\}$. Taking

$$\alpha := \tfrac{1}{2}\left(A_T(x) + \bigwedge\{A_T(x * y), A_T(y), 0.5\}\right)$$

implies that $\alpha \in (0, 0.5)$ and $A_T(x) < \alpha < \bigwedge\{A_T(x * y), A_T(y), 0.5\}$. Then $x * y \in T_\in(A;\alpha)$ and $y \in T_\in(A;\alpha)$, but $x \notin T_\in(A;\alpha)$. This is a contradiction. If

$$\beta := \bigwedge\{A_I(a * b), A_I(b), 0.5\},$$

then $\beta \in (0, 0.5]$, $a * b \in I_\in(A;\beta)$ and $b \in I_\in(A;\beta)$. But $A_I(a) < \beta$ implies $a \notin I_\in(A;\beta)$, which is a contradiction. Taking $\gamma := \bigvee\{A_F(u * v), A_F(v), 0.5\}$ induces $\gamma \in [0.5, 1)$, $u * v \in F_\in(A;\gamma)$ and $v \in F_\in(A;\gamma)$. Since $A_F(u) > \gamma$, we have $u \notin F_\in(A;\gamma)$, a contradiction. Therefore $A_T(x) \geqslant \bigwedge\{A_T(x * y), A_T(y), 0.5\}$, $A_I(x) \geqslant \bigwedge\{A_I(x * y), A_I(y), 0.5\}$ and $A_F(x) \leqslant \bigvee\{A_F(x * y), A_F(y), 0.5\}$ for all $x, y \in X$. It follows from Theorem 3.3 that $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee\, q)$-neutrosophic ideal of X. □

We note that every $(\in, \in)$-neutrosophic ideal is an $(\in, \in \vee\, q)$-neutrosophic ideal. But an $(\in, \in \vee\, q)$-neutrosophic ideal may not be an $(\in, \in)$-neutrosophic ideal as seen in the following example.

Example 3.6. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the binary operation "$*$" which is given in Table 1. Let $A = (A_T, A_I, A_F)$ be a neutrosophic set

Table 1. Tabular representation of the binary operation $*$

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	b	0
c	c	a	c	0	c
d	d	d	d	d	0

in X defined by Table 2. Then

TABLE 2. Tabular representation of the binary operation $*$

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.6	0.5	0.45
a	0.4	0.3	0.95
b	0.3	0.7	0.65
c	0.4	0.3	0.95
d	0.1	0.2	0.75

$$T_\in(A;\alpha) = \begin{cases} \{0\} & \text{if } \alpha \in (0.4, 0.5], \\ \{0, a, c\} & \text{if } \alpha \in (0.3, 0.4], \\ \{0, a, b, c\} & \text{if } \alpha \in (0.1, 0.3], \\ X & \text{if } \alpha \in (0, 0.1], \end{cases}$$

$$I_\in(A;\beta) = \begin{cases} \{0, b\} & \text{if } \beta \in (0.3, 0.5], \\ \{0, a, b, c\} & \text{if } \beta \in (0.2, 0.3], \\ X & \text{if } \beta \in (0, 0.2], \end{cases}$$

$$F_\in(A;\gamma) = \begin{cases} X & \text{if } \gamma \in (0.9, 1), \\ \{0, b, d\} & \text{if } \gamma \in [0.7, 0.9), \\ \{0, b\} & \text{if } \gamma \in [0.6, 0.7), \\ \{0\} & \text{if } \gamma \in [0.5, 0.6), \end{cases}$$

which are ideals of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$. Hence $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q)$-neutrosophic ideal of X by Theorem 3.5. But it is not an $(\in, \in)$-neutrosophic ideal of X since $b \in I_\in(A; 0.6)$ but $0 \in I_\in(A; 0.6)$.

We provide conditions for an $(\in, \in\vee q)$-neutrosophic ideal to be an $(\in, \in)$-neutrosophic ideal.

THEOREM 3.7. *Let $A = (A_T, A_I, A_F)$ be an $(\in, \in\vee q)$-neutrosophic ideal of X such that $A_T(x) < 0.5$, $A_I(x) < 0.5$ and $A_F(x) > 0.5$ for all $x \in X$. Then $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X.*

PROOF. Let $x, y, z \in X$, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$ be such that $x \in T_\in(A;\alpha)$, $y \in I_\in(A;\beta)$ and $z \in F_\in(A;\gamma)$. Then $A_T(x) \geqslant \alpha$, $A_I(y) \geqslant \beta$ and $A_F(z) \leqslant \gamma$, which imply from (3.6) that

$$\begin{aligned} A_T(0) &\geqslant A_T(x) \wedge 0.5 = A_T(x) \geqslant \alpha, \\ A_I(0) &\geqslant A_I(y) \wedge 0.5 = A_I(y) \geqslant \beta, \\ A_F(0) &\leqslant A_F(z) \vee 0.5 = A_F(z) \leqslant \gamma. \end{aligned}$$

It follows that $0 \in T_\in(A;\alpha)$, $0 \in I_\in(A;\beta)$ and $0 \in F_\in(A;\gamma)$. For any $x, y, a, b, u, v \in X$, let α_1, α_2, $\beta_1, \beta_2 \in (0, 1]$ and $\gamma_1, \gamma_2 \in [0, 1)$ be such that $x * y \in T_\in(A;\alpha_1)$, $y \in T_\in(A;\alpha_2)$, $a * b \in I_\in(A;\beta_1)$, $b \in I_\in(A;\beta_2)$, $u * v \in F_\in(A;\gamma_1)$ and $v \in F_\in(A;\gamma_2)$.

Then $A_T(x*y) \geqslant \alpha_1$, $A_T(y) \geqslant \alpha_2$, $A_I(a*b) \geqslant \beta_1$, $A_I(b) \geqslant \beta_2$, $A_F(u*v) \leqslant \gamma_1$ and $A_F(v) \leqslant \gamma_2$. Using (3.7), we have

$$A_T(x) \geqslant \bigwedge\{A_T(x*y), A_T(y), 0.5\} = A_T(x*y) \wedge A_T(y) \geqslant \alpha_1 \wedge \alpha_2,$$
$$A_I(a) \geqslant \bigwedge\{A_I(a*b), A_I(b), 0.5\} = A_I(a*b) \wedge A_I(b) \geqslant \beta_1 \wedge \beta_2,$$
$$A_F(u) \leqslant \bigvee\{A_F(u*v), A_F(v), 0.5\} = A_F(u*v) \vee A_F(b) \leqslant \gamma_1 \vee \gamma_2.$$

Hence $x \in T_\in(A;\alpha_1 \wedge \alpha_2)$, $a \in I_\in(A;\beta_1 \wedge \beta_2)$ and $u \in F_\in(A;\gamma_1 \vee \gamma_2)$. Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X. □

We consider a relation between an $(\in\vee q, \in\vee q)$-neutrosophic ideal and an $(\in, \in\vee q)$-neutrosophic ideal.

THEOREM 3.8. *Every $(\in\vee q, \in\vee q)$-neutrosophic ideal is an $(\in, \in\vee q)$-neutrosophic ideal.*

PROOF. Let $A = (A_T, A_I, A_F)$ be an $(\in\vee q, \in\vee q)$-neutrosophic ideal of X. Let $x, y, z \in X$, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$ be such that $x \in T_\in(A;\alpha)$, $y \in I_\in(A;\beta)$ and $z \in F_\in(A;\gamma)$. Then $x \in T_{\in\vee q}(A;\alpha)$, $y \in I_{\in\vee q}(A;\beta)$ and $z \in F_{\in\vee q}(A;\gamma)$. It follows from (3.4) that $0 \in T_{\in\vee q}(A;\alpha)$, $0 \in I_{\in\vee q}(A;\beta)$ and $0 \in F_{\in\vee q}(A;\gamma)$. For any $x, y, a, b, u, v \in X$, let α_1, α_2, $\beta_1, \beta_2 \in (0, 1]$ and $\gamma_1, \gamma_2 \in [0, 1)$ be such that $x*y \in T_\in(A;\alpha_1)$, $y \in T_\in(A;\alpha_2)$, $a*b \in I_\in(A;\beta_1)$, $b \in I_\in(A;\beta_2)$, $u*v \in F_\in(A;\gamma_1)$ and $v \in F_\in(A;\gamma_2)$. Then $x*y \in T_{\in\vee q}(A;\alpha_1)$, $y \in T_{\in\vee q}(A;\alpha_2)$, $a*b \in I_{\in\vee q}(A;\beta_1)$, $b \in I_{\in\vee q}(A;\beta_2)$, $u*v \in F_{\in\vee q}(A;\gamma_1)$ and $v \in F_{\in\vee q}(A;\gamma_2)$. It follows from (3.5) that $x \in T_{\in\vee q}(A;\alpha)$, $a \in I_{\in\vee q}(A;\beta)$ and $u \in F_{\in\vee q}(A;\gamma)$. Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q)$-neutrosophic ideal of X. □

The converse of Theorem 3.8 is not true in general as seen in the following example.

EXAMPLE 3.9. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the binary operation "$*$" which is given in Table 3.

TABLE 3. Tabular representation of the binary operation $*$

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	b	0
c	c	a	c	0	c
d	d	d	b	d	0

Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X defined by Table 4.

TABLE 4. Tabular representation of the binary operation $*$

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.7	0.6	0.4
a	0.3	0.7	0.3
b	0.2	0.2	0.8
c	0.3	0.7	0.3
d	0.2	0.2	0.8

Then

$$T_{\in}(A;\alpha) = \begin{cases} \{0\} & \text{if } \alpha \in (0.3, 0.5], \\ \{0, a, c\} & \text{if } \alpha \in (0.2, 0.3], \\ X & \text{if } \alpha \in (0, 0.2], \end{cases}$$

$$I_{\in}(A;\beta) = \begin{cases} \{0, a, c\} & \text{if } \beta \in (0.2, 0.5], \\ X & \text{if } \beta \in (0, 0.2], \end{cases}$$

$$F_{\in}(A;\gamma) = \begin{cases} X & \text{if } \gamma \in [0.8, 1), \\ \{0, a, c\} & \text{if } \gamma \in [0.5, 0.8), \end{cases}$$

which are ideals of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$. Hence $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee\, q)$-neutrosophic ideal of X by Theorem 3.5. Note that $b * a \in I_{\in\vee\, q}(A; 0.82)$, $a \in I_{\in\vee\, q}(A; 0.7)$ and $b \notin I_{\in\vee\, q}(A; 0.82 \wedge 0.7)$. Hence $A = (A_T, A_I, A_F)$ is not an $(\in\vee\, q, \in\vee\, q)$-neutrosophic ideal of X.

THEOREM 3.10. *For a neutrosophic set $A = (A_T, A_I, A_F)$ in X, if the nonempty neutrosophic $\in\vee\, q$-subsets $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are ideals of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$, then $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee\, q)$-neutrosophic ideal of X.*

PROOF. Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X such that the nonempty neutrosophic $\in\vee\, q$-subsets $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are ideals of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$. Assume that $A_T(0) < A_T(x) \wedge 0.5 := \alpha_x$, $A_I(0) < A_I(y) \wedge 0.5 := \beta_y$ and $A_F(0) > A_F(z) \vee 0.5 := \gamma_z$ for some $x, y, z \in X$. Then $\alpha_x, \beta_y \in (0, 0.5]$, $\gamma_z \in [0.5, 1)$, $x \in T_{\in}(A;\alpha_x) \subseteq T_{\in\vee\, q}(A;\alpha_x)$, $y \in I_{\in}(A;\beta_y) \subseteq I_{\in\vee\, q}(A;\beta_y)$, $z \in F_{\in}(A;\gamma_z) \subseteq F_{\in\vee\, q}(A;\gamma_z)$, $0 \notin T_{\in}(A;\alpha_x)$, $0 \notin I_{\in}(A;\beta_y)$ and $0 \notin F_{\in}(A;\gamma_z)$. Also, since

$$A_T(0) + \alpha_x < 2\alpha_x \leqslant 1, \text{ i.e., } 0 \notin T_q(A;\alpha_x),$$
$$A_I(0) + \beta_y < 2\beta_y \leqslant 1, \text{ i.e., } 0 \notin I_q(A;\beta_y),$$
$$A_F(0) + \gamma_z > 2\gamma_z \geqslant 1, \text{ i.e., } 0 \notin F_q(A;\gamma_z),$$

we have $0 \notin T_{\in\vee\, q}(A;\alpha_x)$, $0 \notin I_{\in\vee\, q}(A;\beta_y)$ and $0 \notin F_{\in\vee\, q}(A;\gamma_z)$. This is a contradiction, and so (3.6) is valid. Assume that there exist $x, y, a, b, u, v \in X$ such that

$$A_T(x) < \bigwedge\{A_T(x*y), A_T(y), 0.5\} := \alpha,$$
$$A_I(a) < \bigwedge\{A_I(a*b), A_I(b), 0.5\} := \beta,$$
$$A_F(u) > \bigvee\{A_F(u*v), A_F(v), 0.5\} := \gamma.$$

Then $\alpha, \beta \in (0, 0.5]$, $\gamma \in [0.5, 1)$, $x \notin T_\in(A;\alpha)$, $a \notin I_\in(A;\beta)$, $u \notin F_\in(A;\gamma)$, and

$$\begin{aligned} &x*y \in T_\in(A;\alpha) \subseteq T_{\in\vee q}(A;\alpha),\ y \in T_\in(A;\alpha) \subseteq T_{\in\vee q}(A;\alpha),\\ &a*b \in I_\in(A;\beta) \subseteq I_{\in\vee q}(A;\beta),\ b \in I_\in(A;\beta) \subseteq I_{\in\vee q}(A;\beta),\\ &u*v \in F_\in(A;\gamma) \subseteq F_{\in\vee q}(A;\gamma),\ v \in F_\in(A;\gamma) \subseteq F_{\in\vee q}(A;\gamma). \end{aligned} \tag{3.9}$$

Since $T_{\in\vee q}(A;\alpha)$, $I_{\in\vee q}(A;\beta)$ and $F_{\in\vee q}(A;\gamma)$ are ideals of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$, (3.9) implies that $x \in T_\in(A;\alpha) \subseteq T_{\in\vee q}(A;\alpha)$, $a \in I_\in(A;\beta) \subseteq I_{\in\vee q}(A;\beta)$ and $u \in F_\in(A;\gamma) \subseteq F_{\in\vee q}(A;\gamma)$. On the other hand, $A_T(x) + \alpha < 2\alpha \leqslant 1$, $A_I(a) + \beta < 2\beta \leqslant 1$ and $A_F(u) + \gamma > 2\gamma \geqslant 1$, that is, $x \notin T_q(A;\alpha)$, $a \notin I_q(A;\beta)$ and $u \notin F_q(A;\gamma)$. Hence $x \notin T_{\in\vee q}(A;\alpha)$, $a \notin I_{\in\vee q}(A;\beta)$ and $u \notin F_{\in\vee q}(A;\gamma)$, which is a contradiction. Thus (3.7) is valid. Using Theorem 3.3, we know that $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q)$-neutrosophic ideal of X. □

We provide conditions for a neutrosophic set to be a $(q, \in\vee q)$-neutrosophic ideal.

Theorem 3.11. *For a subset J of X, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X such that*

$$(\forall x \in X)\left(A_T(0) \geqslant A_T(x), A_I(0) \geqslant A_I(x), A_F(0) \leqslant A_F(x)\right), \tag{3.10}$$

$$(\forall x \in J)\left(A_T(x) \geqslant 0.5, A_I(x) \geqslant 0.5, A_F(x) \leqslant 0.5\right), \tag{3.11}$$

$$(\forall x \in X \setminus J)\left(A_T(x) = 0, A_I(x) = 0, A_F(x) = 1\right). \tag{3.12}$$

If J is an ideal of X, then $A = (A_T, A_I, A_F)$ is a $(q, \in\vee q)$-neutrosophic ideal of X.

Proof. Assume that J is an ideal of X. Let $x, y, z \in X$, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$ be such that $x \in T_q(A;\alpha)$, $y \in I_q(A;\beta)$ and $z \in F_q(A;\gamma)$. Then

$$A_T(x) + \alpha > 1,\ A_I(y) + \beta > 1 \text{ and } A_F(z) + \gamma < 1. \tag{3.13}$$

Combining (3.10) and (3.13), we have

$$A_T(0) + \alpha \geqslant A_T(x) + \alpha > 1,$$
$$A_I(0) + \beta \geqslant A_I(y) + \beta > 1,$$
$$A_F(0) + \gamma \leqslant A_F(z) + \gamma < 1,$$

that is, $0 \in T_q(A;\alpha) \subseteq T_{\in\vee q}(A;\alpha)$, $0 \in I_q(A;\beta) \subseteq I_{\in\vee q}(A;\beta)$ and $0 \in F_q(A;\gamma) \subseteq F_{\in\vee q}(A;\gamma)$. For any $x, y, a, b, u, v \in X$, let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in (0, 1]$ and $\gamma_1, \gamma_2 \in [0, 1)$ be such that $x*y \in T_q(A;\alpha_1)$, $y \in T_q(A;\alpha_2)$, $a*b \in I_q(A;\beta_1)$, $b \in I_q(A;\beta_2)$, $u*v \in F_q(A;\gamma_1)$ and $v \in F_q(A;\gamma_2)$. Then $A_T(x*y) + \alpha_1 > 1$, $A_T(y) + \alpha_2 > 1$, $A_I(a*b) + \beta_1 > 1$, $A_I(b) + \beta_2 > 1$, $A_F(u*v) + \gamma_1 < 1$ and $A_F(v) + \gamma_2 < 1$. If

$x * y \notin J$ or $y \notin J$ (resp., $a * b \notin J$ or $b \notin J$), then $A_T(x * y) = 0$ or $A_T(y) = 0$ (resp., $A_I(a * b) = 0$ or $A_I(b) = 0$). It follows that $A_T(x * y) + \alpha_1 = \alpha_1 \leqslant 1$ or $A_T(y)+\alpha_2 = \alpha_2 \leqslant 1$ (resp., $A_I(a*b)+\beta_1 = \beta_1 \leqslant 1$ or $A_I(b)+\beta_2 = \beta_2 \leqslant 1$). This is a contradiction, and so $x*y \in J$ and $y \in J$ (resp., $a*b \in J$ and $b \in J$). If $u*v \notin J$ or $v \notin J$, then $A_F(u * v) = 1$ or $A_F(v) = 1$. Hence $A_F(u * v) + \gamma_1 = 1 + \gamma_1 \geqslant 1$ or $A_F(v) + \gamma_2 = 1 + \gamma_2 \geqslant 1$, a contradiction. Thus $u * v \in J$ and $v \in J$. Since J is an ideal of X, we get $x \in J$, $a \in J$ and $u \in J$. Thus $A_T(x) \geqslant 0.5$, $A_I(a) \geqslant 0.5$ and $A_F(u) \leqslant 0.5$. If $\alpha_1 \leqslant 0.5$ or $\alpha_2 \leqslant 0.5$ (resp., $\beta_1 \leqslant 0.5$ or $\beta_2 \leqslant 0.5$), then $A_T(x) \geqslant 0.5 \geqslant \alpha_1 \wedge \alpha_2$ (resp., $A_I(a) \geqslant 0.5 \geqslant \beta_1 \wedge \beta_2$), that is, $x \in T_\in(A; \alpha_1 \wedge \alpha_2)$ (resp., $a \in I_\in(A; \beta_1 \wedge \beta_2)$). If $\alpha_1 > 0.5$ and $\alpha_2 > 0.5$ (resp., $\beta_1 > 0.5$ and $\beta_2 > 0.5$), then $A_T(x)+(\alpha_1 \wedge \alpha_2) > 0.5+0.5 = 1$ (resp., $A_I(a)+(\beta_1 \wedge \beta_2) > 0.5+0.5 = 1$), that is, $x \in T_q(A; \alpha_1 \wedge \alpha_2)$ (resp., $a \in I_q(A; \beta_1 \wedge \beta_2)$). Therefore $x \in T_{\in\vee\, q}(A; \alpha_1 \wedge \alpha_2)$ (resp., $a \in I_{\in\vee\, q}(A; \beta_1 \wedge \beta_2)$). Also, if $\gamma_1 \geqslant 0.5$ or $\gamma_2 \geqslant 0.5$, then $A_F(u) \leqslant 0.5 \leqslant \gamma_1 \vee \gamma_2$ and so $u \in F_\in(A; \gamma_1 \vee \gamma_2) \subseteq F_{\in\vee\, q}(A; \gamma_1 \vee \gamma_2)$. If $\gamma_1 < 0.5$ and $\gamma_2 < 0.5$, then $A_F(u) + (\gamma_1 \vee \gamma_2) < 0.5 + 0.5 = 1$ and thus $u \in F_q(A; \gamma_1 \vee \gamma_2) \subseteq F_{\in\vee\, q}(A; \gamma_1 \vee \gamma_2)$. Consequently $A = (A_T, A_I, A_F)$ is a $(q, \in\vee\, q)$-neutrosophic ideal of X. □

The following example illustrates Theorem 3.11.

Example 3.12. Consider a BCI-algebra $X = \{0, 1, 2, a, b\}$ with the binary operation "$*$" which is given in Table 5.

Table 5. Tabular representation of the binary operation $*$

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	1	b	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	a	b	1	0

Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X defined by Table 6.

Table 6. Tabular representation of $A = (A_T, A_I, A_F)$

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.7	0.8	0.2
1	0.0	0.0	1.0
2	0.5	0.7	0.4
a	0.6	0.6	0.3
b	0.0	0.0	1.0

Then $J = \{0, 2, a\}$ is an ideal of X, and so $A = (A_T, A_I, A_F)$ is a $(q, \in\vee\, q)$-neutrosophic ideal of X.

THEOREM 3.13. *Every $(q, \in\vee q)$-neutrosophic ideal is an $(\in, \in\vee q)$-neutrosophic ideal.*

PROOF. Let $A = (A_T, A_I, A_F)$ be a $(q, \in\vee q)$-neutrosophic ideal of X. For any $x, y, z \in X$, let $\alpha_x, \beta_y \in (0,1]$ and $\gamma_z \in [0,1)$ be such that $x \in T_\in(A;\alpha_x)$, $y \in I_\in(A;\beta_y)$ and $z \in F_\in(A;\gamma_z)$. Then $A_T(x) \geqslant \alpha_x$, $A_I(y) \geqslant \beta_y$ and $A_F(z) \leqslant \gamma_z$. Suppose $0 \notin T_{\in\vee q}(A;\alpha_x)$, $0 \notin I_{\in\vee q}(A;\beta_y)$ and $0 \notin F_{\in\vee q}(A;\gamma_z)$. Then

$$A_T(0) < \alpha_x,\ A_I(0) < \beta_y,\ A_F(0) > \gamma_z, \tag{3.14}$$

$$A_T(0) + \alpha_x \leqslant 1,\ A_I(0) + \beta_y \leqslant 1,\ A_F(0) + \gamma_z \geqslant 1. \tag{3.15}$$

It follows that

$$A_T(0) < 0.5,\ A_I(0) < 0.5,\ A_F(0) > 0.5. \tag{3.16}$$

Combining (3.14) and (3.16), we have

$$A_T(0) < \alpha_x \wedge 0.5,\ A_I(0) < \beta_y \wedge 0.5,\ A_F(0) > \gamma_z \vee 0.5,$$

and so

$$\begin{aligned}
1 - A_T(0) &> 1 - (\alpha_x \wedge 0.5) = (1-\alpha_x) \vee 0.5 \geqslant (1 - A_T(x)) \vee 0.5,\\
1 - A_I(0) &> 1 - (\beta_y \wedge 0.5) = (1-\beta_y) \vee 0.5 \geqslant (1 - A_I(x)) \vee 0.5,\\
1 - A_F(0) &< 1 - (\gamma_z \vee 0.5) = (1-\gamma_z) \wedge 0.5 \leqslant (1 - A_F(x)) \wedge 0.5.
\end{aligned}$$

Hence there exist $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$ such that

$$\begin{aligned}
1 - A_T(0) &\geqslant \alpha > (1 - A_T(x)) \vee 0.5,\\
1 - A_I(0) &\geqslant \beta > (1 - A_I(x)) \vee 0.5,\\
1 - A_F(0) &\leqslant \gamma < (1 - A_F(x)) \vee 0.5.
\end{aligned} \tag{3.17}$$

The right inequalities in (3.17) induces

$$A_T(x) + \alpha > 1,\ A_I(x) + \beta > 1,\ A_F(x) + \gamma < 1,$$

that is, $x \in T_q(A;\alpha)$, $y \in I_q(A;\beta)$ and $z \in F_q(A;\gamma)$. Since $A = (A_T, A_I, A_F)$ is a $(q, \in\vee q)$-neutrosophic ideal of X, we have $0 \in T_{\in\vee q}(A;\alpha)$, $0 \in I_{\in\vee q}(A;\beta)$ and $0 \in F_{\in\vee q}(A;\gamma)$. But the left inequalities in (3.17) implies that

$$\begin{aligned}
A_T(0) + \alpha \leqslant 1 &\text{ and } A_T(0) \leqslant 1 - \alpha < 0.5 < \alpha,\\
A_I(0) + \beta \leqslant 1 &\text{ and } A_I(0) \leqslant 1 - \beta < 0.5 < \beta,\\
A_F(0) + \gamma \geqslant 1 &\text{ and } A_F(0) \geqslant 1 - \gamma > 0.5 > \gamma,
\end{aligned}$$

that is, $0 \notin T_{\in\vee q}(A;\alpha)$, $0 \notin I_{\in\vee q}(A;\beta)$ and $0 \notin F_{\in\vee q}(A;\gamma)$. This is a contradiction, and so $0 \in T_{\in\vee q}(A;\alpha_x)$, $0 \in I_{\in\vee q}(A;\beta_y)$ and $0 \in F_{\in\vee q}(A;\gamma_z)$. For any $x, y, a, b, u, v \in X$, let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in 0,1]$ and $\gamma_1, \gamma_2 \in [0,1)$ be such that $x * y \in T_\in(A;\alpha_1)$, $y \in T_\in(A;\alpha_2)$, $a * b \in I_\in(A;\beta_1)$, $b \in I_\in(A;\beta_2)$, $u * v \in F_\in(A;\gamma_1)$ and $v \in F_\in(A;\gamma_2)$. Then $A_T(x * y) \geqslant \alpha_1$, $A_T(y) \geqslant \alpha_2$, $A_I(a * b) \geqslant \beta_1$, $A_I(b) \geqslant \beta_2$, $A_F(u*v) \leqslant \gamma_1$ and $A_F(v) \leqslant \gamma_2$. Suppose $x \notin T_{\in\vee q}(A;\alpha_1 \wedge \alpha_2)$, $a \notin I_{\in\vee q}(A;\beta_1 \wedge \beta_2)$ and $u \notin F_{\in\vee q}(A;\gamma_1 \vee \gamma_2)$. Then

$$A_T(x) < \alpha_1 \wedge \alpha_2,\ A_I(a) < \beta_1 \wedge \beta_2,\ A_F(u) > \gamma_1 \vee \gamma_2, \tag{3.18}$$

(3.19) $$A_T(x) + (\alpha_1 \wedge \alpha_2) \leqslant 1,\ A_I(a) + (\beta_1 \wedge \beta_2) \leqslant 1,\ A_F(u) + (\gamma_1 \vee \gamma_2) \geqslant 1.$$

It follows that

(3.20) $$A_T(x) < 0.5,\ A_I(a) < 0.5,\ A_F(u) > 0.5.$$

Combining (3.18) and (3.20), we have

$$A_T(x) < \bigwedge\{\alpha_1, \alpha_2, 0.5\},\ A_I(a) < \bigwedge\{\beta_1, \beta_2, 0.5\},\ A_F(u) > \bigvee\{\gamma_1, \gamma_2, 0.5\},$$

and thus

$$\begin{aligned} 1 - A_T(x) &> 1 - \bigwedge\{\alpha_1, \alpha_2, 0.5\} = \bigvee\{1-\alpha_1, 1-\alpha_2, 0.5\} \\ &\geqslant \bigvee\{1 - A_T(x*y), 1 - A_T(y), 0.5\}, \end{aligned}$$

$$\begin{aligned} 1 - A_I(a) &> 1 - \bigwedge\{\beta_1, \beta_2, 0.5\} = \bigvee\{1-\beta_1, 1-\beta_2, 0.5\} \\ &\geqslant \bigvee\{1 - A_I(a*b), 1 - A_I(b), 0.5\}, \end{aligned}$$

$$\begin{aligned} 1 - A_F(u) &< 1 - \bigvee\{\gamma_1, \gamma_2, 0.5\} = \bigwedge\{1-\gamma_1, 1-\gamma_2, 0.5\} \\ &\leqslant \bigwedge\{1 - A_F(u*v), 1 - A_F(v), 0.5\}. \end{aligned}$$

Therefore there exist $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$ such that

(3.21) $$\begin{aligned} 1 - A_T(x) &\geqslant \alpha > \bigvee\{1 - A_T(x*y), 1 - A_T(y), 0.5\}, \\ 1 - A_I(a) &\geqslant \beta > \bigvee\{1 - A_I(a*b), 1 - A_I(b), 0.5\}, \\ 1 - A_F(u) &\leqslant \gamma < \bigwedge\{1 - A_F(u*v), 1 - A_F(v), 0.5\}. \end{aligned}$$

It follows that

$A_T(x*y) + \alpha > 1$ and $A_T(y) + \alpha > 1$, i.e., $x*y \in T_q(A;\alpha)$ and $y \in T_q(A;\alpha)$,
$A_I(a*b) + \beta > 1$ and $A_I(b) + \beta > 1$, i.e., $a*b \in I_q(A;\beta)$ and $b \in I_q(A;\beta)$,
$A_F(u*v) + \gamma < 1$ and $A_F(v) + \gamma < 1$, i.e., $u*v \in F_q(A;\gamma)$ and $v \in F_q(A;\gamma)$,
$A_T(x) + \alpha \leqslant 1$ and $A_T(x) \leqslant 1 - \alpha < \alpha$, i.e., $x \notin T_{\in\vee q}(A;\alpha)$,
$A_I(a) + \beta \leqslant 1$ and $A_I(a) \leqslant 1 - \beta < \beta$, i.e., $a \notin I_{\in\vee q}(A;\beta)$,
$A_F(u) + \gamma \geqslant 1$ and $A_F(u) \geqslant 1 - \gamma > \gamma$, i.e., $u \notin F_{\in\vee q}(A;\gamma)$.

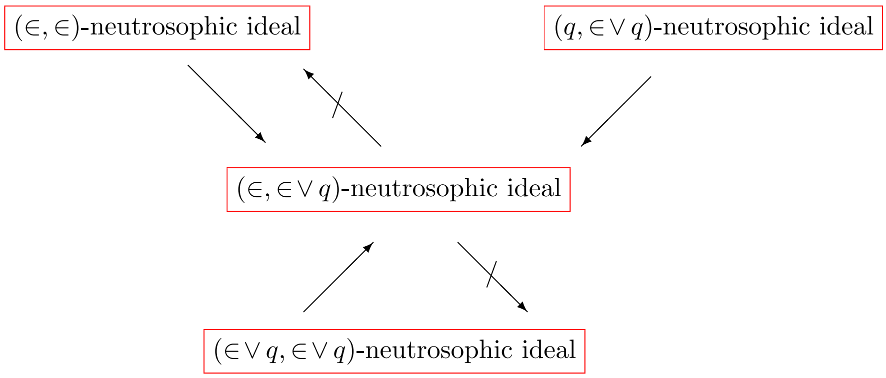
It is a contradiction because $A = (A_T, A_I, A_F)$ is a $(q, \in\vee q)$-neutrosophic ideal of X. Therefore $x \in T_{\in\vee q}(A;\alpha_1 \wedge \alpha_2)$, $a \in I_{\in\vee q}(A;\beta_1 \wedge \beta_2)$ and $u \in F_{\in\vee q}(A;\gamma_1 \vee \gamma_2)$. Consequently, $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q)$-neutrosophic ideal of X. □

Corollary 3.14. *For an ideal J of X, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X satisfying conditions (3.10), (3.11) and (3.12). Then $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q)$-neutrosophic ideal of X.*

Proof. It follows from Theorems 3.11 and 3.13. □

Conclusions

We have introduced the notion of (Φ, Ψ)-neutrosophic ideal of a BCK/BCI-algebra X for $\Phi, \Psi \in \{\in, q, \in\vee q\}$, and have investigated related properties. We have considered characterizations of an $(\in, \in\vee q)$-neutrosophic ideal. We have provided conditions for an $(\in, \in\vee q)$-neutrosophic ideal to be an $(\in, \in)$-neutrosophic ideal. We have considered conditions for a neutrosophic set to be a $(q, \in\vee q)$-neutrosophic ideal and an $(\in, \in\vee q)$-neutrosophic ideal. We have shown that every $(\in\vee q, \in\vee q)$-neutrosophic ideal is an $(\in, \in\vee q)$-neutrosophic ideal, and every $(q, \in\vee q)$-neutrosophic ideal is an $(\in, \in\vee q)$-neutrosophic ideal. We display the relations among $(\in, \in)$-neutrosophic ideal, $(\in, \in\vee q)$-neutrosophic ideal, $(q, \in\vee q)$-neutrosophic ideal and $(\in\vee q, \in\vee q)$-neutrosophic ideal by the following diagram.



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Department of Mathematics, Faculty of Arts and Sciences, Adiyaman University, 02040 Adiyaman, Turkey

E-mail address: mehaliozturk@gmail.com

Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea

E-mail address: skywine@gmail.com