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CHARACTERIZATION OF BIPOLAR FUZZY IDEALS IN ORDERED GAMMA SEMIGROUPS

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Abstract. In this paper, we introduce the notion of (η, δ) bipolar fuzzy ideal, bi-ideal,interior ideal, $(\epsilon, \epsilon \vee q)$-bipolar fuzzy ideal of ordered Γ-semigroups and discuss some of their properties.

1. Introduction

Fuzzy set was introduced by Zadeh [**17**]. Ordered Γ-semigroup was studied by Kehayopula [**8**]. Bipolar fuzzy set was first studied by Lee [**10**]. Bipolar fuzzy set is an extension of fuzzy set whose membership degree range is enlarged from the interval $[0,1]$ to $[-1,1]$. Faiz Muhammad Khan et al [**2**] introduced the concepts of (λ, θ)-fuzzy bi-ideal and (λ, θ)-fuzzy subsemigroup. Kazanci and Yamak [**4**]introduced the concept of a generalized fuzzy bi-ideal in semigroup and established some properties of fuzzy bi-ideals in terms of $(\epsilon, \epsilon \vee q)$-fuzzy bi-ideals. Jun et al [**3**] provided some results on ordered semigroups characterized by their $(\epsilon, \epsilon \vee q)$-fuzzy bi-ideals. Kehayopula and Tsingelies [**7**] initiated the study of fuzzy ordered semigroups. Bhakat and Das [**1**] introduced the concepts of $(\epsilon, \epsilon \vee q)$-fuzzy subgroups using the notion "belongingness ($\in$)" and "quasi-coincidence (q)". In this paper we define the new notions of (η, δ) bipolar fuzzy ideal, bi-ideal,interior ideal, $(\epsilon, \epsilon \vee q)$-bipolar fuzzy ideal of ordered Γ-semigroup and discuss some properties with examples.

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2. Preliminaries

Definition 2.1. ([**14**]) An ordered Γ-semigroup (shortly po-Γ-semigroup) is a Γ-semigroup S together with an order relation $\leqslant$ such that $a \leqslant b$ implies $a\gamma c \leqslant b\gamma c$ and $c\gamma a \leqslant c\gamma b$ for all $a, b, c \in S$ and $\gamma \in \Gamma$.

Definition 2.2. ([**14**]) Let A and B be two non empty subsets of a Γ- semigroup S. We denote
(i) $(A] = \{t \in S \,|\, t \leqslant h \, for \, some \, h \in A\}$,
(ii) $A\Gamma B = \{a\alpha b : a \in A, \, b \in B \; and \; \alpha \in \Gamma\}$,
(iii) $A_x = \{(y, z) \in S \times S \,|\, x \leqslant y\alpha z\}$.

Definition 2.3. ([**9**]) A non-empty subset B of a po Γ-semigroup S is called a bi-ideal of S if
(i) $a \in B$, $b \in S$ and $b \leqslant a$ implies $b \in B$,
(ii) $B\Gamma S\Gamma B \subseteq B$.

Definition 2.4. ([**17**]) Let X be a non-empty set. A fuzzy subset μ of X is a function from X into the closed unit interval $[0, 1]$. The set of all fuzzy subsets of X is called the fuzzy power set of X and is denoted by $FP(X)$.

Definition 2.5. ([**10**]) A bipolar fuzzy set A in a universe U is an object having the form $A = \{\langle x, \mu_A^+(x), \mu_A^-(x)\rangle : x \in X\}$, where $\mu_A^+ : X \to [0, 1]$ and $\mu_A^- : X \to [-1, 0]$. Here $\mu_A^+(x)$ represents the degree of satisfaction of the element x to the property and $\mu_A^-(x)$ represents the degree of satisfaction of x to some implict counter property of A. For simplicity the symbol $\langle \mu_A^+, \mu_A^- \rangle$ is used for the bipolar fuzzy set $A = \{\langle x, \mu_A^+(x), \mu_A^-(x)\rangle : x \in X\}$.

Definition 2.6. ([**2**]) A fuzzy subset μ of an ordered Γ-semigroup S is called a (λ, θ)-fuzzy bi-ideal of S if it satisfies the following conditions
(i) If $x \leqslant y$, then $\mu(x) \geqslant \mu(y)$,
(ii) $\max\{\mu(xy), \lambda\} \geqslant \min\{\mu(x), \mu(y), \theta\}$,
(iii) $\max\{\mu(xyz), \lambda\} \geqslant \min\{\mu(x), \mu(z), \theta\}$, for all $x, y, z \in S$.

Definition 2.7. ([**5**]) A fuzzy subset μ of a po Γ-semigroup S is called a fuzzy bi-ideal of S if
(i) If $x \leqslant y$, then $\mu(x) \geqslant \mu(y)$ and
(ii) $\mu(x\alpha y\beta z) \geqslant \min\{\mu(x), \mu(z)\}$ for every $x, y, z \in S$ and every $\alpha, \beta \in \Gamma$.

Definition 2.8. ([**11**]) A fuzzy subset μ of an ordered Γ-semigroup S is called a fuzzy Γ-subsemigroup of S if
(i) $x \leqslant y \Rightarrow \mu(x) \geqslant \mu(y)$ for all $x, y \in S$, and
(ii) $\mu(x\alpha y) \geqslant \min\{\mu(x), \mu(y)\}$ for all $x, y \in S$ and $\alpha \in \Gamma$.

Definition 2.9. ([**12**]) A fuzzy subset μ of an ordered Γ-semigroup S is called a fuzzy left (resp. right) ideal of S if
(i) $x \leqslant y \Rightarrow \mu(x) \geqslant \mu(y)$ for all $x, y \in S$, and
(ii) $\mu(x\alpha y) \geqslant \mu(y)$ (resp. $\mu(x\alpha y) \geqslant \mu(x)$) for all $x, y \in S$ and $\alpha \in \Gamma$.

A fuzzy subset μ of an ordered Γ-semigroup S is called a fuzzy ideal of S, if it is both fuzzy left ideal and fuzzy right ideal.

Definition 2.10. ([**15**]) Let A be a bipolar fuzzy set, if χ_A is the characteriatic function of A, then $(\chi_A)^{\delta}_{\eta}$ is defined as

$$(\chi_A)^{\beta}_{\alpha}(x) = \begin{cases} \beta \ if \ x \in A, \\ \alpha \ if \ x \notin A. \end{cases}$$

Definition 2.11. ([**13**]) For two bipolar fuzzy subsets $\mu = (\mu^+, \mu^-)$ and $\lambda = (\lambda^+, \lambda^-)$ of S, the product of two bipolar fuzzy subsets is denoted by $\mu \circ \lambda$ and is defined as

$$(\mu^+ \circ \lambda^+)(x) = \begin{cases} \sup\limits_{(s,t)\in A_x} \{\mu^+(s) \wedge \lambda^+(t)\} \ if \ A_x \neq 0 \\ 0 \quad if \qquad A_x = 0 \end{cases}$$

$$(\mu^- \circ \lambda^-)(x) = \begin{cases} \inf\limits_{(s,t)\in A_x} \{\lambda^-(s) \vee \lambda^-(t)\} \ if \, A_x \neq 0 \\ 0 \quad if \qquad A_x = 0 \end{cases}$$

Definition 2.12. A bipolar (η, δ) fuzzy sub Γ-semigroup $B = (\mu^+_B, \mu^-_B)$ of S is called a bipolar $(1, 2)$ fuzzy-Γ-ideal of S if

(i) $\max\{\mu^+_B(p\alpha q\beta(r\gamma s)), \eta^+\} \geqslant \min\{\mu^+_B(p), \mu^+_B(r), \mu^+_B(s), \delta^+\}$ and

(ii) $\min\{\mu^-_B(p\alpha q\beta(r\gamma s)), \eta^-\} \leqslant \max\{\mu^-_B(p), \mu^-_B(r), \mu^+_B(s), \delta^-\}$,

for all $p, q, r, s \in S$ and $\alpha, \beta, \gamma \in \Gamma$.

3. (η, δ)- bipolar fuzzy bi-ideals of ordered Γ-semigroups

In this section S denote as ordered Γ-semigroup.In what follows, $(\eta^+, \delta^+) \in [0, 1]$ and $(\eta^-, \delta^-) \in [-1, 0]$ be such that $0 \leqslant \eta^+ < \delta^+ \leqslant 1$ and $-1 \leqslant \delta^- < \eta^- \leqslant 0$, both $(\eta \ , \ \delta) \in [0, 1]$ are arbitrary but fixed.

Definition 3.1. A fuzzy subset μ of S is called a (η, δ)-bipolar fuzzy subsemigroup of S if it satisfies the following conditions:

(i) $p \leqslant q \Rightarrow \mu^+(p) \geqslant \mu^+(q)$ and $p \leqslant q \Rightarrow \mu^-(p) \leqslant \mu^-(q)$

(ii) $\max\{\mu^+(p\alpha q), \eta^+\} \geqslant \min\{\mu^+(p), \mu^+(q), \delta^+\}$ and

$\min\{\mu^-(p\alpha q), \eta^-\} \leqslant \max\{\mu^-(p), \mu^-(q), \delta^-\}$ for all $p, q \in S$.

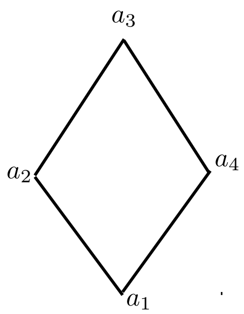
Example 3.1. Let $S = \{a_1, a_2, a_3, a_4\}$ and $\Gamma = \{\alpha\}$ where α is defined on S with the following Cayley table:

α	a_1	a_2	a_3	a_4
a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_2	a_3	a_4
a_3	a_1	a_3	a_3	a_3
a_4	a_1	a_3	a_3	a_3

$\leqslant := \{(a_1, a_1), (a_1, a_2), (a_1, a_3), (a_1, a_4), (a_2, a_2), (a_2, a_3), (a_2, a_4), (a_3, a_3), (a_4, a_3), (a_4, a_4)\}$.

We give the covering relation and the figure of S.

$$\prec = \{(a_1, a_2), (a_1, a_3), (a_1, a_4), (a_2, a_3), (a_2, a_4), (a_4, a_3)\}.$$



Define bipolar fuzzy subset $\tilde{\mu} = [\mu^+, \mu^-] : S \times \Gamma \times S \to [0,1] \times [-1,0]$

$$\mu^+(x) = \begin{cases} 0.7 \;\; if \;\; x = a_1 \\ 0.5 \;\; if \;\; x = a_2 \\ 0.2 \;\; if \;\; x = a_3 \\ 0.3 \;\; if \;\; x = a_4 \end{cases} \quad \mu^-(x) = \begin{cases} -0.9 \;\; if \;\; x = a_1 \\ -0.7 \;\; if \;\; x = a_2 \\ -0.3 \;\; if \;\; x = a_3 \\ -0.6 \;\; if \;\; x = a_4 \end{cases}$$

Then $\tilde{\mu}$ is a $(0.6, 0.8)$ bipolar fuzzy subsemigroup of S.

Definition 3.2. A fuzzy subset μ of S is called a (η, δ)-bipolar fuzzy bi-ideal of S if it satisfies the following conditions:

(i) if $x \leqslant y$, then $\mu^+(x) \geqslant \mu^+(y)$ and $x \leqslant y$, then $\mu^-(x) \leqslant \mu^-(y)$ and $\min\{\mu^-(p), \eta^-\} \leqslant \max\{\mu^-(p), \delta^-\}$ for all $p, q \in S$.

(ii) $\max\{\mu^+(p\alpha q), \eta^+\} \geqslant \min\{\mu^+(p), \mu^+(q), \delta^+\}$
$\min\{\mu^-(p\alpha q), \eta^-\} \leqslant \max\{\mu^-(p), \mu^-(q), \delta^-\}$.

(iii) $\max\{\mu^+(p\alpha q\beta r), \eta^+\} \geqslant \min\{\mu^+(p), \mu^+(r), \delta^+\}$
$\min\{\mu^-(p\alpha q\beta r), \eta^-\} \leqslant \max\{\mu^-(p), \mu^-(r), \delta^-\}$, for all $p, q, r \in S$ and $\alpha, \beta \in \Gamma$.

Example 3.2. Let $S = \{a_1, a_2, a_3, a_4\}$ and $\Gamma = \{\alpha, \beta\}$ where α, β is defined on S with the following Cayley tables:

α	a_1	a_2	a_3	a_4
a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_2	a_3	a_4
a_3	a_1	a_3	a_3	a_3
a_4	a_1	a_3	a_3	a_3

β	a_1	a_2	a_3	a_4
a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_2	a_3	a_4
a_3	a_1	a_3	a_3	a_3
a_4	a_1	a_2	a_3	a_4

$\leqslant := \{(a_1, a_1), (a_1, a_2), (a_1, a_3), (a_1, a_4), (a_2, a_2), (a_2, a_3), (a_2, a_4), (a_3, a_3), (a_4, a_3), (a_4, a_4)\}$.

Define bipolar fuzzy subset $\tilde{\mu} = [\mu^+, \mu^-] : S \times \Gamma \times S \to [0,1] \times [-1,0]$ as

$$\mu^+(x) = \begin{cases} 0.81 \;\; if \;\; x = a_1 \\ 0.62 \;\; if \;\; x = a_2 \\ 0.34 \;\; if \;\; x = a_3 \\ 0.43 \;\; if \;\; x = a_4 \end{cases} \quad \mu^-(x) = \begin{cases} -0.85 \;\; if \;\; x = a_1 \\ -0.65 \;\; if \;\; x = a_2 \\ -0.30 \;\; if \;\; x = a_3 \\ -0.50 \;\; if \;\; x = a_4 \end{cases}$$

Then $\tilde{\mu}$ is a $(0.70, 0.90)$ bipolar fuzzy bi-ideal of S

Theorem 3.1. *A fuzzy subset $\mu_{\tilde{\eta}}$ is a (η, δ)-bipolar fuzzy ordered Γ-sub semigroup (left, right, bi-ideal, interior ideal, (1, 2)-ideal) of S. Then the lower level set $\mu_{\tilde{\eta}} = [\mu^+_\eta, \mu^-_\eta]$ is an ordered Γ-subsemigroup (left, right, bi-ideal, interior ideal, (1, 2)-ideal) of S, where $\mu^+_\eta = \{p \in S|\mu^+(p) > \eta^+\}$ and $\mu^-_\eta = \{p \in S|\mu^-(p) < \eta^-\}$.*

Proof. Suppose that $\mu_{\tilde{\eta}}$ is a (η, δ)-bipolar fuzzy ordered Γ-subsemigroup. Let μ^+_η is a (η^+, δ^+) fuzzy Γ-subsemigroup. Let $p, q \in S$ and $\alpha \in \Gamma$ such that $p, q \in \mu^+_\eta$. Then $\mu^+(p) > \eta^+, \mu^+(q) > \eta^+$. Since μ^+ is a (η^+, δ^+) fuzzy subsemigroup, therefore $\max\{\mu^+(p\alpha q), \eta^+\} \geqslant \min\{\mu^+(p), \mu^+(q), \delta^+\} > \min\{\eta^+, \eta^+, \delta^+\} = \eta^+$. Hence $\mu^+(p\alpha q) > \eta^+$. It shows that $p\alpha q \in \mu^+_\eta$. Therefore μ^+_η is a Γ-subsemigroup of S. Let μ^-_η is a (η^-, δ^-) fuzzy ordered Γ-subsemigroup. Let $p, q \in S$ such that $p, q \in \mu^-_\eta$. Then $\mu^-(p) < \eta^-, \mu^-(q) < \eta^-$.Since μ^- is a (η^-, δ^-) fuzzy ordered Γ-subsemigroup. Therefore $\min\{\mu^-(p\alpha q), \eta^-\} \leqslant \max\{\mu^-(p), \mu^-(q), \delta^-\} < \max\{\eta^-, \eta^-, \delta^-\} = \eta^-$. Hence $\mu^-(p\alpha q) < \eta^-$. It shows that $p\alpha q \in \mu^-_\eta$. Therefore μ^-_η is a Γ-subsemigroup of S. Hence $\mu_{\tilde{\eta}} = [\mu^+_\eta, \mu^-_\eta]$ is a Γ-subsemigroup of S. □

Theorem 3.2. *A non-empty subset A of S is an ordered Γ-subsemigroup (left, right, bi-ideal, interior ideal, (1, 2)-ideal) of S if and only if the bipolar fuzzy subset $\tilde{\mu} = [\mu^+, \mu^-]$ of S defined as*

$$\mu^+(p) = \begin{cases} \geqslant \delta^+ \; for\, all\, p \in (A], \\ \eta^+ \; for\, all p \notin (A], \end{cases} \quad \mu^-(p) = \begin{cases} \leqslant \delta^- \; for\, all\, p \in (A], \\ \eta^- \; for\, all p \notin (A], \end{cases}$$

is a (η, δ)-bipolar fuzzy ordered Γ-subsemigroup (left, right, bi-ideal, interior ideal, (1, 2)-ideal) of S.

Proof. Assume that A is an ordered Γ-subsemigroup of S. Let $p, q \in S$ be such that $p, q \in (A]$ then $p\alpha q \in (A]$ and $\alpha \in \Gamma$. Hence $\mu^+(p\alpha q) \geqslant \delta^+$ and $\mu^-(p\alpha q) \leqslant \delta^-$. Therefore $\max\{\mu^+(p\alpha q), \eta^+\} \geqslant \delta^+ = \min\{\mu^+(p), \mu^+(q), \delta^+\}$ and $\min\{\mu^-(p\alpha q), \eta^-\} \leqslant \delta^- = \max\{\mu^-(p), \mu^-(q), \delta^-\}$. If $p \notin A$ or $q \notin (A]$ then $\min\{\mu^+(p), \mu^+(q), \delta^+\} = \eta^+$, $\max\{\mu^-(p), \mu^-(q), \delta^-\} = \eta^-$.That is $\max\{\mu^+(p\alpha q), \eta^+\} \geqslant \min\{\mu^+(p), \mu^+(q), \delta^+\}$ and $\min\{\mu^-(p\alpha q), \eta^-\} \leqslant \max\{\mu^-(p), \mu^-(q), \delta^-\}$. Therefore $\tilde{\mu} = [\mu^+, \mu^-]$ is a bipolar fuzzy Γ-subsemigroup of S.

Conversely assume that $\tilde{\mu} = [\mu^+, \mu^-]$ is a bipolar fuzzy Γ-subsemigroup of S. Let $p, q \in (A]$. Then $\mu^+(p) \geqslant \delta^+, \mu^+(q) \geqslant \delta^+$ and $\mu^-(p) \leqslant \delta^-, \mu(q) \leqslant \delta^-$. Now μ^+ is (η^+, δ^+) and μ^- is (η^-, δ^-)- fuzzy Γ-subsemigroup of S. Therefore $\max\{\mu^+(p\alpha q), \eta^+\} \geqslant \min\{\mu^+(p), \mu^+(q), \delta^+\} \geqslant \min\{\delta^+, \delta^+, \delta^+,\} = \delta^+$ and $\min\{\mu^-(p\alpha q), \eta^-\} \leqslant \max\{\mu^-(p), \mu^-(q), \delta^-\} \leqslant \max\{\delta^-, \delta^-, \delta^-,\} = \delta^-$. It follows that $p\alpha q \in (A]$. Therefore A is a ordered Γ-subsemigroup of S. □

Corollary 3.1. *A non-empty subset A of S is an ordered Γ-subsemigroup(left, right, bi-ideal, interior ideal,(1, 2)-ideal) of S if and only if the fuzzy subset μ of S defined as*

$$\mu^+(p) = \begin{cases} \geqslant 0.5 \, for\, all\, p \in (A], \\ 0 \, for\, all\, p \notin (A], \end{cases} \quad \mu^-(p) = \begin{cases} \leqslant -0.5 \, for\, all\, p \in (A], \\ 0 \, for\, all\, p \notin (A], \end{cases}$$

is a $(\epsilon, \epsilon \vee q)$-bipolar fuzzy subsemigroup(left, right, bi-ideal, interior ideal, (1, 2)-ideal) of S.

Proof. The proof follows by taking $\eta^+ = 0, \delta^+ = 0.5$ and $\eta^- = 0, \delta^- = -0.5$ in Theorem 3.2 □

Theorem 3.3. *A fuzzy subset $\tilde{\mu}$ of S is a (η, δ)-bipolar fuzzy subsemigroup (left, right, bi-ideal, interior ideal, (1, 2)-ideal) of S if and only if each non-empty level subset $(\tilde{\mu}^{(t,s)})$ is a subsemigroup (left, right, bi-ideal, interior ideal, (1, 2)-ideal) of S for all $t \in (\eta^+, \delta^+]$. and $s \in (\eta^-, \delta^-]$.*

Proof. Assume that $\tilde{\mu}^{(t,s)}$ is an ordered Γ-subsemigroup over S for each $t \in [0,1]$ and $s \in [-1,0]$.For each $p_1, p_2 \in S$ and $a \in (A]$, let $t = \min\{\mu^+(p_1), \mu^+(p_2)\}$ and $s = \max\{\mu^-(p_1), \mu^-(p_2)\}$, then $p_1, p_2 \in \tilde{\mu}^{(t,s)}$. That is $\max\{\mu^+((p\gamma q), \eta^+\} \geqslant t = \min\{\mu^+(p_1), \mu^+(p_2), \delta^+\}$ and $\min\{\mu^-(p_1\gamma p_2), \eta^-\} \leqslant s = \max\{\mu^-(p_1), \mu^-(p_2), \delta^-\}$. This shows that $\tilde{\mu}$ is bipolar fuzzy Γ-subsemigroup over S.

Conversely, assume that $\tilde{\mu}$ is a bipolar fuzzy ordered Γ-subsemigroup of S. For each $a \in (A], t \in [0,1]$ and $s \in [-1,0]$ and $p_1, p_2 \in \tilde{\mu}^{(t,s)}$ we have $\mu^+(p_1) \geqslant t, \mu^+(p_2) \geqslant t$ and $\mu^-(p_1) \leqslant s, \mu^-(p_2) \leqslant s$. Since $\tilde{\mu}$ is a bipolar fuzzy Γ-subsemigroup of S,

$$\max\{\mu^+(p_1\gamma p_2), \eta^+\} \geqslant \min\{\mu^+(p_1), \mu^+(p_2, \delta^+)\} \geqslant t$$
$$\min\{\mu^-(p_1\gamma p_2), \eta^-\} \leqslant \max\{\mu^-(p_1), \mu^-(p_2, \delta^-)\} \leqslant s,$$

$\gamma \in \Gamma$. Therefore $\tilde{\mu}^{(t,s)}$, this implies that $p_1\gamma p_2 \in \tilde{\mu}^{(t,s)}$. Therefore $\tilde{\mu}^{(t,s)}$ is a Γ-subsemigroup of S for each $t \in [0,1]$ and $s \in [-1,0]$. Similar proofs holds for left, right, bi-ideal, interior ideal, (1, 2)-ideal also. □

Example 3.3. Every bipolar fuzzy subsemigroup $\tilde{\mu} = [\mu^+, \mu^-]$ of ordered Γ-semigroup S is a (η, δ)-bipolar fuzzy subsemigroup of S, but converse is not true.

For the Example 3.1, we define bipolar fuzzy subset $\tilde{\mu} = [\mu^+, \mu^-]$ by

$$\mu^+(x) = \begin{cases} 0.65 \ \ if \ x = a_1 \\ 0.58 \ \ if \ x = a_2 \\ 0.51 \ \ if \ x = a_3 \\ 0.53 \ \ if \ x = a_4 \end{cases} \quad \mu^-(x) = \begin{cases} -0.85 \ \ if \ x = a_1 \\ -0.81 \ \ if \ x = a_2 \\ -0.68 \ \ if \ x = a_3 \\ -0.75 \ \ if \ x = a_4 \end{cases}$$

Then $\tilde{\mu}$ is a $(0.56, 0.70)$ bipolar fuzzy ordered Γ-subsemigroup of S, but not a bipolar fuzzy subsemigroup. Since $\mu^+(a_4\alpha a_4) = \mu^+(a_3) = 0.51 \ngeqslant \min\{\mu^+(a_4), \mu^+(a_4)\} = 0.53$ and $\mu^-(a_4\alpha a_4) = \mu^-(a_3) = -0.68 \nleqslant \max\{\mu^-(a_4), \mu^-(a_4)\} = -0.75$

Corollary 3.2. *Every $(\epsilon, \epsilon \vee q)$ bipolar fuzzy ordered Γ-subsemigroup of S is a (η, δ)-bipolar fuzzy ordered Γ-subsemigroup of S, but converse is not true.*

For the Example 3.1, define bipolar fuzzy subset $\tilde{\mu} = [\mu^+, \mu^-]$

$$\mu^+(x) = \begin{cases} 0.42 \ \ if \ x = a_1 \\ 0.38 \ \ if \ x = a_2 \\ 0.26 \ \ if \ x = a_3 \\ 0.30 \ \ if \ x = a_4 \end{cases} \quad \mu^-(x) = \begin{cases} -0.33 \ \ if \ x = a_1 \\ -0.30 \ \ if \ x = a_2 \\ -0.20 \ \ if \ x = a_3 \\ -0.24 \ \ if \ x = a_4 \end{cases}$$

Then $\tilde{\mu}$ is a $(0.35, 0.45)$ bipolar fuzzy ordered Γ-subsemigroup of S, but not a $(\epsilon, \epsilon \vee q)$ bipolar fuzzy ordered Γ-subsemigroup. Since $\mu^+(a_4\alpha a_4) = \mu^+(a_3) = 0.26 \not\geq \min\{\mu^+(a_4), \mu^+(a_4)\} = 0.30$ and $\mu^-(a_4\alpha a_4) = \mu^-(a_3) = -0.20 \not\leq \max\{\mu^-(a_4), \mu^-(a_4)\} = -0.24$.

Example 3.4. Every bipolar fuzzy bi-ideal $\tilde{\mu} = [\mu^+, \mu^-]$ of an ordered Γ-semigroup S is a (η, δ)-bipolar fuzzy bi-ideal of S, but converse is not true.

For the Example 3.2, we define bipolar fuzzy subset $\tilde{\mu} = [\mu^+, \mu^-]$

$$\mu^+(x) = \begin{cases} 0.81 & if\ x = a_1 \\ 0.62 & if\ x = a_2 \\ 0.34 & if\ x = a_3 \\ 0.43 & if\ x = a_4 \end{cases} \quad \mu^-(x) = \begin{cases} -0.85 & if\ x = a_1 \\ -0.65 & if\ x = a_2 \\ -0.30 & if\ x = a_3 \\ -0.50 & if\ x = a_4 \end{cases}$$

Then $\tilde{\mu}$ is a $(0.70, 0.85)$ bipolar fuzzy bi-ideal of S, but not a bipolar fuzzy bi-ideal, since $\mu^+(a_4\alpha a_4\beta a_4) = \mu^+(a_3) = 0.34 \not\geq \min\{\mu^+(a_4), \mu^+(a_4)\} = 0.43$ and $\mu^-(a_4\alpha a_4\beta a_4) = \mu^-(a_3) = -0.30 \not\leq \max\{\mu^-(a_4), \mu^-(a_4)\} = -0.50$

Corollary 3.3. *Every $(\epsilon, \epsilon \vee q)$ bipolar fuzzy bi-ideal of S is a (η, δ)-bipolar fuzzy bi-ideal of S, but converse is not true.*

For the Example 3.2, we define bipolar fuzzy subset $\tilde{\mu} = [\mu^+, \mu^-]$

$$\mu^+(x) = \begin{cases} 0.43 & if\ x = a_1 \\ 0.38 & if\ x = a_2 \\ 0.25 & if\ x = a_3 \\ 0.30 & if\ x = a_4 \end{cases} \quad \mu^-(x) = \begin{cases} -0.35 & if\ x = a_1 \\ -0.30 & if\ x = a_2 \\ -0.20 & if\ x = a_3 \\ -0.25 & if\ x = a_4 \end{cases}$$

Then $\tilde{\mu}$ is a $(0.40, 0.47)$ fuzzy bi-ideal of S, but not a fuzzy bi-ideal, since

$$\mu^+(a_4\alpha a_4\beta a_4) = \mu^+(a_3) = 0.25 \not\geq \min\{\mu^+(a_4), \mu^+(a_4)\} = 0.30$$

$$\mu^-(a_4\alpha a_4\beta a_4) = \mu^-(a_3) = -0.20 \not\leq \min\{\mu^-(a_4), \mu^-(a_4)\} = -0.25.$$

Definition 3.3. If χ_A is the characteristic function of A, then $(\chi_A)^\delta_\eta$ is defined as

$$(\chi^+_A)^\delta_\eta(x) = \begin{cases} \delta^+\ if\ x \in (A], \\ \eta^+\ if\ x \notin (A]. \end{cases} \quad (\chi^-_A)^\delta_\eta(x) = \begin{cases} \delta^-\ if\ x \in (A], \\ \eta^-\ if\ x \notin (A]. \end{cases}$$

Theorem 3.4. *A non empty subset A of S is a subsemigroup (left, right, bi-ideal, interior ideal, (1, 2)-ideal) of S if and only if fuzzy subset $\tilde{\chi}_A = [\chi^+_{(A]}, \chi^-_{(A]}]$ is a (η, δ)-bipolar fuzzy subsemigroup(left, right, bi-ideal, interior ideal, (1, 2)-ideal) of S.*

Proof. Assume that A is a subsemigroup of S. Then $\tilde{\chi}_{(A]}$ is a bipolar fuzzy subsemigroup of S and hence $\tilde{\chi}_{(A]}$ is an (η, δ)-bipolar fuzzy subsemigroup of S.

Conversely, let $p, q \in S$ be such that $p, q \in (A]$. Then $\chi^+_{(A]}(p) = \delta^+ = \chi^+_{(A]}(q) = \delta^+$ and $\chi^-_{(A]}(p) = \delta^- = \chi^-_{(A]}(q) = \delta^-$. Since $\tilde{\chi}_{(A]}$ is a (η, δ)-bipolar fuzzy subsemigroup. Consider

$$\begin{aligned}\max\{\chi^+_{(A]}(p\alpha q), \eta^+\} &\geqslant \min\{\chi^+_{(A]}(p), \chi^+_{(A]}(q), \delta^+\}\\ &= \min\{\delta^+, \delta^+, \delta^+\}\\ &= \delta^+\end{aligned}$$

as $\eta^+ < \delta^+$, this implies that $\{\chi^+_{(A]}(p\alpha q)\} \geqslant \delta^+$. Thus $p\alpha q \in (A]$.
Therefore A is a subsemigroup of S. And

$$\begin{aligned}\min\{\chi^-_{(A]}(p\alpha q), \eta^-\} &\leqslant \max\{\chi^-_{(A]}(p), \chi^-_{(A]}(q), \delta^-\}\\ &= \max\{\delta^-, \delta^-, \delta^-\}\\ &= \delta^-\end{aligned}$$

as $\delta^- < \eta^-$, this implies that $\{\chi^-_{(A]}(p\alpha q)\} \leqslant \delta^-$. Thus $p\alpha q \in (A]$.
Therefore $(A]$ is a subsemigroup of S.

Let $p, q \in S$ be such that $p, q \notin (A]$. Then $\chi^+_{(A]}(p) = \eta^+ = \chi^+_{(A]}(q) = \eta^+$ and $\chi^-_{(A]}(p) = \eta^+ = \chi^-_{(A]}(q) = \eta^+$. Since $\tilde{\chi}_{(A]}$ is a (η, δ)-bipolar fuzzy subsemigroup.

$$\begin{aligned}\max\{\chi^+_{(A]}(p\alpha q), \eta^+\} &\geqslant \min\{\chi^+_{(A]}(p), \chi^+_{(A]}(q), \delta^+\}\\ &= \min\{\eta^+, \eta^+, \delta^+\}\\ &= \eta^+\end{aligned}$$

as $\eta^+ < \delta^+$, this implies that $\{\chi^+_{(A]}(p\alpha q)\} \geqslant \eta^+$. Thus $p\alpha q \in (A]$. Therefore $(A]$ is a subsemigroup of S. And

$$\begin{aligned}\min\{\chi^-_{(A]}(p\alpha q), \eta^-\} &\leqslant \max\{\chi^-_{(A]}(p), \chi^-_{(A]}(q), \delta^-\}\\ &= \max\{\eta^-, \eta^-, \delta^-\}\\ &= \eta^-\end{aligned}$$

as $\delta^- < \eta^-$, this implies that $\{\chi^-_{(A]}(p\alpha q)\} \leqslant \eta^-$. Thus $p\alpha q \in (A]$. Therefore $(A]$ is a subsemigroup of S. Similar to proof holds for left, right, bi-ideal, interior ideal, (1, 2)-ideal also. □

Definition 3.4. Let $\tilde{\mu}$ be a bipolar fuzzy subset of an ordered semigroup S. We define the bipolar fuzzy subsets $(\mu^+)^\delta_\eta(p) = \{\mu^+(p) \wedge \delta^+\} \vee \eta^+$ and $(\mu^-)^\delta_\eta(p) = \{\mu^-(p) \vee \delta^-\} \wedge \eta^-$ for all $p \in S$.

Definition 3.5. Let $\tilde{\mu}_1$ and $\tilde{\mu}_2$ be two bipolar fuzzy subsets of an ordered semigroup S. Then we define the bipolar fuzzy subset

(i) $(\mu_1^+ \wedge^\delta_\eta \mu_2^+)(x) = \{\mu_1^+ \wedge \mu_2^+(x) \wedge \delta^+\} \vee \eta^+$.
(ii) $(\mu_1^- \wedge^\delta_\eta \mu_2^-)(x) = \{\mu_1^- \wedge \mu_2^-(x) \vee \delta^-\} \wedge \eta^-$.
(iii) $(\mu_1^+ \vee^\delta_\eta \mu_2^+)(x) = \{\mu_1^+ \vee \mu_2^+(x) \wedge \delta^+\} \vee \eta^+$.
(iv) $(\mu_1^- \vee^\delta_\eta \mu_2^-)(x) = \{\mu_1^- \vee \mu_2^-(x) \vee \delta^-\} \wedge \eta^-$.
(v) $(\mu_1^+ \circ^\delta_\eta \mu_2^+)(x) = \{\mu_1^+ \circ \mu_2^+(x) \wedge \delta^+\} \vee \eta^+$.

(vi) $(\mu_1^- \circ_\eta^\delta \mu_2^-)(x) = \{\mu_1^- \circ \mu_2^-(x) \vee \delta^-\} \wedge \eta^-$.

LEMMA 3.1. *Let A and B be non-empty subsets of S. Then the following hold:*
(i) $((\mu_1^+)\wedge_\eta^\delta(\mu_2^+))(x) = ((\mu_1^+)_\eta^\delta \wedge (\mu_2^+)_\eta^\delta)$ *and* $((\mu_1^-)\wedge_\eta^\delta(\mu_2^-))(x) = ((\mu_1^-)_\eta^\delta \wedge (\mu_2^-)_\eta^\delta)$
(ii) $((\mu_1^+)\vee_\eta^\delta(\mu_2^+))(x) = ((\mu_1^+)_\eta^\delta \vee (\mu_2^+)_\eta^\delta)$, $((\mu_1^-)\vee_\eta^\delta(\mu_2^-))(x) = ((\mu_1^-)_\eta^\delta \vee (\mu_2^-)_\eta^\delta)$
(iii) $((\mu_1^+)\circ_\eta^\delta(\mu_2^+))(x) = ((\mu_1^+)_\eta^\delta \circ (\mu_2^+)_\eta^\delta)$ *and* $((\mu_1^-)\circ_\eta^\delta(\mu_2^-))(x) = ((\mu_1^-)_\eta^\delta \circ (\mu_2^-)_\eta^\delta)$

LEMMA 3.2. *Let A and B be non-empty subsets of S. Then the following hold:*
(i) $(\chi_{(A]}^+ \wedge_\eta^\delta \chi_{(B]}^+) = (\chi_{A\cap B}^+)_\eta^\delta$ *and* $(\chi_{(A]}^- \wedge_\eta^\delta \chi_{(B]}^-) = (\chi_{A\cap B}^-)_\eta^\delta$.
(ii) $(\chi_{(A]}^+ \vee_\eta^\delta \chi_{(B]}^+) = (\chi_{A\cup B}^+)_\eta^\delta$ *and* $(\chi_{(A]}^- \vee_\eta^\delta \chi_{(A]}^-) = (\chi_{A\cup B}^-)_\eta^\delta$.
(iii) $(\chi_{(A]}^+ \circ_\Gamma{}_\eta^\delta \chi_B^+) = (\chi_{(A\Gamma B]}^+)_\eta^\delta$ *and* $(\chi_A^- \circ_\Gamma{}_\eta^\delta \chi_{(B]}^-) = (\chi_{(A\Gamma B]}^-)_\eta^\delta$.

PROOF. (i) and (ii) Straightforward.

(iii) Let $p \in S$. If $p \in (A\Gamma B]$, then $(\chi_{(A\Gamma B]}^+)(p) = \delta^+$ and $(\chi_{(A\Gamma B]}^-)(p) = \delta^-$. Since $p \leqslant a\alpha b$ for some $a \in (A]$, $b \in (B]$ and $\alpha \in \Gamma$, we have $(a, b) \in A_p$ and $A_p \neq 0$. We have

$$\begin{aligned}
(\chi_{(A]}^+ \circ_\Gamma \chi_{(A]}^+)(p) &= \sup_{p=y\alpha z} \min\{\chi_{(A]}^+(y), \chi_{(A]}^+(z)\} \\
&\geqslant \min\{\chi_{(A]}^+(a), \chi_{(A]}^+(b)\} \\
&= \delta^+ \\
(\chi_A^- \circ_\Gamma \chi_{(B]}^-)(p) &= \inf_{p=y\alpha z} \min\{\chi_{(A]}^-(y), \chi_{(A]}^-(z)\} \\
&\leqslant \max\{\chi_{(A]}^-(a), \chi_{(A]}^-(b)\} \\
&= \delta^-
\end{aligned}$$

Therefore $(\chi_{(A]}^+ \circ_\Gamma \chi_{(B]}^+)(p) = \delta^+ = (\chi_{(A\Gamma B]}^+(p)$ and $(\chi_{(A]}^- \circ_\Gamma \chi_{(B]}^-)(p) = \delta^- = (\chi_{(A\Gamma B]}^-(p)$. If $p \notin (A\Gamma B]$ then $(\chi_{(A\Gamma B]}^+)(p) = \eta^+$ and $(\chi_{(A\Gamma B]}^-)(p) = \eta^-$. Since $p \leqslant a\alpha b$ for some $a \notin (A]$, $b \notin (B]$ and $\alpha \in \Gamma$. We have

$$\begin{aligned}
(\chi_{(A]}^+ \circ_\Gamma \chi_{(B]}^+)(p) &= \sup_{p=y\alpha z} \min\{\chi_{(A]}^+(y), \chi_{(A]}^+(z)\} \\
&\geqslant \min\{\chi_{(A]}^+(a), \chi_{(A]}^+(b)\} \\
&= \eta^+ \\
(\chi_{(A]}^- \circ_\Gamma \chi_{(A]}^-)(p) &= \inf_{p=y\alpha z} \min\{\chi_{(A]}^-(y), \chi_{(A]}^-(z)\} \\
&\leqslant \max\{\chi_{(A]}^+(a), \chi_{(A]}^+(b)\} \\
&= \eta^-
\end{aligned}$$

Hence $(\chi_{(A]}^+ \circ_\Gamma \chi_{(B]}^+)(p) = \eta^+ = (\chi_{(A\Gamma B]}^+)(p)$ and $(\chi_{(A]}^- \circ_\Gamma \chi_{(B]}^-)(p) = \eta^- = (\chi_{(A\Gamma B]}^-)(p)$ □

THEOREM 3.5. *Let S be an (η, δ) ordered Γ-semigroup. Let $A, B \subseteq S$ and $\{A_i | i \in I\}$ be a family of subsets of S then*

(i) $(A] \subseteq (B]$ *if and only if* $(\chi_{(A]}^+)_\eta^\delta \leqslant (\chi_{(B]}^+)_\eta^\delta$ *and* $(\chi_{(A]}^-)_\eta^\delta \geqslant (\chi_{(B]}^-)_\eta^\delta$.

(ii) $(\cap_{i\in I}\chi_{(A_i]}^+)_\eta^\delta = (\chi_{\cap_{i\in I}(A_i]}^+)_\eta^\delta$ *and* $(\cup_{i\in I}\chi_{(A_i]}^-)_\eta^\delta = (\chi_{\cap_{i\in I}(A_i]}^-)_\eta^\delta$.

(iii) $(\cup_{i\in I}\chi^{+}_{(A_i]})^{\delta}_{\eta} = (\chi^{+}_{\cup_{i\in I}(A_i]})^{\delta}_{\eta}$ *and* $(\cap_{i\in I}\chi^{-}_{(A_i]})^{\delta}_{\eta} = (\chi^{-}_{\cup_{i\in I}(A_i]})^{\delta}_{\eta}$.

Proof. The proof follows from Proposition 2.4 [16]. □

Proposition 3.1. *If A is a (η, δ)- bipolar fuzzy left(subsemigroup, right, interior, (1, 2)-ideal)ideal of S, then $A = [(\mu^{+})^{\delta}_{\eta}, (\mu^{-})^{\delta}_{\eta}]$ is a bipolar fuzzy left(subsemi group, right, interior, (1, 2)-ideal) ideal of S.*

Proof. Assume that A is a (η, δ)-bipolar fuzzy left ideal of S. If there exist $p, q \in S$, and $\alpha \in \Gamma$ then

$$\begin{aligned}
\max\{(\mu^{+})^{\delta}_{\eta}(p\alpha q), \eta^{+}\} &= \max\{(\{\mu^{+}(p\alpha q) \wedge \delta^{+}\} \vee \eta^{+}), \eta^{+}\} \\
&= \{\mu^{+}(p\alpha q) \wedge \delta^{+}\} \vee \eta^{+} \\
&= \{\mu^{+}(p\alpha q) \vee \eta^{+}\} \wedge \{\delta^{+} \vee \eta^{+}\} \\
&= \{(\mu^{+}(p\alpha q) \vee \eta^{+}) \vee \eta^{+}\} \wedge \delta^{+} \\
&\geqslant \{(\mu^{+}(q) \wedge \delta^{+}) \vee \eta^{+}\} \wedge \delta^{+} \\
&\geqslant (\mu^{+})^{\delta}_{\eta}(q) \wedge \delta^{+}.
\end{aligned}$$

and

$$\begin{aligned}
\min\{(\mu^{-})^{\delta}_{\eta}(p\alpha q), \eta^{-}\} &= \min\{(\{\mu^{-}(p\alpha q) \vee \delta^{-}\} \wedge \eta^{-}), \eta^{-}\} \\
&= \{\mu^{-}(p\alpha q) \vee \delta^{-}\} \wedge \eta^{-} \\
&= \{\mu^{-}(p\alpha q) \wedge \eta^{-}\} \vee \{\delta^{-} \wedge \eta^{-}\} \\
&= \{(\mu^{-}(p\alpha q) \wedge \eta^{-}) \wedge \eta^{-}\} \vee \delta^{-} \\
&\leqslant \{(\mu^{-}(q) \vee \delta^{-}) \wedge \eta^{-}\} \vee \delta^{-} \\
&\leqslant (\mu^{-})^{\delta}_{\eta}(q) \vee \delta^{-}\}
\end{aligned}$$

Hence $A = [(\mu^{+})^{\delta}_{\eta}, (\mu^{-})^{\delta}_{\eta}]$ is a bipolar fuzzy left ideal of S.
Similar to proofs hold for subsemigroup, right ideals and interior ideal, (1, 2)-ideal also. □

Proposition 3.2. *If A is a (η, δ)-bipolar fuzzy bi-ideal,then $A = [(\mu^{+})^{\delta}_{\eta}, (\mu^{-})^{\delta}_{\eta}]$ is a bipolar fuzzy bi-ideal of S.*

Proof. Assume that A is a (η, δ)-bipolar fuzzy bi-ideal of S. If there exist $p, q, r \in S$, and $\alpha, \beta \in \Gamma$ then

$$\begin{aligned}
\max\{(\mu^{+})^{\delta}_{\eta}(p\alpha q\beta r), \eta^{+}\} &= \max\{(\{\mu^{+}(p\alpha q\beta r) \wedge \delta^{+}\} \vee \eta^{+}), \eta^{+}\} \\
&= \{\mu^{+}(p\alpha q\beta r) \wedge \delta^{+}\} \vee \eta^{+} \\
&= \{\mu^{+}(p\alpha q\beta r) \vee \eta^{+}\} \wedge \{\delta^{+} \vee \eta^{+}\} \\
&= \{\mu^{+}(p\alpha q\beta r) \vee \eta^{+}\} \wedge \delta^{+} \\
&= \{(\mu^{+}(p\alpha q\beta r) \vee \eta^{+}) \vee \eta^{+}\} \wedge \delta^{+}
\end{aligned}$$

$$\begin{aligned}
&\geqslant \{(\mu^+(p) \wedge \mu^+(r) \wedge \delta^+) \vee \eta^+\} \wedge \delta^+ \\
&= \{(\mu^+(p) \wedge \mu^+(r) \wedge \delta^+ \wedge \delta^+) \vee \eta^+ \vee \eta^+\} \wedge \delta^+ \\
&= \{\{(\mu^+(p) \wedge \delta^+) \vee \eta^+\} \wedge \{(\mu^+(r) \wedge \delta^+) \vee \eta^+\}\} \wedge \delta^+ \\
&= \{(\mu^+)^\delta_\eta(p) \wedge (\mu^+)^\delta_\eta(r)\} \wedge \delta^+.
\end{aligned}$$

and

$$\begin{aligned}
\min\{(\mu^-)^\delta_\eta(p\alpha q\beta r), \eta^-\} &= \min\{(\{\mu^-(p\alpha q\beta r) \vee \delta^-\} \wedge \eta^-), \eta^-\} \\
&= \{\mu^-(p\alpha q\beta r) \vee \delta^-\} \wedge \eta^- \\
&= \{\mu^-(p\alpha q\beta r) \wedge \eta^-\} \vee \{\delta^- \wedge \eta^-\} \\
&= \{\mu^-(p\alpha q\beta r) \wedge \eta^-\} \vee \delta^- \\
&= \{(\mu^-(p\alpha q\beta r) \wedge \eta^-) \wedge \eta^-\} \vee \delta^- \\
&\leqslant \{(\mu^-(p) \vee \mu^-(r) \vee \delta^-) \wedge \eta^-\} \vee \delta^- \\
&= \{(\mu^-(p) \vee \mu^-(r) \vee \delta^- \vee \delta^-) \wedge \eta^- \wedge \eta\} \vee \delta^- \\
&= \{\{(\mu^-(p) \vee \delta^-) \wedge \eta^-\} \vee \{(\mu^-(r) \vee \delta^-) \wedge \eta^-\}\} \vee \delta^- \\
&= \{(\mu^-)^\delta_\eta(p) \vee (\mu^-)^\delta_\eta(r)\} \vee \delta^-.
\end{aligned}$$

By similar way we can show the remaining part of the proposition. □

THEOREM 3.6. *Let $A = (\mu^+_A, \mu^-_A)$ be a bipolar (η, δ)- fuzzy right ideal and $B = (\mu^+_B, \mu^-_B)$ be a bipolar (η, δ)- fuzzy left ideal of S then $((A \circ_\Gamma B])^\delta_\eta \subseteq A \cap^\delta_\eta B$ and $((A \circ_\Gamma B])^\delta_\eta \supseteq A \cup^\delta_\eta B$.*

PROOF. Let $A = (\mu^+_A, \mu^-_A)$ be a bipolar (η, δ)- fuzzy right ideal and $B = (\mu^+_B, \mu^-_B)$ be a bipolar (η, δ)- fuzzy left ideal of S. Let $(p, q) \in I_r$. If $I_r \neq \emptyset$, then $r \leqslant p\gamma q$. Thus $\mu^+_A(r) \geqslant \mu^+_A(p\alpha q) \geqslant \mu^+_A(p)$ and $\mu^-_A(r) \leqslant \mu^-_A(p\alpha q) \leqslant \mu^-_A(p)$.
Similarly $\mu^+_B(r) \geqslant \mu^+_B(p\alpha q) \geqslant \mu^+_B(q)$ and $\mu^-_B(r) \leqslant \mu^-_B(p\alpha q) \leqslant \mu^-_B(q)$. we have

$$\begin{aligned}
(\mu^+_{(A\circ_\Gamma B]})_\eta{}^\delta(r) &= (\mu^+_{(A\circ_\Gamma B]}(r) \wedge \delta^+) \vee \eta^+ \\
&= (\max\{\mu^+_A(p) \wedge \mu^+_B(q)\} \wedge \delta^+) \vee \eta^+ \\
&= (\max\{\mu^+_A(p) \wedge \mu^+_B(q)\} \wedge \delta^+ \wedge \delta^+) \vee \eta^+ \\
&= (\max\{(\mu^+_A(p) \wedge \delta^+) \wedge (\mu^+_B(q) \wedge \delta^+)\} \wedge \delta^+) \vee \eta^+ \\
&\leqslant (\{(\mu^+_A(r) \vee \eta^+) \wedge (\mu^+_A(r) \vee \eta^+)\} \wedge \delta^+) \vee \eta^+ \\
&= ((\mu^+_A(r) \vee \eta^+) \wedge (\mu^+_B(r) \vee \eta^+) \wedge \delta^+) \vee \eta^+ \\
&= \{((\mu^+_A(r) \wedge \mu^+_B(r)) \vee \eta^+) \wedge \delta^+\} \vee \eta^+ \\
&= \{((\mu^+_A \wedge \mu^+_B)(r) \wedge \delta^+\} \vee \eta^+ \\
&= (\mu^+_{A\cap^\delta_\eta B})(r)
\end{aligned}$$

and

$$
\begin{aligned}
(\mu^-_{(A\circ_\Gamma B]})^\delta_\eta(r) &= (\mu^-_{(A\circ_\Gamma B]}(r)\vee\delta^-)\wedge\eta^- \\
&= (\min\{\mu^-_A(p)\vee\mu^-_B(q)\}\vee\delta^-)\vee\eta^- \\
&= (\min\{\mu^-_A(p)\vee\mu^-_B(q)\}\vee\delta^-\vee\delta^-)\wedge\eta^- \\
&= (\min\{(\mu^-_A(p)\vee\delta^-)\vee(\mu^-_B(q)\vee\delta^-)\}\vee\delta^-)\wedge\eta^- \\
&\geqslant (\{(\mu^-_A(r)\wedge\eta^-)\vee(\mu^-_A(r)\wedge\eta^-)\}\vee\delta^-)\wedge\eta^- \\
&= ((\mu^-_A(r)\wedge\eta^-)\vee(\mu^-_B(r)\wedge\eta^-)\vee\delta^-)\wedge\eta^- \\
&= \{((\mu^-_A(r)\vee\mu^-_B(r))\wedge\eta^-)\vee\delta^-\}\wedge\eta^- \\
&= \{((\mu^-_A\vee\mu^-_B)(r)\vee\delta^-\}\wedge\eta^- \\
&= (\mu^-_{A\cup^\delta_\eta B})(r)
\end{aligned}
$$

Let $p,q\notin I$. If $I_r=\emptyset$, then $(\mu^+_A\circ_\Gamma\mu^+_B)(r)=0=(\mu^-_A\circ_\Gamma\mu^-_B)(r)$ and $\alpha\in\Gamma$ such that $r\leqslant p\alpha q$.We have

$$
\begin{aligned}
(\mu^+_{(A\circ_\Gamma B]})^\delta_\eta(r) &= (\mu^+_{(A\circ_\Gamma B]}(p)\wedge\delta^+)\vee\eta^+ \\
&= 0\vee\eta^+ \\
&= \eta^+ \\
&\leqslant (\mu^+_{A\cap B}(p)\wedge\delta^+)\vee\eta^+ \\
&= (\mu^+_{A\cap B}(p)\wedge\delta^+)
\end{aligned}
$$

and

$$
\begin{aligned}
(\mu^-_{(A\circ_\Gamma B]})^\delta_\eta(r) &= (\mu^-_{(A\circ_\Gamma B]}(r)\vee\delta^-)\wedge\eta^- \\
&= 0\wedge\eta^- \\
&= \eta^- \\
&\geqslant (\mu^-_{A\cup B}(p)\vee\delta^-)\wedge\eta^- \\
&= (\mu^-_{A\cup B}(p)\vee\delta^-)
\end{aligned}
$$

Therefore $((A\circ_\Gamma B])^\delta_\eta\subseteq A\cap^\delta_\eta B$ and $((A\circ_\Gamma B])^\delta_\eta\supseteq A\cup^\delta_\eta B$. □

COROLLARY 3.4. *Let $A=(\mu^+_A,\mu^-_A)$ be a bipolar $(\epsilon,\epsilon\vee q)$- fuzzy right ideal and $B=(\mu^+_B,\mu^-_B)$ be a bipolar $(\epsilon,\epsilon\vee q)$- fuzzy left ideal of S then $((A\circ_\Gamma B])\subseteq A\cap B$ and $((A\circ_\Gamma B])\supseteq A\cup B$.*

PROOF. The proof follows taking $\eta^+=0,\delta^+=0.5$ and $\eta^-=0,\delta^-=-0.5$ in Theorem 3.6. □

COROLLARY 3.5 ([**6**]). *Let S be an ordered Γ-semigroup is regular if and only if every right ideal A and every left ideal B of S then $A\cap B=(A\circ_\Gamma B]$.*

THEOREM 3.7. *An ordered Γ-semigroup S is regular,let $A=(\mu^+_A,\mu^-_A)$ be a bipolar (η,δ)-fuzzy right ideal and $B=(\mu^+_B,\mu^-_B)$ be a bipolar (η,δ)- fuzzy left ideal of S if and only if $((A\circ_\Gamma B])^\delta_\eta=A\cap^\delta_\eta B$ and $((A\circ_\Gamma B])^\delta_\eta=A\cup^\delta_\eta B$.*

PROOF. Let S be an ordered Γ-regular semigroup and $A = (\mu_A^+, \mu_A^-)$ be a bipolar (η, δ)-fuzzy right ideal and $B = (\mu_B^+, \mu_B^-)$ be a bipolar (η, δ)- fuzzy left ideal of S. Let I be a non-empty set, then $I_r = \{(p, q) \in S \times S | r \leqslant p\gamma q\}$ from definition 2.2 in (iii). Thus $\mu_A^+(r) \geqslant \mu_A^+(p\alpha q) \geqslant \mu_A^+(p)$ and $\mu_A^-(r) \leqslant \mu_A^-(p\alpha q) \leqslant \mu_A^-(p)$. Similarly $\mu_B^+(r) \geqslant \mu_B^+(p\alpha q) \geqslant \mu_B^+(q)$ and $\mu_B^-(r) \leqslant \mu_B^-(p\alpha q) \leqslant \mu_B^-(q)$.

$$\begin{aligned}
(\mu^+_{(A\circ_\Gamma B]})^\delta_\eta(r) &= (\mu^+_{(A\circ_\Gamma B]}(r) \wedge \delta^+) \vee \eta^+ \\
&= (\max\{\mu_A^+(p) \wedge \mu_B^+(q)\} \wedge \delta^+) \vee \eta^+ \\
&= (\max\{\mu_A^+(p) \wedge \mu_B^+(q)\} \wedge \delta^+ \wedge \delta^+) \vee \eta^+ \\
&= (\max\{(\mu_A^+(p) \wedge \delta^+) \wedge (\mu_B^+(q) \wedge \delta^+)\} \wedge \delta^+) \vee \eta^+ \\
&\geqslant (\{(\mu_A^+(r\alpha x) \vee \eta^+) \wedge (\mu_A^+(r) \vee \eta^+)\} \wedge \delta^+) \vee \eta^+ \\
&\geqslant ((\mu_A^+(r) \vee \eta^+) \wedge (\mu_B^+(r) \vee \eta^+) \wedge \delta^+) \vee \eta^+ \\
&= \{((\mu_A^+(r) \wedge \mu_B^+(r)) \vee \eta^+) \wedge \delta^+\} \vee \eta^+ \\
&= \{((\mu_A^+ \wedge \mu_B^+)(r) \wedge \delta^+\} \vee \eta^+ \\
&= (\mu^+_{A\cap^\delta_\eta B})(r)
\end{aligned}$$

and

$$\begin{aligned}
(\mu^-_{(A\circ_\Gamma B]})^\delta_\eta(r) &= (\mu^-_{(A\circ_\Gamma B]}(r) \vee \delta^-) \wedge \eta^- \\
&= (\min\{\mu_A^-(p) \vee \mu_B^-(q)\} \vee \delta^-) \vee \eta^- \\
&= (\min\{\mu_A^-(p) \vee \mu_B^-(q)\} \vee \delta^- \vee \delta^-) \wedge \eta^- \\
&= (\min\{(\mu_A^-(p) \vee \delta^-) \vee (\mu_B^-(q) \vee \delta^-)\} \vee \delta^-) \wedge \eta^- \\
&\leqslant (\{(\mu_A^-(r\alpha x) \wedge \eta^-) \vee (\mu_A^-(r) \wedge \eta^-)\} \vee \delta^-) \wedge \eta^- \\
&\leqslant ((\mu_A^-(r) \wedge \eta^-) \vee (\mu_B^-(r) \wedge \eta^-) \vee \delta^-) \wedge \eta^- \\
&= \{((\mu_A^-(r) \vee \mu_B^-(r)) \wedge \eta^-) \vee \delta^-\} \wedge \eta^- \\
&= \{((\mu_A^- \vee \mu_B^-)(r) \vee \delta^-\} \wedge \eta^- \\
&= (\mu^-_{A\cup^\delta_\eta B})(r)
\end{aligned}$$

Thus $((A \circ {}_\Gamma B])^\delta_\eta \supseteq A \cap^\delta_\eta B$ and $((A \circ {}_\Gamma B])^\delta_\eta \subseteq A \cup^\delta_\eta B$, by Theorem 3.7 and hence $((A \circ {}_\Gamma B])^\delta_\eta = A \cap^\delta_\eta B$ and $((A \circ {}_\Gamma B])^\delta_\eta = A \cup^\delta_\eta B$. Conversely assume that $((A \circ {}_\Gamma B])^\delta_\eta = A \cap^\delta_\eta B$ and $((A \circ {}_\Gamma B])^\delta_\eta = A \cup^\delta_\eta B$.
Let $A = (\mu_A^+, \mu_A^-)$ be a bipolar (η, δ)-fuzzy right ideal and $B = (\mu_B^+, \mu_B^-)$ be a bipolar (η, δ)- fuzzy left ideal of S. Then by Theorem 3.4, χ_A be a bipolar (η, δ)-fuzzy right ideal and χ_A be a bipolar (η, δ)- fuzzy left ideal of S. By Lemma 3.2 and Theorem 3.5, we have $(\chi^+_{(A\cap B]})^\delta_\eta = (\chi_A^+ \cap^\delta_\eta \chi_B^+) = (\chi_A^+ \circ_\Gamma \chi_B^+)^\delta_\eta = (\chi^+_{(A\circ_\Gamma B]})^\delta_\eta$ and $(\chi^-_{(A\cap B]})^\delta_\eta = (\chi_A^- \cup^\delta_\eta \chi_B^-) = (\chi_A^- \circ_\Gamma \chi_B^-)^\delta_\eta = (\chi^-_{(A\circ_\Gamma B]})^\delta_\eta$. This implies $(A \cap B)^\delta_\eta = ((A \circ_\Gamma B])^\delta_\eta$. Hence by Corollary 3.5 S is regular. □

COROLLARY 3.6. *Let $A = (\mu_A^+, \mu_A^-)$ be a bipolar $(\epsilon, \epsilon \vee q)$-fuzzy right ideal and $B = (\mu_B^+, \mu_B^-)$ be a bipolar $(\epsilon, \epsilon \vee q)$- fuzzy left ideal of an ordered Γ-semigroup S. S is regular if and only if $((A \circ_\Gamma B]) = A \cap B$ and $((A \circ_\Gamma B]) = A \cup B$.*

PROOF. Taking $\eta^+ = 0, \delta^+ = 0.5$ and $\eta^- = 0, \delta^- = -0.5$ in Theorem 3.7 the proof follows. □

THEOREM 3.8. *Let $A = (\mu_A^+, \mu_A^-)$ be a bipolar (η, δ)-fuzzy bi-ideal and $B = (\mu_B^+, \mu_B^-)$ be a bipolar (η, δ)- fuzzy left ideal of an ordered Γ-semigroup S. S is regular if and only if $(A \circ_\Gamma B)_\eta^\delta = A \cap_\eta^\delta B$ and $(A \circ_\Gamma B)_\eta^\delta = A \cup_\eta^\delta B$.*

PROOF. Let S be an ordered Γ-regular semigroup and $A = (\mu_A^+, \mu_A^-)$ be a bipolar (η, δ)-fuzzy bi-ideal and $B = (\mu_B^+, \mu_B^-)$ be a bipolar (η, δ)- fuzzy left ideal of S. Let I be a non-empty set, then $I_r = \{(p, q) \in S \times S | r \leqslant p\gamma q\}$. Thus $\mu_A^+(r) \geqslant \mu_A^+(p\alpha q) \geqslant \mu_A^+(p)$ and $\mu_A^-(r) \leqslant \mu_A^-(p\alpha q) \leqslant \mu_A^-(p)$. Similarly $\mu_B^+(r) \geqslant \mu_B^+(p\alpha q) \geqslant \mu_B^+(q)$ and $\mu_B^-(r) \leqslant \mu_B^-(p\alpha q) \leqslant \mu_B^-(q)$.
For $r \in S$, there exists $x \in S$ such that $r \leqslant r\alpha x\beta r = r\alpha(x\beta r) \leqslant (r\alpha x\beta r)\alpha(x\beta r)$. Then $(r\alpha x\beta r), (x\beta r) \in I_r$. We have

$$\begin{aligned}
(\mu_{A\circ_\Gamma B}^+)_\eta^\delta(r) &= (\mu_{A\circ_\Gamma B}^+(r) \wedge \delta^+) \vee \eta^+ \\
&= (\max\{\mu_A^+(p) \wedge \mu_B^+(q)\} \wedge \delta^+) \vee \eta^+ \\
&= (\max\{\mu_A^+(p) \wedge \mu_B^+(q)\} \wedge \delta^+ \wedge \delta^+) \vee \eta^+ \\
&= (\max\{(\mu_A^+(p) \wedge \delta^+) \wedge (\mu_B^+(q) \wedge \delta^+)\} \wedge \delta^+) \vee \eta^+ \\
&\geqslant (\{(\mu_A^+(r\alpha x\beta r) \vee \eta^+) \wedge (\mu_A^+(x\beta r) \vee \eta^+)\} \wedge \delta^+) \vee \eta^+ \\
&\geqslant ((\mu_A^+(r) \vee \eta^+) \wedge (\mu_B^+(r) \vee \eta^+) \wedge \delta^+) \vee \eta^+ \\
&= \{((\mu_A^+(r) \wedge \mu_B^+(r)) \vee \eta^+) \wedge \delta^+\} \vee \eta^+ \\
&= \{((\mu_A^+ \wedge \mu_B^+)(r) \wedge \delta^+\} \vee \eta^+ \\
&= (\mu_{A\cap_\eta^\delta B}^+)(r)
\end{aligned}$$

and

$$\begin{aligned}
(\mu_{A\circ_\Gamma B}^-)_\eta^\delta(r) &= (\mu_{A\circ_\Gamma B}^-(r) \vee \delta^-) \wedge \eta^- \\
&= (\min\{\mu_A^-(p) \vee \mu_B^-(q)\} \vee \delta^-) \vee \eta^- \\
&= (\min\{\mu_A^-(p) \vee \mu_B^-(q)\} \vee \delta^- \vee \delta^-) \wedge \eta^- \\
&= (\min\{(\mu_A^-(p) \vee \delta^-) \vee (\mu_B^-(q) \vee \delta^-)\} \vee \delta^-) \wedge \eta^- \\
&\leqslant (\{(\mu_A^-(r\alpha x\beta r) \wedge \eta^-) \vee (\mu_A^-(x\beta r) \wedge \eta^-)\} \vee \delta^-) \wedge \eta^- \\
&\leqslant ((\mu_A^-(r) \wedge \eta^-) \vee (\mu_B^-(r) \wedge \eta^-) \vee \delta^-) \wedge \eta^- \\
&= \{((\mu_A^-(r) \vee \mu_B^-(r)) \wedge \eta^-) \vee \delta^-\} \wedge \eta^- \\
&= \{((\mu_A^- \vee \mu_B^-)(r) \vee \delta^-\} \wedge \eta^- \\
&= (\mu_{A\cup_\eta^\delta B}^-)(r)
\end{aligned}$$

Thus $(A \circ_\Gamma B)^\delta_\eta \supseteq A \cap^\delta_\eta B$ and $(A \circ_\Gamma B)^\delta_\eta \subseteq A \cup^\delta_\eta B$.
Conversely assume that $(A \circ_\Gamma B)^\delta_\eta \supseteq A \cap^\delta_\eta B$ and $(A \circ_\Gamma B)^\delta_\eta \subseteq A \cup^\delta_\eta B$.
Let $A = (\mu^+_A, \mu^-_A)$ be a bipolar (η, δ)-fuzzy bi ideal and $B = (\mu^+_B, \mu^-_B)$ be a bipolar (η, δ)- fuzzy left ideal of S. Since every bipolar fuzzy (η, δ)-right ideal of S is a bipolar fuzzy (η, δ)-bi-ideal of S. Thus $(A \circ_\Gamma B)^\delta_\eta \supseteq A \cap^\delta_\eta B$ and $(A \circ_\Gamma B)^\delta_\eta \subseteq A \cup^\delta_\eta B$, by Theorem 3.7. Hence S is regular. □

Corollary 3.7. *Let $A = (\mu^+_A, \mu^-_A)$ be a bipolar $(\epsilon, \epsilon \vee q)$-fuzzy bi-ideal and $B = (\mu^+_B, \mu^-_B)$ be a bipolar $(\epsilon, \epsilon \vee q)$- fuzzy left ideal of an ordered Γ-semigroup S. S is regular if and only if $(A \circ_\Gamma B) = A \cap^\delta_\eta B$ and $(A \circ_\Gamma B) = A \cup^\delta_\eta B$.*

Proof. Follows from Theorem 3.8 □

Theorem 3.9. *Let $A = (\mu^+_A, \mu^-_A)$ be a bipolar (η, δ)-fuzzy right ideal and $B = (\mu^+_B, \mu^-_B)$ be a bipolar (η, δ)- fuzzy left ideal of an ordered Γ-semigroup S. S weakly regular if and only if $(A \circ_\Gamma B)^\delta_\eta \supseteq A \cap^\delta_\eta B$ and $(A \circ_\Gamma B)^\delta_\eta \subseteq A \cup^\delta_\eta B$.*

Proof. Straightforward. □

Corollary 3.8. *Let $A = (\mu^+_A, \mu^-_A)$ be a bipolar $(\epsilon, \epsilon \vee q)$-fuzzy right ideal and $B = (\mu^+_B, \mu^-_B)$ be a bipolar $(\epsilon, \epsilon \vee q)$- fuzzy left ideal of an ordered Γ-semigroup S. S is weakly regular if and only if $(A \circ_\Gamma B) \supseteq A \cap B$ and $(A \circ_\Gamma B) \subseteq A \cup B$.*

Proof. Straightforward. □

Theorem 3.10. *Let $A = (\mu^+_A, \mu^-_A)$ be a bipolar (η, δ)-fuzzy bi-ideal and $B = (\mu^+_B, \mu^-_B)$ be a bipolar (η, δ)- fuzzy left ideal of an ordered Γ-semigroup S. S is weakly regular if and only if $(A \circ_\Gamma B)^\delta_\eta \supseteq A \cap^\delta_\eta B$ and $(A \circ_\Gamma B)^\delta_\eta \subseteq A \cup^\delta_\eta B$.*

Proof. Straightforward. □

Corollary 3.9. *Let $A = (\mu^+_A, \mu^-_A)$ be a bipolar $(\epsilon, \epsilon \vee q)$-fuzzy bi-ideal and $B = (\mu^+_B, \mu^-_B)$ be a bipolar $(\epsilon, \epsilon \vee q)$- fuzzy left ideal of an ordered Γ-semigroup S. S is weakly regular if and only if $(A \circ_\Gamma B) \supseteq A \cap B$ and $(A \circ_\Gamma B) \subseteq A \cup B$.*

Proof. Straightforward. □

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