JOURNAL OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4866, ISSN (o) 2303-4947
www.imvibl.org /JOURNALS / JOURNAL
Vol. 8(2018), 121-139
DOI: 10.7251/JIMVI1801121K

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

EXISTENCE CRITERIA OF POSITIVE SOLUTION FOR A SYSTEM OF RIENANN - LIOUVILLE TYPE p-LAPLACIAN PRACTIONAL ORDER BOUNDARY VALUE PROBLEMS

Boddu Muralee Bala Krushna

Abstract. This paper is concerned with determining the eigenvalue intervals of λ_1 and λ_2 for which there exist positive solutions to a coupled system of Riemann–Liouville type p-Laplacian fractional order boundary value problems by utilizing a fixed point theorem on a cone under suitable conditions.

1. Introduction

The main aim of this paper is to study the existence of eigenvalue intervals of λ_1, λ_2 for which there exist positive solutions to the coupled system of p-Laplacian fractional order boundary value problems

$$D_{0^+}^{\beta_1}\Big(\phi_p\Big(D_{0^+}^{\alpha_1}y_1(t)\Big)\Big) = \lambda_1 f\Big(t, y_1(t), y_2(t)\Big),\ t \in (0,1), \tag{1.1}$$

$$D_{0^+}^{\beta_2}\Big(\phi_p\Big(D_{0^+}^{\alpha_2}y_2(t)\Big)\Big) = \lambda_2 g\Big(t, y_1(t), y_2(t)\Big),\ t \in (0,1), \tag{1.2}$$

$$\begin{cases} y_1(0) = y_1'(0) = D_{0^+}^{q_1}y_1(0) = 0, \\ \zeta y_1(1) + \vartheta D_{0^+}^{q_2}y_1(1) = 0, \\ \phi_p\Big(D_{0^+}^{\alpha_1}y_1(0)\Big) = \phi_p\Big(D_{0^+}^{\alpha_1}y_1(1)\Big) = 0, \end{cases} \tag{1.3}$$

2010 *Mathematics Subject Classification.* 26A33, 34B18, 35J05.

Key words and phrases. Fractional derivative, p-Laplacian, eigenvalue, boundary value problem, Green's function, positive solution.

The author expresses his deep sense of gratitude to his guide Dr. K.R. Prasad, Professor of Applied Mathematics, Andhra University, Visakhapatnam.

$$(1.4)\qquad \begin{cases} y_2(0) = y_2'(0) = D_{0^+}^{q_1} y_2(0) = 0, \\ \zeta y_2(1) + \vartheta D_{0^+}^{q_2} y_2(1) = 0, \\ \phi_p\Big(D_{0^+}^{\alpha_2} y_2(0)\Big) = \phi_p\Big(D_{0^+}^{\alpha_2} y_2(1)\Big) = 0, \end{cases}$$

where

$$\lambda_1, \lambda_2 > 0,\ \phi_p(s) = |s|^{p-2}s,\ p > 1,\ \phi_p^{-1} = \phi_q,\ \tfrac{1}{p} + \tfrac{1}{q} = 1,\ \zeta, \vartheta$$

are positive real numbers, $\alpha_i \in (3,4]$, $q_2 \in (2,3]$, $1 < q_1, \beta_i \leqslant 2$ and $D_{0^+}^{\alpha_i}, D_{0^+}^{\beta_i}, D_{0^+}^{q_i}$, for $i = 1, 2$ are the standard Riemann–Liouville fractional order derivatives.

The theory of differential equations provides a broad mathematical basis to understand the problems of modern society which are complex and interdisciplinary by nature. Fractional order differential equations have attained much importance due to their applications to almost all areas of science, engineering and technology [**2, 4, 8, 10, 13, 14, 16, 17, 19**]. Among all the theories, the most applicable operator is the classical p-Laplacian, given by $\phi_p(s) = |s|^{p-2}s,\ p > 1$ [**3, 5, 6, 11**]. These types of problems have a wide range of applications in physics and related sciences such as biophysics, plasma physics, and chemical reaction design, we refer [**7, 18**].

Recently, Prasad and Krushna [**15**] established the existence of at least three positive solutions for a coupled system of p-Laplacian fractional order two-point boundary value problems,

$$D_{0^+}^{\beta_1}(\phi_p(D_{0^+}^{\alpha_1} u(t))) = f_1(t, u(t), v(t)),\ t \in (0,1),$$

$$D_{0^+}^{\beta_2}(\phi_p(D_{0^+}^{\alpha_2} v(t))) = f_2(t, u(t), v(t)),\ t \in (0,1),$$

$$u(0) = D_{0^+}^{q_1} u(0) = 0,\ \gamma u(1) + \delta D_{0^+}^{q_2} u(1) = 0,\ D_{0^+}^{\alpha_1} u(0) = D_{0^+}^{\alpha_1} u(1) = 0,$$

$$v(0) = D_{0^+}^{q_1} v(0) = 0,\ \gamma v(1) + \delta D_{0^+}^{q_2} v(1) = 0,\ D_{0^+}^{\alpha_2} v(0) = D_{0^+}^{\alpha_2} v(1) = 0,$$

where $\phi_p(s) = |s|^{p-2}s,\ p > 1,\ \gamma, \delta$ are constants, $2 < \alpha_i \leqslant 3,\ 1 < \beta_i, q_i \leqslant 2$ and f_1, f_2 are given functions, by applying five functionals fixed point theorem.

We assume the following conditions hold throughout the paper:

(A1) $f, g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^+$ are continuous,

(A2) each of $f_0, g_0, f^0, g^0, f_\infty, g_\infty, f^\infty, g^\infty$ by

$$f_0 = \lim_{y_1+y_2 \to 0^+} \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \frac{f(t, y_1, y_2)}{\phi_p(y_1 + y_2)}, \quad g_0 = \lim_{y_1+y_2 \to 0^+} \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \frac{g(t, y_1, y_2)}{\phi_p(y_1 + y_2)},$$

$$f^0 = \lim_{y_1+y_2 \to 0^+} \max_{t \in [\frac{1}{4}, \frac{3}{4}]} \frac{f(t, y_1, y_2)}{\phi_p(y_1 + y_2)}, \quad g^0 = \lim_{y_1+y_2 \to 0^+} \max_{t \in [\frac{1}{4}, \frac{3}{4}]} \frac{g(t, y_1, y_2)}{\phi_p(y_1 + y_2)},$$

$$f_\infty = \lim_{y_1+y_2 \to \infty} \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \frac{f(t, y_1, y_2)}{\phi_p(y_1 + y_2)}, \quad g_\infty = \lim_{y_1+y_2 \to \infty} \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \frac{g(t, y_1, y_2)}{\phi_p(y_1 + y_2)},$$

$$f^\infty = \lim_{y_1+y_2 \to \infty} \max_{t \in [\frac{1}{4}, \frac{3}{4}]} \frac{f(t, y_1, y_2)}{\phi_p(y_1 + y_2)}, \quad g^\infty = \lim_{y_1+y_2 \to \infty} \max_{t \in [\frac{1}{4}, \frac{3}{4}]} \frac{g(t, y_1, y_2)}{\phi_p(y_1 + y_2)},$$

exists as positive real number.

The rest of the paper is organized as follows. In Section 2, we compute the Green functions for the homogeneous boundary value problems corresponding to (1.1), (1.3) and (1.2), (1.4) and estimate the bounds for these Green functions. In Section 3, we establish criteria to determine the eigenvalue intervals of λ_1, λ_2 for which the p-Laplacian fractional order boundary value problem (1.1)-(1.4) has at least one positive solution on a cone by utilizing Guo–Krasnosel'skii fixed point theorem. In Section 4, as an application, we demonstrate our results with an example.

2. Green Functions and Bounds

In this section, the Green functions for the homogeneous boundary value problems are constructed and the bounds for the Green functions are estimated, which are essential to establish the main results.

Let $G_1(t,s)$ be the Green's function for the homogeneous boundary value problem

$$-D_{0^+}^{\alpha_1} y_1(t) = 0,\ t \in (0,1), \tag{2.1}$$

$$y_1(0) = 0, y_1'(0) = 0, D_{0^+}^{q_1} y_1(0) = 0,\ \zeta y_1(1) + \vartheta D_{0^+}^{q_2} y_1(1) = 0. \tag{2.2}$$

LEMMA 2.1. *Let*

$$\Delta_1 = \vartheta \Gamma(\alpha_1) + \zeta \Gamma(\alpha_1 - q_2) \neq 0.$$

If $h_1(t) \in C[0,1]$, *then the fractional order differential equation*

$$D_{0^+}^{\alpha_1} y_1(t) + h_1(t) = 0,\ t \in (0,1), \tag{2.3}$$

satisfying (2.2) *has a unique solution,*

$$y_1(t) = \int_0^1 G_1(t,s) h_1(s) ds,$$

where

$$G_1(t,s) = \begin{cases} G_{11}(t,s),\ 0 \leqslant t \leqslant s \leqslant 1, \\ G_{12}(t,s),\ 0 \leqslant s \leqslant t \leqslant 1, \end{cases} \tag{2.4}$$

$$\begin{aligned} G_{11}(t,s) =& \frac{1}{\Delta_1}\Big[\frac{\zeta\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} + \vartheta(1-s)^{-q_2}\Big][t(1-s)]^{\alpha_1 - 1}, \\ G_{12}(t,s) =& G_{11}(t,s) - \frac{1}{\Delta_1}\Big[\vartheta + \frac{\zeta\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)}\Big](t-s)^{\alpha_1 - 1}. \end{aligned}$$

PROOF. Let $y_1 \in C^4[0,1]$ be the solution of fractional order boundary value problem given by (2.3) and (2.2). An equivalent integral equation for (2.3) is given by

$$y_1(t) = \frac{-1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1 - 1} h_1(s) ds + c_1 t^{\alpha_1 - 1} + c_2 t^{\alpha_1 - 2} + c_3 t^{\alpha_1 - 3} + c_4 t^{\alpha_1 - 4}.$$

Using the boundary conditions (2.2), one can determine $c_4 = c_3 = c_2 = 0$ and

$$c_1 = \frac{1}{\Delta_1}\int_0^1 \Big[\frac{\zeta\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} + \vartheta(1-s)^{-q_2}\Big](1-s)^{\alpha_1-1}h_1(s)ds.$$

Hence, the unique solution of the fractional order boundary value problem given by (2.3) and (2.2) is

$$\begin{aligned} y_1(t) &= \int_0^t \Bigg\{\frac{1}{\Delta_1}\Big[\frac{\zeta\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} + \vartheta(1-s)^{-q_2}\Big][t(1-s)]^{\alpha_1-1} - \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}\Bigg\}h_1(s)ds \\ &+ \int_t^1 \frac{1}{\Delta_1}\Big[\frac{\zeta\Gamma(\alpha_1 - q_2)}{\Gamma(\alpha_1)} + \vartheta(1-s)^{-q_2}\Big]\,[t(1-s)]^{\alpha_1-1}\,h_1(s)ds \\ &= \int_0^1 G_1(t,s)h_1(s)ds, \end{aligned}$$

□

LEMMA 2.2. *Let $g_1(t) \in C[0,1]$. Then the fractional order differential equations*

$$D_{0^+}^{\beta_1}\Big(\phi_p\Big(D_{0^+}^{\alpha_1}y_1(t)\Big)\Big) = g_1(t),\ \ t \in (0,1), \tag{2.5}$$

satisfying

$$\phi_p\Big(D_{0^+}^{\alpha_1}y_1(0)\Big) = \phi_p\Big(D_{0^+}^{\alpha_1}y_1(1)\Big) = 0, \tag{2.6}$$

has a unique solution,

$$y_1(t) = \int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)g_1(\tau)d\tau\Big)ds,$$

where

$$H_1(t,s) = \begin{cases} \dfrac{[t(1-s)]^{\beta_1-1}}{\Gamma(\beta_1)}, & 0 \leqslant t \leqslant s \leqslant 1, \\ \dfrac{[t(1-s)]^{\beta_1-1} - (t-s)^{\beta_1-1}}{\Gamma(\beta_1)}, & 0 \leqslant s \leqslant t \leqslant 1. \end{cases} \tag{2.7}$$

PROOF. An equivalent integral equation for (2.5) is given by

$$\phi_p\Big(D_{0^+}^{\alpha_1}y_1(t)\Big) = \frac{1}{\Gamma(\beta_1)}\int_0^t (t-\tau)^{\beta_1-1}g_1(\tau)d\tau + k_1t^{\beta_1-1} + k_2t^{\beta_1-2}.$$

Applying (2.6), one can get $k_2 = 0$ and $k_1 = \dfrac{-1}{\Gamma(\beta_1)}\displaystyle\int_0^1 (1-\tau)^{\beta_1-1}g_1(\tau)d\tau$. Then,

$$\begin{aligned} \phi_p\Big(D_{0^+}^{\alpha_1}y_1(t)\Big) &= \int_0^t \frac{(t-\tau)^{\beta_1-1}}{\Gamma(\beta_1)}g_1(\tau)d\tau - \int_0^1 \frac{[t(1-\tau)]^{\beta_1-1}}{\Gamma(\beta_1)}g_1(\tau)d\tau \\ &= -\int_0^1 H_1(t,\tau)g_1(\tau)d\tau. \end{aligned}$$

Hence, $y_1(t) = \int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)g_1(\tau)d\tau\Big)ds$ is the solution of fractional order boundary value problem given by (2.5) and (1.3). □

LEMMA 2.3. *Assume that* $\vartheta(q_2-1) - \dfrac{\zeta\Gamma(\alpha_1-q_2)}{\Gamma(\alpha_1)} > 0$. *Then the Green's function* $G_1(t,s)$ *given in* (2.4) *is nonnegative, for all* $(t,s)\in[0,1]\times[0,1]$.

PROOF. Consider the Green's function $G_1(t,s)$ given by (2.4). Let $0\leqslant t\leqslant s\leqslant 1$. Then, we have

$$G_{11}(t,s) = \frac{1}{\Delta_1}\Big[\frac{\zeta\Gamma(\alpha_1-q_2)}{\Gamma(\alpha_1)} + \vartheta(1-s)^{-q_2}\Big][t(1-s)]^{\alpha_1-1} \geqslant 0.$$

Let $0\leqslant s\leqslant t\leqslant 1$. Then, we have

$$\begin{aligned}
&G_{12}(t,s)\\
&=\Big[\frac{\zeta\Gamma(\alpha_1-q_2)}{\Gamma(\alpha_1)} + \vartheta(1-s)^{-q_2}\Big]\frac{[t-ts]^{\alpha_1-1}}{\Delta_1} - \Big[\vartheta + \frac{\zeta\Gamma(\alpha_1-q_2)}{\Gamma(\alpha_1)}\Big]\frac{(t-s)^{\alpha_1-1}}{\Delta_1}\\
&\geqslant\Big[\frac{\zeta\Gamma(\alpha_1-q_2)}{\Gamma(\alpha_1)} + \vartheta(1-s)^{-q_2}\Big]\frac{[t-ts]^{\alpha_1-1}}{\Delta_1} - \Big[\vartheta + \frac{\zeta\Gamma(\alpha_1-q_2)}{\Gamma(\alpha_1)}\Big]\frac{(t-ts)^{\alpha_1-1}}{\Delta_1}\\
&\geqslant\frac{\vartheta}{\Delta_1}\big[(1-s)^{-q_2}-1\big]\,[t-ts]^{\alpha_1-1} \geqslant 0.
\end{aligned}$$

□

LEMMA 2.4. *Assume that* $\vartheta(q_2-1) - \dfrac{\zeta\Gamma(\alpha_1-q_2)}{\Gamma(\alpha_1)} > 0$. *Then the Green's function* $G_1(t,s)$ *given in* (2.4) *satisfies the inequality*

$$G_1(t,s)\leqslant G_1(1,s),\ \textit{for all}\ (t,s)\in[0,1]\times[0,1].$$

PROOF. Consider the Green's function $G_1(t,s)$ given by (2.4).

Let $0\leqslant t\leqslant s\leqslant 1$. Then, we have

$$\frac{\partial G_{11}(t,s)}{\partial t} = \frac{1}{\Delta_1}\Big[\frac{\zeta\Gamma(\alpha_1-q_2)}{\Gamma(\alpha_1)} + \vartheta(1-s)^{-q_2}\Big](1-s)^{\alpha_1-1}(\alpha_1-1)t^{\alpha_1-2} \geqslant 0.$$

Therefore, $G_{11}(t,s)$ is increasing in t, which implies $G_{11}(t,s)\leqslant G_{11}(1,s)$.

Let $0\leqslant s\leqslant t\leqslant 1$. Then, we have

$$\begin{aligned}
\frac{\partial G_{12}(t,s)}{\partial t} &=\Big[\frac{\zeta\Gamma(\alpha_1-q_2)}{\Gamma(\alpha_1)} + \vartheta(1-s)^{-q_2}\Big]\frac{(\alpha_1-1)[t-ts]^{\alpha_1-2}(1-s)}{\Delta_1}\\
&\quad - \Big[\vartheta + \frac{\zeta\Gamma(\alpha_1-q_2)}{\Gamma(\alpha_1)}\Big]\frac{(\alpha_1-1)(t-s)^{\alpha_1-2}}{\Delta_1}\\
&\geqslant\frac{(\alpha_1-1)(t-ts)^{\alpha_1-2}}{\Delta_1}\Big[\vartheta\Big((1-s)^{-(q_2-1)}-1\Big) - \frac{\zeta\Gamma(\alpha_1-q_2)}{\Gamma(\alpha_1)}s\Big]\\
&=\frac{(\alpha_1-1)(t-ts)^{\alpha_1-2}}{\Delta_1}\Big[\Big(\vartheta(q_2-1) - \frac{\zeta\Gamma(\alpha_1-q_2)}{\Gamma(\alpha_1)}\Big)s + O(s^2)\Big]\\
&\geqslant 0.
\end{aligned}$$

Therefore, $G_{12}(t,s)$ is increasing in t, which implies $G_{12}(t,s) \leqslant G_{12}(1,s)$. $\square$

Lemma 2.5. *Assume that* $\vartheta(q_2-1) - \dfrac{\zeta\Gamma(\alpha_1-q_2)}{\Gamma(\alpha_1)} > 0$. *Then the Green's function* $G_1(t,s)$ *given in* (2.4) *satisfies the inequality*

$$G_1(t,s) \geqslant \left(\frac{1}{4}\right)^{\alpha_1-1} G_1(1,s), \ \textit{for all} \ (t,s) \in I \times [0,1],$$

where $I = \left[\frac{1}{4}, \frac{3}{4}\right]$.

Proof. Consider the Green's function $G_1(t,s)$ given by (2.4).
Let $0 \leqslant t \leqslant s \leqslant 1$ and $t \in I$. Then

$$\begin{aligned} G_{11}(t,s) =& \Big[\frac{\zeta\Gamma(\alpha_1-q_2)}{\Gamma(\alpha_1)} + \vartheta(1-s)^{-q_2}\Big][t(1-s)]^{\alpha_1-1} \\ =& t^{\alpha_1-1}\Big[\frac{\zeta\Gamma(\alpha_1-q_2)}{\Gamma(\alpha_1)} + \vartheta(1-s)^{-q_2}\Big](1-s)^{\alpha_1-1} \\ \geqslant& \left(\frac{1}{4}\right)^{\alpha_1-1} G_{11}(1,s). \end{aligned}$$

Let $0 \leqslant s \leqslant t \leqslant 1$ and $t \in I$. Then

$$\begin{aligned} &G_{12}(t,s) \\ =& \Big[\frac{\zeta\Gamma(\alpha_1-q_2)}{\Gamma(\alpha_1)} + \vartheta(1-s)^{-q_2}\Big][t(1-s)]^{\alpha_1-1} - \Big[\vartheta + \frac{\zeta\Gamma(\alpha_1-q_2)}{\Gamma(\alpha_1)}\Big](t-s)^{\alpha_1-1} \\ \geqslant& t^{\alpha_1-1}\Big[\frac{\zeta\Gamma(\alpha_1-q_2)}{\Gamma(\alpha_1)} + \vartheta(1-s)^{-q_2} - \Big(\vartheta + \frac{\zeta\Gamma(\alpha_1-q_2)}{\Gamma(\alpha_1)}\Big)\Big](1-s)^{\alpha_1-1} \\ \geqslant& \left(\frac{1}{4}\right)^{\alpha_1-1} G_{12}(1,s). \end{aligned}$$

$\square$

Lemma 2.6. [?] *For* $t, s \in [0,1]$, *the Green's function* $H_1(t,s)$ *satisfies the following inequalities*

$$(i)\ H_1(t,s) \geqslant 0,$$
$$(ii)\ H_1(t,s) \leqslant H_1(s,s).$$

Lemma 2.7. *Let* $\xi \in (\frac{1}{4}, \frac{3}{4})$. *Then the Green's function* $H_1(t,s)$ *holds the inequality*

$$\min_{t\in I} H_1(t,s) \geqslant \psi_1^*(s) H_1(s,s), \ \textit{for} \ 0 < s < 1, \tag{2.8}$$

where

$$\psi_1^*(s) = \begin{cases} \dfrac{[\frac{3}{4}(1-s)]^{\beta_1-1} - (\frac{3}{4}-s)^{\beta_1-1}}{[s(1-s)]^{\beta_1-1}}, & s \in (0,\xi], \\ \dfrac{1}{(4s)^{\beta_1-1}}, & s \in [\xi, 1). \end{cases} \tag{2.9}$$

PROOF. We define

$$h_1(t,s)=\frac{[t(1-s)]^{\beta_1-1}-(t-s)^{\beta_1-1}}{\Gamma(\beta_1)}, h_2(t,s)=\frac{[t(1-s)]^{\beta_1-1}}{\Gamma(\beta_1)} \text{ and}$$

$$H_1(s,s)=\frac{1}{\Gamma(\beta_1)}[s(1-s)]^{\beta_1-1}.$$

Then

$$\begin{aligned}
\min_{t\in I} H_1(t,s)=&\begin{cases} h_1(\frac{3}{4},s), & s\in(0,\frac{1}{4}],\\ \min\Big\{h_1(\frac{3}{4},s),h_2(\frac{1}{4},s)\Big\}, & s\in[\frac{1}{4},\frac{3}{4}],\\ h_2(\frac{1}{4},s), & s\in[\frac{3}{4},1),\end{cases}\\
=&\begin{cases} h_1(\frac{3}{4},s), & s\in(0,\xi],\\ h_2(\frac{1}{4},s), & s\in[\xi,1),\end{cases}\\
\geqslant&\begin{cases} \dfrac{[\frac{3}{4}(1-s)]^{\beta_1-1}-(\frac{3}{4}-s)^{\beta_1-1}}{[s(1-s)]^{\beta_1-1}}H_1(s,s), & s\in(0,\xi],\\ \dfrac{1}{(4s)^{\beta_1-1}}H_1(s,s), & s\in[\xi,1),\end{cases}\\
=&\psi_1^*(s)H_1(s,s).
\end{aligned}$$

□

In a similar way, we construct the Green's function $G_2(t,s)$ for the homogeneous fractional order boundary value problem

$$-D_{0^+}^{\alpha_2}y_2(t)=0,\ t\in(0,1), \tag{2.10}$$

$$y_2(0)=y_2'(0)=D_{0^+}^{q_1}y_2(0)=0,\ \zeta y_2(1)+\vartheta D_{0^+}^{q_2}y_2(1)=0. \tag{2.11}$$

LEMMA 2.8. *Let $\Delta_2=\vartheta\Gamma(\alpha_2)+\zeta\Gamma(\alpha_2-q_2)\neq 0$. If $h_2(t)\in C[0,1]$, then the fractional order differential equation*

$$D_{0^+}^{\alpha_2}y_2(t)+h_2(t)=0,\ t\in(0,1), \tag{2.12}$$

satisfying (2.11) *has a unique solution,*

$$y_2(t)=\int_0^1 G_2(t,s)h_2(s)ds,$$

where

$$G_2(t,s)=\begin{cases} G_{21}(t,s), & 0\leqslant t\leqslant s\leqslant 1,\\ G_{22}(t,s), & 0\leqslant s\leqslant t\leqslant 1,\end{cases} \tag{2.13}$$

$$\begin{aligned}
G_{21}(t,s)=&\frac{1}{\Delta_2}\Big[\frac{\zeta\Gamma(\alpha_2-q_2)}{\Gamma(\alpha_2)}+\vartheta(1-s)^{-q_2}\Big][t(1-s)]^{\alpha_2-1},\\
G_{22}(t,s)=&G_{21}(t,s)-\frac{1}{\Delta_2}\Big[\vartheta+\frac{\zeta\Gamma(\alpha_2-q_2)}{\Gamma(\alpha_2)}\Big](t-s)^{\alpha_2-1}.
\end{aligned}$$

PROOF. Proof is similar to Lemma 2.1. □

LEMMA 2.9. *Let $g_2(t) \in C[0,1]$. Then the fractional order differential equation*

$$D_{0^+}^{\beta_2}\Big(\phi_p\Big(D_{0^+}^{\alpha_2}y_2(t)\Big)\Big) = g_2(t),\ t \in (0,1), \tag{2.14}$$

satisfying

$$\phi_p\Big(D_{0^+}^{\alpha_2}y_2(0)\Big) = \phi_p\Big(D_{0^+}^{\alpha_2}y_2(1)\Big) = 0, \tag{2.15}$$

has a unique solution,

$$y_2(t) = \int_0^1 G_2(t,s)\phi_q\Big(\int_0^1 H_2(s,\tau)g_2(\tau)d\tau\Big)ds,$$

where

$$H_2(t,s) = \begin{cases} \dfrac{[t(1-s)]^{\beta_2-1}}{\Gamma(\beta_2)}, & 0 \leqslant t \leqslant s \leqslant 1, \\ \dfrac{[t(1-s)]^{\beta_2-1} - (t-s)^{\beta_2-1}}{\Gamma(\beta_2)}, & 0 \leqslant s \leqslant t \leqslant 1. \end{cases} \tag{2.16}$$

PROOF. Proof is similar to Lemma 2.2. □

LEMMA 2.10. *Assume that $\vartheta(q_2-1) - \dfrac{\zeta\Gamma(\alpha_2-q_2)}{\Gamma(\alpha_2)} > 0$. Then the Green's function $G_2(t,s)$ given in* (2.13) *is nonnegative, for all $(t,s) \in [0,1]\times[0,1]$.*

PROOF. Proof is similar to Lemma 2.3. □

LEMMA 2.11. *Assume that $\vartheta(q_2-1) - \dfrac{\zeta\Gamma(\alpha_2-q_2)}{\Gamma(\alpha_2)} > 0$. Then the Green's function $G_2(t,s)$ given in* (2.13) *satisfies the inequality*

$$G_2(t,s) \leqslant G_2(1,s),\ \textit{for all}\ (t,s) \in [0,1]\times[0,1].$$

PROOF. Proof is similar to Lemma 2.4. □

LEMMA 2.12. *Assume that $\vartheta(q_2-1) - \dfrac{\zeta\Gamma(\alpha_2-q_2)}{\Gamma(\alpha_2)} > 0$. Then the Green's function $G_2(t,s)$ given in* (2.13) *satisfies the inequality*

$$G_2(t,s) \geqslant \Big(\frac{1}{4}\Big)^{\alpha_2-1}G_2(1,s),\ \textit{for all}\ (t,s) \in I\times[0,1],$$

where $I = \left[\frac{1}{4}, \frac{3}{4}\right]$.

PROOF. Proof is similar to Lemma 2.5. □

LEMMA 2.13. [?] *For $t,s \in [0,1]$, the Green's function $H_2(t,s)$ satisfies the following inequalities*

$$(i)\ H_2(t,s) \geqslant 0,$$
$$(ii)\ H_2(t,s) \leqslant H_2(s,s).$$

LEMMA 2.14. *Let* $\xi \in (\frac{1}{4}, \frac{3}{4})$. *Then the Green's function* $H_2(t,s)$ *holds the inequality*

$$\min_{t\in I} H_2(t,s) \geqslant \psi_2^*(s) H_2(s,s), \ \textit{for } 0 < s < 1, \tag{2.17}$$

where

$$\psi_2^*(s) = \begin{cases} \dfrac{\left[\frac{3}{4}(1-s)\right]^{\beta_2-1} - (\frac{3}{4}-s)^{\beta_2-1}}{[s(1-s)]^{\beta_2-1}}, & s \in (0,\xi], \\ \dfrac{1}{(4s)^{\beta_2-1}}, & s \in [\xi,1). \end{cases} \tag{2.18}$$

PROOF. Proof is similar to Lemma 2.7. □

3. Existence of Positive Solutions

In this section, we establish criteria to determine the eigenvalue intervals of λ_1, λ_2 for which the p-Laplacian fractional order boundary value problem (1.1)-(1.4) has at least one positive solution on a cone by utilizing Guo–Krasnosel'skii fixed point theorem.

THEOREM 3.1. [**9, 12**] *Let* X *be a Banach Space,* $P \subseteq X$ *be a cone and suppose that* Ω_1, Ω_2 *are open subsets of* X *with* $0 \in \Omega_1$ *and* $\overline{\Omega}_1 \subset \Omega_2$. *Suppose further that* $T : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ *is completely continuous operator such that either*

(i) $\| Tu \| \leqslant \| u \|,\ u \in P \cap \partial\Omega_1$ *and* $\| Tu \| \geqslant \| u \|,\ u \in P \cap \partial\Omega_2$, *or*

(ii) $\| Tu \| \geqslant \| u \|,\ u \in P \cap \partial\Omega_1$ *and* $\| Tu \| \leqslant \| u \|,\ u \in P \cap \partial\Omega_2$ *holds.*

Then T *has a fixed point in* $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Consider the Banach space $B = E \times E$, where $E = \{y_1 : y_1 \in C[0,1]\}$ equipped with the norm $\|(y_1,y_2)\| = \|y_1\|_0 + \|y_2\|_0$, for $(y_1,y_2) \in B$ and the norm is defined as

$$\|y_1\|_0 = \max_{t\in[0,1]} |y_1(t)|.$$

Define a cone $P \subset B$ by

$$P = \Big\{(y_1,y_2) \in B \mid y_1(t) \geqslant 0,\ y_2(t) \geqslant 0,\ t \in [0,1] \text{ and } \min_{t\in I} [y_1(t) + y_2(t)] \geqslant \eta \|(y_1,y_2)\| \Big\},$$

where

$$\eta = \min\left\{ \Big(\frac{1}{4}\Big)^{\alpha_1-1}, \Big(\frac{1}{4}\Big)^{\alpha_2-1} \right\}. \tag{3.1}$$

Let $T_1, T_2 : P \to E$ and $T : P \to B$ be the operators defined by

$$T_1(y_1,y_2)(t) = \lambda_1 \int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau) f(\tau, y_1(\tau), y_2(\tau)) d\tau\Big) ds,$$

$$T_2(y_1,y_2)(t) = \lambda_2 \int_0^1 G_2(t,s)\phi_q\Big(\int_0^1 H_2(s,\tau) g(\tau, y_1(\tau), y_2(\tau)) d\tau\Big) ds,$$

and

$$T(y_1,y_2)(t)=\Big(T_1(y_1,y_2)(t),T_2(y_1,y_2)(t)\Big),\ \text{for } (y_1,y_2)\in B. \tag{3.2}$$

LEMMA 3.1. *The operator T defined by* (3.2) *is a self map on P.*

PROOF. Let $(y_1,y_2)\in P$. Clearly, $T_1(y_1,y_2)(t)\geqslant 0$ and $T_2(y_1,y_2)(t)\geqslant 0$, for $t\in[0,1]$. Also, for $(y_1,y_2)\in P$,

$$\begin{aligned}\|T_1(y_1,y_2)\|_0 &\leqslant \lambda_1\int_0^1 G_1(1,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f(\tau,y_1(\tau),y_2(\tau))d\tau\Big)ds,\\ \|T_2(y_1,y_2)\|_0 &\leqslant \lambda_2\int_0^1 G_2(1,s)\phi_q\Big(\int_0^1 H_2(s,\tau)g(\tau,y_1(\tau),y_2(\tau))d\tau\Big)ds,\end{aligned}$$

and

$$\begin{aligned}\min_{t\in I}T_1(y_1,y_2)(t) &= \min_{t\in I}\lambda_1\int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f(\tau,y_1(\tau),y_2(\tau))d\tau\Big)ds\\ &\geqslant \eta\lambda_1\int_0^1 G_1(1,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f(\tau,y_1(\tau),y_2(\tau))d\tau\Big)ds\\ &\geqslant \eta\|T_1(y_1,y_2)\|_0.\end{aligned}$$

Similarly, $\min_{t\in I}T_2(y_1,y_2)(t)\geqslant \eta\|T_2(y_1,y_2)\|_0$. Therefore,

$$\begin{aligned}\min_{t\in I}\Big\{T_1(y_1,y_2)(t)+T_2(y_1,y_2)(t)\Big\} &\geqslant \eta\|T_1(y_1,y_2)\|_0+\eta\|T_2(y_1,y_2)\|_0\\ &= \eta\|(T_1(y_1,y_2),T_2(y_1,y_2))\|\\ &= \eta\|T(y_1,y_2)\|.\end{aligned}$$

Hence, $T(y_1,y_2)\in P$ and so $T:P\to P$. Standard arguments involving the Arzela–Ascoli theorem shows that T is completely continuous. □

Let

$$\begin{aligned}\Lambda_1=\max\Bigg\{&\frac{1}{2\Big[\int_{s\in I}\eta^2G_1(1,s)\phi_q\Big(\int_{\tau\in I}\psi^*(\tau)H_1(\tau,\tau)d\tau\Big)ds\Big](f_\infty)^{q-1}},\\ &\frac{1}{2\Big[\int_{s\in I}\eta^2G_2(1,s)\phi_q\Big(\int_{\tau\in I}\psi^*(\tau)H_2(\tau,\tau)d\tau\Big)ds\Big](g_\infty)^{q-1}}\Bigg\}\end{aligned}$$

and

$$\begin{aligned}\Lambda_2=\min\Bigg\{&\frac{1}{2\Big[\int_0^1 G_1(1,s)\phi_q\Big(\int_0^1 H_1(\tau,\tau)d\tau\Big)ds\Big](f^0)^{q-1}},\\ &\frac{1}{2\Big[\int_0^1 G_2(1,s)\phi_q\Big(\int_0^1 H_2(\tau,\tau)d\tau\Big)ds\Big](g^0)^{q-1}}\Bigg\},\end{aligned}$$

where

$$\psi^*(\tau)=\min\Big\{\psi_1^*(\tau),\psi_2^*(\tau)\Big\}. \tag{3.3}$$

THEOREM 3.2. *Assume that the conditions* $(A1)$-$(A2)$ *are satisfied. Then, for each* λ_i, *for* $i = 1, 2$ *satisfying*

$$\lambda_1, \lambda_2 \in (\Lambda_1, \Lambda_2), \tag{3.4}$$

there exists at least one positive solution to the coupled system of p*-Laplacian fractional order boundary value problem* (1.1)-(1.4) *that lies in* P.

PROOF. Let λ_1, λ_2 be given as in (3.4). Now, let $\epsilon > 0$ be chosen such that

$$\max\left\{\frac{1}{2\Big[\int_{s\in I}\eta^2 G_1(1,s)\phi_q\Big(\int_{\tau\in I}\psi^*(\tau)H_1(\tau,\tau)d\tau\Big)ds\Big](f_\infty-\epsilon)^{q-1}},\right.$$
$$\left.\frac{1}{2\Big[\int_{s\in I}\eta^2 G_2(1,s)\phi_q\Big(\int_{\tau\in I}\psi^*(\tau)H_2(\tau,\tau)d\tau\Big)ds\Big](g_\infty-\epsilon)^{q-1}}\right\} \leqslant \lambda_1, \lambda_2$$

and

$$\lambda_1, \lambda_2 \leqslant \min\left\{\frac{1}{2\Big[\int_0^1 G_1(1,s)\phi_q\Big(\int_0^1 H_1(\tau,\tau)d\tau\Big)ds\Big](f^0+\epsilon)^{q-1}},\right.$$
$$\left.\frac{1}{2\Big[\int_0^1 G_2(1,s)\phi_q\Big(\int_0^1 H_2(\tau,\tau)d\tau\Big)ds\Big](g^0+\epsilon)^{q-1}}\right\}.$$

Let T be the cone preserving, completely continuous operator defined by (3.2). Now from the definition of f^0, we may choose $H^{1^*} > 0$ so that

$$f(t, y_1, y_2) \leqslant (f^0+\epsilon)\phi_p(y_1+y_2), \ \text{for } 0 < (y_1, y_2) \leqslant H^{1^*}. \tag{3.5}$$

Similarly from the definition of g^0, there exists an $H^{1^{**}} > 0$ such that

$$g(t, y_1, y_2) \leqslant (g^0+\epsilon)\phi_p(y_1+y_2), \ \text{for } 0 < (y_1, y_2) \leqslant H^{1^{**}}. \tag{3.6}$$

In particular, then by putting $H^1 = \min\{H^{1^*}, H^{1^{**}}\}$. We find both (3.5) and (3.6) hold for $0 < (y_1, y_2) \leqslant H^1$. So, define Ω_1 by

$$\Omega_1 = \Big\{(y_1, y_2) \in B : \|(y_1, y_2)\| < H^1\Big\}. \tag{3.7}$$

Then for $(y_1, y_2) \in P \cap \partial\Omega_1$, we have

$$\begin{aligned}
\|T_1(y_1,y_2)(t)\| &= \lambda_1\int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f(\tau,y_1(\tau),y_2(\tau))d\tau\Big)ds\\
&\leqslant \lambda_1\int_0^1 G_1(1,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f(\tau,y_1,y_2)d\tau\Big)ds\\
&\leqslant \lambda_1\int_0^1 G_1(1,s)\phi_q\Big(\int_0^1 H_1(\tau,\tau)(f^0+\epsilon)\phi_p(y_1+y_2)d\tau\Big)ds\\
&\leqslant \lambda_1\int_0^1 G_1(1,s)\phi_q\Big(\int_0^1 H_1(\tau,\tau)d\tau\Big)(f^0+\epsilon)^{q-1}\|(y_1,y_2)\|ds\\
&\leqslant \frac{1}{2}\|(y_1,y_2)\|.
\end{aligned}$$

Therefore,

$$\|T_1(y_1,y_2)(t)\| \leqslant \frac{1}{2}\|(y_1,y_2)\|, \quad \text{for } t\in[0,1]. \tag{3.8}$$

Similarly,

$$\|T_2(y_1,y_2)(t)\| \leqslant \frac{1}{2}\|(y_1,y_2)\|, \quad \text{for } t\in[0,1]. \tag{3.9}$$

For $(y_1,y_2)\in P\cap\partial\Omega_1$, we have

$$\begin{aligned}\|T(y_1,y_2)(t)\| &= \|(T_1(y_1,y_2)(t),T_2(y_1,y_2)(t))\|\\ &\leqslant \frac{1}{2}\|(y_1,y_2)\|+\frac{1}{2}\|(y_1,y_2)\|\\ &= \|(y_1,y_2)\|.\end{aligned}$$

Hence,

$$\|T(y_1,y_2)\| \leqslant \|(y_1,y_2)\|, \text{ for } (y_1,y_2)\in P\cap\partial\Omega_1. \tag{3.10}$$

By the definition of f_∞ and g_∞ there exists $\overline{H}^2>0$ such that

$$f(t,y_1,y_2) \geqslant (f_\infty-\epsilon)\phi_p(y_1+y_2), \text{ for } 0<(y_1,y_2)\geqslant \overline{H}^2, \tag{3.11}$$

$$g(t,y_1,y_2) \geqslant (g_\infty-\epsilon)\phi_p(y_1+y_2), \text{ for } 0<(y_1,y_2)\geqslant \overline{H}^2. \tag{3.12}$$

If we set $H^2=\max\Big\{2H^1,\dfrac{\overline{H}^2}{\eta}\Big\}$ and define

$$\Omega_2=\Big\{(y_1,y_2)\in B:\|(y_1,y_2)\|<H^2\Big\}. \tag{3.13}$$

If $(y_1,y_2)\in P\cap\partial\Omega_2$ then it follows that $\|(y_1,y_2)\|=H^2$, we have

$$\min_{t\in I}[y_1(t)+y_2(t)] \geqslant \eta\|(y_1,y_2)\| \geqslant \overline{H}^2, \tag{3.14}$$

and so

$$\begin{aligned}&T_1(y_1,y_2)(t)\\ &=\lambda_1\int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f(\tau,y_1(\tau),y_2(\tau))d\tau\Big)ds\\ &\geqslant\lambda_1\int_{s\in I}\eta G_1(1,s)\phi_q\Big(\int_{\tau\in I}\psi^*(\tau)H_1(\tau,\tau)(f_\infty-\epsilon)\phi_p(y_1+y_2)d\tau\Big)ds\\ &\geqslant\lambda_1\int_{s\in I}\eta G_1(1,s)\phi_q\Big(\int_{\tau\in I}\psi^*(\tau)H_1(\tau,\tau)d\tau\Big)(f_\infty-\epsilon)^{q-1}(y_1+y_2)ds\\ &\geqslant\lambda_1\int_{s\in I}\eta^2 G_1(1,s)\phi_q\Big(\int_{\tau\in I}\psi^*(\tau)H_1(\tau,\tau)d\tau\Big)(f_\infty-\epsilon)^{q-1}\|(y_1,y_2)\|ds\\ &\geqslant\frac{1}{2}\|(y_1,y_2)\|.\end{aligned}$$

Thus,

$$\|T_1(y_1,y_2)\| \geqslant \frac{1}{2}\|(y_1,y_2)\|, \text{ for } (y_1,y_2)\in P\cap\partial\Omega_2. \tag{3.15}$$

Similarly,

$$\|T_2(y_1,y_2)\| \geqslant \frac{1}{2}\|(y_1,y_2)\|, \text{ for } (y_1,y_2)\in P\cap\partial\Omega_2. \tag{3.16}$$

Thus, (3.15)-(3.16) imply that

$$\begin{aligned}\|T(y_1,y_2)(t)\| &= \|(T_1(y_1,y_2)(t),T_2(y_1,y_2)(t))\|\\ &= \|T_1(y_1,y_2)(t)\|+\|T_2(y_1,y_2)(t)\|\\ &\geqslant \frac{1}{2}\|(y_1,y_2)\|+\frac{1}{2}\|(y_1,y_2)\|\\ &= \|(y_1,y_2)\|.\end{aligned}$$

Hence,

$$\|T(y_1,y_2)\| \geqslant \|(y_1,y_2)\|, \text{ for } (y_1,y_2)\in P\cap\partial\Omega_2. \tag{3.17}$$

An application of Theorem 3.1 to (3.10) and (3.17) yields a fixed point (y_1,y_2) of T that lies in $P\cap(\overline{\Omega}_2\backslash\Omega_1)$. This fixed point is the solution of the fractional order boundary value problem (1.1)-(1.4). □

Let

$$\Lambda_3=\max\left\{\frac{1}{2\Big[\int_{s\in I}\eta^2G_1(1,s)\phi_q\Big(\int_{\tau\in I}\psi^*(\tau)H_1(\tau,\tau)d\tau\Big)ds\Big](f_0)^{q-1}},\right.$$
$$\left.\frac{1}{2\Big[\int_{s\in I}\eta^2G_2(1,s)\phi_q\Big(\int_{\tau\in I}\psi^*(\tau)H_2(\tau,\tau)d\tau\Big)ds\Big](g_0)^{q-1}}\right\}$$

and

$$\Lambda_4=\min\left\{\frac{1}{2\Big[\int_0^1G_1(1,s)\phi_q\Big(\int_0^1H_1(\tau,\tau)d\tau\Big)ds\Big](f^\infty)^{q-1}},\right.$$
$$\left.\frac{1}{2\Big[\int_0^1G_2(1,s)\phi_q\Big(\int_0^1H_2(\tau,\tau)d\tau\Big)ds\Big](g^\infty)^{q-1}}\right\},$$

where $\psi^*(\tau)$ is given in (3.3).

THEOREM 3.3. *Assume that the conditions (A1)-(A2) are satisfied. Then, for each λ_i, for $i=1,2$ satisfying*

$$\lambda_1,\lambda_2\in(\Lambda_3,\Lambda_4), \tag{3.18}$$

there exists at least one positive solution to the coupled system of p-Laplacian fractional order boundary value problem (1.1)-(1.4) that lies in P.

Proof. Let λ_1, λ_2 be given as in (3.18). Now, let $\epsilon > 0$ be chosen such that

$$\max\left\{\frac{1}{2\Big[\int_{s\in I}\eta^2 G_1(1,s)\phi_q\Big(\int_{\tau\in I}\psi^*(\tau)H_1(\tau,\tau)d\tau\Big)ds\Big](f_0-\epsilon)^{q-1}},\right.$$
$$\left.\frac{1}{2\Big[\int_{s\in I}\eta^2 G_2(1,s)\phi_q\Big(\int_{\tau\in I}\psi^*(\tau)H_2(\tau,\tau)d\tau\Big)ds\Big](g_0-\epsilon)^{q-1}}\right\}\leqslant \lambda_1,\lambda_2$$

and

$$\lambda_1,\lambda_2\leqslant\min\left\{\frac{1}{2\Big[\int_0^1 G_1(1,s)\phi_q\Big(\int_0^1 H_1(\tau,\tau)d\tau\Big)ds\Big](f^\infty+\epsilon)^{q-1}},\right.$$
$$\left.\frac{1}{2\Big[\int_0^1 G_2(1,s)\phi_q\Big(\int_0^1 H_2(\tau,\tau)d\tau\Big)ds\Big](g^\infty+\epsilon)^{q-1}}\right\}.$$

Let T be the cone preserving, completely continuous operator defined by (3.2). Now from the definition of f_0, g_0 there exist $J^{1*} > 0, J^{1**} > 0$ such that

$$f(t,y_1,y_2)\geqslant(f_0-\epsilon)\phi_p(y_1+y_2),\ \text{for } 0<(y_1,y_2)\leqslant J^{1*}, \tag{3.19}$$

$$g(t,y_1,y_2)\geqslant(g_0-\epsilon)\phi_p(y_1+y_2),\ \text{for } 0<(y_1,y_2)\leqslant \overline{J}^{1**}. \tag{3.20}$$

In particular, then by putting $J^1=\min\Big\{J^{1^*},J^{1^{**}}\Big\}$. In this case, we define

$$\Omega_1=\Big\{(y_1,y_2)\in B:\|(y_1,y_2)\|<J^1\Big\}. \tag{3.21}$$

Then for $(y_1,y_2)\in P\cap\partial\Omega_1$ and $t\in I$, we have

$$\begin{aligned}
&T_1(y_1,y_2)(t)\\
&\quad=\lambda_1\int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f(\tau,y_1,y_2)d\tau\Big)ds\\
&\quad\geqslant\lambda_1\int_{s\in I}\eta G_1(1,s)\phi_q\Big(\int_{\tau\in I}\psi^*(\tau)H_1(\tau,\tau)(f_0-\epsilon)\phi_p(y_1+y_2)d\tau\Big)ds\\
&\quad\geqslant\lambda_1\int_{s\in I}\eta G_1(1,s)\phi_q\Big(\int_{\tau\in I}\psi^*(\tau)H_1(\tau,\tau)d\tau\Big)(f_0-\epsilon)^{q-1}\eta\|(y_1,y_2)\|ds\\
&\quad\geqslant\lambda_1\eta^2\int_{s\in I}G_1(1,s)\phi_q\Big(\int_{\tau\in I}\psi^*(\tau)H_1(\tau,\tau)d\tau\Big)ds(f_0-\epsilon)^{q-1}\|(y_1,y_2)\|\\
&\quad\geqslant\frac{1}{2}\|(y_1,y_2)\|.
\end{aligned}$$

Therefore,

$$\|T_1(y_1,y_2)(t)\|\geqslant\frac{1}{2}\|(y_1,y_2)\|,\quad\text{for } t\in[0,1]. \tag{3.22}$$

Similarly,

$$\|T_2(y_1,y_2)(t)\|\geqslant\frac{1}{2}\|(y_1,y_2)\|,\quad\text{for } t\in[0,1]. \tag{3.23}$$

And so

$$\begin{aligned}\|T(y_1,y_2)(t)\| &= \|(T_1(y_1,y_2)(t),T_2(y_1,y_2)(t))\|\\ &= \|T_1(y_1,y_2)(t)\|+\|T_2(y_1,y_2)(t)\|\\ &\geqslant \frac{1}{2}\|(y_1,y_2)\|+\frac{1}{2}\|(y_1,y_2)\|\\ &= \|(y_1,y_2)\|.\end{aligned}$$

Hence,

$$\|T(y_1,y_2)\| \geqslant \|(y_1,y_2)\|,\ \text{for } (y_1,y_2)\in P\cap\partial\Omega_1. \tag{3.24}$$

By the definition of f^∞, g^∞ there exists $\overline{J}^2>0$ such that

$$f(t,y_1,y_2)\leqslant (f^\infty+\epsilon)\phi_p(y_1+y_2),\ \text{for } (y_1,y_2)\geqslant \overline{J}^2, \tag{3.25}$$

$$g(t,y_1,y_2)\leqslant (g^\infty+\epsilon)\phi_p(y_1+y_2),\ \text{for } (y_1,y_2)\geqslant \overline{J}^2. \tag{3.26}$$

There are two subcases.

Case(i) : Suppose $L>0$ is such that $\max_{t\in[0,1]} f(t,y_1,y_2)\leqslant L$, $\max_{t\in[0,1]} g(t,y_1,y_2)\leqslant L$, for all $0<(y_1,y_2)<\infty$.

Let

$$\begin{aligned}J^2=\max\Big\{&2J^1, 2L^{q-1}\lambda_1\int_0^1 G_1(1,s)\phi_q\Big(\int_0^1 H_1(\tau,\tau)d\tau\Big)ds,\\ &2L^{q-1}\lambda_2\int_0^1 G_2(1,s)\phi_q\Big(\int_0^1 H_2(\tau,\tau)d\tau\Big)ds\Big\},\end{aligned}$$

and

$$\Omega_2=\Big\{(y_1,y_2)\in B:\|(y_1,y_2)\|<J^2\Big\}.$$

Then for $(y_1,y_2)\in P\cap\partial\Omega_2$ with $\|(y_1,y_2)\|=J^2$, we have

$$\begin{aligned}T_1(y_1,y_2)(t)&=\lambda_1\int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f(\tau,y_1,y_2)d\tau\Big)ds\\ &\leqslant \lambda_1\int_0^1 G_1(1,s)\phi_q\Big(\int_0^1 H_1(s,\tau)Ld\tau\Big)ds\\ &\leqslant \lambda_1\int_0^1 G_1(1,s)\phi_q\Big(\int_0^1 H_1(\tau,\tau)d\tau\Big)L^{q-1}ds\\ &\leqslant \lambda_1 L^{q-1}\int_0^1 G_1(1,s)\phi_q\Big(\int_0^1 H_1(\tau,\tau)d\tau\Big)ds\\ &\leqslant \frac{1}{2}J^2=\frac{1}{2}\|(y_1,y_2)\|.\end{aligned}$$

Therefore,

$$\|T_1(y_1,y_2)(t)\|\leqslant\frac{1}{2}\|(y_1,y_2)\|,\quad \text{for } t\in[0,1]. \tag{3.27}$$

Similarly,

$$\|T_2(y_1,y_2)(t)\| \leqslant \frac{1}{2}\|(y_1,y_2)\|, \quad \text{for } t\in[0,1]. \tag{3.28}$$

Thus by putting (3.27)-(3.28) together we find that for $(y_1,y_2)\in P\cap\partial\Omega_2$,

$$\begin{aligned}\|T(y_1,y_2)(t)\| &= \|(T_1(y_1,y_2)(t),T_2(y_1,y_2)(t))\|\\ &= \|T_1(y_1,y_2)(t)\|+\|T_2(y_1,y_2)(t)\|\\ &\leqslant \frac{1}{2}\|(y_1,y_2)\|+\frac{1}{2}\|(y_1,y_2)\|\\ &= \|(y_1,y_2)\|.\end{aligned}$$

Hence,

$$\|T(y_1,y_2)\| \leqslant \|(y_1,y_2)\|, \text{ for } (y_1,y_2)\in P\cap\partial\Omega_2. \tag{3.29}$$

Case(ii) : Let $J^2 > \max\left\{2J^1,\overline{J}^2\right\}$ be such that $f(t,y_1,y_2)\leqslant f(t,J^2,J^2)$, $g(t,y_1,y_2)\leqslant g(t,J^2,J^2)$ and

$$\Omega_2=\left\{(y_1,y_2)\in B:\|(y_1,y_2)\|<J^2\right\}.$$

Choosing $(y_1,y_2)\in P\cap\partial\Omega_2$ with $\|(y_1,y_2)\|=J^2$, we have

$$\begin{aligned}T_1(y_1,y_2)(t) &= \lambda_1\int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f(\tau,y_1,y_2)d\tau\Big)ds\\ &\leqslant \lambda_1\int_0^1 G_1(1,s)\phi_q\Big(\int_0^1 H_1(\tau,\tau)f(\tau,J^2,J^2)d\tau\Big)ds\\ &\leqslant \lambda_1\int_0^1 G_1(1,s)\phi_q\Big(\int_0^1 H_1(\tau,\tau)(f^\infty+\epsilon)\phi_p(J^2)d\tau\Big)ds\\ &\leqslant \lambda_1\int_0^1 G_1(1,s)\phi_q\Big(\int_0^1 H_1(\tau,\tau)d\tau\Big)(f^\infty+\epsilon)^{q-1}\|(y_1,y_2)\|ds\\ &\leqslant \frac{1}{2}\|(y_1,y_2)\|.\end{aligned}$$

Therefore,

$$\|T_1(y_1,y_2)(t)\| \leqslant \frac{1}{2}\|(y_1,y_2)\|, \quad \text{for } t\in[0,1]. \tag{3.30}$$

Similarly,

$$\|T_2(y_1,y_2)(t)\| \leqslant \frac{1}{2}\|(y_1,y_2)\|, \quad \text{for } t\in[0,1]. \tag{3.31}$$

Thus by putting (3.30)-(3.31) together we find that for $(y_1,y_2)\in P\cap\partial\Omega_2$,

$$\begin{aligned}\|T(y_1,y_2)(t)\| &= \|(T_1(y_1,y_2)(t),T_2(y_1,y_2)(t))\|\\ &\leqslant \frac{1}{2}\|(y_1,y_2)\|+\frac{1}{2}\|(y_1,y_2)\|\\ &= \|(y_1,y_2)\|.\end{aligned}$$

And so

$$\|T(y_1,y_2)\| \leqslant \|(y_1,y_2)\|, \text{ for } (y_1,y_2) \in P \cap \partial\Omega_2. \tag{3.32}$$

An application of Theorem 3.1 to (3.24), (3.29) and (3.32) yields a fixed point (y_1,y_2) of T that lies in $P\cap(\overline{\Omega}_2\backslash\Omega_1)$. This fixed point is the solution of the fractional order boundary value problem (1.1)-(1.4). □

4. Example

In this section, as an application, the result is demonstrated with an example.

Consider a coupled system of p-Laplacian fractional order boundary value problem,

$$D_{0^+}^{1.8}\Big(\phi_p\Big(D_{0^+}^{3.7}y_1(t)\Big)\Big) = \lambda_1 f(t,y_1,y_2),\ t\in(0,1), \tag{4.1}$$

$$D_{0^+}^{1.7}\Big(\phi_p\Big(D_{0^+}^{3.6}y_2(t)\Big)\Big) = \lambda_2 g(t,y_1,y_2),\ t\in(0,1), \tag{4.2}$$

$$\begin{cases} y_1(0)=y_1'(0)=D_{0^+}^{1.5}y_1(0)=0,\\ 6y_1(1)+8D_{0^+}^{2.6}y_1(1)=0,\\ \phi_p\Big(D_{0^+}^{3.7}y_1(0)\Big)=\phi_p\Big(D_{0^+}^{3.7}y_1(1)\Big)=0, \end{cases} \tag{4.3}$$

$$\begin{cases} y_2(0)=y_2'(0)=D_{0^+}^{1.5}y_2(0)=0,\\ 6y_2(1)+8D_{0^+}^{2.6}y_2(1)=0,\\ \phi_p\Big(D_{0^+}^{3.6}y_2(0)\Big)=\phi_p\Big(D_{0^+}^{3.6}y_2(1)\Big)=0, \end{cases} \tag{4.4}$$

where

$$\begin{aligned} f(t,y_1,y_2) =&(y_1+y_2)\Big(6370000-6369960e^{-3(y_1+y_2)}\Big),\\ g(t,y_1,y_2) =&(y_1+y_2)\Big(4560000-4559980e^{-2(y_1+y_2)}\Big). \end{aligned}$$

Then the Green's function $G_1(t,s)$ for the homogeneous boundary value problem,

$$-D_{0^+}^{3.7}y_1(t)=0,\ t\in(0,1),$$
$$y_1(0)=y_1'(0)=D_{0^+}^{1.5}y_1(0)=0,\ 6y_1(1)+8D_{0^+}^{2.6}y_1(1)=0,$$

is given by

$$G_1(t,s)=\begin{cases} \frac{1}{39.07}\left[\frac{6\Gamma(1.1)}{\Gamma(3.7)}+8(1-s)^{-2.6}\right][t(1-s)]^{2.7},\ t\leqslant s,\\ \frac{1}{39.07}\left[\frac{6\Gamma(1.1)}{\Gamma(3.7)}+8(1-s)^{-2.6}\right][t(1-s)]^{2.7}-\frac{(t-s)^{2.7}}{\Gamma(3.7)},\ s\leqslant t, \end{cases}$$

and the Green's function $G_2(t,s)$ for the homogeneous boundary value problem,

$$-D_{0^+}^{3.6}y_2(t)=0,\ t\in(0,1),$$
$$y_2(0)=y_2'(0)=D_{0^+}^{1.5}y_2(0)=0,\ 6y_2(1)+8D_{0^+}^{2.6}y_2(1)=0,$$

is given by

$$G_2(t,s)=\begin{cases}\frac{1}{35.74}\left[\frac{6}{\Gamma(3.6)}+8(1-s)^{-2.6}\right][t(1-s)]^{2.6},\ t\leqslant s,\\ \frac{1}{35.74}\left[\frac{6}{\Gamma(3.6)}+8(1-s)^{-2.6}\right][t(1-s)]^{2.6}-\frac{(t-s)^{2.6}}{\Gamma(3.6)},\ s\leqslant t.\end{cases}$$

Also the Green's function $H_1(t,s)$ for the boundary value problem,

$$D_{0^+}^{1.8}\Big(\phi_p\Big(D_{0^+}^{3.7}y_1(t)\Big)\Big)=0,\ t\in(0,1),$$
$$\phi_p\Big(D_{0^+}^{3.7}y_1(0)\Big)=\phi_p\Big(D_{0^+}^{3.7}y_1(1)\Big)=0,$$

is given by

$$H_1(t,s)=\begin{cases}\frac{[t(1-s)]^{0.8}}{\Gamma(1.8)},\ t\leqslant s,\\ \frac{[t(1-s)]^{0.8}-(t-s)^{0.8}}{\Gamma(1.7)},\ s\leqslant t,\end{cases}$$

and the Green's function $H_2(t,s)$ for the boundary value problem,

$$D_{0^+}^{1.7}\Big(\phi_p\Big(D_{0^+}^{3.6}y_2(t)\Big)\Big)=0,\ t\in(0,1),$$
$$\phi_p\Big(D_{0^+}^{3.6}y_2(0)\Big)=\phi_p\Big(D_{0^+}^{3.6}y_2(1)\Big)=0,$$

is given by

$$H_2(t,s)=\begin{cases}\frac{[t(1-s)]^{0.7}}{\Gamma(1.7)},\ t\leqslant s,\\ \frac{[t(1-s)]^{0.7}-(t-s)^{0.7}}{\Gamma(1.7)},\ s\leqslant t.\end{cases}$$

Clearly, the Green functions $G_i(t,s)$ and $H_i(t,s)$, for $i=1,2$ are positive, Let $p=2$. By direct calculations, one can determine $\eta=0.0237$, $f^0=40, g^0=20$, $f_\infty=6370000$, $g_\infty=4560000$. Applying Theorem 3.3, we get an eigenvalue interval $0.0000087214<\lambda_1,\lambda_2<0.8901146789$, for which the p-Laplacian fractional order boundary value problem (4.1)-(4.4) has at least one positive solution.

References

[1] R. P. Agarwal, D. O'Regan and P. J. Y. Wong. *Positive Solutions of Differential, Difference and Integral Equations.* Kluwer Academic Publishers, Dordrecht, The Netherlands, 1999.

[2] B. Ahmed and J. J. Nieto. Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions. *Comput. Math. Appl.*, **58**(9)(2009), 1838–1843.

[3] R. I. Avery and J. Henderson. Existence of three positive pseudo-symmetric solutions for a one-dimensional p-Laplacian. *J. Math. Anal. Appl.*, **277**(2)(2003), 395–404.

[4] Z. Bai and H. Lü. Positive solutions for boundary value problem of nonlinear fractional differential equation. *J. Math. Anal. Appl.*, **311**(2)(2005), 495–505.

[5] G. Chai. Positive solutions for boundary value problem of fractional differential equation with p-Laplacian operator. *Bound. Value Probl.*, **2012**(2012): 18.

[6] T. Chen and W. Liu. An anti-periodic boundary value problem for the fractional differential equation with a p-Laplacian operator. *Appl. Math. Lett.*, **25**(11)(2012), 1671–1675.

[7] L. Diening, P. Lindqvist and B. Kawohl. Mini-Workshop: The p-Laplacian Operator and Applications. *Oberwolfach Reports*, **10**(2013) 433–482.

[8] C. S. Goodrich. Existence of a positive solution to systems of differential equations of fractional order. *Comput. Math. Appl.*, **62**(3)(2011), 1251–1268.

[9] D. Guo and V. Lakshmikantham. *Nonlinear Problems in Abstract Cones.* Acadamic Press, San Diego, 1988.

[10] A. A. Kilbas, H. M. Srivasthava and J. J. Trujillo. *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies, **204**, Elsevier Science, Amsterdam, 2006.

[11] L. Kong and J. Wang. Multiple positive solutions for the one-dimensional p-Laplacian. *Nonlinear Anal.*, **42**(8)(2000), 1327–1333.

[12] M. A. Krasnosel'skii. *Positive Solutions of Operator Equations.* Noordhoff, Groningen, 1964.

[13] I. Podulbny. *Fractional Diffrential Equations,* Academic Press, San Diego, 1999.

[14] K. R. Prasad and B. M. B. Krushna. Multiple positive solutions for a coupled system of Riemann–Liouville fractional order two-point boundary value problems. *Nonlinear Stud.*, **20**(2013), 501–511.

[15] K. R. Prasad and B. M. B. Krushna. Multiple positive solutions for a coupled system of p-Laplacian fractional order two-point boundary value problems. *Int. J. Differ. Equ.*, **2014**(2014), 1–10, DOI: 10.1155/2014/485647.

[16] K. R. Prasad and B. M. B. Krushna. Eigenvalues for iterative systems of Sturm–Liouville fractional order two-point boundary value problems. *Fract. Calc. Appl. Anal.*, **17**(2014), 638–653, DOI: 10.2478/s13540-014-0190-4.

[17] K. R. Prasad and B. M. B. Krushna. Multiple positive solutions for the system of (n, p)-type fractional order boundary value problems. *Bull. Int. Math. Virtual Inst.*, **5** (2015), 1–12.

[18] K. R. Prasad and B. M. B. Krushna. Solvability of p-Laplacian fractional higher order two-point boundary value problems. *Commun. Appl. Anal.*, **19**(2015), 659–678.

[19] K. R. Prasad, B. M. B. Krushna and L. T. Wesen. Existence results for multiple positive solutions of Riemann–Liouville fractional order three-point boundary value problems. *Bull. Int. Math. Virtual Inst.*, **6**(2016), 25–36.

Receibed by editors 03.04.2017; Accepted 19.12.2017; Available online 02.01.2018.

B.M.B. Krushna: Department of Mathematics, MVGR College of Engineering (Autonomous), Vizianagaram, 535 005, India

E-mail address: muraleebalu@yahoo.com