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GENERALIZED DEGREE DISTANCE FOR FOUR TRANSFORMATION OF GRAPH

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ABSTRACT. The *generalized degree distance*, denoted by $H_\lambda(G)$, is defined as

$$H_\lambda(G) = \frac{1}{2} \sum_{u,v \in V(G)} (d_G(u) + d_G(v)) d_G^\lambda(u,v),$$

where λ is any real number. In this paper, we discuss the mathematical properties for the generalized degree distance of four edge grafting transformations of graph.

1. Introduction

A single number that can be used to characterize some property of the graph of a molecule is called a *topological index*. Topological index is a graph theoretical property that is preserved by isomorphism.The chemical information derived through *topological index* has been found useful in chemical documentation,isomer discrimination, structure property correlations.The interest in topological in-dices is mainly related to their use in non-empirical quantitative structure-property relationships and quantitative structural-activity relationships. One of the oldest and well-studied distance based graph invariants associated with a connected graph G is the *Wiener number* $W(G)$, also termed as *Wiener index* in chemical or mathematical chemistry literature, which is defined Wiener as the sum of distance over all unordered vertex pairs in G, namely,

$$W(G) = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u,v).$$

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Another distance based graph invariant, defined in a fully analogous manner to winner index, is the*Harary index*, which is equal to the sum of the the reciprocal distances overall unordered vertex pairs in G that is,

$$H(G) = \frac{1}{2} \sum_{u,v \in V(G)} \frac{1}{d_G(u,v)}.$$

Dobrynin and Kochetova [**3**] and Gutman [**4**] independently proposed a vertex-degree-weighted version of Wiener index called *degree distance*, which is defined for a connected graph G as

$$DD(G) = \frac{1}{2} \sum_{u,v \in V(G)} (d_G(u) + d_G(v)) d_G(u,v).$$

The *additively weighted Harary index*(H_A) or *reciprocal degree distance*(RDD) is defined in [**1**] as

$$H_A(G) = RDD(G) = \frac{1}{2} \sum_{u,v \in V(G)} \frac{(d_G(u) + d_G(v))}{d_G(u,v)}.$$

Hua and Zhang [**7**] have obtained lower and upper bounds for the reciprocal degree distance of graph in terms of other graph invariants. The chemical applications and mathematical properties of the reciprocal degree distance are well studied in [**1, 11, 14**].

The *generalized degree distance*, denoted by $H_\lambda(G)$, is defined as

$$H_\lambda(G) = \frac{1}{2} \sum_{u,v \in V(G)} (d_G(u) + d_G(v)) d_G^\lambda(u,v),$$

where λ is any real number. If $\lambda = 1$, then $H_\lambda(G) = DD(G)$ and if $\lambda = -1$, then $H_\lambda(G) = RDD(G)$. The generalized degree distance of unicyclic and bicyclic graphs are studied by Hamzeh et. al [**5, 6**]. Also they are given the generalized degree distance of Cartesian product, join, symmetric difference, composition and disjunction of two graphs. The generalized degree distance of the strong and tensor product of graphs are obtained in [**12, 13**]. In this sequence, here we find the bounds for the generalized degree distance of four edge grafting transformations of graph.

2. α-transformation

LetG_1be a simple graph,as depicted in Fig.1, where H_1, H_2 are two connected graphs. Let

$$G_2 = G_1 - \{v_l x : x \in N_{H_2}(v_l)\} + \{v_1 x : x \in N_{H_2}(v_l)\}.$$

We call that G_2 is obtained by α - transformation on G_1. In particular, if G_1 is a tree, Kelmas [**9**] used this tree-transformation to prove the some results on the number of spanning trees of graph in [**9**]. Recently, Bollobas and Tyomkyn [**2**] used this tree-transformation to count the total number of walks(resp. closed walks, paths) of trees. Here we obtain the generalized degree distance of α-transformation.

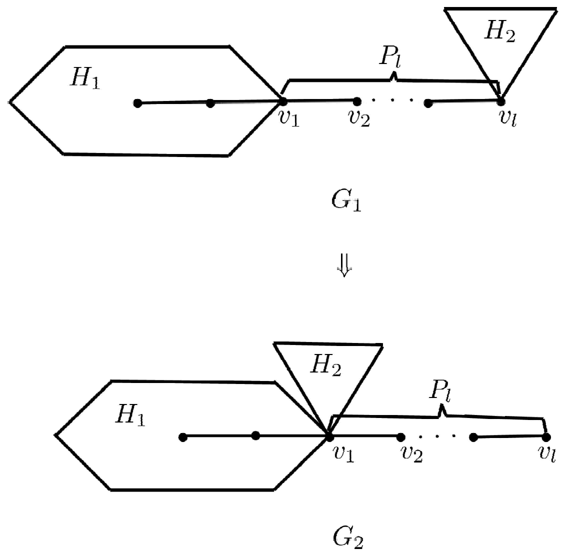

Fig 1. α-transformation

THEOREM 2.1. *Let G_2 be the graph obtained from G_1 by α-transformation. Then $H_\lambda(G_1) < H_\lambda(G_2)$ if $\lambda < 0$ and $H_\lambda(G_1) > H_\lambda(G_2)$ if $\lambda > 0$.*

PROOF. For each x in $V(H_1)\setminus\{v_1\}$ and for each vertex y in $V(H_2)\setminus\{v_l\}$, one can easily check that $d_{G_1}(x) = d_{H_1}(x) = d_{G_2}(x)$ and $d_{G_1}(y) = d_{H_2}(y) = d_{G_2}(y)$.

Moreover, for each $y \in V(H_2)$ we have $d^\lambda_{H_2}(v_l, y) = t$ in G_1 if and only if $d^\lambda_{H_2}(v_1, y) = t$ in G_2.

For each vertex $x \in V(H_1)\setminus\{v_1\}$, we have

$$\begin{aligned} D^\lambda_{G_1}(x) = & \sum_{y\in V(H_1)\setminus\{v_1,x\}} d^\lambda_{H_1}(x,y) + \sum_{k=0}^{l-1}\big(d_{H_1}(x,v_1)+k\big)^\lambda \\ & + \sum_{y\in V(H_2)\setminus\{v_l\}} \big(d_{H_1}(x,v_1)+l-1+d_{H_2}(v_l,y)\big)^\lambda \end{aligned}$$

and

$$\begin{aligned} D^\lambda_{G_2}(x) = & \sum_{y\in V(H_1)\setminus\{v_1,x\}} d^\lambda_{H_1}(x,y) + \sum_{k=0}^{l-1}(d_{H_1}(x,v_1)+k)^\lambda \\ & + \sum_{y\in V(H_2)\setminus\{v_1\}} \big(d_{H_1}(x,v_1)+d_{H_2}(v_1,y)\big)^\lambda . \end{aligned}$$

Now we obtain $D^{\lambda}_{G_1}(x) - D^{\lambda}_{G_2}(x)$.

$$D^{\lambda}_{G_1}(x) - D^{\lambda}_{G_2}(x) = \sum_{y \in V(H_2)\setminus\{v_l\}} \big(d_{H_1}(x,v_1) + l - 1 + d_{H_2}(v_l,y)\big)^{\lambda} - \sum_{y \in V(H_2)\setminus\{v_1\}} \big(d_{H_1}(x,v_1) + d_{H_2}(v_1,y)\big)^{\lambda}.$$

$$= \sum_{y \in V(H_2)\setminus\{v_l\}} \sum_{a=0}^{\lambda} \binom{\lambda}{a} \big(d_{H_1}(x,v_1) + l - 1\big)^{\lambda-a} (d_{H_2}(v_l,y))^{a} - \sum_{y \in V(H_2)\setminus\{v_1\}} \sum_{a=0}^{\lambda} \binom{\lambda}{a} (d_{H_1}(x,v_1))^{\lambda-a} (d_{H_2}(v_1,y))^{a}.$$

By direct calculation, we have $D^{\lambda}_{G_1}(x) < D^{\lambda}_{G_2}(x)$, when $\lambda < 0$ and $D^{\lambda}_{G_1}(x) > D^{\lambda}_{G_2}(x)$, when $\lambda > 0$. Observe that whenever $d_{G_1}(x) = d_{H_1}(x) = d_{G_2}(x)$ for all x in $V(H_1)\setminus\{v_1\}$. If $\lambda > 0$ then

$$\sum_{x \in V(H_1)\setminus\{v_1\}} d_{G_1}(x) D^{\lambda}_{G_1}(x) > \sum_{x \in V(H_1)\setminus\{v_1\}} d_{G_2}(x) D^{\lambda}_{G_2}(x)$$

and if $\lambda < 0$, we get

$$\sum_{x \in V(H_1)\setminus\{v_1\}} d_{G_1}(x) D^{\lambda}_{G_1}(x) < \sum_{x \in V(H_1)\setminus\{v_1\}} d_{G_2}(x) D^{\lambda}_{G_2}(x). \tag{1}$$

For each vertex x in $V(H_2)\setminus\{v_l\}$, we have

$$D^{\lambda}_{G_1}(x) = \sum_{y \in V(H_2)\setminus\{(v_l),x\}} d^{\lambda}_{H_2}(x,y) + \sum_{k=0}^{l-1} \big(d_{H_2}(x,v_l) + k\big)^{\lambda} + \sum_{y \in V(H_1)\setminus\{v_1\}} \big(d_{H_2}(x,v_l) + l - 1 + d_{H_1}(v_1,y)\big)^{\lambda}$$

and

$$D^{\lambda}_{G_2}(x) = \sum_{y \in V(H_2)\setminus\{v_l,x\}} d^{\lambda}_{H_2}(x,y) + \sum_{k=0}^{l-1} (d_{H_2}(x,v_1) + k)^{\lambda} + \sum_{y \in V(H_1)\setminus\{v_1\}} (d_{H_2}(x,v_1) + d_{H_1}(v_1,y))^{\lambda}.$$

Next we compute

$$D^{\lambda}_{G_1}(x) - D^{\lambda}_{G_2}(x) = \sum_{y \in V(H_1)\setminus\{v_1\}} \big(d_{H_2}(x,v_l) + l - 1 + d_{H_1}(v_1,y)\big)^{\lambda} - \sum_{y \in V(H_1)\setminus\{v_1\}} (d_{H_2}(x,v_1) + d_{H_1}(v_1,y))^{\lambda}.$$

once again using simple calculation we get

$$D^{\lambda}_{G_1}(x) < D^{\lambda}_{G_2}(x) \text{ when } \lambda < 0$$

and

$$D^{\lambda}_{G_1}(x) > D^{\lambda}_{G_2}(x) \text{ when } \lambda > 0$$

one can note that $d_{G_1}(x) = d_{H_2}(x) = d_{G_2}(x)$,for all x in $V(H_2)\setminus\{v_l\}$. Hence whenever $\lambda < 0$ we get

$$\sum_{x\in V(H_2)\setminus\{v_l\}} d_{G_1}(x)D^{\lambda}_{G_1}(x) < \sum_{x\in V(H_2)\setminus\{v_l\}} d_{G_2}(x)D^{\lambda}_{G_2}(x),$$

and if $\lambda > 0$ we get

$$\sum_{x\in V(H_2)\setminus\{v_l\}} d_{G_1}(x)D^{\lambda}_{G_1}(x) > \sum_{x\in V(H_2)\setminus\{v_l\}} d_{G_2}(x)D^{\lambda}_{G_2}(x) \tag{2}$$

For each $v_j \in V(P_l) = \{v_1, v_2, \ldots, v_l\}$, we have

$$\begin{aligned} D^{\lambda}_{G_1}(v_j) = &\sum_{x\in V(H_1)\setminus\{v_1\}} \big(d_{H_1}(x, v_1) + j - 1\big)^{\lambda} \\ &+ D^{\lambda}_{P_l}(v_j) + \sum_{x\in V(H_2)\setminus\{v_l\}} \big(d_{H_2}(x, v_l) + l - j\big)^{\lambda} \end{aligned} \tag{3}$$

$$\begin{aligned} D^{\lambda}_{G_2}(v_j) = &\sum_{x\in V(H_1)\setminus\{v_1\}} \big(d_{H_1}(x, v_1) + j - 1\big)^{\lambda} \\ &+ D^{\lambda}_{P_l}(v_j) + \sum_{x\in V(H_2)\setminus\{v_l\}} \big(d_{H_2}(x, v_1) + j - 1\big)^{\lambda} \end{aligned} \tag{4}$$

One can easily observe that $d_{G_1}(v_j) = d_{G_2}(v_j) = 2$, for each $v_j \in V(P_l)\setminus\{v_1, v_l\}$, and it is easily to find that $d_{G_1}(v_1) = d_{H_1}(v_1) + 1$, $d_{G_1}(v_l) = d_{H_2}(v_l) + 1$, $d_{G_2}(v_1) = d_{H_1}(v_1) + d_{H_2}(v_1) + 1$ and $d_{G_2}(v_l) = 1$.

thus

$$\sum_{j=1}^{l} d_{G_1}(v_j) D^{\lambda}_{G_1}(v_j) = (d_{H_1}(v_1)+1)\Bigg(\sum_{x\in V(H_1)\setminus\{v_1\}} d^{\lambda}_{H_1}(x,v_1) + D^{\lambda}_{P_l}(v_1)$$

$$+ \sum_{x\in V(H_2)\setminus\{v_l\}} (d_{H_2}(x,v_l)+l-1)^{\lambda}\Bigg)$$

$$+ (d_{H_2}(v_l)+1)\Bigg(\sum_{x\in V(H_2)\setminus(v_l)} d^{\lambda}_{H_2}(x,v_l) + D^{\lambda}_{P_l}(v_l)$$

$$+ \sum_{x\in V(H_1)\setminus(v_1)} (d_{H_2}(x,v_1)+l-1)^{\lambda}\Bigg)$$

$$+ 2\sum_{j=2}^{l-1}\Bigg(\sum_{x\in V(H_1)\setminus\{v_1\}} (d_{H_1}(x,v_1)+j-1)^{\lambda} + D^{\lambda}_{P_l}(v_j)$$

$$+ \sum_{x\in V(H_2)\setminus(v_l)} (d_{H_2}(x,v_l)+l-j)^{\lambda}\Bigg),$$

by (3), and

$$\sum_{j=1}^{l} d_{G_2}(v_j)\, D^{\lambda}_{G_2}(v_j) = (d_{H_1}(v_1)+d_{H_2}(v_l)+1)\Bigg(\sum_{x\in V(H_1)\setminus\{v_1\}} d^{\lambda}_{H_1}(x,v_1) + D^{\lambda}_{P_l}(v_1)$$

$$+ \sum_{x\in V(H_2)\setminus\{v_1\}} d^{\lambda}_{H_2}(x,v_1)\Bigg)$$

$$+ 2\sum_{j=2}^{l-1}\Bigg(\sum_{x\in V(H_1)\setminus(v_1)} (d_{H_1}(x,v_1)+j-1)^{\lambda} + D^{\lambda}_{P_l}(v_j)$$

$$+ \sum_{x\in V(H_2)\setminus(v_1)} (d_{H_2})(x,v_1+j-1)^{\lambda}\Bigg)$$

$$\sum_{x\in V(H_2)\setminus(v_1)} \big(d_{H_2}(x,v_1)+l-1)^{\lambda} + D^{\lambda}_{P_l}(v_l)$$

$$+ \sum_{x\in V(H_1)\setminus(v_1)} (d_{H_1}(x,v_1)+l-1)^{\lambda}\Big),$$

by (4), where $D^{\lambda}_{P_l}(v_1) = D^{\lambda}_{P_1}(v_l)$.
Hence

$$\sum_{j=1}^{l} d_{G_2}(v_j) D^{\lambda}_{G_2}(v_j) - \sum_{j=1}^{l} d_{G_1}(v_j) D^{\lambda}_{G_1}(v_j)$$

$$= d_{H_1}(v_1)\Big(\sum_{x\in V(H_2)\setminus(v_l)} \big(d_{H_2}(x,v_l)\big)^{\lambda} - \sum_{x\in V(H_2)\setminus(v_l)} \big(d_{H_2}(x,v_l)+l-1\big)^{\lambda}\Big)$$
$$+ d_{H_2}(v_l)\Big(\sum_{x\in V(H_1)\setminus(v_1)} \big(d_{H_1}(x,v_1)\big)^{\lambda} - \sum_{x\in V(H_1)\setminus(v_1)} \big(d_{H_1}(x,v_1)+l-1\big)^{\lambda}\Big)$$
$$= d_{H_1}(v_1)\Big(\sum_{x\in V(H_2)\setminus(v_l)} \big(d^{\lambda}_{H_2}(x,v_l)\big) - \sum_{x\in V(H_2)\setminus(v_l)} \sum_{a=0}^{\lambda} \tbinom{\lambda}{a}(d_{H_2}(x,v_l))^{\lambda-a}(l-1))^{a}\Big)$$
$$+ d_{H_2}(v_l)\Big(\sum_{x\in V(H_1)\setminus(v_1)} \big(d^{\lambda}_{H_1}(x,v_1)\big) - \sum_{x\in V(H_1)\setminus(v_1)} \sum_{a=0}^{\lambda} \tbinom{\lambda}{a}(d_{H_1}(x,v_1))^{\lambda-a}(l-1)^{a}\Big)$$

Simplify the above equation, we have:
if $\lambda > 0$, then

$$\sum_{j=1}^{l} d_{G_2}(v_j)D^{\lambda}_{G_2}(v_j) > \sum_{j=1}^{l} d_{G_1}(v_j)D^{\lambda}_{G_1}(v_j)$$

and if $\lambda < 0$, we have

$$\sum_{j=1}^{l} d_{G_2}(v_j)D^{\lambda}_{G_2}(v_j) < \sum_{j=1}^{l} d_{G_1}(v_j)D^{\lambda}_{G_1}(v_j). \tag{5}$$

From (1), (2) and (5), we have $H_\lambda(G_1) < H_\lambda(G_2)$ if $\lambda < 0$ and $H_\lambda(G_1) > H_\lambda(G_2)$ if $\lambda > 0$. □

3. ρ-transformation

Let T be an n-vertex tree as shown in Fig. 2, where $wv \in E(T)$ with $d_T(v) = m+1 (m \geqslant 2)$,$T_1, T_2, \ldots, T_m$ are subtrees under v with root vertices $v_1, v_2, \ldots, v_m$ such that the tree T_m is actually a path P_l, H_1 contains a path $P_k = us\ldots$ satisfying $|V(H_1)| \geqslant l+2$ and $|V(P_k)| \geqslant |V(P_l)|$. Let

$$T' = T - \{vv_1, vv_2, \ldots, vv_{m-1}\} + \{wv_1, wv_2, \ldots, wv_{m-1}\},$$

see Fig. 2. Clearly T' is obtained from T by ρ- transformation. Ilić [8] used ρ - transformation to study the Laplacian coefficient of trees. Here we obtain the generalized degree distance of ρ - transformation.

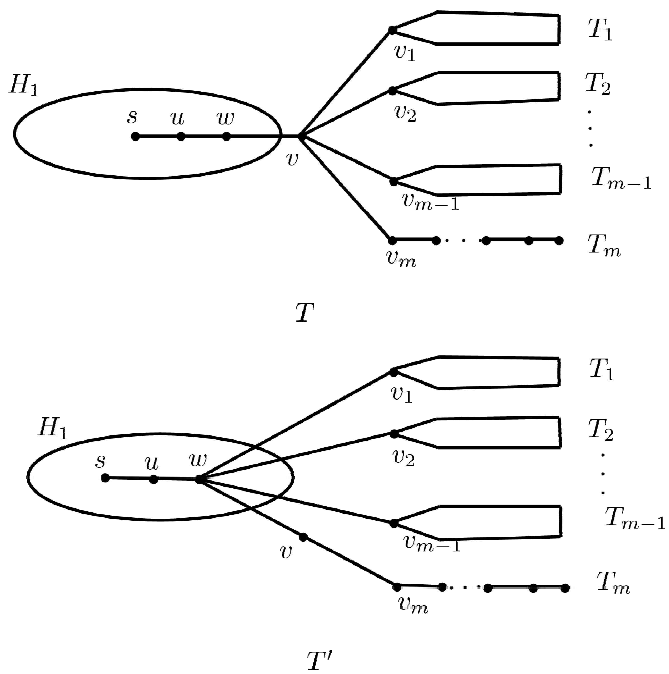

Fig. 2 ρ-Transformtion

THEOREM 3.1. *Let T' be the n-vertex tree obtained from T by ρ-transformation. Then we have $H_\lambda(T) < H_\lambda(T')$ if $\lambda < 0$ and $H_\lambda(T) > H_\lambda(T')$ if $\lambda > 0$.*

PROOF. Let $H_2 = T_1 \cup T_2 \cup \ldots \cup T_{m-1}$. Let $P_l = u_1 u_2 \ldots u_l$ be a path on length L. One can see that H_1 contains a path $P_k = us \ldots$ whose length is no less than that of $T_m = P_l$.

Observe that for all $x \in V(H_1) \backslash V(P_k)$, we have

$$
\begin{aligned}
D_T^\lambda(x) = & \sum_{y \in V(H_1) \backslash \{x,w\}} d_{H_1}^\lambda(x,y) + d_{H_1}^\lambda(x,w) + \Big(d_{H_1}(x,w) + 1\Big)^\lambda \\
& + \sum_{j=1}^{l} \Big(d_{H_1}(x,w) + 1 + j\Big)^\lambda \\
& + \sum_{y \in V(H_2)} \Big(d_{H_1}(x,w) + 1 + d_{T[V(H_2) \cup \{v\}]}(y,v)\Big)^\lambda
\end{aligned}
$$

and

$$D^{\lambda}_{T'}(x) = \sum_{y\in V(H_1)\setminus\{x,w\}} d^{\lambda}_{H_1}(x,y) + d^{\lambda}_{H_1}(x,w) + \Big(d_{H_1}(x,w)+1\Big)^{\lambda}$$
$$+\sum_{j=1}^{l}\Big(d_{H_1}(x,w)+j+1\Big)^{\lambda}$$
$$+\sum_{y\in V(H_2)}\Big(d_{H_1}(x,w)+d_{T'[V(H_2)\cup\{w\}]}(y,w)\Big)^{\lambda},$$

where for each y in $V(H_2)$ we have $(d_{T[V(H_2)\cup\{v\}]}(y,v))^{\lambda} = (d_{T'[V(H_2)\cup\{w\}]}(y,w))^{\lambda}$

Now we obtain $D^{\lambda}_{T'}(x) - D^{\lambda}_{T}(x)$.

$$D^{\lambda}_{T'}(x) - D^{\lambda}_{T}(x) = \sum_{y\in V(H_2)} (d_{H_1}(x,w)+d_{T'[V(H_2)\cup\{w\}]}(y,w))^{\lambda}$$
$$- \sum_{y\in V(H_2)} \big(d_{H_1}(x,w)+1+(d_{T[V(H_2)\cup\{v\}]}(y,v)\big)^{\lambda}.$$
$$= \sum_{y\in V(H_2)}\sum_{a=0}^{\lambda}\tbinom{\lambda}{a}(d^{\lambda-a}_{H_1}(x,w)+d^{a}_{T'[V(H_2)\cup\{w\}]}(y,w))^{\lambda}$$
$$- \sum_{y\in V(H_2)}\sum_{a=0}^{\lambda}\tbinom{\lambda}{a}\Big(d_{H_1}(x,w)+1\Big)^{\lambda-a} d^{a}_{T[V(H_2)\cup\{v\}]}(y,v))^{\lambda}.$$
$$= \sum_{y\in V(H_2)}\sum_{a=0}^{\lambda}\tbinom{\lambda}{a}d^{a}_{T'[V(H_2)\cup\{w\}]}(y,w)\Big[d^{\lambda-a}_{H_1}(x,w)$$
$$-\Big(d_{H_1}(x,w)+1\Big)^{\lambda-a}\Big]$$

If $\lambda < 0$, then $(d^{\lambda-a}_{H_1}(x,w) < \Big(d_{H_1}(x,w)+1\Big)^{\lambda-a}$ thus $D^{\lambda}_{T'}(x) < D^{\lambda}_{T}(x)$.
If $\lambda > 0$, then

$$(d^{\lambda-a}_{H_1}(x,w) > \Big(d_{H_1}(x,w)+1\Big)^{\lambda-a} thus\ D^{\lambda}_{T'}(x) > D^{\lambda}_{T'}(x).$$

Observe that for each vertex $x\ inV(H_1)\setminus V(P_k)$, we have $d_T(x) = d_{H_1}(x) = d'_T(x)$.
Hence

$$\sum_{x\in V(H_1)\setminus V(P_k)} d_T(x)D^{\lambda}_{T}(x) < \sum_{x\in V(H_1)\setminus V(P_k)} d_T(x)D^{\lambda}_{T'}(x).\ If\ \lambda < 0$$

and

$$\sum_{x\in V(H_1)\setminus V(P_k)} d_T(x)D^{\lambda}_{T}(x) > \sum_{x\in V(H_1)\setminus V(P_k)} d_T(x)D^{\lambda}_{T'}(x).\ if\ \lambda > 0. \qquad (6)$$

Similarly, for each vertex x in $V(H_2)$, we have

$$\begin{aligned}D_T^{\lambda}(x) = & \sum_{y\in V(H_2)\backslash\{x\}} d_{H_2}^{\lambda}(x,y) + \Big(d_{T[V(H_2)\cup\{v\}]}(x,v)\Big)^{\lambda} + \\ & \Big(d_{T[V(H_2)\cup\{v\}]}(x,v)+1\Big)^{\lambda} \\ & + \sum_{j=1}^{l}\Big(d_{T[V(H_2)\cup\{v\}]}(x,v)+j\Big)^{\lambda} \\ & + \sum_{y\in V(H_1)\backslash\{V(P_k),w\}}\Big(d_{T[V(H_2)\cup\{v\}]}(x,v)+1+d_{H_1}(y,w)\Big)^{\lambda} \\ & + \sum_{j=1}^{k}\Big(d_{T[V(H_2)\cup\{v\}]}(x,v)+j+1\Big)^{\lambda}\end{aligned}$$

and

$$\begin{aligned}D_{T'}^{\lambda}(x) = & \sum_{y\in V(H_2)\backslash\{x\}} d_{H_2}^{\lambda}(x,y) + \Big(d_{T'[V(H_2)\cup\{w\}]}(x,w)\Big)^{\lambda} \\ & \Big(d_{T'[V(H_2)\cup\{w\}]}(x,w)+1)^{\lambda} \\ & + \sum_{j=1}^{l}\Big(d_{T'[V(H_2)\cup\{w\}]}(x,w)+j+1\Big)^{\lambda}, \\ & + \sum_{y\in V(H_1)\backslash\{V(P_k),w\}}\Big(d_{H_2\cup\{w\}}(x,w)+d_{H_1}(y,w)\Big)^{\lambda} \\ & + \sum_{j=1}^{k}(d_{T'[V(H_2)\cup\{w\}]}(x,w)+j)^{\lambda}.\end{aligned}$$

By using similar calculation on (6) we have

$$D^{\lambda}_{T'}(x) - D^{\lambda}_{T}(x) = \sum_{j=1}^{l}\Big(d_{T'[V(H_2)\cup\{w\}]}(x,w)+j+1\Big)^{\lambda}$$
$$-\sum_{j=1}^{l}\Big(d_{T[V(H_2)\cup\{v\}]}(x,v)+j\Big)^{\lambda}$$
$$+\sum_{j=1}^{k}\Big(d_{T'[V(H_2)\cup\{w\}]}(x,w)+j\Big)^{\lambda}$$
$$-\sum_{j=1}^{k}\Big(d_{T[V(H_2)\cup\{v\}]}(x,v)+j+1\Big)^{\lambda}$$
$$+\sum_{y\in V(H_1)\setminus\{V(P_k),w\}}\Big(d_{H_2\cup\{w\}}(x,w)$$
$$+d_{H_1}(y,w)-d_{T[V(H_2)\cup\{v\}]}(x,v)+1+d_{H_1}(y,w)\Big)^{\lambda}.$$

If H_1 is a path , that is , k $\geqslant l+1$, then

$$D^{\lambda}_{T'}(x) - D^{\lambda}_{T}(x) = \sum_{j=l+1}^{k}\Big(d_{T'[V(H_2)\cup\{w\}]}(x,w)+j\Big)^{\lambda}$$
$$-\sum_{j=l+1}^{k}\Big(d_{T'[V(H_2)\cup\{w\}]}(x,w)+j+1\Big)^{\lambda}$$

$$=\sum_{j=l+1}^{k}\sum_{a=0}^{\lambda}\tbinom{\lambda}{a}d^{\lambda-a}_{T'[V(H_2)\cup\{w\}]}(x,w)(j)^{a}$$
$$-\sum_{j=l+1}^{k}\sum_{a=0}^{\lambda}\tbinom{\lambda}{a}\Big(d_{T'[V(H_2)\cup\{w\}]}(x,w)+1\Big)^{\lambda-a}(j)^{a}$$
$$=\sum_{j=l+1}^{k}\sum_{a=0}^{\lambda}\tbinom{\lambda}{a}(j)^{a}\Big(d^{\lambda-a}_{T'[V(H_2)\cup\{w\}]}(x,w)-\Big(d_{T'[V(H_2)\cup\{w\}]}(x,w)+1\Big)^{\lambda-a}\Big).$$

If $\lambda>0$, then

$$\Big(d^{\lambda-a}_{T'[V(H_2)\cup\{w\}]}(x,w)\Big) > \Big(d_{T'[V(H_2)\cup\{w\}]}(x,w)+1\Big)^{\lambda-a}$$

and if $\lambda<0$, then

$$\Big(d^{\lambda-a}_{T'[V(H_2)\cup\{w\}]}(x,w)\Big) < \Big(d_{T'[V(H_2)\cup\{w\}]}(x,w)+1\Big)^{\lambda-a}. \tag{7}$$

Hence $D^{\lambda}_{T'}(x)-D^{\lambda}_{T}(x)<0$ when $\lambda>0$ and $D^{\lambda}_{T'}(x)-D^{\lambda}_{T}(x)>0$ when $\lambda<0$

Again if H_1 is a path , that is , $k \geqslant l+1$ then we have

$$D^{\lambda}_{T'}(x) - D^{\lambda}_{T}(x) = \sum_{y \in V(H_1)\setminus\{V(P_k),w\}} \Big[\Big(d_{H_2\cup\{w\}}(x,w) + d_{H_1}(y,w)\Big)^{\lambda} - \Big(d_{T[V(H_2)\cup\{v\}]}(x,v) + 1 + d_{H_1}(y,w)\Big)^{\lambda}\Big]$$

A similar argument in (7) , we have if $\lambda < 0$, then $D^{\lambda}_{T'}(x) - D^{\lambda}_{T}(x) < 0$ and if $\lambda > 0$, then $D^{\lambda}_{T'}(x) - D^{\lambda}_{T}(x) > 0$.

One can easily observe that for each $x \in V(H_2)$, $d_T(x) = d_{H_2}(x) = d_{T'}(x)$. Then

$$\sum_{x\in V(H_2)} d_T(x) D^{\lambda}_{T}(x) > \sum_{x\in V(H_2)} d_{T'}(x) D^{\lambda}_{T'}(x)$$

when $\lambda > 0$ and

$$\sum_{x\in V(H_2)} d_T(x) D^{\lambda}_{T}(x) < \sum_{x\in V(H_2)} d_{T'}(x) D^{\lambda}_{T'}(x) \tag{8}$$

when $\lambda < 0$.

For each $u_j \in P_l$, $1 \leqslant j \leqslant l$, let $d(u_j) = d_T(u_j) = d_{T'}(u_j)$. Then

$$D^{\lambda}_{T}(u_j) = D^{\lambda}_{P_l}(u_j) + j^{\lambda} + (j+1)^{\lambda} + \sum_{x\in V(H_2)} \Big(d_{T[V(H_2)\cup\{v\}]}(x,v) + j\Big)^{\lambda} + \sum_{i=1}^{k}\Big(j+1+i\Big)^{\lambda} + \sum_{x\in V(H_1)\setminus\{V(P_k),w\}} \Big(j+1+d_{H_1}(x,w)\Big)^{\lambda}$$

and

$$D^{\lambda}_{T'}(u_j) = D^{\lambda}_{P_l}(u_j) + j^{\lambda} + (j+1)^{\lambda} + \sum_{x\in V(H_2)} \Big(d_{T'[V(H_2)\cup\{w\}]}(x,w) + j + 1\Big)^{\lambda} + \sum_{i=1}^{k}\Big(j+1+i\Big)^{\lambda} + \sum_{x\in V(H_1)\setminus\big\{V_{(P_k)},w\big\}} \Big(j+1+d_{H_1}(x,w)\Big)^{\lambda}.$$

This implies

$$\begin{aligned} &\sum_{j=1}^{l} d_{T'}(u_j) D^{\lambda}_{T'}(u_j) - \sum_{j=1}^{l} d_{T}(u_j) D^{\lambda}_{T}(u_j) \\ &= \sum_{j=1}^{l} d(u_j) \sum_{x\in V(H_2)} \Big(d_{T'[V(H_2)\cup\{w\}]}(x,w) + j + 1\Big)^{\lambda} \\ &- \sum_{j=1}^{l} d(u_j) \sum_{x\in V(H_2)} \Big(d_{T[V(H_2)\cup\{v\}]}(x,v) + j\Big)^{\lambda}. \end{aligned} \tag{9}$$

For each $w_j \in P_k$, $1 \leqslant j \leqslant k$, let $d(w_j) = d_T(w_j) = d_{T'}(w_j)$, then we have

$$D_T^\lambda(w_j) = D_{P_k}^\lambda(w_j) + j^\lambda + (j+1)^\lambda + \sum_{x \in V(H_2)} \Big(d_{T[V(H_2)\cup\{v\}]}(x,v) + j + 1\Big)^\lambda$$
$$+ \sum_{i=1}^{l} (j+1+i)^\lambda + \sum_{x \in V(H_1)\setminus\{V(P_k),w\}} d_{H_1}^\lambda(x, w_j).$$

and

$$D_{T'}^\lambda(w_j) = D_{P_k}^\lambda(w_j) + j^\lambda + (j+1))^\lambda + \sum_{x \in V(H_2)} \Big(d_{T'[V(H_2)\cup\{w\}]}(x,w) + j\Big)^\lambda$$
$$+ \sum_{i=1}^{l} (j+1+i)^\lambda + \sum_{x \in V(H_1)\setminus\big\{V_{(P_k)},w\big\}} d_{H_1}^\lambda(x, w_j).$$

This gives

$$\sum_{j=1}^{k} d_{T'}(w_j) D_{T'}^\lambda(w_j) - \sum_{j=1}^{k} d_T(w_j) D_T^\lambda(w_j)$$
$$= \sum_{j=1}^{k} d(w_j) \times \sum_{x \in V(H_2)} \Big(d_{T'[V(H_2)\cup\{w\}]}(x,w) + j\Big)^\lambda$$
$$-\sum_{j=1}^{k} d(w_j) \times \sum_{x \in V(H_2)} \Big(d_{T[V(H_2)\cup\{v\}]}(x,v) + j + 1\Big)^\lambda \tag{10}$$

If H_1 is not a path, then, for each $1 \leqslant j \leqslant l \leqslant k$, we have $d(w_j) \geqslant d(u_j)$ there exists $j \in \{1, 2, \ldots, k\}$ such that $d(w_j) > d(u_j)$. Hence,a similar of above results we have at $\lambda > 0$

$$\sum_{j=1}^{l} d_{T'}(u_j) D_{T'}^\lambda(u_j) - \sum_{j=1}^{l} d_T(u_j) D_T^\lambda(u_j) + \sum_{j=1}^{k} d_{T'}(w_j) D_{T'}^\lambda(w_j) - \sum_{j=1}^{k} d_T(w_j) D_T^\lambda(w_j)$$

$$> \sum_{j=l+1}^{k} d(w_j) \sum_{x \in V(H_2)} \Big(d_{T'[V(H_2)\cup\{w\}]}(x,w) + j\Big)^\lambda$$

$$- \sum_{j=l+1}^{k} d(w_j) \sum_{x \in V(H_2)} \Big(_{T'[V(H_2)\cup\{w\}]}(x,w) + j + 1\Big)^\lambda > 0.$$

Thus

$$\sum_{j=1}^{l} d_{T'}(u_j)D^{\lambda}_{T'}(u_j) + \sum_{j=1}^{k} d_{T'}(w_j)D^{\lambda}_{T'}(w_j) > \sum_{j=1}^{l} d_T(u_j)D^{\lambda}_T(u_j) + \sum_{j=1}^{k} d_T(w_j)D^{\lambda}_T(w_j). \tag{11}$$

In case of $\lambda < 0$, H_1 is not a path, then, for each $1 \leqslant j \leqslant l \leqslant k$, we have $d(w_j) \geqslant d(u_j)$ there exists $j \in \{1, 2, \ldots, k\}$ such that $d(w_j) > d(u_j)$.

Hence, in view of above results, we have

$$\begin{aligned}
&\sum_{j=1}^{l} d_{T'}(u_j)D^{\lambda}_{T'}(u_j) - \sum_{j=1}^{l} d_T(u_j)D^{\lambda}_T(u_j) + \sum_{j=1}^{k} d_{T'}(w_j)D^{\lambda}_{T'}(w_j) - \sum_{j=1}^{k} d_T(w_j)D^{\lambda}_T(w_j)\\
&< \sum_{j=l+1}^{k} d(w_j) \sum_{x \in V(H_2)} \Big(d_{T'[V(H_2)\cup\{w\}]}(x,w) + j\Big)^{\lambda}\\
&- \sum_{j=l+1}^{k} d(w_j) \sum_{x \in V(H_2)} \Big(d_{T'[V(H_2)\cup\{w\}]}(x,w) + j + 1\Big)^{\lambda} < 0.
\end{aligned}$$

Thus,

$$\sum_{j=1}^{l} d_{T'}(u_j)D^{\lambda}_{T'}(u_j) + \sum_{j=1}^{k} d_{T'}(w_j)D^{\lambda}_{T'}(w_j) < \sum_{j=1}^{l} d_{(T)}(u_j)D^{\lambda}_{(T)}(u_j) + \sum_{j=1}^{k} d_{(T)}(w_j)D^{\lambda}_{(T)}(w_j).$$

However, if H_1 is a path and $\lambda > 0$ we have

$$\sum_{j=1}^{l} d_{T'}(u_j)D^{\lambda}_{T'}(u_j) + \sum_{j=1}^{k} d_{T'}(w_j)D^{\lambda}_{T'}(w_j) \geqslant \sum_{j=1}^{l} d_T(u_j)D^{\lambda}_T(u_j) + \sum_{j=1}^{k} d_T(w_j)D^{\lambda}_T(w_j)$$

and $\lambda < 0$ we have

$$\sum_{j=1}^{l} d_{T'}(u_j)D^{\lambda}_{T'}(u_j) + \sum_{j=1}^{k} d_{T'}(w_j)D^{\lambda}_{T'}(w_j) \leqslant \sum_{j=1}^{l} d_T(u_j)D^{\lambda}_T(u_j) + \sum_{j=1}^{k} d_T(w_j)D^{\lambda}_T(w_j),$$

By definition of $D^{\lambda}_G(u)$, we have

$$\begin{aligned}
D^{\lambda}_T(w) = &\sum_{x \in V(H_1)\setminus\{V(P_k),w\}} d^{\lambda}_{H_1}(x,w) + \sum_{j=1}^{k}(j+1)^{\lambda} + \sum_{j=1}^{l}(j)^{\lambda} + 1\\
&+ \sum_{x \in V(H_2)} \Big(d_{T[V_{(H_2)}\cup\{v\}]}(x,v) + 1\Big)^{\lambda}
\end{aligned}$$

and

$$D^{\lambda}_{T'}(w) = \sum_{x\in V(H_1)\setminus\{V(P_k),w\}} d^{\lambda}_{H_1}(x,w) + \sum_{j=1}^{l}(j+1)^{\lambda} + \sum_{j=1}^{k}(j)^{\lambda} + 1 + \sum_{x\in V(H_2)} d^{\lambda}_{T'[V(H_2)\cup\{w\}]}(x,w).$$

$$D^{\lambda}_{T}(v) = \sum_{x\in V(H_1)\setminus\{V(P_k),w\}} \Big(d_{H_1}(x,w)+1\Big)^{\lambda} + \sum_{j=l}^{k}(j+1)^{\lambda} + 1 + \sum_{j=1}^{l}(j)^{\lambda} + \sum_{x\in V(H_2)} d^{\lambda}_{T[V(H_2)\cup\{v\}]}(x,v)$$

and

$$D^{\lambda}_{T'}(v) = \sum_{x\in V(H_1)\setminus\{V(P_k),w\}} \Big(d_{H_1}(x,w)+1\Big)^{\lambda} + \sum_{j=1}^{k}(j+1)^{\lambda} + 1 + \sum_{j=1}^{l}(j)^{\lambda} + \sum_{x\in V(H_2)} \Big(d_{T'[V(H_2)\cup\{w\}]}(x,w)+1\Big)^{\lambda}.$$

Substituting $d_T(w) = d_{H_1}(w)+1$, $d_T(v) = m+1$, $d_{T'}(w) = d_{H_1}(w)+m$ and $d_{T'}(v) = 2$. Hence
$d_{T'}(w)D^{\lambda}_{T'}(w) + d_{T'}(v)D^{\lambda}_{T'}(v) - d_T(w)D^{\lambda}_T(w) - d_T(v)D^{\lambda}_T(v)$

$$= (m-1)\Big(\sum_{j=l+1}^{k}(j)^{\lambda} - \sum_{j=l+1}^{k}(j+1)^{\lambda}\Big) + (d_{H_1}(w)-1)\Big(\sum_{x\in V(H_2)} d_{T'[V_{H_2}\cup\{w\}]}(x,w)^{\lambda} - \sum_{x\in V(H_2)}\Big(d_{T'[V(H_2)\cup\{w\}]}(x,w)+1)^{\lambda}\Big).$$

Observe that $m > 1$,we have $d_{H_1}(w) > 1$. If $\lambda > 0$ we have $d_{T'}(w)D^{\lambda}_{T'}(w) + d_{T'}(v)D^{\lambda}_{T'}(v) > d_T(w)D^{\lambda}_T(w) + d_T(v)D^{\lambda}_T(v)$
and if $\lambda < 0$ we have

$$d_{T'}(w)D^{\lambda}_{T'}(w) + d_{T'}(v)D^{\lambda}_{T'}(v) < d_T(w)D^{\lambda}_T(w) - d_T(v)D^{\lambda}_T(v).$$

Therefore from (6),(7),(8),(9),(10) and (11), we obtain

$$when\ \lambda < 0\ H_{\lambda}(T) < H_{\lambda}(T'),$$
$$and\ if\ \lambda > 0, H_{\lambda}(T) > H_{\lambda}(T').$$

Hence get desired result . □

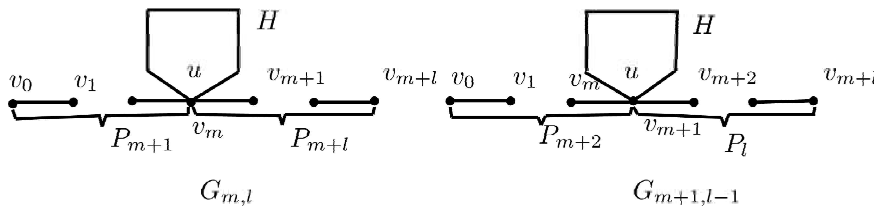

Fig.3 Graph $G_{m,l}$ and $G_{m+1,l-1}$

THEOREM 3.2. *Let u be a vertex of connected graph H and for non-negative integers m and l, let $G_{m,l}$ be the graph obtained from H by adding two pendent paths of length m and l, respectively, to u of H ; see Fig. 3. If $l \geqslant m+2$, we have $H_\lambda(G_{m,l}) < H_\lambda(G_{m+l,l-1})$ if $\lambda < 0$ and if $\lambda > 0$ we have*
$H_\lambda(G_{m,l}) > H_\lambda(G_{m+l,l-1})$

PROOF. Let $G_1 := G_{m,l}$ and $G_2 := G_{m+1,l-1}$.
For all $x \in V(H)\setminus\{u\}$, we have

$$\begin{aligned}
D^\lambda_{G_1}(x) &= \sum_{y\in V(H)\setminus\{u,x\}} d^\lambda_H(x,y) + \sum_{j=0}^{m+l}\Big(d_H(x,u)+d_{P_k}(v_m,v_j)\Big)^\lambda \\
&= \sum_{y\in V(H)\setminus\{u,x\}} d^\lambda_H(x,y) + \sum_{j=0}^{m-1}\Big(d_H(x,u)+m-j\Big)^\lambda + d^\lambda_H(x,u) \\
&+ \sum_{j=m+1}^{m+l}\Big(d_H(x,u)+j-m\Big)^\lambda \\
&= \sum_{y\in V(H)\setminus\{u,x\}} d^\lambda_H(x,y) + \sum_{j=0}^{m-1}\Big(d_H(x,u)+m-j\Big)^\lambda + d^\lambda_H(x,u) \\
&+ \sum_{j=1}^{l}\Big(d_H(x,u)+j\Big)^\lambda
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
D^{\lambda}_{G_2}(x) &= \sum_{y\in V(H)\setminus\{u,x\}} d^{\lambda}_{H}(x,y) + \sum_{j=0}^{m+l}\Big(d_H(x,u)+d_{P_k}(v_{m+1},v_j)\Big)^{\lambda}\\
&= \sum_{y\in V(H)\setminus\{u,x\}} d^{\lambda}_{H}(x,y) + \sum_{j=0}^{m}\Big(d_H(x,u)+m+1-j\Big)^{\lambda} + d^{\lambda}_{H}(x,u)\\
&+ \sum_{j=m+2}^{m+l}\Big(d_H(x,u)+j-(m+1))^{\lambda}\\
&= \sum_{y\in V(H)\setminus\{u,x\}} d^{\lambda}_{H}(x,y) + \sum_{j=0}^{m}\Big(d_H(x,u)+m+1-j\Big)^{\lambda} + d^{\lambda}_{H}(x,u)\\
&+ \sum_{j=1}^{l-1}\Big(d_H(x,u)+j\Big)^{\lambda}.
\end{aligned}$$

Now on subtracting $D^{\lambda}_{G_2}(x)$ - $D^{\lambda}_{G_1}(x) = \Big(d_H(x,u)+m+1\Big)^{\lambda} - \Big(d_H(x,u)+l)^{\lambda} > 0$ whenever $\lambda < 0, l < m+1$ and $\Big(d_H(x,u)+m+1\Big)^{\lambda} - \Big(d_H(x,u)+l)^{\lambda} < 0$ whenever $\lambda > 0$

The above inequality are obtained from $l > m+1$. Note that , for each x in $V(H)\setminus\{u\}$, we have
$d_{G_1}(x) = d_H(x) = d_{G_2}(x)$. Hence,

$$\sum_{x\in V(H)\setminus\{u\}} d_{G_1}(x)D^{\lambda}_{G_1}(x) > \sum_{x\in V(H)\setminus\{u\}} d_{G_2}(x)D^{\lambda}_{G_2}(x), when\ \lambda > 0$$

and

$$\sum_{x\in V(H)\setminus\{u\}} d_{G_1}(x)D^{\lambda}_{G_1}(x) < \sum_{x\in V_H\setminus\{u\}} d_{G_2}(x)D^{\lambda}_{G_2}(x), when\ \lambda < 0.$$

Next we focus the vertex on the two pendent paths in $G_1(resp.G_2)$.
In fact ,$G_1(resp.G_2)$ can be obtained by identifying the vertex u of H with the vertex $v_m(resp.v_{m+1})$ of path $P_k := P_{m+l+1} = v_0v_1\dots v_mv_{m+1}\dots v_{m+l}$. For each $v_j \in V(P_K)$, we have

$$D^{\lambda}_{G_1}(v_j) = D^{\lambda}_{P_k}(v_j) + \sum_{x\in V(H)\setminus\{u\}}\Big(d_H(x,u)+d_{P_k}(v_j,v_m)\Big)^{\lambda}$$

and

$$D^{\lambda}_{G_2}(v_j) = D^{\lambda}_{P_k}(v_j) + \sum_{x\in V(H)\setminus\{u\}}\Big(d_H(x,u)+d_{P_k}(v_j,v_{m+1})\Big)^{\lambda}.$$

Note that $d_{G_1}(v_j) = d_{G_2}(v_j)$ if $j \in \{0,1,\dots,m+l\}\setminus\{m,m+1\}$, whereas
$d_{G_1}(v_m) \leqslant d_H(u)+2, d_{G_1}(v_{m+1}) = 2$, $d_{G_2}(v_m) = 2$ and $d_{G_2}(v_{m+1}) = d_H(u)+2$,

Hence,

$$
\begin{aligned}
\sum_{j=0}^{m+l} d_{G_1}(v_j) D^{\lambda}_{G_1}(v_j) &= \sum_{j=0}^{m+l} d_{G_1}(v_j) D^{\lambda}_{P_k}(v_j) \\
&+ \sum_{j=0}^{m+l} d_{G_1}(v_j) \sum_{x\in V(H)\setminus\{u\}} \Big(d_H(x,u) + d_{P_k}(v_j, v_m)\Big)^{\lambda} \\
&\leqslant (d_H(u)+2) D^{\lambda}_{P_k}(v_m) + 2D^{\lambda}_{P_k}(v_{m+1}) \\
&+ \sum_{j=0, j\neq m, m+1}^{m+l} d_{G_1}(v_j) D^{\lambda}_{P_k}(v_j) \\
&+ \sum_{x\in V(H)\setminus\{u\}} \Big(d_H(x,u) + m\Big)^{\lambda} \\
&+ 2\sum_{j=1}^{m-1} \sum_{x\in V(H)\setminus\{u\}} \Big(d_H(x,u) + m - j\Big)^{\lambda} \\
&+ (d_H(u)+2) d^{\lambda}_H(x,u) \\
&+ 2\sum_{j=1}^{l-1} \sum_{x\in V(H)\setminus\{u\}} \Big(d_H(x,u) + j\Big)^{\lambda} \\
&+ \sum_{x\in V(H)\setminus\{u\}} \Big(d_H(x,u) + l\Big)^{\lambda}
\end{aligned}
$$

and

$$\begin{aligned}\sum_{j=0}^{m+l} d_{G_2}(v_j)D^{\lambda}_{G_2}(v_j) &= \sum_{j=0}^{m+l} d_{G_2}(v_j)D^{\lambda}_{P_k}(v_j)\\ &\quad+\sum_{j=0}^{m+l} d_{G_2}(v_j)\sum_{x\in V(H)\backslash\{u\}}(d_H(x,u)+d_{P_k}(v_j,v_{m+1}))^{\lambda}\\ &= 2D^{\lambda}_{P_k}(v_m)+(d_H(u)+2)D^{\lambda}_{P_k}(v_{m+1})\\ &+\sum_{j=0,j\neq m,m+1}^{m+l} d_{G_2}(v_j)D^{\lambda}_{P_k}(v_j)\\ &+\sum_{x\in V(H)\backslash\{u\}}\Big(d_H(x,u)+m+1\Big)^{\lambda}\\ &+2\sum_{j=0}^{m-1}\sum_{x\in V(H)\backslash\{u\}}\Big(d_H(x,u)+m-j\Big)^{\lambda}\\ &+(d_H(u)+2)d^{\lambda}_H(x,u)\\ &+2\sum_{j=1}^{l-2}\sum_{x\in V(H)\backslash\{u\}}\Big(d_H(x,u)+j\Big)^{\lambda}\\ &+\sum_{x\in V(H)\backslash\{u\}}\Big(d_H(x,u)+l-1\Big)^{\lambda}.\end{aligned}$$

Observe that one can find

$$\begin{aligned}\sum_{j=0}^{m+l} d_{G_2}(v_j)D^{\lambda}_{G_2}(v_j)-\sum_{j=0}^{m+l} d_{G_1}(v_j)D^{\lambda}_{G_1}(v_j) &\leqslant d_H(u)(D^{\lambda}_{P_k}(v_{m+1})-D^{\lambda}_{P_k}(v_m))\\ &+\sum_{x\in V(H)\backslash\{u\}}(d^{\lambda}_H(x,u)+m+1)^{\lambda}\\ &-\sum_{x\in V(H)\backslash\{u\}}\Big(d_H(x,u)+l-1\Big)^{\lambda}\\ &+\sum_{x\in V(H)\backslash\{u\}}(d_H(x,u)+m)^{\lambda}\\ &-\sum_{x\in V(H)\backslash\{u\}}\Big(d_H(x,u)+l\Big)^{\lambda}.\end{aligned}$$

By simple calculation ,we have

$$\sum_{j=0}^{m+l} d_{G_2}(v_j)D^{\lambda}_{G_2}(v_j) > \sum_{j=0}^{m+l} d_{G_1}(v_j)D^{\lambda}_{G_1}k(v_j), when \lambda<0$$

and

$$\sum_{j=0}^{m+l} d_{G_2}(v_j) D^{\lambda}_{G_2}(v_j) < \sum_{j=0}^{m+l} d_{G_1}(v_j) D^{\lambda}_{G_1} k(v_j), when \lambda > 0.$$

On above observations we find $H_\lambda(G_{m,l}) < H_\lambda(G_{m+l,l-1})$ if $\lambda < 0$ and if $\lambda > 0$ we have $H_\lambda(G_{m,l}) > H_\lambda(G_{m+l,l-1})$. Hence desired □

4. θ-transformation

Let uw be a cut edge of a bipartite graph U with $d_U(w) \geqslant 2$. G is obtained from U and the star S_{k+2} by identifying u with a pendent vertex of S_{k+2} whose center is v. Let $G[v \to w; 2]$ be the graph obtained from G by deleting all edges vz, $z \in w$ and adding all edges wz, $z \in w$, where $w = N_G(v) \backslash \{u\}$.

In notation, $G[v \to w; 2] = G - \{vz : z \in w\} + \{wz : z \in w\}$ and we say $G[v \to w; 2]$ is obtained from G by θ-transformation. Graphs G , $G[v \to w; 2]$ are depicted in Fig. 4. The Laplacian permanent of trees with given bi-partition using above θ-transformation is studied by Li and Zhang [**10**]. Here we are to use the θ-transformation as a tool to study the generalized degree distance of trees.

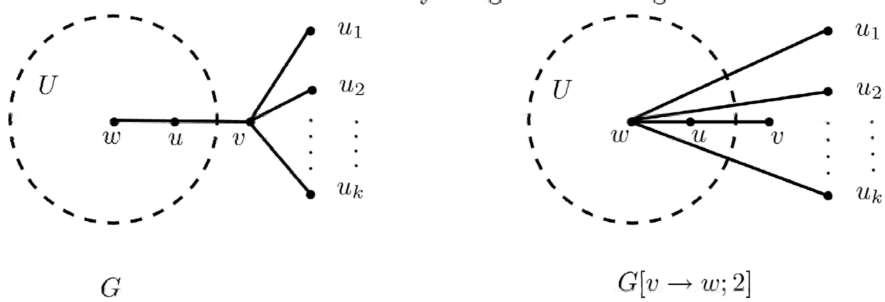

Fig.4 $G \Rightarrow G[v \to w; 2]$ by θ-transformation

THEOREM 4.1. *Let G and $G[v \to w; 2]$ be the bipartite graphs with some labelled vertices as above. Then*

$$H_\lambda G[v \to w; 2] > H_\lambda(G), \ If \ \lambda > 0$$
$$and \ if \ \lambda < 0 \ we \ have \ H_\lambda G[v \to w; 2] < H_\lambda(G).$$

PROOF. Let $G' = G[v \to w; 2]$ and let T_1 be the component in $G - \{wu, uv\}$ which contains u. Define $A = V(U) \backslash \big(V(T_1) \cup \{w\}\big)$. One can observe that, for all $x \in A$, we have

$$D^{\lambda}_G(x) = \sum_{y \in A \backslash \{x\}} (d_U(x,y))^{\lambda} + (d_U(x,w))^{\lambda}$$
$$+ \sum_{y \in V(T_1)} (d_G(x,y))^{\lambda} + (d_U(x, w+2))^{\lambda} + k(d_U(x,w)+3)^{\lambda},$$

$$D_{G'}^{\lambda}(x) = \sum_{y\in A\setminus\{x\}} (d_U(x,y))^{\lambda} + (d_U(x,w))^{\lambda}$$
$$+ \sum_{y\in V(T_1)} (d_{G'}(x,y))^{\lambda} + (d_U(x,w+2))^{\lambda} + k(d_U(x,w)+1)^{\lambda}.$$

Note that

$$\sum_{y\in V(T_1)} d_G^{\lambda}(x,y) = \sum_{y\in V(T_1)} d_{G'}^{\lambda}(x,y)$$

. Hence $D_{G'}^{\lambda}(x) < D_G^{\lambda}(x)$, if $\lambda < 0$ and $D_{G'}^{\lambda}(x) > D_G^{\lambda}(x)$ when $\lambda > 0$. It is easy to check that for all x in A, one has $d_G(x) = d_U(x) = d_{G'}(x)$. So we obtain that Whenever $\lambda < 0$

$$\sum_{x\in A} d_G(x) D_G^{\lambda}(x) < \sum_{x\in A} d_{G'}(x) D_{G'}^{\lambda}(x)$$

and if $\lambda > 0$,

$$\sum_{x\in A} d_G(x) D_G^{\lambda}(x) > \sum_{x\in A} d_{G'}(x) D_{G'}^{\lambda}(x).$$

For each vertex x in $V(T_1)$, it is easily check that $d_G(x) = d_{T_1}(x) = d_{G'}(x)$ and $D_G^{\lambda}(x) = D_{G'}^{\lambda}(x)$. Hence

$$\sum_{x\in V(T_1)} d_G(x) D_G^{\lambda}(x) = \sum_{x\in V(T_1)} d_{G'}(x) D_{G'}^{\lambda}(x).$$

For each $u_j \in X' = \{u_1, u_2, \ldots, u_k\}$, by direct calculation we have

$$D_G^{\lambda}(u_j) = 1 + (2)^{\lambda}(k-1) + \sum_{x\in V(T_1)} (d_G(x,v)+1)^{\lambda} + (3)^{\lambda} + \sum_{x\in A} d_U(x,w)+3)^{\lambda},$$

$$D_{G'}^{\lambda}(u_j) = 1 + (2)^{\lambda}(k-1) + \sum_{x\in V(T_1)} (d_G(x,w)+1)^{\lambda} + (3)^{\lambda} + \sum_{x\in A} (d_U(x,w)+1)^{\lambda}.$$

It is routine to check that $d_G^{\lambda}(x,v) = d_{G'}^{\lambda}(x,w)$ for $x \in V(T')$.
Hence we have $D_{G'}^{\lambda}(u_j) < D_G^{\lambda}(u_j)$. Note that $d_G(u_j) = d_{G'}(u_j) = 1$ for $u_j \in X'$. Hence when $\lambda < 0$ we have

$$\sum_{u_j\in X'} d_G(u_j) D_G^{\lambda}(u_j) < \sum_{u_j\in X'} d_{G'}(u_j) D_{G'}^{\lambda}(u_j)$$

and if $\lambda > 0$ we have

$$\sum_{u_j\in X'} d_G(u_j) D_G^{\lambda}(u_j) > \sum_{u_j\in X'} d_{G'}(u_j) D_{G'}^{\lambda}(u_j).$$

By overall calculation, we have

$$D_G^{\lambda}(w) = \sum_{x\in A} (d_U(x,w))^{\lambda} + \sum_{x\in V(T_1)} (d_G(x,w))^{\lambda} + (2)^{\lambda} + k(3)^{\lambda}$$

$$and\ D_G^{\lambda}(v) = \sum_{x\in A} (d_U(x,w)+2)^{\lambda} + \sum_{x\in V(T_1)} (d_G(x,v))^{\lambda} + (2)^{\lambda} + k$$

Similarly, we obtain

$$D^{\lambda}_{G'}(w) = \sum_{x\in A}(d_U(x,w))^{\lambda} + \sum_{x\in V(T_1)}(d_{G'}(x,w))^{\lambda} + (2)^{\lambda} + k$$

$$and\ D^{\lambda}_{G'}(v) = \sum_{x\in A}(d_U(x,w)+2)^{\lambda} + \sum_{x\in V(T_1)}(d_{G'}(x,v))^{\lambda} + (2)^{\lambda} + k(3)^{\lambda}.$$

One can see that $d_G(w) = d_U(w)+1$, $d_G(v) = k+1$, $d_{G'}(w) = d_U(w)+k+1$, $d_{G'}(v) = 1$ and for each vertex x in T_1, $d_G(x,w) = d_G(x,v) = d_{G'}(x,w) = d_{G'}(x,v)$. Observe that the difference of

$$\begin{aligned}
&d_{G'}(w)D^{\lambda}_{G'}(w) + d_{G'}(v)D^{\lambda}_{G'}(v) - d_G(w)D^{\lambda}_G(w) - d_G(v)D^{\lambda}_G(v)\\
&= k\Big(\sum_{x\in A}(d_U(x,w))^{\lambda} + \sum_{x\in V(T_1)}(d_{G'}(x,w))^{\lambda} + (2)^{\lambda}\Big) + k\Big(d_U(w)+k+1\Big)\\
&- k(3)^{\lambda}(d_U(w)+1) - k\Big(\sum_{x\in A}(d_U(x,w)+2)^{\lambda} + \sum_{x\in V_{T_1}}(d_{G'}(x,v))^{\lambda} + (2)^{\lambda}\Big)\\
&+ k(3)^{\lambda} - k(k-1)\\
&> \frac{(2)^{\lambda}}{(3)^{\lambda}}kd_U(w) + \frac{(1)^{\lambda}}{(3)^{\lambda}}k^2 + \frac{(5)^{\lambda}}{(3)^{\lambda}}k > 0,
\end{aligned}$$

i.e., when $\lambda > 0$ we have

$$d_{G'}(w)D^{\lambda}_{G'}(w) + d_{G'}(v)D^{\lambda}_{G'}(v) > d_G(w)D^{\lambda}_G(w) + d_G(v)D^{\lambda}_G(v)$$

but when $\lambda < 0$ we have

$$d_{G'}(w)D^{\lambda}_{G'}(w) + d_{G'}(v)D^{\lambda}_{G'}(v) < d_G(w)D^{\lambda}_G(w) - d_G(v)D^{\lambda}_G(v).$$

In view of the above results we get the following inequality

$$H_{\lambda}G[v\to w;2] > H_{\lambda}(G), If \lambda > 0$$

and

$$H_{\lambda}G[v\to w;2] < H_{\lambda}(G), if \lambda < 0\ as\ desired$$

□

References

[1] Y. Alizadeh, A. Iranmanesh, T. Došlić. Additively weighted Harary index of some composite graphs. *Discrete Math.*, **313**(1)(2013), 26–34.

[2] B. Bollobás and M. Tyomkyn. Walks and paths in trees. *J. Graph Theory*, **70**(1)(2012), 54–66.

[3] A. A. Dobrynin and A.A. Kochetova. Degree distance of a graph: a degree analogue of the Wiener index. *J. Chem. Inf. Comput. Sci.*, **34**(5)(1994), 1082–1086.

[4] I. Gutman. Selected properties of the Schultz molecular topological index. *J. Chem. Inf. Comput. Sci.*, **34**(5) (1994), 1087–1089.

[5] A. Hamzeh, A. Iranmanesh, S. Hossein-Zadeh and M. V. Diudea. Generalized degree distance of trees, unicyclic and bicyclic graphs. *Studia Ubb Chemia*, **LVII**(4)(2012), 73–85.

[6] A. Hamzeh, A. Iranmanesh and S. Hossein-Zadeh. Some results on generalized degree distance. *Open J. Discrete Math.*, **3**(2013), 143–150.

[7] H. Hua and S. Zhang. On the reciprocal degree distance of graphs. *Discrete Appl. Math.*, **160**(7-8) (2012), 1152–1163.
[8] A. Ilić . Trees with minimal Laplacian coefficient. *Comput. Math. Appl.*, **59**(8)(2010), 2776–2783.
[9] A. K. Kelmans. Comparison of graphs by their number of spanning trees. *Discrete Math.*, **16**(3)(1976), 241–261.
[10] S. C. Li and Y. Li amd X. X. Zhang. Edge - grafting theorems on permanents of the Laplacian matrices of graphs and their applications. *Electronic J. Linear Algebra*, **26**(2013), 28–48.
[11] S. C. Li and X. Meng. Four edge-grafting theorems on the reciprocal degree distance of graphs and their applications. *J. Comb. Optim.*, **30**(3)(2015), 468488.
[12] K. Pattabiraman and P. Kandan. Generalized degree distance of strong product of graphs. *Iran. J. Math. Sci. & Inf.*, **10**(2)(2015), 87–98.
[13] K. Pattabiraman and P. Kandan. Generalization on the degree distance of tensor product of graphs. *Aus. J. Comb.*, **62**(3)(2015), 211–227.
[14] G. F. Su, L.M. Xiong, X.F. Su and X. L. Chen. Some results on the reciprocal sum-degree distance of graphs. *J. Comb. Optim.*, **30**(3)(2015), 435–446.

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