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HIGH ORDER VARIABLE MESH EXPONENTIAL FINITE DIFFERENCE METHOD FOR THE NUMERICAL SOLUTION OF THE TWO POINTS BOUNDARY VALUE PROBLEMS

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Abstract. In this article, we present a non-uniform mesh size high order exponential finite difference scheme for the numerical solutions of two point boundary value problems with Dirichlet's boundary conditions. Under appropriate conditions, we have discussed the local truncation error and the convergence of the proposed method. Numerical experiments have been carried out to demonstrate the use and high order computational efficiency of the present method in several model problems. Numerical results showed that the proposed method is accurate and convergent. The order of accuracy is at least cubic which is in good agreement with the theoretically established order of the method.

1. Introduction

In this article we considered a method for the numerical solution of the two-point boundary value problems of the form

$$y''(x) = f(x, y), \quad a < x < b, \tag{1.1}$$

subject to the boundary conditions

$$y(a) = \alpha \quad and \quad y(b) = \beta,$$

where α and β are real constants and f is continuous on (x, y) for all $x \in [a, b]$ $y \in \Re$.

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Two point boundary value problems are of common occurrence in many areas of sciences and engineering. This class of problems has gained importance in the literature for the variety of their applications. In most cases it is impossible to obtain solutions of these problems using analytical methods which satisfy the given boundary conditions. In these cases we resort to approximate solution of the problems and the last few decades have seen substantial progress in the development of approximate solutions of these problems. In all numerical solutions the continuous ordinary differential equation is replaced with a discrete approximation. In the literature, there are many different methods and approaches such as method of integration and discretization that are used to derive the approximate solutions of these problems [**1, 2, 3, 4**]. The finite difference method is one of several discretization techniques for obtaining approximate numerical solutions to problem (1.1).

The existence and uniqueness of the solution to problem (1.1) is assumed. We further assumed that problem (1.1) is well posed with continuous derivatives and that the solution depends differentially on the boundary conditions. The specific assumption on $f(x, y)$ to ensure existence and uniqueness will not be considered [**3, 4, 5**].

Over the last few decades, finite difference methods [**6, 7, 8**] have generated renewed interest and in recent years, variety of specialized techniques [**9, 10**] for the numerical solution of boundary value problems in ODEs have been reported in the literature. Recently, an exponential finite difference method with non-uniform step size was proposed in [**11**] for the numerical solution of two point boundary value problem. This method generated impressive numerical results for the problem (1). Hence, the purpose of this article is to propose an exponential finite difference method with non-uniform mesh size of higher order for problem (1). The main advantage for high order method is computationally more efficient than the second order method. In order to achieve a higher order method more points are required for discretization approximation. But in proposed higher order method we have used only three points.

A method of at least cubic order is proposed for the numerical solution of linear boundary value problems (1.1). Our idea is to apply and extend the exponential finite difference method reported in [**10**] to discretize equation (1) in order to get a system of algebraic equations. In addition, if we apply a linearization technique and approximations, the method results in a tridiagonal matrix for the nodal values. The elements of this matrix depend on the source function i.e. right-hand side of the ordinary differential equation as well as on its partial derivatives with respect to the dependent variable and its first-order derivative.

We hope that others may find the proposed method an improvement and appealing to those existing finite difference methods for two-point boundary value problems.

We have presented our work in this article as follows. In the next section we derive a new variable mesh size exponential finite difference method. We discuss in

Section 3, local truncation error and convergence of the new method in Section 4. The application of the proposed method to the problems (1.1) has been presented and illustrative numerical results have been produced to show the efficiency of the new method in Section 5. Discussion and conclusion on the performance of the new method are presented in Section 6.

2. The Variable Mesh Size Exponential Difference Method

We define N finite numbers of nodal points of the domain $[a,b]$, in which the solution of the problem (1.1) is desired, as

$$a \leqslant x_0 < x_1 < x_2 < \ ...\ < x_N < x_{N+1} = b,$$

using nonuniform step length h_i such that $x_{i+1} = x_i + h_{i+1}, \quad i = 0, 1, 2, ..., N$ and $r_i = \frac{h_{i+1}}{h_i}$. Suppose that we wish to determine the numerical approximation of the theoretical solution $y(x)$ of the problem (1.1) at the nodal point $x_i, \quad i = 1, 2, ..., N$. We denote the numerical approximation of $y(x)$ at node $x = x_i$ as y_i . Let us denote f_i as the approximation of the theoretical value of the source function $f(x, y(x))$ at node $x = x_i, \quad i = 0, 1, 2, ..., N+1$. We can define other notations used in this article i.e. $f_{i\pm1}$, and $y_{i\pm1}$, in the similar way. Thus differential equation (1) at the mesh point $x = x_i$ may be written as,

$$y_i'' = f(x_i, y_i)$$

Hence following the ideas in [**10**], we propose an approximation to the theoretical solution $y(x_i)$ of the problem (1.1) by the exponential finite difference scheme as,

$$a_2 y_{i+1} + a_1 y_i + a_0 y_{i-1} = b_0 h_i^2 f_i \exp(\phi(x_i)), \quad i = 1, 2, ..., N. \tag{2.1}$$

where a_0, a_1, a_2 and b_0 are unknown functions of the argument r_i and $\phi(x_i)$ is an unknown sufficiently differentiable function of x. Let us define a function $F(h, y)$ in which arguments are $h = h_i$, $y = y_i$ at mesh point $x = x_i$ and associate $F(h, y)$ with (2.1) as,

$$F_i(h, y) \equiv a_2 y_{i+1} + a_1 y_i + a_0 y_{i-1} - b_0 h_i^2 f_i \exp(\phi(x_i)) = 0. \tag{2.2}$$

Assume that $\phi(x_i)$ can be expanded in Taylor series about the point $x = x_{i-1}$. Hence we write $\phi(x_i)$ in Taylor series ,

$$\phi(x_i) = \phi(x_{i-1}) + h_i \phi'(x_{i-1}) + \frac{h_i^2}{2}\phi''(x_{i-1}) + O(h_i^3). \tag{2.3}$$

The application of (2.3) in the expansion of $\exp(\phi(x_i))$ will provide an $O(h_i^3)$ approximation of the form as,

$$\exp(\phi(x_i)) = \exp(\phi(x_{i-1}))(1+h_i\phi'(x_{i-1})+\frac{h_i^2}{2}(\phi'^2(x_{i-1})+\phi''(x_{i-1})))+O(h_i^3) \tag{2.4}$$

Expand $F_i(h,y)$ in Taylor series about mesh point $x = x_i$ and using (2.4) in it, we have

(2.5)

$$F_i(h,y) \equiv \{(a_0+a_1+a_2)y_i+h_i(r_ia_2-a_0)y_i'+\frac{h_i^2}{2}(r_i^2a_2+a_0)y_i''+\frac{h_i^3}{6}(r_i^3a_2-a_0)y_i^{(3)}+\frac{h_i^4}{24}(r_i^4a_2+a_0)y_i^{(4)}\}-b_0h_i^2f_i\exp(\phi(x_{i-1}))(1+h_i\phi'(x_{i-1})+\frac{h_i^2}{2}(\phi'^2(x_{i-1})+\phi''(x_{i-1})))=0.$$

On comparing the coefficients of $h_i^p, p = 0,1,2,3,4$ both sides in (2.5), we get the following system of nonlinear equations

$$\begin{aligned} a_0 + a_1 + a_2 &= 0, \\ r_ia_2 - a_0 &= 0, \\ (r_i^2a_2 + a_0)y_i'' - 2b_0f_i\exp(\phi(x_{i-1})) &= 0, \\ (r_i^3a_2 - a_0)y_i^{(3)} - 6b_0f_i\exp(\phi(x_{i-1}))\phi'(x_{i-1}) &= 0, \\ (r_i^4a_2 + a_0)y_i^{(4)} - 12b_0f_i\exp(\phi(x_{i-1}))(\phi'^2(x_{i-1}) + \phi''(x_{i-1})) &= 0. \end{aligned} \tag{2.6}$$

To determine the unknown functions a_0, a_1, b_0 , $\phi(x_{i-1})$, $\phi'(x_{i-1})$ and $\phi''(x_{i-1})$ in (2.6), we have to assign arbitrary value to some unknown functions. To simplify the system of equations in (2.6), we have considered the following assumption:

$$\phi(x_{i-1}) = 0. \tag{2.7}$$

Using (2.7) in (2.6) and solved the reduced system of equations, we obtained

$$\begin{aligned} a_0 &= r_ia_2, \\ a_1 &= -(r_i+1)a_2, \\ b_0 &= \frac{r_i(r_i+1)a_2}{2}, \\ \phi'(x_{i-1}) &= \frac{(r_i-1)y_i^{(3)}}{3f_i}, \\ \phi''(x_{i-1}) &= \frac{(r_i^2-r_i+1)y_i^{(4)}}{6f_i} - \frac{((r_i-1)y_i^{(3)})^2}{(3f_i)^2}. \end{aligned} \tag{2.8}$$

Write f_i' for $y_i^{(3)}$ and f_i'' for $y_i^{(4)}$ in (2.8) and substituting the values of $\phi(x_{i-1})$, $\phi'(x_{i-1})$ and $\phi''(x_{i-1})$ from (2.7) and (2.8) in (2.3), we have

$$\phi(x_i) = \frac{h_i(r_i-1)f_i'}{3f_i} + \frac{h_i^2}{2}\{\frac{(r_i^2-r_i+1)f_i''}{6f_i} - \frac{((r_i-1)f_i')^2}{(3f_i)^2}\} + O(h_i^3). \tag{2.9}$$

Finally substitute the values of a_0, a_1, b_0 and $\phi(x_i)$ from (2.8) and (2.9) in (2.1), we obtain our proposed exponential finite difference method as

$$\begin{aligned}(2.10)\quad & y_{i+1} - (1+r_i)y_i + r_i y_{i-1} = \\ & \frac{h_i^2 r_i(r_i+1)}{2} f_i \exp\Big(\frac{h_i(r_i-1)f_i'}{3f_i} + \frac{h_i^2}{2}\Big\{\frac{(r_i^2-r_i+1)f_i''}{6f_i} - \frac{((r_i-1)f_i')^2}{(3f_i)^2}\Big\}\Big).\end{aligned}$$

For each nodal point, we will obtain the nonlinear system of equations given by (2.10) or a linear system of equations if the source function is *f(x).* In the derived numerical method (2.10), the exponential function

$$\exp\Big(\frac{h_i(r_i-1)f_i'}{3f_i} + \frac{h_i^2}{2}\Big\{\frac{(r_i^2-r_i+1)f_i''}{6f_i} - \frac{((r_i-1)f_i')^2}{(3f_i)^2}\Big\}\Big)$$

has the argument

$$\frac{h_i(r_i-1)f_i'}{3f_i} + \frac{h_i^2}{2}\Big\{\frac{(r_i^2-r_i+1)f_i''}{6f_i} - \frac{((r_i-1)f_i')^2}{(3f_i)^2}\Big\}.$$

If f_i in the denominator of the argument becomes zero in the domain of the solution, we take the series expansion of the function

$$\exp\Big(\frac{h_i(r_i-1)f_i'}{3f_i} + \frac{h_i^2}{2}\Big\{\frac{(r_i^2-r_i+1)f_i''}{6f_i} - \frac{((r_i-1)f_i')^2}{(3f_i)^2}\Big\}\Big).$$

Neglecting $O(h_i^3)$ and higher order terms in the expansion, the method (2.10) becomes

$$(2.11)\quad y_{i+1}-(1+r_i)y_i+r_i y_{i-1} = \frac{h_i^2}{24} r_i(r_i+1)(12f_i+4h_i(r_i-1)f_i'+h_i^2(r_i^2-r_i+1)f_i'').$$

The formulas (2.10) or (2.11) use up to the second derivative of the source function $f(x,y)$ which bring in more computational complexity in the system. Thus for the computational purpose in Section 4, let us define following $O(h_i^3)$ approximations :

$$(2.12)\qquad \overline{y}_{i+\frac{1}{2}} = c_{10}y_{i+1} + c_{11}y_i + c_{12}y_{i-1} + h_{i+1}^2 c_{13} f_i,$$

where

$$c_{10} = \tfrac{r_i^2-4}{8(r_i^2-1)},\ c_{11} = \tfrac{3r_i^2+4r_i-4}{8(r_i-1)},\ c_{12} = \tfrac{-3r_i^3}{8(r_i^2-1)} \quad \text{and} \quad c_{13} = \tfrac{r_i+2}{16(r_i-1)}.$$

$$(2.13)\qquad \overline{y}_{i-\frac{1}{2}} = c_{20}y_{i+1} + c_{21}y_i + c_{22}y_{i-1} + h_i^2 c_{23} f_i,$$

where

$$c_{20} = \frac{3}{8r_i(r_i^2-1)},\ c_{21} = \frac{4r_i^2-4r_i-3}{8r_i(r_i-1)},\ c_{22} = \frac{4r_i^2-1}{8(r_i^2-1)},\ c_{23} = \frac{-(2r_i+1)}{16(r_i-1)}.$$

Thus from (13)-(14), $\overline{y}_{i\pm\frac{1}{2}}$ will provide $O(h_i^3)$ for $y_{i\pm\frac{1}{2}}$.

Let us define following $O(h_i^5)$ approximations :

$$(2.14)\qquad \overline{\overline{y}}_{i+\frac{1}{2}} = c_{30}y_{i+1} + c_{31}y_i + c_{32}y_{i-1} + h_{i+1}^2(c_{33}f_{i+1} + c_{34}f_i),$$

where

$c_{30} = \frac{3r_i^3-16r_i-8}{16(r_i^3-2r_i-1)}$, $c_{31} = \frac{5r_i^4+13r_i^3-16r_i-8}{16(r_i^3-2r_i-1)}$, $c_{32} = \frac{-5r_i^4}{16(r_i^3-2r_i-1)}$, $c_{33} = \frac{r_i^3-7r_i-6}{96(1+2r_i-r_i^3)}$, and $c_{34} = \frac{4r_i^3+15r_i^2+17r_i+6}{96(r_i^3-2r_i-1)}$.

$$\bar{\bar{y}}_{i-\frac{1}{2}} = c_{40}y_{i+1} + c_{41}y_i + c_{42}y_{i-1} + h_i^2(c_{43}f_i + c_{44}f_{i-1}), \tag{2.15}$$

where

$c_{40} = \frac{5}{16r_i(r_i^3+2r_i^2-1)}$, $c_{41} = \frac{8r_i^4+16r_i^3-13r_i-5}{16r_i(r_i^3+2r_i^2-1)}$, $c_{42} = \frac{8r_i^4+16r_i^3-3r_i}{16r_i(r_i^3+2r_i^2-1)}$, $c_{43} = \frac{-(6r_i^3+17r_i^2+15r_i+4)}{96(r_i^3+2r_i^2-1)}$ and $c_{44} = \frac{-6r_i^3-7r_i^2+1}{96(r_i^3+2r_i-1)}$.

Thus from (2.14)-(2.15), $\bar{\bar{y}}_{i\pm\frac{1}{2}}$ will provide $O(h_i^5)$ for $y_{i\pm\frac{1}{2}}$.

Let define

$$\bar{f}_{i+\frac{1}{2}} = f(x_{i+\frac{1}{2}}, \bar{y}_{i+\frac{1}{2}}), \quad \bar{f}_{i-\frac{1}{2}} = f(x_{i-\frac{1}{2}}, \bar{y}_{i-\frac{1}{2}}) \tag{2.16}$$

$$\bar{\bar{f}}_{i+\frac{1}{2}} = f(x_{i+\frac{1}{2}}, \bar{\bar{y}}_{i+\frac{1}{2}}), \quad \textit{and} \quad \bar{\bar{f}}_{i-\frac{1}{2}} = f(x_{i-\frac{1}{2}}, \bar{\bar{y}}_{i-\frac{1}{2}}). \tag{2.17}$$

Thus from (2.12),(2.13) and (2.16), we find that $\bar{f}_{i\pm\frac{1}{2}}$ will provide $O(h_i^3)$ approximation for $f_{i\pm\frac{1}{2}}$. Similarly from (2.14), (2.15) and (2.17), we can find that $\bar{\bar{f}}_{i\pm\frac{1}{2}}$ will provide $O(h_i^5)$ approximation for $f_{i\pm\frac{1}{2}}$.

Using (2.16) and (2.17), let we define the following third order finite difference approximation for the terms h_if_i' , $h_i^2f_i''$ in (2.10) and (2.11) as :

$$h_if_i' = c_{50}f_{i+1} + c_{51}f_{i-1} + c_{52}\bar{f}_{i+\frac{1}{2}} + c_{53}\bar{f}_{i-\frac{1}{2}}, \tag{2.18}$$

where

$c_{50} = \frac{2-3r_i}{r_i(r_i+1)(2r_i+1)}$, $c_{51} = \frac{r_i(3-2r_i)}{(r_i+1)(r_i+2)}$, $c_{52} = \frac{4(3r_i-1)}{r_i(r_i+1)(r_i+2)}$ and $c_{53} = \frac{4r_i(r_i-3)}{(r_i+1)(2r_i+1)}$.

$$h_i^2f_i'' = c_{60}f_{i+1} + c_{61}f_i + c_{62}f_{i-1} + c_{63}\bar{\bar{f}}_{i+\frac{1}{2}} + c_{64}\bar{\bar{f}}_{i-\frac{1}{2}}, \tag{2.19}$$

where

$c_{60} = \frac{4-6r_i}{r_i^2(r_i+1)(2r_i+1)}$, $c_{61} = \frac{2(4r_i^3-16r_i^2-5r_i+2)}{r_i^2(2r_i+1)}$, $c_{62} = \frac{2r_i(2r_i-3)}{(r_i+1)(r_i+2)}$, $c_{63} = \frac{16(3r_i-1)}{r_i^2(r_i+1)(r_i+2)}$ and $c_{64} = \frac{16r_i(3-r_i)}{(r_i+1)(2r_i+1)}$.

Thus substituting the terms h_if_i' and $h_i^2f_i''$ from (2.18) and (2.19) in (2.10) or (2.11), we obtained $O(h_i^3)$ method for computation of numerical solution of problems (1.1).

3. Local Truncation Error

We can write the following expression for the term in (2.10) :

(3.1)
$$\exp\Big(\frac{h_i(r_i-1)f_i'}{3f_i}+\frac{h_i^2}{2}\{\frac{(r_i^2-r_i+1)f_i''}{6f_i}-\frac{((r_i-1)f_i')^2}{(3f_i)^2}\}\Big)=1+\frac{h_i(r_i-1)f_i'}{3f_i}$$
$$+\frac{h_i^2(r_i^2-r_i+1)f_i''}{12f_i}+\frac{h_i^3f_i'}{324f_i^3}(9(r_i^3+1)f_if_i''-4(r_i-1)^3(f_i')^2)+O(h_i^4).$$

From (2.10) and (3.1), the truncation error T_i at the nodal point $x=x_i$ may be written as [**8, 12, 13**],

$$T_i=y_{i+1}-(1+r_i)y_i+r_iy_{i-1}-\frac{h_i^2(r_i^2+r_i)f_i}{2}(1+\frac{h_i(r_i-1)f_i'}{3f_i}+$$
$$\frac{h_i^2(r_i^2-r_i+1)f_i''}{12f_i}+\frac{h_i^3f_i'}{324f_i^3}(9(r_i^3+1)f_if_i''-4(r_i-1)^3(f_i')^2))+O(h_i^6).$$

By the Taylor series expansion of y at nodal point $x=x_i$ and using $y_i''=f_i$, $y_i^{(3)}=f_i'$ and $y^{(4)}=f_i''$ etc., we have

(3.2)
$$T_i=(\frac{h_{i+1}^5}{120}-\frac{r_ih_i^5}{120})y_i^{(5)}+\frac{h_i^5r_i(r_i+1)y_i^{(3)}}{648(y_i'')^3}(9(r_i^3+1)y_i''y_i^{(4)}-4(r_i-1)^3(y_i^{(3)})^2))+O(h_i^6)$$
$$=\frac{h_i^5}{3240}(27r_i(r_i^4-1)y^{(5)}+\frac{5r_i(r_i+1)y_i^{(3)}}{(y_i'')^3}(9(r_i^3+1)y_i''y_i^{(4)}-4(r_i-1)^3(y_i^{(3)})^2))+O(h_i^6).$$

Thus from (3.2), T_i can be written as :

(3.3)
$$T_i=O(h_i^5).$$

Thus we have obtained a truncation error at each node of $O(h_i^5)$.

4. Convergence of the Method

Let us substitute (2.18) and (2.19) into (2.11) and then simplify, we have

$$y_{i+1}-(1+r_i)y_i+r_iy_{i-1}=\frac{h_i^2r_i(r_i+1)}{24}((12+(r_i^2-r_i+1)c_{61})f_i+$$
$$(4(r_i-1)c_{50}+(r_i^2-r_i+1)c_{60})f_{i+1}+(4(r_i-1)c_{51}+(r_i^2-r_i+1)c_{62})f_{i-1}$$
$$+4(r_i-1)c_{52}\overline{f}_{i+\frac{1}{2}}+(r_i^2-r_i+1)c_{63}\overline{\overline{f}}_{i+\frac{1}{2}}+4(r_i-1)c_{53}\overline{f}_{i-\frac{1}{2}}+(r_i^2-r_i+1)c_{64}\overline{\overline{f}}_{i-\frac{1}{2}}).$$

Thus

(4.1)
$$-y_{i+1}+(1+r_i)y_i-r_iy_{i-1}+\frac{h_i^2}{24}(\alpha_if_i+\gamma_if_{i+1}+\beta_if_{i-1}+\overline{\delta}_i\overline{f}_{i+\frac{1}{2}}+$$
$$\overline{\overline{\delta}}_i\overline{\overline{f}}_{i+\frac{1}{2}}+\overline{\lambda}_i\overline{f}_{i-\frac{1}{2}}+\overline{\overline{\lambda}}_i\overline{\overline{f}}_{i-\frac{1}{2}})=0,$$

where

$$\alpha_i = r_i(r_i+1)(12+(r_i^2-r_i+1)c_{61}),$$
$$\beta_i = r_i(r_i+1)(4(r_i-1)c_{50}+(r_i^2-r_i+1)c_{60}),$$
$$\gamma_i = r_i(r_i+1)(4(r_i-1)c_{51}+(r_i^2-r_i+1)c_{62}), \quad \overline{\delta}_i = 4r_i(r_i^2-1)c_{52},$$
$$\overline{\overline{\delta}}_i = (r_i^2-r_i+1)c_{63}, \quad \overline{\lambda}_i = 4r_i(r_i^2-1)c_{53}, \quad \text{and } \overline{\overline{\lambda}}_i = (r_i^2-r_i+1)c_{64}.$$

Let us define

$$\phi_1 = \frac{h_1^2}{24}(\alpha_1 f(x_1,y_1)+\gamma_1 f(x_2,y_2)+\overline{\delta}_1 f(x_{\frac{3}{2}},\overline{y}_{\frac{3}{2}})+\overline{\overline{\delta}}_1 f(x_{\frac{3}{2}},\overline{\overline{y}}_{\frac{3}{2}})$$
$$+\overline{\lambda}_1 f(x_{\frac{1}{2}},\overline{y}_{\frac{1}{2}})+\overline{\overline{\lambda}}_1 f(x_{\frac{1}{2}},\overline{\overline{y}}_{\frac{1}{2}}))+\frac{h_1^2}{24}\beta_1 f(x_0,y_0)+r_1 y_0, \quad i=1$$

$$\phi_i = \frac{h_i^2}{24}(\alpha_i f(x_i,y_i)+\gamma_i f(x_{i+1},y_{i+1})+\beta_i f(x_{i-1},y_{i-1})+\overline{\delta}_i f(x_{i+\frac{1}{2}},\overline{y}_{i+\frac{1}{2}})$$
$$+\overline{\overline{\delta}}_i f(x_{i+\frac{1}{2}},\overline{\overline{y}}_{i+\frac{1}{2}})+\overline{\lambda}_i f(x_{i-\frac{1}{2}},\overline{y}_{i-\frac{1}{2}})+\overline{\overline{\lambda}}_i f(x_{i-\frac{1}{2}},\overline{\overline{y}}_{i-\frac{1}{2}})), \quad 2\leqslant i\leqslant N-1$$

$$\phi_N = \frac{h_N^2}{24}(\alpha_N f(x_N,y_N)+\beta_N f(x_{N-1},y_{N-1})+\overline{\delta}_N f(x_{N+\frac{1}{2}},\overline{y}_{N+\frac{1}{2}})$$
$$+\overline{\overline{\delta}}_N f(x_{N+\frac{1}{2}},\overline{\overline{y}}_{N+\frac{1}{2}})+\overline{\lambda}_N f(x_{N-\frac{1}{2}},\overline{y}_{N-\frac{1}{2}})+\overline{\overline{\lambda}}_N f(x_{N+\frac{1}{2}},\overline{\overline{y}}_{N+\frac{1}{2}}))$$
$$+\frac{h_N^2}{24}\gamma_N f(x_{N+1},y_{N+1})+y_{N+1}. \quad i=N$$

Let us define column matrix $\boldsymbol{\phi}_{N\times 1}$ and $\mathbf{y}_{N\times 1}$as

$$\boldsymbol{\phi} = [\phi_1,\phi_2,...,\phi_N]'_{1\times N}, \quad \mathbf{y} = [y_1,y_2,...,y_N]'_{1\times N},$$

where $[.....]'$ is the transpose of a column matrix.

The difference method (4.1) represents a system of nonlinear equations in unknown $y_i, i=1,2,...,N$. Let us write (4.1) in matrix form as,

$$\mathbf{D}\mathbf{y}+\boldsymbol{\phi}(\mathbf{y}) = \mathbf{0}, \tag{4.2}$$

where

$$\mathbf{D} = \begin{pmatrix} 1+r_1 & -1 & & & 0 \\ -r_2 & 1+r_2 & -1 & & \\ & -r_3 & 1+r_3 & -1 & \\ .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. \\ 0 & & & -r_N & 1+r_N \end{pmatrix}_{N\times N}$$

is a tridiagonal matrix. Let $\mathbf{Y}$ be the exact solution of (4.1), so it will satisfy the matrix equation

$$\mathbf{D}\mathbf{Y}+\boldsymbol{\phi}(\mathbf{Y})+\mathbf{T} = \mathbf{0}, \tag{4.3}$$

where $\mathbf{Y}$ is a column matrix of order $N\times 1$ which can be obtained by replacing y with Y in matrix $\mathbf{y}$ and $\mathbf{T}$ is a truncation error matrix in which each element has $O(h_i^5)$.

Let us define

$$F_{i+\frac{1}{2}} = f(x_{i+\frac{1}{2}}, Y_{i+\frac{1}{2}}), \quad F_{i-\frac{1}{2}} = f(x_{i-\frac{1}{2}}, Y_{i-\frac{1}{2}}),$$
$$F_{i+1} = f(x_{i+1}, Y_{i+1}), \quad f_{i+1} = f(x_{i+1}, y_{i+1}), \quad F_{i-1} = f(x_{i-1}, Y_{i-1}),$$
$$f_{i-1} = f(x_{i-1}, y_{i-1}), \quad F_i = f(x_i, Y_i), and \quad f_i = f(x_i, y_i).$$

After linearization of $\overline{f}_{i+\frac{1}{2}}$ and using (2.12) into it, we have

$$\overline{f}_{i+\frac{1}{2}} = F_{i+\frac{1}{2}} + (\overline{y}_{i+\frac{1}{2}} - Y_{i+\frac{1}{2}})G_{i+\frac{1}{2}} \tag{4.4}$$

where $G_{i+\frac{1}{2}} = (\frac{\partial f}{\partial Y})_{i+\frac{1}{2}}$. But $\overline{y}_{i+\frac{1}{2}}$ provide $O(h_i^3)$ approximation for $Y_{i+\frac{1}{2}}$. Thus from (4.4), we have

$$\overline{f}_{i+\frac{1}{2}} - F_{i+\frac{1}{2}} = O(h_i^3). \tag{4.5}$$

Similarly, we can linearize $\overline{\overline{f}}_{i+\frac{1}{2}}$, $\overline{f}_{i-\frac{1}{2}}$, $\overline{\overline{f}}_{i-\frac{1}{2}}$, f_{i+1}, f_{i-1}, f_i and use (2.13-2.15), to obtain the following results :

$$\overline{\overline{f}}_{i+\frac{1}{2}} - F_{i+\frac{1}{2}} = O(h_i^5). \tag{4.6}$$

$$\overline{f}_{i-\frac{1}{2}} - F_{i-\frac{1}{2}} = O(h_i^3). \tag{4.7}$$

$$\overline{\overline{f}}_{i-\frac{1}{2}} - F_{i-\frac{1}{2}} = O(h_i^5). \tag{4.8}$$

$$f_{i+1} - F_{i+1} = (y_{i+1} - Y_{i+1})G_{i+1}. \tag{4.9}$$

$$f_{i-1} - F_{i-1} = (y_{i-1} - Y_{i-1})G_{i-1}, \tag{4.10}$$

$$f_i - F_i = (y_i - Y_i)G_i. \tag{4.11}$$

From the difference of (4.2) and (4.3), by taking the Taylor series expansion of $G_{i\pm1}$ about $x = x_i$. Neglecting the terms $O(h_i^3)$ and higher orders, we can write

$$\boldsymbol{\phi}(\mathbf{y}) - \boldsymbol{\phi}(\mathbf{Y}) = \mathbf{PE}, \tag{4.12}$$

where $\mathbf{P} = (P_{lm})_{N\times N}$ is a tri-diagonal matrix defined as

$$P_{lm} = \frac{h_i^2}{24}(\alpha_i G_i), \quad i = l = m, \quad l = 1, 2, ..., N,$$
$$P_{lm} = \frac{h_i^2}{24}\gamma_i(G_i + h_{i+1}(\frac{\partial G}{\partial x})_i), \quad m = l + 1, \quad i = l = 1, 2, ..., N - 1,$$
$$P_{lm} = \frac{h_i^2}{24}\beta_i(G_i - h_i(\frac{\partial G}{\partial x})_i), \quad i = l = m + 1, \quad m = 1, 2, ..., N - 2,$$

and $\mathbf{E} = [E_1, E_2, ..., E_N]'_{1\times N}$, where $E_i = (y_i - Y_i), i = 1, 2, ..., N$.

Let us assume that the solution of difference equation (2.10) has no roundoff error. So from (4.2), (4.3) and (4.12) we have

$$(\mathbf{D} + \mathbf{P})\mathbf{E} = \mathbf{JE} = \mathbf{T}. \tag{4.13}$$

Let us define $G^0 = \{G_i : i = 1, 2, ..., N\}$,

$$G_* = \min_{x\in[a,b]} \frac{\partial f}{\partial Y}, and \quad G^* = \max_{x\in[a,b]} \frac{\partial f}{\partial Y},$$

such that

$$0 \leqslant G_* \leqslant t \leqslant G^*, \quad \forall t \in G^0.$$

We further define $H^0 = \{(\frac{\partial G}{\partial x})_i, \quad i = 1, 2, ..., N\}$. Let there exist some positive constant W such that $\left|t^0\right| \leqslant W, \quad \forall \quad t^0 \in H^0$. So it is possible for very small $h_i, \forall i = 1, 2, ..., N$,

$$|P_{lm}| \leqslant 1 + r_i, \quad \forall \quad i = l = m \quad l = 1, 2, ..., N,$$

$$|P_{lm}| \leqslant 1, \quad \forall \quad m = l + 1, \quad i = l = 1, 2, ..., N - 1,$$

$$|P_{lm}| \leqslant r_i, \quad \forall \quad i = l = m + 1, \quad m = 1, 2, ..., N - 2.$$

Let $\mathbf{R} = [R_1, R_2, ..., R_N]'_{1\times N}$, denotes the row sum of the matrix $\mathbf{J} = (J_{lm})_{N\times N}$ where

$$R_1 = r_1 + \frac{h_1^2}{24}(\alpha_1 + \gamma_1)G_1 + r_1\gamma_1\frac{h_1^3}{24}(\frac{\partial G}{\partial x})_i, \quad l = i = 1,$$

$$R_l = \frac{h_i^2}{24}(\alpha_i+\gamma_i+\beta_i)G_i+\frac{h_i^3}{24}(r_i\gamma_i-\beta_i)(\frac{\partial G}{\partial x})_i, \quad l = i = k, and \quad 2 \leqslant k \leqslant N-1,$$

$$R_N = 1 + \frac{h_N^2}{24}(\alpha_N + \beta_N)G_N - \frac{h_N^3}{24}\beta_N(\frac{\partial G}{\partial x})_N, \quad l = i = N.$$

On neglecting the higher order terms i.e. $O(h_i^3)$ in R_i then it is easy to see that $\mathbf{J}$ is irreducible [**12**]. By the row sum criterion and for sufficiently small $h_i, \forall i = 1, 2, ..., N$, $\mathbf{J}$ is monotone [**14**]. Thus $\mathbf{J}^{-1}$ exist and $\mathbf{J}^{-1} \geqslant 0$. For the bound of $\mathbf{J}$, we define [**15, 16**]

$$d_l(\mathbf{J}) = |J_{ll}| - \sum_{l\neq m}^{N} |J_{lm}|, \quad l = 1, 2, ..., N,$$

where

$$d_1(\mathbf{J}) = r_1 + \frac{h_1^2}{24}(\alpha_1 - \gamma_1)G_1 - \frac{h_1^3}{6}r_1\gamma_1(\frac{\partial G}{\partial x})_1,$$

$$d_l(\mathbf{J}) = \frac{h_i^2}{24}(\alpha_i - \beta_i - \gamma_i)G_i + \frac{h_i^3}{24}(\beta_i - r_i\gamma_i)(\frac{\partial G}{\partial x})_i, \quad l = i = k, and \quad 2 \leqslant k \leqslant N - 1,$$

$$d_N(\mathbf{J}) = 1 + \frac{h_N^2}{24}(\alpha_N - \beta_N)G_N + \frac{h_N^3}{24}\beta_N(\frac{\partial G}{\partial x})_N, \quad l = i = N.$$

We note that higher order terms i.e. $O(h_i^3)$ in the above expressions are neglected. Let $d_l(\mathbf{J}) \geqslant 0, \quad \forall \quad l$ and

$$d_*(\mathbf{J}) = \min_{1\leqslant l\leqslant N} d_l(\mathbf{J}).$$

Then

$$\|\mathbf{J}^{-1}\| \leqslant \frac{1}{d_*(\mathbf{J})}. \tag{4.14}$$

Thus from (4.13) and (4.14), we have

$$\|\mathbf{E}\| \leqslant \frac{1}{d_*(\mathbf{J})}\|\mathbf{T}\|. \tag{4.15}$$

It follows from (3.3) and (4.15) that $\|\mathbf{E}\| \to 0$ as $h_i \to 0$. Thus we conclude that method (2.10) converges and the order of the convergence of method (2.10) is at least cubic.

5. Numerical Results

To illustrate our method and demonstrate its computational efficiency, we considered some model problems. In each model problem, we took non-uniform step size h_i. In Table 1 - Table 3, we have shown the maximum absolute error (MAY), computed for different values of N and is defined as

$$MAY = \max_{1\leqslant i\leqslant N} |y(x_i) - y_i|.$$

The starting value of the step length h_1 is calculated by formula

$$h_1 = \begin{cases} \frac{(b-a)(r-1)}{r^N-1} & \text{if} \quad r > 1 \\ \frac{(b-a)(1-r)}{1-r^N} & \text{if} \quad r < 1 \end{cases}$$

where $r = r_i, \quad \forall \quad i = 1, 2, ..., N$ in computation. The order of the convergence (O_N) of the method (2.10) is estimated by the formula

$$(O_N) = \log_m\left(\frac{MAY_N}{MAY_{mN}}\right),$$

where m can be estimated by considering the ratio of different values of N.

The approximations for f' and f'' in formula (2.10) or (2.11) bring in more complexity in the system and thus more functional evaluation and computational cost. However we have used Newton-Raphson iteration method to solve the system of nonlinear equations arisen from equation (2.10) or (2.11). All computations were performed on a Windows 7 Ultimate operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Core i3-2330M 2.20 Ghz PC. The solutions are computed on N nodes and iteration is continued until either the maximum difference between two successive iterates is less than 10^{-10} or the number of iteration reached 10^5.

Problem 1. The first model problem is a linear problem [**17**] given by

$$y''(x) = \frac{-3\epsilon}{(\epsilon + x^2)^2}y, \quad y(-0.1) = \frac{-0.1}{\sqrt{(\epsilon + 0.01)}}, \quad y(0.1) = \frac{0.1}{\sqrt{(\epsilon + 0.01)}}, \quad x \in [-0.1, 0.1].$$

The analytical solution is $y(x) = \frac{x}{\sqrt{(\epsilon+x^2)}}$. The MAY computed by method (2.10) for different values of N and ϵ are presented in Table 1.

TABLE 1. Maximum absolute errors (Problem 1).

r_1	N	Maximum absolute error $\epsilon = 1.0$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-6}$	$\epsilon = 10^{-8}$	$\epsilon = 10^{-10}$
1.01	200	.49479455(-2)	.18858910(-3)	.95963478(-5)	.26226044(-5)	.41723251(-6)
	400	.49697235(-2)	.15079975(-4)	.66757202(-5)	.71525574(-6)	.47683716(-6)
	800	.50188862(-2)	.91791153(-5)	.67353249(-5)	.77486038(-6)	.10728836(-5)
	1600	.50241202(-2)	.81062317(-5)	.31590462(-5)	.71525574(-6)	.23841858(-6)
1.06	200	.49364120(-2)	.29802322(-6)	.77486038(-6)	.11324883(-5)	.17881393(-6)
	400	.49356222(-2)	.35762787(-6)	.77486036(-6)	.11324883(-5)	.17881393(-6)
	800	.49356222(-2)	.35762787(-6)	.77486036(-6)	.11324883(-5)	.17881393(-6)

Problem 2. The second model problem is a nonlinear problem

$$\epsilon y''(x) = \frac{3}{2}y^2, \quad y(0) = 4, \quad y(1) = \frac{4}{(1+\frac{1}{\sqrt{\varepsilon}})^2}, \quad x \in [0, 1].$$

The analytical solution is $y(x) = \frac{4}{(1+\frac{x}{\sqrt{\varepsilon}})^2}$. The MAY computed by method (2.10) for different values of N are presented in Table 2.

TABLE 2. Maximum absolute errors with comparison to [**11**] (Problem 2).

$r_1 = 1.08$	N	Maximum absolute error $\epsilon = 1.0$	$\epsilon = 10.0$	$\epsilon = 100.0$	$\epsilon = 1000.0$
method (2.10)	4	.21700012(-2)	.81062317(-5)	.23841858(-6)	.00000000
	8	.75817108(-4)	.23841858(-6)	.23841858(-6)	.23841858(-6)
	16	.95367432(-6)	.23841858(-6)	.23841858(-6)	.23841858(-6)
	32	.23841858(-6)	.23841858(-6)	.23841858(-6)	.23841858(-6)
	64	.23841858(-6)	.23841858(-6)	.23841858(-6)	.23841858(-6)
	128	.23841858(-6)	.23841858(-6)	.23841858(-6)	.23841858(-6)
	256	.11920929(-5)	.11920929(-5)	.11920929(-5)	.11920929(-5)
method in [**11**]	4	.24600029(-1)	.12617111(-2)	.23126602(-4)	.23841858(-6)
	8	.60970783(-2)	.31232834(-3)	.52452087(-5)	.23841858(-6)
	16	.14939308(-2)	.87022781(-4)	.23841858(-6)	.23841858(-6)
	32	.53775311(-3)	.31709671(-4)	.23841858(-6)	.23841858(-6)
	64	.36931038(-3)	.18835068(-4)	.23841858(-6)	.23841858(-6)
	128	.35333633(-3)	.19550323(-4)	.23841858(-6)	.23841858(-6)
	256	.35226345(-3)	.18119812(-4)	.23841858(-6)	.23841858(-6)

Problem 3. The third model problem is a linear problem [**18**] given by

$$\epsilon y''(x) = \frac{4}{(x+1)^4}(1+\sqrt{\epsilon}(x+1))y - f(x), \quad y(0) = 2, \quad y(1) = -1, \quad x \in [0, 1],$$

where $f(x)$ is calculated so that

$$y(x) = -\cos(\frac{4\pi x}{x+1}) + \frac{3[\exp(\frac{-2x}{\sqrt{\epsilon}(x+1)}) - \exp(\frac{-1}{\sqrt{\epsilon}})]}{1 - \exp(\frac{-1}{\sqrt{\epsilon}})}$$

is the analytical solution. The MAY computed by method (2.10) for different values of N and ϵ are presented in Table 3

TABLE 3. Maximum absolute errors with comparison to [**11**] (Problem 3).

$r_1 = 1.06$	ϵ	Maximum absolute error $N = 100$	$N = 500$	$N = 1000$	$N = 1500$
method (2.10)	1.0	.32806396(-3)	.81896782(-4)	.8428968(-4)	.81658363(-4)
	2^{-4}	.97751617(-5)	.11980534(-4)	.11861324(-4)	.12099743(-4)
	2^{-6}	.54240227(-5)	.56922436(-5)	.56624413(-5)	.54836273(-5)
	2^{-8}	.14781952(-4)	.14960766(-4)	.14930964(-4)	.14662743(-4)
	2^{-10}	.21040440(-4)	.20861626(-4)	.20891428(-4)	.20593405(-4)
	2^{-12}	.29563904(-4)	.29027462(-4)	.28967857(-4)	.28938055(-4)
	2^{-14}	.68480372(-1)	.65816164(-1)	.65812349(-1)	.65810800(-1)
	2^{-16}	.97610575(0)	.94299066(0)	.94299060(0)	.94299066(0)
	2^{-18}	.27835369(-4)	.12516975(-5)	.26702881(-4)	.12516975(-5)
	2^{-20}	.22050738(-3)	.11920929(-5)	.11920929(-5)	.11920929(-5)
	2^{-22}	.26870966(-2)	.10728836(-5)	.10728836(-5)	.10728836(-5)
	2^{-24}	.48569411(-1)	.11920929(-5)	.11920929(-5)	.11920929(-5)
method in [**11**]	1.0	.38763934(-2)	.33957958(-2)	.33957660(-2)	.33954034(-2)
	2^{-4}	.19446164(-2)	.19327551(-2)	.19329339(-2)	.19328594(-2)
	2^{-6}	.11941493(-2)	.11839569(-2)	.11841953(-2)	.11840165(-2)
	2^{-8}	.48494339(-3)	.48014522(-3)	.48032403(-3)	.48032403(-3)
	2^{-10}	.43356419(-3)	.36138296(-3)	.36138296(-3)	.36132336(-3)
	2^{-12}	.49465895(-3)	.34594536(-3)	.34594536(-3)	.34594536(-3)
	2^{-14}	.67013502(-3)	.33903122(-3)	.33909082(-3)	.33909082(-3)
	2^{-16}	.11677146(-2)	.33509731(-3)	.33515692(-3)	.33521652(-3)
	2^{-18}	.27157217(-2)	.33360720(-3)	.33360720(-3)	.33372641(-3)
	2^{-20}	.79871528(-2)	.33271313(-3)	.33271313(-3)	.33271313(-3)
	2^{-22}	.25293380(-1)	.33223629(-3)	.33217669(-3)	.33223629(-3)
	2^{-24}	.85839853(-1)	.33134222(-3)	.33128262(-3)	.33128262(-3)

We have described a high order method for numerically solving two-point boundary value problems and several model problems considered to demonstrate the performance of the proposed method. Numerical result for examples 1 which is presented in table 1, for different values of r_i show as ϵ decreases and N increases, for the non-uniform mesh size, maximum absolute error decreases but not substantial. The numerical results for examples 2, is maximum absolute error either decreases or remains same as there are change in N but ϵ remains same. The results for examples 3, the maximum absolute error increases as ϵ decreases and N increases. Over all method (2.10) is convergent and convergence of the method depends on choice of mesh ratio r_i. The main advantage of the proposed high order method over method in [**11**] is its computational efficiency in considered model problems.

6. Conclusion

A new non-uniform mesh size high order method to find the numerical solution of two point boundary value problems has been developed. At each nodal point $x = x_i, i = 1, 2, ..., N$, we will obtain a system of algebraic equations given by (2.10). If the source function is f(x) then the system of equations from (2.10) is linear otherwise we will obtain nonlinear system of equations. It is obvious that special method required for some special problem where the solution is not regular and varies rapidly, so non-uniform mesh size method is natural choice. The new high order method produces good numerical approximate solutions for variety of model problems with non-uniform mesh i.e. $r_i \neq 1$. The numerical results of the model problems showed that the new high order method is computationally efficient. The rate of convergence of the present method is cubic. The idea presented in this article leads to the possibility to develop non-uniform mesh size difference methods to solve third order and fourth order boundary value problems in ordinary differential equations. Works in these directions are in progress.

References

[1] U. Ascher, R. M. M. Mattheij and R. D. Russell. *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations*. Prentice-Hall, Englewood Cliffs, NJ (1988).

[2] L. Collatz. *Numerical Treatment of Differential Equations* (3/ed.). Springer Verlag, Berlin (1966).

[3] H. B. Keller. *Numerical Methods for Two Point Boundary Value Problems*. Blaisdell Publ. Co., New York, Waltham, Mass (1968).

[4] J. Stoer and R. Bulirsch. *Introduction to Numerical Analysis*. Springer-Verlag, Berlin Heidelberg (2002).

[5] J. V. Baxley. Nonlinear Two Point Boundary Value. In W.N.Everrit and B.D.Sleeman (Eds.). *Problems in Ordinary and Partial Differential Equations* (pp. 46-54). Springer Lecture Notes in Mathematics, No. 846, Springer-Verlag, New York (1981).

[6] H. G. Roos, M. Stynes and L. Tobiska. Numerical Methods for Singularly Perturbed Differential Equations. Springer, New York (2006).

[7] E. Hairer, S. P. Nrsett and G. Wanner. Solving Ordinary Differential Equations I Nonstiff Problems (Second Revised Edition). Springer-Verlag New York, Inc. New York, NY, USA (1993).

[8] M. K. Jain. Numerical Solution of Difierential Equations. Wiley Eastern Limited, New Delhi, (1984).

[9] F. D. Van Niekerk. A nonlinear one step methods for initial value problems. *Comp. Math. Appl.* **13**(4)(1987), 367-371.

[10] P. K. Pandey. Solving Two Point Boundary Value Problems for Ordinary Differential Equations Using Exponential Finite Difference Method. *Bol. Soc. Paran. Mat.*, **34**(1)(2016), 45-52.

[11] P. K. Pandey and B. D. Pandey. Variable Mesh Size Exponential Finite Difference Method for the Numerical Solutions of Two Point Boundary Value Problems. *Bol. Soc. Paran. Mat.*, **34**(2)(2016), 9-27.

[12] R. S. Varga. Matrix Iterative Analysis, Second Revised and Expanded Edition. Springer-Verlag, Heidelberg, (2000).

[13] G. H. Golub and C. F. Van Loan. Matrix Computations. John Hopkins University Press, Baltimore, USA (1989).

[14] P. Henrici. *Discrete Variable Methods in Ordinary Differential Equations*. John Wiley and Sons, New York, (1962).

[15] Yu. S. Volkov and V. L. Miroshnichenko. Norm estimates for the inverses of matrices of monotone type and totally positive matrics. *Sib. Math. J.*, (2009) 50: 982 [Translated from *Sibirski Matematicheski Zhurnal*, **50**(6)(2009), 1248-1254].
[16] J. H. Ahlberg and E. N. Nilson. Convergence properties of the spline fit. *J. Soc. Indus Appl. Math.* , **11**(1)(1963), 95-104.
[17] M. R. Scott and H. A. Watts. Computational solution of linear two point boundary value problems via orthonormalization. *SIAM J. Numer. Anal.*, **14**(1)(1977), 40-70.
[18] E. O'Riordan and M. Stynes. A uniformely accurate finite element method for a singularly perturbed one-dimensional reaction-diffusion problem. *Math. Comp.*, **47**(1986), 555-570.

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