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SOME COINCIDENCE AND COMMON FIXED POINT THEOREMS CONCERNING F-CONTRACTION AND APPLICATIONS

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Abstract. The aim of this paper is to establish coincidence and common fixed point theorems for a discontinuous noncompatible pair of self-maps in noncomplete metric space without containment requirement of range space of involved maps acknowledging the notion of F-contraction introduced by Wardowski [D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory and Applications (2012) 2012:94, 6 pages, doi: 10.1186/1687-1812-2012-94]. Our results generalize, extend and improve analogous results existing in literature and are supported with the help of illustrative examples associated with pictographic validations to demonstrate the authenticity of the postulates. Solution of two-point boundary value problem of the second order differential equation arising in electric circuit equation and Volterra type integral equation using Ćirić type as well as Hardy-Rogers-type F-contaction are also given to exhibit the usability of results obtained.

1. Introduction

Bisht and Shahzad [**4**] introduced the notion of faint compatibility as an improvement of conditional compatibility introduced by Pant and Bisht [**16**], which allowed the existence of a common fixed point, multiple common fixed points, coincidence points and multiple coincidence points and is suitable for contraction, strict contractive, contractive and non-contractive conditions. Recently Tomar et al. [**25**] extended the notion of faint compatibility to a hybrid pair of maps. However it is well known that weak compatibility is most widely used concept among all weaker forms of commuting maps in fixed-point considerations but is not applicable when

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a pair of self maps has more than one coincidence points. For details on a brief development of weaker forms of commuting maps, one may refer to Singh and Tomar [**20**]. The aim of this paper is to establish the existence and uniqueness of coincidence and common fixed point of discontinuous non-compatible faintly compatible pair of self maps in non-complete metric space without using containment requirement of range space of involved maps via Ćirić type F-contraction and Hardy-Roger type F-contraction, which are more general than the F-contraction introduced by Wardowski [**26**]. Results obtained are verified with the help of illustrative examples associated with pictographic validations to demonstrate the authenticity of the postulates. Also inspired by the fact that the study of two-point boundary value problem related with second order differential equation plays a significant role in the real world problems and scientific research, we solve two-point boundary value problem of the second order differential equation arising in electric circuit equation. Further Volterra type integral equation is solved using Ćirić type F-contraction as well as Hardy Roger type F-contraction.

2. Preliminaries

A pair of self maps f and g have a coincidence point at $x \in X$ if $fx = gx$. Further, a point $x \in X$ is a common fixed point of f and g if $fx = gx = x$. In this paper we denote the set of all real numbers by $\mathbb{R}$, the set of all positive real numbers by $\mathbb{R}^+$ and the set of all natural numbers by $\mathbb{N}$.

Definition 2.1. ([**26**]) Let (X, d) be a metric space. A map $f : X \to X$ is an F-contraction if there exists $\tau > 0$ such that $\tau + F(d(fx, fy)) \leqslant F(d(x, y))$ for all $x, y \in X$ with $fx \neq fy$, where $F : \mathbb{R}^+ \to \mathbb{R}$ is a function satisfying:

(1) F is strictly increasing, i.e., for all $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta, F(\alpha) < F(\beta)$;
(2) For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(3) There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

We denote by $\mathcal{F}$, the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the conditions (1)-(3).

Every F-contraction is a contractive map i.e., $d(fx, fy) < d(x, y)$ for all $x, y \in X, fx \neq fy$ and hence is necessarily continuous. In fact Banach contraction [**1**] is a particular case of F-contraction. Meanwhile there exist F-contractions, which are not Banach contractions (Wardowski [**26**]).

Taking different functions F, we obtain a variety of F-contractions, some of them being already known in the literature. Some examples of the functions belonging to F are:

(1) $F(\alpha) = ln\alpha$;
(2) $F(\alpha) = ln\alpha + \alpha, \alpha > 0$;
(3) $F(\alpha) = \frac{-1}{\sqrt{\alpha}}, \alpha > 0$;
(4) $F(\alpha) = ln(\alpha^2 + \alpha), \alpha > 0$.

Definition 2.2. A pair of self-maps (f, g) on a metric space (X, d) is

(1) **Compatible** ([**10**]), if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$, for some $u \in X$.
(2) **non − compatible**, if (f, g) is not compatible, i.e., if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$, but either $\lim_{n\to\infty} d(fgx_n, gfx_n) \neq 0$ or non existent.
(3) **weakly compatible** ([**11**]), if the pair commute on the set of their coincidence points, i.e., for $x \in X, fx = gx$ implies $fgx = gfx$.
(4) **conditionally compatible** ([**16**]), if whenever the set of sequences $\{x_n\}$ satisfying $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$ is non-empty, there exists a sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = u$, for some $u \in X$ and $\lim_{n\to\infty} d(fgy_n, gfy_n) = 0$.
(5) **reciprocally continuous** ([**15**]), if $\lim_{n\to\infty} fgx_n = fx, \lim_{n\to\infty} gfx_n = gx$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$, for some $u \in X$.
(6) **conditionally reciprocally continuous** ([**3**]), iff whenever the set of sequences $\{x_n\}$ satisfying $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$ is non empty, there exists a sequence $\{y_n\}$ satisfying $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = u$ (say) for some $u \in X$ such that $\lim_{n\to\infty} fgy_n = fu$ and $\lim_{n\to\infty} gfy_n = gu$.
(7) **faintly compatible** ([**4**]), if (f, g) is conditional compatible and f and g commute on a non-empty subset of the set of coincidence points, whenever the set of coincidence points is nonempty.

Faint compatibility does not reduce to the class of compatibility in the presence of unique common fixed point (or unique coincidence point) like most of the weaker forms of compatibility existing in literature ([**4, 20**]) and is independent of compatibility and non-compatibility.

3. Main results

Following Minak et al. [**14**] and Wardowski and Dung [**27**], we now extend notion of Ćirić type F-contraction to a pair of maps. For a single valued map Minak et al. [**14**] introduced it as Ćirić type generalized F-contraction and independently Wardowski and Dung [**27**] introduced it as F-weak contraction.

Definition 3.1. A pair of self maps (f, g) of a metric space (X, d) is said to be Ćirić type F-contraction if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$:

(1) $d(fx, fy) > 0 \Rightarrow \tau + F(d(fx, fy)) \leqslant F(M(x, y))$,

where

$$M(x, y) = \max\{d(gx, gy), d(gx, fx), d(gy, fy), \frac{[d(gx, fy) + d(gy, fx)]}{2}\}.$$

Notice that every Ćirić type F-contraction for a pair of self maps is also a F-contraction but the reverse implication does not always hold.

Now we prove our main result using Ćirić type F-contraction for a faintly compatible pair of maps using conditional reciprocal continuity which is weaker than continuity of even single map.

THEOREM 3.1. *Let a faintly compatible pair of self maps (f, g) of a metric space (X, d) be conditionally reciprocally continuous. Then f and g have a coincidence point. Moreover f and g have a unique common fixed point provided that the pair of self maps (f, g) satisfies Ćirić type F-contraction(1).*

PROOF. Let $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t,$$

for some $t \in X$. Since pair (f, g) is also faintly compatible, there exists a sequence $\{y_n\}$ in X satisfying

$$\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = u,$$

for some $u \in X$ such that

$$\lim_{n\to\infty} d(fgy_n, gfy_n) = 0.$$

As pair (f, g) is also conditionally reciprocally continuous, we get

$$\lim_{n\to\infty} fgy_n = fu$$

and

$$\lim_{n\to\infty} gfy_n = gu.$$

Hence $fu = gu$, i.e., f and g have a coincidence point. Since the pair (f, g) is faintly compatible, we get $fgu = gfu$. So $ffu = fgu = gfu = ggu$. If $fu \neq ffu$, by (1),

$$\begin{aligned}\tau + F(d(fu, ffu)) \leqslant F(\max\{&d(gu, gfu), d(gu, fu), d(gfu, ffu),\\ &\frac{[d(gu, ffu) + d(gfu, fu)]}{2}\}),\end{aligned}$$

i.e., $\tau + F(d(fu, ffu)) \leqslant F(d(fu, ffu))$, a contradiction.

Hence fu is a common fixed point of f and g. The uniqueness of the common fixed point is an easy consequence of the condition (1). □

EXAMPLE 3.1. Let $X = (2, 8)$ and d be the usual metric on X. Define $f, g : X \to X$ as follows:

$$fx = \begin{cases} 3, & x \leqslant 3 \\ 5, & x > 3, \end{cases} \qquad gx = \begin{cases} 6 - x, & x \leqslant 3 \\ 7, & x > 3. \end{cases}$$

(1) Let $\{x_n\}$ be a sequence in X such that $x_n = 3 - \frac{1}{n}$, then $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 3$ and $\lim_{n\to\infty} fgx_n = \lim_{n\to\infty} f(3+\frac{1}{n}) = 5, \lim_{n\to\infty} gfx_n = \lim_{n\to\infty} g3 = 3$, i.e., $\lim_{n\to\infty} d(fgx_n, gfx_n) \neq 0$, i.e., pair (f, g) is non compatible. Also $\lim_{n\to\infty} fgx_n = 5 \neq f3$, i.e., pair (f, g) is not reciprocally continuous.

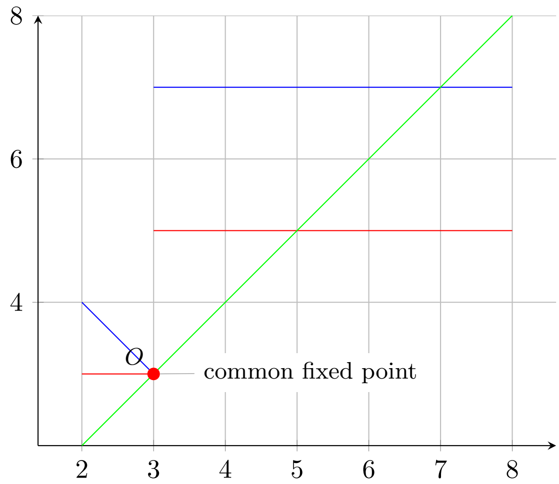
(2) Let $\{y_n\}$ be a sequence in X such that $y_n = 3$, then $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = 3$ and $\lim_{n\to\infty} gfy_n = 3, \lim_{n\to\infty} fgy_n = 3$, i.e., $\lim_{n\to\infty} d(fgy_n, gfy_n) = 0$. Therefore pair (f, g) is conditionally compatible. Also $f3 = g3, fg3 = gf3$, i.e., pair (f, g) is faintly compatible.

(3) Further $\lim_{n\to\infty} fgy_n = 3 = f3, \lim_{n\to\infty} gfy_n = 3 = g3$, pair (f, g) is conditionally reciprocally continuous.

(4) Also f and g satisfy Ćirić type F-contraction for $\tau = 0.04$ and $F(\alpha) = log\alpha$.

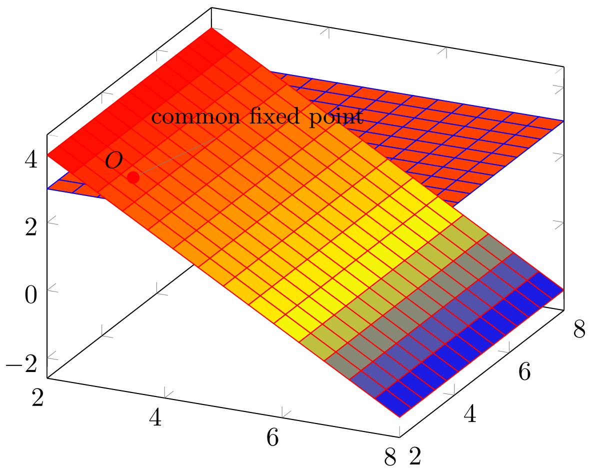
Hence all the conditions of Theorem 3.1 are satisfied and $x = 3$ is a unique coincidence and common fixed point of f and g. Moreover both the self maps are discontinuous at common fixed point and are neither compatible nor reciprocally continuous. Further $fX \not\subseteq gX$.

Fig.1 (2D-View)



- In fig.1:(2D-view), the red line denotes fx, the blue line denotes gx and the green line denotes the line $y = x$. Clearly, the functions f and g intersect on the line $y = x$ only at $x = 3$, i.e., $x = 3$ is the unique common fixed point of f and g.

Fig.2(3D-view)



- In fig.2:(3D-view), the plane with the blue lines denotes fx and the plane with the red lines denotes gx. Clearly both of the planes intersect at $x = 3$. Hence $x = 3$ is unique common fixed point of f and g.

Now we extend Hardy-Rogers-type F-contraction introduced by Cosentino and Vetro [**7**] to a pair of map.

DEFINITION 3.2. A pair of self maps (f, g) of a metric space (X, d) is said to be Hardy-Rogers-type F-contraction if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$:

$$(2)\ d(fx, fy) > 0 \Rightarrow \tau + F(d(fx, fy)) \leqslant F(M(x, y)),$$

where

$$M(x, y) = \alpha d(gx, gy) + \beta d(gx, fx) + \gamma d(gy, fy) + \delta[d(gx, fy) + d(gy, fx)],$$

for $\alpha, \beta, \gamma \geqslant 0, \alpha + \beta + \gamma + 2\delta < 1$.

Notice that every Hardy-Rogers-type F-contraction for a pair of self maps is also a F-contraction but the reverse implication does not always hold.

Now we prove the next result using Hardy-Rogers-type F-contraction.

THEOREM 3.2. *Let a faintly compatible pair of self maps (f, g) of a metric space (X, d) be conditionally reciprocally continuous. Then f and g have a coincidence point. Moreover f and g have a unique common fixed point provided that the pair of self maps (f, g) satisfies Hardy-Rogers-type F-contraction(2).*

PROOF. For all $x, y \in X$, we have

$$\alpha d(gx, gy) + \beta d(gx, fx) + \gamma d(gy, fy) + \delta[d(gx, fy) + d(gy, fx)]$$

$$\leqslant (\alpha + \beta + \gamma + 2\delta) \max\{d(gx, gy), d(gx, fx), d(gy, fy), \frac{[d(gx, fy) + d(gy, fx)]}{2}\}$$

$$< \max\{d(gx,gy), d(gx,fx), d(gy,fy), \frac{[d(gx,fy)+d(gy,fx)]}{2}\}.$$

Rest of the proof is similar to Theorem 3.1. □

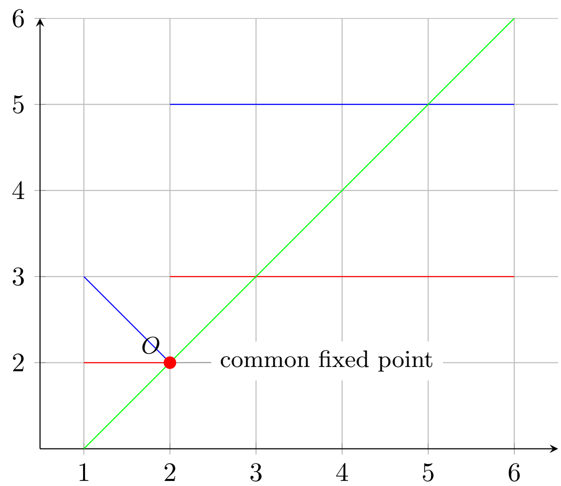
It is interesting to point out here that Theorem 3.1 is an easy consequence of Theorem 3.2.

EXAMPLE 3.2. Let $X = (1,6)$ and d be the usual metric in X. Define $f, g : X \to X$ as follows:

$$fx = \begin{cases} 2, & x \leqslant 2 \\ 3, & x > 2, \end{cases} \qquad gx = \begin{cases} 4-x, & x \leqslant 2 \\ 5, & x > 2. \end{cases}$$

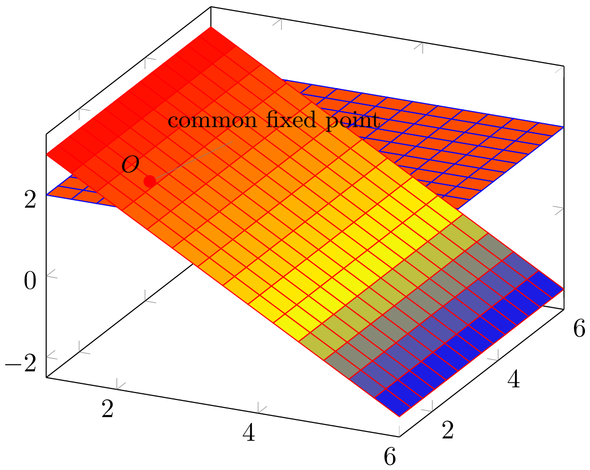
(1) Let $\{x_n\}$ be a sequence in X such that $x_n = 2 - \frac{1}{n}$, then $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 2$ and $\lim_{n\to\infty} fgx_n = 3, \lim_{n\to\infty} gfx_n = 2$, i.e., $\lim_{n\to\infty} d(fgx_n, gfx_n) \neq 0$, i.e., pair (f,g) is non compatible. Also $\lim_{n\to\infty} fgx_n = 3 \neq f2$, i.e., pair (f,g) is not reciprocally continuous.
(2) Let $\{y_n\}$ be a sequence in X such that $y_n = 2$, then $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = 2$, $\lim_{n\to\infty} gfy_n = \lim_{n\to\infty} fgy_n = 2$, i.e., $\lim_{n\to\infty} d(fgy_n, gfy_n) = 0$. Therefore f and g are conditionally compatible. Also $f2 = g2, fg2 = gf2$, i.e., pair (f,g) is faintly compatible.
(3) Further $\lim_{n\to\infty} fgy_n = 2 = f2, \lim_{n\to\infty} gfy_n = 2 = g2$.
Hence (f,g) is conditionally reciprocally continuous.
(4) Further f and g satisfy Hardy-Rogers type F-contraction (2) for $\tau = 0.01$ and $F(x) = logx, \alpha = \frac{1}{5}, \beta = \frac{1}{5}, \gamma = \frac{1}{6}, \delta = \frac{1}{12}$.
Hence all the conditions of Theorem 3.2 are satisfied and $x = 2$ is a unique coincidence and common fixed point of f and g. Moreover both the self maps are discontinuous at common fixed point and are neither compatible nor reciprocally continuous. Further $fX \not\subseteq gX$.

Fig.3(2D-View)



- In fig.3:(2D-view), the red line denotes fx, the blue line denotes gx and the green line denotes the line $y = x$. Clearly, the functions f and g intersect on the line $y = x$ only at $x = 2$, i.e., $x = 2$ is the unique common fixed point of f and g.

Fig.4(3D-View)



- In fig.4:(3D-view), the plane with the blue lines denotes fx and the plane with the red lines denotes gx. Clearly both of the planes intersect at $x = 2$. Hence $x = 2$ is unique common fixed point of f and g.

Now we extend Hardy-Rogers-type F-contraction introduced by Cosentino and Vetro [**7**] to a pair of self map.

Definition 3.3. A pair of self maps (f, g) of a metric space (X, d) is said to be weak Hardy-Rogers-type F-contraction if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$:

$$(3)\ d(fx, fy) > 0 \Rightarrow \tau + F(d(fx, fy)) \leqslant F(M(x, y)),$$

where

$$M(x, y) = \alpha d(gx, gy) + \beta d(gx, fx) + \gamma d(gy, fy) + \delta d(gx, fy) + Ld(gy, fx),$$

for $\alpha, \beta, \gamma, \delta, L \geqslant 0$ such that $\alpha + \beta + \gamma + \delta + L < 1$ and $\gamma \neq 1$.

Theorem 3.3. *Let a faintly compatible pair of self maps (f, g) of a metric space (X, d) be conditionally reciprocally continuous. Then f and g have a coincidence point. Moreover f and g have a unique common fixed point provided that the pair of self maps (f, g) satisfies weak Hardy-Rogers-type F-contraction (3).*

Proof. Proof follows on the same lines as of Theorem 3.2. □

Example 3.3. Let $X = (3, 15)$ and d be the usual metric in X. Define $f, g : X \to X$ as follows:

$$fx = \begin{cases} 4, & x \leqslant 4 \\ 8, & x > 4, \end{cases} \qquad gx = \begin{cases} 8 - x, & x \leqslant 4 \\ 14, & x > 4. \end{cases}$$

(1) Let $\{x_n\}$ be a sequence in X such that $x_n = 4 - \frac{1}{n}$, then $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 4$ and $\lim_{n\to\infty} fgx_n = 8, \lim_{n\to\infty} gfx_n = 4$, i.e., $\lim_{n\to\infty} d(fgx_n, gfx_n) \neq 0$. Hence pair (f, g) is non compatible. Also $\lim_{n\to\infty} fgx_n = 8 \neq g4$, i.e., pair (f, g) is not reciprocally continuous.

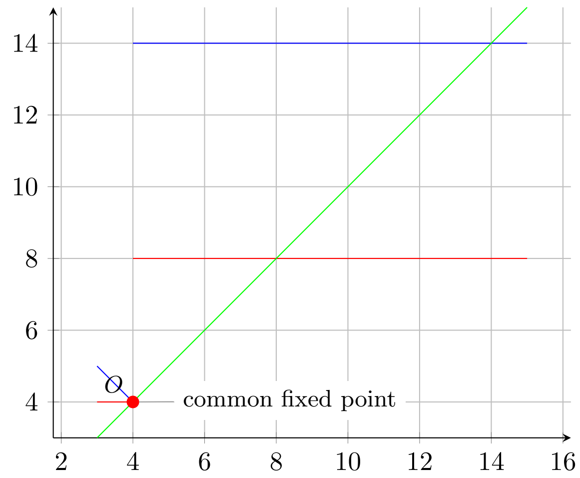
(2) Let $\{y_n\}$ be a sequence in X such that $y_n = 4$, then $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = 4$, $\lim_{n\to\infty} gfy_n = 4, \lim_{n\to\infty} fgy_n = 4$, i.e., $\lim_{n\to\infty} d(fgy_n, gfy_n) = 0$. Therefore f and g are conditionally compatible. Also $f4 = g4, fg4 = gf4$, i.e., pair (f, g) is faintly compatible.

(3) Further $\lim_{n\to\infty} fgy_n = 4 = f4, \lim_{n\to\infty} gfy_n = 4 = g4$, i.e., (f, g) is conditionally reciprocally continuous.

(4) Also f and g satisfy Hardy-Rogers type F-contraction condition (3) for $\tau = \frac{1}{20}$ and $F(x) = -\frac{1}{\sqrt{x}}, \alpha = \frac{1}{9}, \beta = \frac{1}{9}, \gamma = \frac{1}{9}, \delta = \frac{1}{3}, L = \frac{1}{9}$.

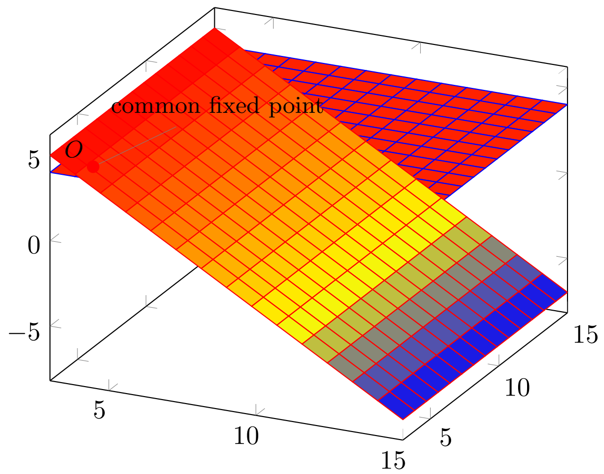
Hence all the conditions of Theorem 3.3 are satisfied and $x = 4$ is a unique coincidence and common fixed point of f and g. Moreover both the self maps are discontinuous at common fixed point and are neither compatible nor reciprocally continuous. Further $fX \not\subseteq gX$.

Fig.5(2D-View)



- In fig.5:(2D-view), the red line denotes fx, the blue line denotes gx and the green line denotes the line $y = x$. Clearly, the functions f and g intersect on the line $y = x$ only at $x = 4$, i.e., $x = 4$ is the unique common fixed point of f and g.

Fig.6(3D-View)



- In fig.6:(3D-view), the plane with the blue lines denotes fx and the plane with the red lines denotes gx. Clearly both of the planes intersect at $x = 4$. Hence $x = 4$ is unique common fixed point of f and g.

Further, putting $\alpha = \delta = L = 0$ and $\beta = \gamma \in [0, \frac{1}{2}]$ in (3) we obtain the following version of Kannan's result [**12**].

Corollary 3.1. *Let a faintly compatible pair of self maps (f, g) of a metric space (X, d) be conditionally reciprocally continuous. Then f and g have a coincidence point. Moreover f and g have a unique common fixed point provided that the pair of self maps (f, g) satisfies:*

$$(4)\ d(fx, fy) > 0 \Rightarrow \tau + F(d(fx, fy)) \leqslant F(M(x, y)),$$

where

$$M(x, y) = \beta[d(gx, fx) + d(gy, fy)]$$

for all $x, y \in X$, $\beta \in [0, \frac{1}{2}]$, $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$.

A following version of the Chatterjee's [**5**] fixed point Theorem is obtained from the Theorem 3.3 putting $\alpha = \beta = \gamma = 0$ and $\delta = L \in [0, \frac{1}{2}]$.

Corollary 3.2. *Let a faintly compatible pair of self maps (f, g) of a metric space (X, d) be conditionally reciprocally continuous. Then f and g have a coincidence point. Moreover f and g have a unique common fixed point provided that the pair of self maps (f, g) satisfies:*

$$(5)\ d(fx, fy) > 0 \Rightarrow \tau + F(d(fx, fy)) \leqslant F(M(x, y)),$$

where

$$M(x, y) = \delta[(gx, fy) + d(gy, fx)]$$

for all $x, y \in X$, $\delta \in [0, \frac{1}{2}]$, $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$.

If we choose $\delta = L = 0$, we obtain a Reich [**17**] type result.

COROLLARY 3.3. *Let a faintly compatible pair of self maps (f, g) of a metric space (X, d) be conditionally reciprocally continuous. Then f and g have a coincidence point. Moreover f and g have a unique common fixed point provided that the pair of self maps (f, g) satisfies:*

$$\text{(6) } d(fx, fy) > 0 \Rightarrow \tau + F(d(fx, fy)) \leqslant F(M(x, y)),$$

where

$$M(x, y) = \alpha d(gx, gy) + \beta d(gx, fx) + \gamma d(gy, fy)$$

for all $x, y \in X$, $\alpha + \beta + \gamma < 1$, $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$.

Finally, taking $g = I$(the identity map of X) in Theorem 3.3 and $\alpha < 1$, $\beta = \gamma = \delta = L = 0$, we obtain Theorem 2.1 of Wardowski [**26**] as a corollary to our result.

COROLLARY 3.4. *Let a faintly compatible pair of self maps (f, g) of a metric space (X, d) be conditionally reciprocally continuous. Then f and g have a coincidence point. Moreover f and g have a unique common fixed point provided that the pair of self maps (f, g) satisfies:*

$$\text{(7) } d(fx, fy) > 0 \Rightarrow \tau + F(d(fx, fy)) \leqslant F(d(x, y)),$$

where $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$.

Remarks (**1**). In all the results, unique coincidence and common fixed point is established for a pair of discontinuous self maps without using containment of range space of involved maps and completeness (or closedness) of underlying space. Moreover the notion of faint compatibility, which is weaker than commutativity of a pair of maps is used, thereby extending and improving the result of Batra et al. [**2**] which establishes only unique coincidence point for F-g-contraction by taking containment of range space of involved maps, completeness of space along with continuity and commutativity of both the maps.

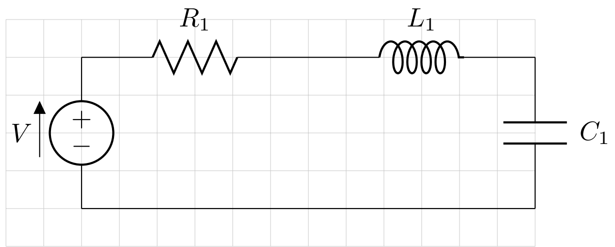
(**2**). The notion of conditional reciprocal continuity used is weaker than most of the variants of continuity. For details on this aspect one may refer to Tomar and Karapinar [**23**].

(**3**). It is interesting to note that Ćirić type F-contraction, Hardy-Rogers type F-contractions and weak Hardy-Rogers type F-contraction are more general than the analogous contractions existing in the literature, since F-contraction is proper generalization of ordinary contraction. Our results generalize, extend and improve multitude of common fixed point results existing in the literature (for instance Banach [**1**], Batra et al. [**2**], Bisht and Pant [**3**], Bisht and Shahzad [**4**], Chatterjea [**5**], Ćirić [**6**], Cosentino and Vetro [**7**], Djoric et al. [**8**], Hardy-Rogers [**9**], Kannan [**12**], Manro and Tomar [**13**], Minak et al. [**14**], Pant [**15**], Pant and Bisht [**16**], Reich [**17**], Shukla and Radenovic [**18**], Shukla et al. [**19**], Tomar et al. [**21**]-[**22**], Tomar and Upadhyay [**24**], Wardowski [**26**], Wardowski and Dung [**27**] and references there in).

4. Applications

4.1. Application to electric circuit equation. As an application of our main result now we solve electric circuit equation which is in the form of second order differential equation. It is well known that electric circuit contains an electromotive force E (supplied by a battery or generator), a resistor R, an inductor L and a capacitor C in series. If the current I is the rate of change of charge q with respect to time t, $I = \frac{dq}{dt}$. We are familiar with the following relations:

(1) $V = IR$;
(2) $V = \frac{q}{C}$;
(3) $V = L\frac{dI}{dt}$,



where V = voltage. Now by Kirchhoffs voltage law, the sum of these voltage drops is equal to the supplied voltage, i.e.,

$$IR + \frac{q}{c} + L\frac{dI}{dt} = V(t),$$

or

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = V(t);$$

$$q(0) = 0, q'(0) = 0. \tag{4.1}$$

The Green function associated to (4.1) is given by

$$G(t,s) = \begin{cases} -se^{\tau(s-t)}, & 0 \leqslant s \leqslant t \leqslant 1 \\ -te^{\tau(s-t)}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \tag{4.2}$$

where $\tau > 0$ is a constant, calculated in terms of R and L. Let $X = C([0,a], \mathbb{R}^+)$ be the set of all non negative real valued functions defined on $[0,a]$. For an arbitrary $u \in X$, we define

$$\|u\|_\tau = \sup_{t\in[0,a]} \{|x(t)|e^{-2\tau t}\}. \tag{4.3}$$

Define $d : X \times X \to \mathbb{R}^+$ by

$$d(u,v) = \|u-v\|_\tau = \sup_{t\in[0,a]} \{|u(t)-v(t)|e^{-2\tau t}\}. \tag{4.4}$$

Then clearly (X,d) is a metric space. We now state and the prove the result for the existence of a solution of the LCR-circuit equation of the second order differential equation:

Theorem 4.1. *Let $f,g : C([0,a]) \to C([0,a])$ be self maps of a metric space (X,d) such that the following conditions hold:*

(1) *there exists a function $K : [0,a]\times[0,a]\times\mathbb{R} \to \mathbb{R}$ such that $|K(t,s,u) - K(t,s,v)| \leqslant \tau^2 e^{-\tau} M(u,v)$, where $M(u,v) = \max\{d(gu,gv), d(gu,fu), d(gv,fv), \frac{[d(gu,fv)+d(gv,fu)]}{2}\}$, for $F \in \mathcal{F}$, $\tau \in \mathbb{R}^+$, $t,s\in[0,a]$ and u, $v \in \mathbb{R}^+$;*

(2) *$\lim_n fx_n = t = \lim_n gx_n$ for some $t \in C([0,a])$, there exists a sequence $\{y_n\}$ satisfying $\lim_n fy_n = u = \lim_n gy_n$ for some $u \in C([0,a])$ such that $\lim_n fgy_n = fu$, $\lim_n gfy_n = gu$ and $\lim_n fgy_n = \lim_n gfy_n$;*

(3) *for all $x\in X, fx = gx$ implies $fgx = gfx$.*

Then equation (4.1) *has a solution.*

Proof. Above problem is equivalent to the integral equation

$$u(t) = \int_0^t G(t,s)K(t,s,u(s))ds, \tag{4.5}$$

$t,s\in[0,a]$.
Consider the self-maps $f : X \to X$, defined by

$$fx(t) = \int_0^t G(t,s)K(t,s,u(s))ds, \tag{4.6}$$

$t\in[0,a]$, $a>0$. Then clearly u^* is a solution of (4.5), if and only if u^* is a common fixed point of f and g. From (1), for all $u,v\in X$, we have

$$|fu(t)-fv(t)| \leqslant \int_0^t G(t,s)|K(t,s,u(s)) - K(t,s,v(s))|ds$$

$$\leqslant \int_0^t G(t,s)\tau^2 e^{-\tau} M(u,v)ds$$

$$|fu(t)-fv(t)| \leqslant \int_0^t \tau^2 e^{-\tau} e^{2\tau s} e^{-2\tau s} M(u,v)G(t,s)ds$$

$$\leqslant \tau^2 e^{-\tau}\|M(u,v)\|_\tau \times \int_0^t e^{2\tau s}G(t,s)ds$$

$$\leqslant \tau^2 e^{-\tau}\|M(u,v)\|_\tau \times [-\frac{e^{2\tau t}}{\tau^2}(2\tau t - \tau t e^{-\tau t} + e^{-\tau t} - 1)]$$

$$|fu(t) - fv(t)|e^{-2\tau t} \leqslant e^{-\tau}\|M(u,v)\|_\tau \times [(1 - 2\tau t + \tau t e^{-\tau t} - e^{-\tau t})]$$

$$\|fu(t) - fv(t)\|_\tau \leqslant e^{-\tau}\|M(u,v)\|_\tau \times [(1 - 2\tau t + \tau t e^{-\tau t} - e^{-\tau t})].$$

Clearly, $(1 - 2\tau t + \tau t e^{-\tau t} - e^{-\tau t}) \leqslant 1$. Hence

$$\|fu(t) - fv(t)\|_\tau \leqslant e^{-\tau}\|M(u,v)\|_\tau,$$

or

$$d(fu, fv) \leqslant e^{-\tau}\|M(u,v)\|_\tau.$$

Taking logarithm,

$$ln(d(fu, fv)) \leqslant ln[e^{-\tau}\|M(u,v)\|_\tau].$$

or

$$\tau + ln(d(fu, fv)) \leqslant ln\|M(u,v)\|_\tau.$$

Clearly using (2) and (3), all conditions of Theorem 3.1 are satisfied by operators f and g taking $F(x) = lnx$. Hence f and g has a common fixed point which is the solution of differential equation arising in electric circuit equation. □

4.2. Application to Volterra type integral equation for Ćirić type F-contraction. Motivated by the fact that a large class of boundary value problem can be converted to a Volterra integral equation, in this section, we discuss the application of Theorem 3.1 to the Volterra type integral equation:

$$u(t) = \int_0^t K(t, s, u(s))ds + h(t); \tag{4.7}$$

for $t, s \in [0, a]$, where $a > 0$. Let $X = C([0, a], \mathbb{R})$ be the space of all functions defined on $[0, a]$.
For $u \in C([0, a], \mathbb{R})$, define supremum norm as: $\|u\| = sup_{t\in[0,a]}\{u(t)e^{-\tau t}\}$, where $\tau > 0$ is arbitrary. Let (X, d) be a metric space endowed with the metric

$$d(u, v) = sup_{t\in[0,a]}\{|(u(t) - v(t))|e^{-\tau t}\}, \tag{4.8}$$

for all $u, v \in C([0, a], \mathbb{R})$.

Theorem 4.2. *Let $f, g : C([0, a]) \to C([0, a])$ be self maps of a metric space (X, d) such that the following conditions hold:*

(1) *there exists a function $K : [0, a] \times [0, a] \times \mathbb{R} \to \mathbb{R}$ such that* $\|K(t, s, u) - K(t, s, v)\| \leqslant \tau e^{-\tau}[M(u, v)]$, *where* $M(u, v) = \max\{d(gu, gv), d(gu, fu), d(gv, fv), \frac{[d(gu,fv)+d(gv,fu)]}{2}\}$, *for $F \in \mathcal{F}$, $\tau \in \mathbb{R}^+$, $t, s \in [0, a]$ and $u, v \in C([0, a], \mathbb{R})$;*
(2) $\lim_n fx_n = t = \lim_n gx_n$ *for some* $t \in C([0, a])$, *there exists a sequence* $\{y_n\}$ *satisfying* $\lim_n fy_n = u = \lim_n gy_n$ *for some* $u \in C([0, a])$ *such that* $\lim_n fgy_n = fu$, $\lim_n gfy_n = gu$ *and* $\lim_n fgy_n = \lim_n gfy_n$.

(3) *for all* $x \in X, fx = gx$ *implies* $fgx = gfx$.

Then the integral equation (4.7) has a solution.

PROOF. Let

$$fu(t) = \int_0^t K(t, s, u(s))ds + h(t), t \in [0, a], a > 0.$$

Now by assumption (2)

$$|fu(t) - fv(t)| = \int_0^t |K(t, s, u(s)) - K(t, s, v(s))|ds$$

$$\leqslant \int_0^t \tau e^{-\tau}([M(u, v)]e^{-\tau s})e^{\tau s}ds$$

$$\leqslant \int_0^t \tau e^{-\tau}\|M(u, v)\|_\tau e^{\tau s}ds$$

$$\leqslant \tau e^{-\tau}\|M(u, v)\|_\tau \int_0^t e^{\tau s}ds$$

$$\leqslant \tau e^{-\tau}\|M(u, v)\|_\tau \frac{1}{\tau}.e^{\tau t}$$

$$\leqslant e^{-\tau}\|M(u, v)\|_\tau e^{\tau t}.$$

So

$$|fu(t) - fv(t)|e^{-\tau t} \leqslant e^{-\tau}\|M(u, v)\|_\tau$$

or

$$\|fu(t) - fv(t)\|_\tau \leqslant e^{-\tau}\|M(u, v)\|_\tau.$$

Taking logarithm on both sides, we get

$$\tau + ln\|fu(t) - fv(t)\|_\tau \leqslant ln\|M(u, v)\|_\tau.$$

Using (2) and (3), all the conditions of Theorem 3.1 are satisfied for $F(x) = lnx$. Hence integral equations given in (4.7) has a unique solution. □

4.3. Application to Volterra type integral equation for Hardy-Rogers -type F-contraction. In this section, we solve Volterra type integral equation using Hardy-Rogers-type F-contraction.
Define supremum norm as: $\|u\| = sup_{t\in[0,a]}\{u(t)e^{-\tau t}\}$, where $\tau > 0$ is arbitrary. Let $X = (C[0,a], \mathbb{R})$ and (X, d) be the metric space of all real-valued functions endowed with the metric $d(u,v) = sup_{t\in[0,a]}\{|u(t) - v(t)|e^{-\tau t}\}, a > 0$.

Consider an integral equation

$$u(t) = \int_0^t K(t,s,u(s))ds + h(t); \tag{4.9}$$

for $t, s \in [0,a]$.

THEOREM 4.3. *Let $f, g : C([0,a]) \to C([0,a])$ be self maps of a metric space (X, d) such that*

(1) *there exists a function $K : [0,a] \times [0,a] \times \mathbb{R} \to \mathbb{R}$ and $\tau > 0$ such that*

$$|K(t,s,u) - K(t,s,v)| \leqslant \frac{e^{-\tau}}{\alpha(u+v)}|u-v|,$$

$|\int_0^t \frac{e^{\tau s}}{\alpha(u(s)+v(s))}ds| \leqslant e^{\tau t}$, *for all $t, s \in [0,a]$ and $u, v \in X$;*

(2) $\lim_n fx_n = t = \lim_n gx_n$ *for some $t \in C([0,a])$, there exists a sequence $\{y_n\}$ satisfying $\lim_n fy_n = u = \lim_n gy_n$ such that for some $u \in C([0,a])$ $\lim_n fgy_n = fu$, $\lim_n gfy_n = gu$ and $\lim_n fgy_n = \lim_n gfy_n$;*

(3) $e^{-\tau s}|u(s) - v(s)| = \|M(u,v)\|$, *where $M(u,v) = \alpha d(gu,gv) + \beta d(gu,fu) + \gamma d(gv,fv) + \delta[d(gu,fv) + d(gv,fu)]$, $\alpha, \beta, \gamma \geqslant 0$, $\alpha + \beta + \gamma + 2\delta < 1$, $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$;*

(4) *for all $x \in X, fx = gx$ implies $fgx = gfx$.*

Then the integral equation (4.9) has a solution in X.

PROOF. Let

$$fu(t) = \int_0^t K(t,s,u(s))ds + h(t), \tag{4.10}$$

for $t, s \in [0,a]$, $h : [0,a] \to \mathbb{R}$ are functions for each $t \in [0,a], a > 0$.
First we show that f is an Hardy-Rogers-type F-contraction.

$$|fu(t) - fv(t)| = \int_0^t |K(t,s,u(s)) - K(t,s,v(s))|ds$$

$$\leqslant \int_0^t \frac{e^{-\tau}}{\alpha(u(s)+v(s))}|u(s) - v(s)|ds$$

$$= \int_0^t \frac{e^{-\tau} e^{\tau s}}{\alpha(u(s)+v(s))} |u(s) - v(s)| e^{-\tau s} ds$$

$$\leqslant e^{-\tau} \|M(u,v)\|_\tau \int_0^t \frac{e^{\tau s}}{\alpha(u(s)+v(s))} ds$$

$$\leqslant e^{\tau t} e^{-\tau} \|M(u,v)\|_\tau .$$

Hence

$$|fu(t) - fv(t)| e^{-\tau t} \leqslant e^{-\tau} \|M(u,v)\|_\tau$$

or

$$d(fu, fv) \leqslant e^{-\tau} \|M(u,v)\|_\tau .$$

Taking logarithm,

$$ln(d(fu, fv)) \leqslant lne^{-\tau} \|M(u,v)\|_\tau ,$$

i.e.,

$$\tau + ln(d(fu, fv)) \leqslant ln\|M(u,v)\|_\tau .$$

Thus f is an F-contraction of Hardy-Rogers-type with $\alpha < 1$, $\beta = \gamma = \delta = 0$ and $F(x) = lnx$. Using (2) and (3) all other conditions of Theorem 3.2 immediately hold. Therefore, the operators f and g have a common fixed point, i.e., the integral equation (4.9) has a solution in X.

□

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