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A NEW SUZUKI TYPE COMMON COUPLED FIXED POINT RESULT FOR FOUR MAPS IN S_b-METRIC SPACES

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Abstract. In this paper we prove a Suzuki type unique common coupled fixed point theorem for two pairs of w-compatible mappings in S_b-metric spaces. We also furnish an example to support our main result.

1. Introduction

In the year 2008, Suzuki [**11**] generalized the Banach contraction principle [**2**].

Theorem 1.1 ([**11**]). *Let (X, d) be a complete metric space and let T be a mapping on X. Define a non-increasing function $\theta : [0, 1) \to \left(\frac{1}{2}, 1\right]$ by*

$$\theta(r) = \begin{cases} 1 & if \quad 0 \leqslant r \leqslant \frac{(\sqrt{5}-1)}{2}, \\ (1-r)r^{-2} & if \quad \frac{(\sqrt{5}-1)}{2} \leqslant r \leqslant \quad 2^{-\frac{1}{2}}, \\ (1+r)^{-1} & if \quad 2^{-\frac{1}{2}} \leqslant r < 1. \end{cases}$$

Assume that there exists $r \in [0, 1)$ such that

$$\theta(r)d(x, Tx) \leqslant d(x, y) \Rightarrow d(Tx, Ty) \leqslant rd(x, y)$$

for all $x, y \in X$. Then there exists a unique fixed point z of T. Moreover

$$\lim_n T^n x = z$$

for all $x \in X$.

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Bhaskar and Lakshmikantham [**4**] introduced the notion of coupled fixed point and proved some coupled fixed point results .

Recently, Sedghi et al. [**8**] defined S_b-metric spaces using the concept of S-metric spaces [**9**].

The aim of this paper is to prove Suzuki type unique common coupled fixed point theorem for four mappings satisfying generalized contractive condition in S_b-metric spaces. Throughout this paper $\mathcal{R}, \mathcal{R}^+$ and $\mathcal{N}$ denote the set of all real numbers, non-negative real numbers and positive integers respectively.
First we recall some definitions, lemmas and examples.

Definition 1.1. ([**9**]) Let X be a non-empty set. A S-metric on X is a function $S : X^3 \to \mathcal{R}^+$ that satisfies the following conditions for each $x, y, z, a \in X$,

$(S1)$: $0 < S(x, y, z)$ for all $x, y, z \in X$ with $x \neq y \neq z$,

$(S2)$: $S(x, y, z) = 0 \Leftrightarrow x = y = z$,

$(S3)$: $S(x, y, z) \leqslant S(x, x, a) + S(y, y, a) + S(z, z, a)$ for all $x, y, z, a \in X$.

Then the pair (X, S) is called a S-metric space.

Definition 1.2. ([**8**]) Let X be a non-empty set and $b \geqslant 1$ be given real number. Suppose that a mapping $S_b : X^3 \to \mathcal{R}^+$ be a function satisfying the following properties :

(S_b1) $0 < S_b(x, y, z)$ for all $x, y, z \in X$ with $x \neq y \neq z$,

(S_b2) $S_b(x, y, z) = 0 \Leftrightarrow x = y = z$,

(S_b3) $S_b(x, y, z) \leqslant b(S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a))$ for all $x, y, z, a \in X$.

Then the function S_b is called a S_b-metric on X and the pair (X, S_b) is called a S_b-metric space.

Remark 1.1. ([**8**]) It should be noted that, the class of S_b-metric spaces is effectively larger than that of S-metric spaces. Indeed each S-metric space is a S_b-metric space with $b = 1$.

Following example shows that a S_b-metric on X need not be a S-metric on X.

Example 1.1. ([**8**]) Let (X, S) be a S-metric space, and

$$S_*(x, y, z) = S(x, y, z)^p$$

, where $p > 1$ is a real number. Note that S_* is a S_b-metric with $b = 2^{2(p-1)}$. Also, (X, S_*) is not necessarily a S-metric space.

Definition 1.3. ([**8**]) Let (X, S_b) be a S_b-metric space. Then, for $x \in X$ and $r > 0$, we define the open ball $B_{S_b}(x, r)$ and closed ball $B_{S_b}[x, r]$ with center x and radius r as follows respectively:

$$\begin{aligned} B_{S_b}(x, r) &= \{y \in X : S_b(y, y, x) < r\}, \\ B_{S_b}[x, r] &= \{y \in X : S_b(y, y, x) \leqslant r\}. \end{aligned}$$

Lemma 1.1 ([**8**]). *In a S_b-metric space, we have*

$$S_b(x, x, y) \leqslant b\ S_b(y, y, x)$$

and

$$S_b(y,y,x) \leqslant b\ S_b(x,x,y)$$

LEMMA 1.2 ([**8**]). *In a S_b-metric space, we have*

$$S_b(x,x,z) \leqslant 2b\ S_b(x,x,y) + b^2\ S_b(y,y,z)$$

DEFINITION 1.4 ([**8**]). If (X, S_b) be a S_b-metric space. A sequence $\{x_n\}$ in X is said to be:

(1) S_b-Cauchy sequence if, for each $\epsilon > 0$, there exists $n_0 \in \mathcal{N}$ such that $S_b(x_n, x_n, x_m) < \epsilon$ for each $m, n \geqslant n_0$.

(2) S_b-convergent to a point $x \in X$ if, for each $\epsilon > 0$, there exists a positive integer n_0 such that $S_b(x_n, x_n, x) < \epsilon$ or $S_b(x, , x, x_n) < \epsilon$ for all $n \geqslant n_0$ and we denote by $\lim\limits_{n\to\infty} x_n = x$.

DEFINITION 1.5. ([**8**]) A S_b-metric space (X, S_b) is called complete if every S_b-Cauchy sequence is S_b-convergent in X.

LEMMA 1.3 ([**8**]). *If (X, S_b) be a S_b-metric space with $b \geqslant 1$ and suppose that $\{x_n\}$ is a S_b-convergent to x, then we have*

(i) $\frac{1}{2b}\ S_b(y,x,x) \leqslant \lim\limits_{n\to\infty} inf S_b(y,y,x_n) \leqslant \lim\limits_{n\to\infty} sup S_b(y,y,x_n) \leqslant 2b\ S_b(y,y,x)$

and

(ii) $\frac{1}{b^2} S_b(x,x,y) \leqslant \lim\limits_{n\to\infty} inf S_b(x_n,x_n,y) \leqslant \lim\limits_{n\to\infty} sup S_b(x_n,x_n,y) \leqslant b^2 S_b(x,x,y)$ *for all $y \in X$.*

In particular, if $x = y$, then we have $\lim\limits_{n\to\infty} S_b(x_n,x_n,y) = 0$.

DEFINITION 1.6. ([**4**]) Let X be a non-empty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

DEFINITION 1.7. ([**5**]) Let X be a non-empty set. An element $(x, y) \in X \times X$ is called

(i) a coupled coincident point of mappings $F : X \times X \to X$ and $f : X \to X$ if $fx = F(x, y)$ and $fy = F(y, x)$.

(ii) a common coupled fixed point of mappings $F : X \times X \to X$ and $f : X \to X$ if $x = fx = F(x, y)$ and $y = fy = F(y, x)$.

Now we give our main result.

2. Main Result

Let Φ denote the class of all functions $\phi : \mathcal{R}^+ \to \mathcal{R}^+$ such that ϕ is non-decreasing, continuous, $\phi(t) < \frac{t}{4b^4}$ for all $t > 0$ and $\phi(0) = 0$.

THEOREM 2.1. *Let (X, S_b) be a S_b-metric space. Suppose that $A, B : X \times X \to X$ and $P, Q : X \to X$ be satisfying*

(2.1.1) $A(X \times X) \subseteq Q(X), B(X \times X) \subseteq P(X)$, (2.1.2) *$\{A, P\}$ and $\{B, Q\}$ are w-compatible pairs,*

(2.1.3) *One of $P(X)$ or $Q(X)$ is S_b-complete subspace of X,*

(2.1.4) $$\frac{1}{8b^3}\min\left\{\begin{array}{c} S_b(A(x,y),A(x,y),Px), \\ S_b(B(u,v),B(u,v),Qu), \\ S_b(A(y,x),A(y,x),Py), \\ S_b(B(v,u),B(v,u),Qv) \end{array}\right\} \leqslant \max\left\{\begin{array}{c} S_b(Px,Px,Qu), \\ S_b(Py,Py,Qv) \end{array}\right\}$$

implies that

$$2b^5\ S_b(A(x,y),A(x,y),B(u,v))$$

$$\leqslant \phi\left(\max\left\{\begin{array}{c} S_b(Px,Px,Qu), S_b(Py,Py,Qv), \\ S_b(A(x,y),A(x,y),Px), S_b(A(y,x),A(y,x),Py), \\ S_b(B(u,v),B(u,v),Qu), S_b(B(v,u),B(v,u),Qv), \\ \frac{S_b(A(x,y),A(x,y),Qu)\ S_b(B(u,v),B(u,v),Px)}{1+S_b(Px,Px,Qu)}, \\ \frac{S_b(A(y,x),A(y,x),Qv)\ S_b(B(v,u),B(v,u),Py)}{1+S_b(Py,Py,Qv)} \end{array}\right\}\right),$$

for all $x, y, u, v \in X$, $\phi \in \Phi$. Then A, B, P and Q have a unique common coupled fixed point in $X \times X$.

Proof. Let $x_0, y_0 \in X$. From (2.1.1), we can construct the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ such that

$$\begin{aligned} A(x_{2n},y_{2n}) &= Qx_{2n+1} = z_{2n}, \\ A(y_{2n},x_{2n}) &= Qy_{2n+1} = w_{2n}, \\ B(x_{2n+1},y_{2n+1}) &= Px_{2n+2} = z_{2n+1}, \\ B(y_{2n+1},x_{2n+1}) &= Py_{2n+2} = w_{2n+1},\ n = 0,1,2,\cdots \end{aligned}$$

Case (i). Suppose $z_{2m} = z_{2m+1}$ and $w_{2m} = w_{2m+1}$ for some m. Assume that $z_{2m+1} \neq z_{2m+2}$ or $w_{2m+1} \neq w_{2m+2}$. Since

$$\frac{1}{8b^3}\quad\min\left\{\begin{array}{c} S_b(A(x_{2m+2},y_{2m+2}),A(x_{2m+2},y_{2m+2}),Px_{2m+2}), \\ ]S_b(B(x_{2m+1},y_{2m+1}),B(x_{2m+1},y_{2m+1}),Qx_{2m+1}), \\ S_b(A(y_{2m+2},x_{2m+2}),A(y_{2m+2},x_{2m+2}),Py_{2m+2}), \\ S_b(B(y_{2m+1},x_{2m+1}),B(y_{2m+1},x_{2m+1}),Qy_{2m+1}) \end{array}\right\}$$

$$\leqslant \max\left\{\ S_b(Px_{2m+2},Px_{2m+2},Qx_{2m+1}), S_b(Py_{2m+2},Py_{2m+2},Qy_{2m+1})\ \right\}.$$

From (2.1.4), we have

$S_b(z_{2m+2}, z_{2m+2}, z_{2m+1})$

$$\leqslant 2b^5 S_b(A(x_{2m+2}, y_{2m+2}), A(x_{2m+2}, y_{2m+2}), B(x_{2m+1}, y_{2m+1}))$$

$$\leqslant \phi\left(\max\left\{\begin{array}{l} S_b(z_{2m+1}, z_{2m+1}, z_{2m}), S_b(w_{2m+1}, w_{2m+1}, w_{2m}), \\ S_b(z_{2m+2}, z_{2m+2}, z_{2m+1}), S_b(w_{2m+2}, w_{2m+2}, w_{2m+1}), \\ S_b(z_{2m+1}, z_{2m+1}, z_{2m}), S_b(w_{2m+1}, w_{2m+1}, w_{2m}), \\ \frac{S_b(z_{2m+2}, z_{2m+2}, z_{2m+1})\ S_b(z_{2m+1}, z_{2m+1}, z_{2m})}{1+S_b(z_{2m+1}, z_{2m+1}, z_{2m})}, \\ \frac{S_b(w_{2m+2}, w_{2m+2}, w_{2m+1})\ S_b(w_{2m+1}, w_{2m+1}, w_{2m})}{1+S_b(w_{2m+1}, w_{2m+1}, w_{2m})} \end{array}\right\}\right)$$

$$= \phi\left(\max\left\{\ S_b(z_{2m+2}, z_{2m+2}, z_{2m+1}), S_b(w_{2m+2}, w_{2m+2}, w_{2m+1})\ \right\}\right).$$

Similarly, we can prove

$$S_b(w_{2m+2}, w_{2m+2}, w_{2m+1}) \leqslant \phi\left(\max\left\{\begin{array}{l} S_b(z_{2m+2}, z_{2m+2}, z_{2m+1}), \\ S_b(w_{2m+2}, w_{2m+2}, w_{2m+1}) \end{array}\right\}\right).$$

Thus

$$\max\left\{\begin{array}{l} S_b(z_{2m+2}, z_{2m+2}, z_{2m+1}), \\ S_b(w_{2m+2}, w_{2m+2}, w_{2m+1}) \end{array}\right\} \leqslant \phi\left(\max\left\{\begin{array}{l} S_b(z_{2m+2}, z_{2m+2}, z_{2m+1}), \\ S_b(w_{2m+2}, w_{2m+2}, w_{2m+1}) \end{array}\right\}\right)$$

It follows that $z_{2m+2} = z_{2m+1}$ and $w_{2m+2} = w_{2m+1}$.

Continuing in this process we can conclude that $z_{2m+k} = z_{2m}$ and $w_{2m+k} = w_{2m}$ for all $k \geqslant 0$. It follows that $\{z_n\}$ and $\{w_n\}$ are Cauchy sequences.

Case (ii). Assume that $z_{2n} \neq z_{2n+1}$ and $w_{2n} \neq w_{2n+1}$ for all n. Put

$$S_n = \max\{S_b(z_{n+1}, z_{n+1}, z_n), S_b(w_{n+1}, w_{n+1}, w_n)\}.$$

Since

$$\frac{1}{8b^3} \quad \min\left\{\begin{array}{l} S_b(A(x_{2n+2}, y_{2n+2}), A(x_{2n+2}, y_{2n+2}), Px_{2n+2}), \\ S_b(B(x_{2n+1}, y_{2n+1}), B(x_{2n+1}, y_{2n+1}), Qx_{2n+1}), \\ S_b(A(y_{2n+2}, x_{2n+2}), A(y_{2n+2}, x_{2n+2}), Py_{2n+2}), \\ S_b(B(y_{2n+1}, x_{2n+1}), B(y_{2n+1}, x_{2n+1}), Qy_{2n+1}) \end{array}\right\}$$

$$\leqslant \max\left\{\ S_b(Px_{2n+2}, Px_{2n+2}, Qx_{2n+1}), S_b(Py_{2n+2}, Py_{2n+2}, Qy_{2n+1})\ \right\}.$$

From (2.1.4), we have

$S_b(z_{2n+2}, z_{2n+2}, z_{2n+1})$

$$\leqslant 2b^5 S_b(A(x_{2n+2}, y_{2n+2}), A(x_{2n+2}, y_{2n+2}), B(x_{2n+1}, y_{2n+1}))$$

$$\leqslant \phi\left(\max\left\{\begin{array}{l} S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(w_{2n+1}, w_{2n+1}, w_{2n}), \\ S_b(z_{2n+2}, z_{2n+2}, z_{2n+1}), S_b(w_{2n+2}, w_{2n+2}, w_{2n+1}), \\ S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(w_{2n+1}, w_{2n+1}, w_{2n}), \\ \frac{S_b(z_{2n+2}, z_{2n+2}, z_{2n})\ S_b(z_{2n+1}, z_{2n+1}, z_{2n+1})}{1+S_b(z_{2n+1}, z_{2n+1}, z_{2n})}, \\ \frac{S_b(w_{2n+2}, w_{2n+2}, w_{2n})\ S_b(w_{2n+1}, w_{2n+1}, w_{2n+1})}{1+S_b(w_{2n+1}, w_{2n+1}, w_{2n})} \end{array}\right\}\right)$$

$$= \phi\left(\max\left\{\begin{array}{l} S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(z_{2n+2}, z_{2n+2}, z_{2n+1}), \\ S_b(w_{2n+1}, w_{2n+1}, w_{2n}), S_b(w_{2n+2}, w_{2n+2}, w_{2n+1}) \end{array}\right\}\right)$$

$$= \phi\left(\max\left\{\ S_{2n+1}, S_{2n}\ \right\}\right).$$

Similarly, we can prove that

$$S_b(w_{2n+2}, w_{2n+2}, w_{2n+1}) \leqslant \phi\left(\max\left\{\ S_{2n+1}, S_{2n}\ \right\}\right).$$

Thus

$$S_{2n+1} \leqslant \phi(\max\{S_{2n}, S_{2n+1}\}).$$

If S_{2n+1} is maximum then we get contradiction so that S_{2n} is maximum. Thus

$$\begin{array}{rcl} S_{2n+1} & \leqslant & \phi(S_{2n}) \\ & < & S_{2n}. \end{array} \tag{2.1}$$

Similarly we can conclude that $S_{2n} < S_{2n-1}$.

It is clear that $\{S_n\}$ is a non-increasing sequence of non-negative real numbers and must converge to a real number, say $r \geqslant 0$. Suppose $r > 0$. Letting $n \to \infty$, in (2.1), we have $r \leqslant \phi(r) < r$. It is contradiction. Hence $r = 0$ Thus

$$\lim_{n\to\infty} S(z_{n+1}, z_{n+1}, z_n) = 0 \tag{2.2}$$

and

$$\lim_{n\to\infty} S(w_{n+1}, w_{n+1}, w_n) = 0. \tag{2.3}$$

Now we prove that $\{z_{2n}\}$ and $\{w_{2n}\}$ are Cauchy sequences in (X, S_b). On contrary we suppose that $\{z_{2n}\}$ or $\{w_{2n}\}$ is not Cauchy. Then there exist $\epsilon > 0$ and monotonically increasing sequence of natural numbers $\{2m_k\}$ and $\{2n_k\}$ such that $n_k > m_k$.

$$\max\{S_b(z_{2m_k}, z_{2m_k}, z_{2n_k}), S_b(w_{2m_k}, w_{2m_k}, w_{2n_k})\} \geqslant \epsilon \tag{2.4}$$

and

$$\max\{S_b(z_{2m_k}, z_{2m_k}, z_{2n_{k-2}}), S_b(w_{2m_k}, w_{2m_k}, w_{2n_{k-2}})\} < \epsilon. \tag{2.5}$$

From (2.4) and (2.5), we have

$$
\begin{aligned}
(2.6)\ \epsilon \ &\leqslant \max\{S_b(z_{2m_k}, z_{2m_k}, z_{2n_k}), S_b(w_{2m_k}, w_{2m_k}, w_{2n_k})\} \\
&\leqslant 2b \max\{S_b(z_{2m_k}, z_{2m_k}, z_{2m_k+2}), S_b(w_{2m_k}, w_{2m_k}, w_{2m_k+2})\} \\
&\quad + b \max\{S_b(z_{2n_k}, z_{2n_k}, z_{2m_k+2}), S_b(w_{2n_k}, w_{2n_k}, w_{2m_k+2})\} \\
&\leqslant 2b\,(2b \max\{S_b(z_{2m_k}, z_{2m_k}, z_{2m_k+1}), S_b(w_{2m_k}, w_{2m_k}, w_{2m_k+1})\}) \\
&\quad + 2b\,(b \max\{S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1})\}) \\
&\quad + b\,(2b \max\{S_b(z_{2n_k}, z_{2n_k}, z_{2n_k+1}), S_b(w_{2n_k}, w_{2n_k}, w_{2n_k+1})\}) \\
&\quad + b\,(b \max\{S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k+1})\}) \\
&\leqslant 4b^3 \max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\} \\
&\quad + 2b^2 \max\{S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1})\} \\
&\quad + 2b^3 \max\{S_b(z_{2n_k+1}, z_{2n_k}, z_{2n_k}), S_b(w_{2n_k+1}, w_{2n_k}, w_{2n_k})\} \\
&\quad + b^2 \max\{S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k+1})\}.
\end{aligned}
$$

Now first we claim that

$$
(2.7) \quad \tfrac{1}{8b^3} \quad \min \left\{ \begin{array}{c} S_b(A(x_{2m_k+2}, y_{2m_k+2}), A(x_{2m_k+2}, y_{2m_k+2}), Px_{2m_k+2}), \\ S_b(B(x_{2n_k+1}, y_{2n_k+1}), B(x_{2n_k+1}, y_{2n_k+1}), Qx_{2n_k+1}), \\ S_b(A(y_{2m_k+2}, x_{2m_k+2}), A(y_{2m_k+2}, x_{2m_k+2}), Py_{2m_k+2}), \\ S_b(B(y_{2n_k+1}, x_{2n_k+1}), B(y_{2n_k+1}, x_{2n_k+1}), Qy_{2n_k+1}) \end{array} \right\}
$$

$$
\leqslant \max \left\{ \begin{array}{c} S_b(Px_{2m_k+2}, Px_{2m_k+2}, Qx_{2n_k+1}), \\ S_b(Py_{2m_k+2}, Py_{2m_k+2}, Qy_{2n_k+1}) \end{array} \right\}.
$$

On contrary suppose that

$$
\tfrac{1}{8b^3} \quad \min \left\{ \begin{array}{c} S_b(A(x_{2m_k+2}, y_{2m_k+2}), A(x_{2m_k+2}, y_{2m_k+2}), Px_{2m_k+2}), \\ S_b(B(x_{2n_k+1}, y_{2n_k+1}), B(x_{2n_k+1}, y_{2n_k+1}), Qx_{2n_k+1}), \\ S_b(A(y_{2m_k+2}, x_{2m_k+2}), A(y_{2m_k+2}, x_{2m_k+2}), Py_{2m_k+2}), \\ S_b(B(y_{2n_k+1}, x_{2n_k+1}), B(y_{2n_k+1}, x_{2n_k+1}), Qy_{2n_k+1}) \end{array} \right\}
$$

$$
> \max \left\{ \begin{array}{c} S_b(Px_{2m_k+2}, Px_{2m_k+2}, Qx_{2n_k+1}), \\ S_b(Py_{2m_k+2}, Py_{2m_k+2}, Qy_{2n_k+1}) \end{array} \right\}.
$$

Now from (2.4), we have

$$\begin{aligned}
\epsilon &\leqslant \max\{S_b(z_{2m_k}, z_{2m_k}, z_{2n_k}), S_b(w_{2m_k}, w_{2m_k}, w_{2n_k})\} \\
&\leqslant 2b\max\{S_b(z_{2m_k}, z_{2m_k}, z_{2m_k+1}), S_b(w_{2m_k}, w_{2m_k}, w_{2m_k+1})\} \\
&\quad +b\max\{S_b(z_{2n_k}, z_{2n_k}, z_{2m_k+1}), S_b(w_{2n_k}, w_{2n_k}, w_{2m_k+1})\} \\
&\leqslant 2b^2\max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\} \\
&\quad +b^2\max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k})\} \\
&< 2b^2\max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\} \\
&\quad +b^2\frac{1}{8b^3}\min\left\{\begin{array}{c} S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), \\ S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1}), \\ S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), \\ S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2n_k}) \end{array}\right\}.
\end{aligned}$$

Letting $k \to \infty$, we have $\epsilon \leqslant 0$. It is a contradiction. Hence (2.7) holds.

Now from (2.1.4), we have

$$2b^5 S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1})$$

$$\leqslant \phi\left(\max\left\{\begin{array}{l} S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k}), \\ S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1}), \\ S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2n_k}), \\ \frac{S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k})\ S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1})}{1+S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})}, \\ \frac{S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k})\ S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2m_k+1})}{1+S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k})} \end{array}\right\}\right)$$

Similarly

$$2b^5 S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1})$$

$$\leqslant \phi\left(\max\left\{\begin{array}{l} S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k}), \\ S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1}), \\ S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2n_k}), \\ \frac{S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k})\ S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1})}{1+S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})}, \\ \frac{S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k})\ S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2m_k+1})}{1+S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k})} \end{array}\right\}\right).$$

Thus

$$(2.8)\quad 2b^5\max\left\{\ S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1})\ \right\}$$

$$\leqslant \phi\left(\max\left\{\begin{array}{l} S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k}), \\ S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1}), \\ S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2n_k}), \\ \frac{S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k})\ S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1})}{1+S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})}, \\ \frac{S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k})\ S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2m_k+1})}{1+S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k})} \end{array}\right\}\right).$$

But
$\max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k})\}$

$$\begin{aligned}
&\leqslant 2b\max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\}\\
&\quad +b\max\{S_b(z_{2n_k}, z_{2n_k}, z_{2m_k}), S_b(w_{2n_k}, w_{2n_k}, w_{2m_k})\}\\
&\leqslant 2b\max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\}\\
&\quad +b^2\max\{S_b(z_{2m_k}, z_{2m_k}, z_{2n_k}), S_b(w_{2m_k}, w_{2m_k}, w_{2n_k})\}\\
&\leqslant 2b\max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\}\\
&\quad +b^2\left(2b\max\{S_b(z_{2m_k}, z_{2m_k}, z_{2n_{k-2}}), S_b(w_{2m_k}, w_{2m_k}, w_{2n_{k-2}})\}\right)\\
&\quad +b^2\left(b\max\{S_b(z_{2n_k}, z_{2n_k}, z_{2n_{k-2}}), S_b(w_{2n_k}, w_{2n_k}, w_{2n_{k-2}})\}\right)\\
&< 2b\max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\}\\
&\quad +2b^3\epsilon + b^3\left(2b\max\{S_b(z_{2n_k}, z_{2n_k}, z_{2n_k-1}), S_b(w_{2n_k}, w_{2n_k}, w_{2n_k-1})\}\right)\\
&\quad +b^3\left(b\max\{S_b(z_{2n_{k-2}}, z_{2n_k-2}, z_{2n_k-1}), S_b(w_{2n_k-2}, w_{2n_k-2}, w_{2n_k-1})\}\right)\\
&\leqslant 2b\max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\}\\
&\quad +2b^3\epsilon + 2b^4\max\{S_b(z_{2n_k}, z_{2n_k}, z_{2n_k-1}), S_b(w_{2n_k}, w_{2n_k}, w_{2n_k-1})\}\\
&\quad +b^5\max\{S_b(z_{2n_k-1}, z_{2n_k-1}, z_{2n_k-2}), S_b(w_{2n_k-1}, w_{2n_k-1}, w_{2n_k-2})\}.
\end{aligned}$$

Letting $k \to \infty$, we have

$$\lim_{k\to\infty}\max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k})\} \leqslant 2b^3\epsilon. \tag{2.9}$$

Also

$$\begin{aligned}
&\lim_{k\to\infty}\frac{S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k})\ S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1})}{1+S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})}\\
&\leqslant \lim_{k\to\infty}\frac{\left[\begin{array}{l}[2bS_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}) + bS_b(z_{2n_k}, z_{2n_k}, z_{2m_k+1})]\\ {[2bS_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}) + bS_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})]}\end{array}\right]}{1+S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})}\\
&\leqslant \lim_{k\to\infty}\frac{b^3S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})\ S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})}{1+S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})}\\
&\leqslant \lim_{k\to\infty} b^3S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k})\\
&\leqslant 2b^6\epsilon\ from(2.9).
\end{aligned}$$

Similarly

$$\lim_{k\to\infty}\frac{S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k})\ S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2m_k+1})}{1+S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k})} \leqslant 2b^6\epsilon.$$

Letting $k \to \infty$ in (2.8), we have

$$\text{(2.10)} \quad \begin{aligned} \lim_{k\to\infty} \quad & \max\{S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k+1})\} \\ & \leqslant \frac{1}{2b^5}\phi\left(\max\{2b^3\epsilon, 0, 0, 0, 0, 2b^6\epsilon, 2b^6\epsilon\}\right) \\ & = \frac{1}{2b^5}\phi(2b^6\epsilon). \end{aligned}$$

Now letting $n \to \infty$ in (2.6), from (2.2),(2.3) and (2.10), we have

$$\epsilon \leqslant 0 + 0 + 0 + b^2 \frac{1}{2b^5}\phi(2b^6\epsilon) < \epsilon.$$

It is a contradiction.

Hence $\{z_{2n}\}$ and $\{w_{2n}\}$ are S_b-Cauchy sequences in (X, S_b). In addition

$$\begin{aligned} & \max\{S_b(z_{2n+1}, z_{2n+1}, z_{2m+1}), S_b(w_{2n+1}, w_{2n+1}, w_{2m+1})\} \\ & \leqslant 2b\max\{S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(w_{2n+1}, w_{2n+1}, w_{2n})\} \\ & \quad + b\max\{S_b(z_{2m+1}, z_{2m+1}, z_{2n}), S_b(w_{2m+1}, w_{2m+1}, w_{2n})\} \\ & \leqslant 2b\max\{S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(w_{2n+1}, w_{2n+1}, w_{2n})\} \\ & \quad + 2b^2\max\{S_b(z_{2m+1}, z_{2m+1}, z_{2m}), S_b(w_{2m+1}, w_{2m+1}, w_{2m})\} \\ & \quad + b^2\max\{S_b(z_{2n}, z_{2n}, z_{2m}), S_b(w_{2n}, w_{2n}, w_{2m})\}. \end{aligned}$$

Since $\{z_{2n}\}$ and $\{w_{2n}\}$ are Cauchy and using (2.2), (2.3), we have $\{z_{2n+1}\}$ and $\{w_{2n+1}\}$ are also S_b-Cauchy sequences in (X, S_b). Thus $\{z_n\}$ and $\{w_n\}$ are S_b-Cauchy sequences in (X, S).

Suppose assume that $P(X)$ is a S_b- complete subspace of (X, S_b). Then the sequences $\{z_n\}$ and $\{w_n\}$ converge to α and β in $P(X)$. Thus there exist a and b in $P(X)$ such that

$$\text{(2.11)} \quad \lim_{n\to\infty} z_n = \alpha = Pa \text{ and } \lim_{n\to\infty} w_n = \beta = Pb.$$

Before going to prove common coupled fixed point for the mappings A, B, P and Q, first we claim that for each $n \geqslant 1$ at least one of the following assertions holds.

$$\tfrac{1}{8b^3}\min\left\{\begin{array}{c} S_b(z_{2n+1}, z_{2n+1}, z_{2n}), \\ S_b(w_{2n+1}, w_{2n+1}, w_{2n}) \end{array}\right\} \leqslant \max\left\{\ S_b(\alpha, \alpha, z_{2n}), S_b(\beta, \beta, w_{2n})\ \right\}$$

or

$$\tfrac{1}{8b^3}\min\left\{\begin{array}{c} S_b(z_{2n}, z_{2n}, z_{2n-1}), \\ S_b(w_{2n}, w_{2n}, w_{2n-1}) \end{array}\right\} \leqslant \max\left\{\ S_b(\alpha, \alpha, z_{2n-2}), S_b(\beta, \beta, w_{2n-2})\ \right\}.$$

On contrary, suppose that

$$\tfrac{1}{8b^3}\min\left\{\begin{array}{c} S_b(z_{2n+1}, z_{2n+1}, z_{2n}), \\ S_b(w_{2n+1}, w_{2n+1}, w_{2n}) \end{array}\right\} > \max\left\{\ S_b(\alpha, \alpha, z_{2n}), S_b(\beta, \beta, w_{2n})\ \right\}$$

and

$$\tfrac{1}{8b^3}\min\left\{\begin{array}{c} S_b(z_{2n}, z_{2n}, z_{2n-1}), \\ S_b(w_{2n}, w_{2n}, w_{2n-1}) \end{array}\right\} > \max\left\{\ S_b(\alpha, \alpha, z_{2n-1}), S_b(\beta, \beta, w_{2n-1})\ \right\}.$$

Now consider

$$\begin{aligned}
\min &\left\{ \begin{array}{c} S_b(z_{2n}, z_{2n}, z_{2n-1}), \\ S_b(w_{2n}, w_{2n}, w_{2n-1}) \end{array} \right\} \\
&\leqslant \min \left\{ \begin{array}{c} 2bS_b(z_{2n}, z_{2n}, \alpha) + b^2 S_b(\alpha, \alpha, z_{2n-1}), \\ 2bS_b(w_{2n}, w_{2n}, \beta) + b^2 S_b(\beta, \beta, z_{2n-1}) \end{array} \right\} \\
&\leqslant 2b^2 \min \left\{ \begin{array}{c} S_b(\alpha, \alpha, z_{2n}), \\ S_b(\beta, \beta, w_{2n}) \end{array} \right\} + b^2 \min \left\{ \begin{array}{c} S_b(\alpha, \alpha, z_{2n-1}), \\ S_b(\beta, \beta, z_{2n-1}) \end{array} \right\} \\
&< \tfrac{1}{4b} \min \left\{ \begin{array}{c} S_b(z_{2n+1}, z_{2n+1}, z_{2n}), \\ S_b(w_{2n+1}, w_{2n+1}, w_{2n}) \end{array} \right\} + \tfrac{1}{8b} \min \left\{ \begin{array}{c} S_b(z_{2n}, z_{2n}, z_{2n-1}), \\ S_b(w_{2n}, w_{2n}, w_{2n-1}) \end{array} \right\} \\
&\leqslant \tfrac{1}{4b} \min \left\{ \begin{array}{c} S_b(z_{2n}, z_{2n}, z_{2n-1}), \\ S_b(w_{2n}, w_{2n}, w_{2n-1}) \end{array} \right\} + \tfrac{1}{8b} \min \left\{ \begin{array}{c} S_b(z_{2n}, z_{2n}, z_{2n-1}), \\ S_b(w_{2n}, w_{2n}, w_{2n-1}) \end{array} \right\} \\
&= \tfrac{3}{8b} \min \left\{ \begin{array}{c} S_b(z_{2n}, z_{2n}, z_{2n-1}), \\ S_b(w_{2n}, w_{2n}, w_{2n-1}) \end{array} \right\} \\
&< \min \left\{ \begin{array}{c} S_b(z_{2n}, z_{2n}, z_{2n-1}), \\ S_b(w_{2n}, w_{2n}, w_{2n-1}) \end{array} \right\}.
\end{aligned}$$

It is a contradiction. Hence our assertion holds.
Sub case(a).

$$\tfrac{1}{8b^3} \min \left\{ \begin{array}{c} S_b(z_{2n+1}, z_{2n+1}, z_{2n}), \\ S_b(w_{2n+1}, w_{2n+1}, w_{2n}) \end{array} \right\} \quad \leqslant \max \left\{ \begin{array}{c} S_b(\alpha, \alpha, z_{2n}), \\ S_b(\beta, \beta, w_{2n}) \end{array} \right\}$$

holds. Since

$$\tfrac{1}{8b^3} \min \left\{ \begin{array}{c} S_b(A(a,b), A(a,b), \alpha), \\ S_b(z_{2n+1}, z_{2n+1}, z_{2n}) \\ S_b(A(b,a), A(b,a), \beta), \\ S_b(w_{2n+1}, w_{2n+1}, w_{2n}) \end{array} \right\} \quad \leqslant \max \left\{ \begin{array}{c} S_b(Pa, Pa, z_{2n}), \\ S_b(Pb, Pb, w_{2n}) \end{array} \right\}.$$

That is

$$\begin{aligned}
\tfrac{1}{8b^3} \min &\left\{ \begin{array}{c} S_b(A(a,b), A(a,b), \alpha), S_b(z_{2n+1}, z_{2n+1}, z_{2n}), \\ S_b(A(b,a), A(b,a), \beta), S_b(w_{2n+1}, w_{2n+1}, w_{2n}) \end{array} \right\} \\
&\leqslant \max \left\{ \begin{array}{c} S_b(\alpha, \alpha, z_{2n}), \\ S_b(\beta, \beta, w_{2n}) \end{array} \right\}.
\end{aligned}$$

From (2.1.4) and Lemma (1.11), we have

$$
\begin{aligned}
\tfrac{1}{2b} \quad & S_b(A(a,b),A(a,b),\alpha) \\
& \leqslant \lim_{n\to\infty} \inf \; S_b(A(a,b),A(a,b),B(x_{2n+1},y_{2n+1})) \\
& \leqslant \lim_{n\to\infty} \inf 2\, b^5 \; S_b(A(a,b),A(a,b),B(x_{2n+1},y_{2n+1})) \\
& \leqslant \lim_{n\to\infty} \inf \; \phi\left(\max \left\{ \begin{array}{c} S_b(\alpha,\alpha,z_{2n}), S_b(\beta,\beta,w_{2n}), \\ S_b(A(a,b),A(a,b),\alpha), S_b(A(b,a),A(b,a),\beta), \\ S_b(z_{2n+1},z_{2n+1},z_{2n}), S_b(w_{2n+1},w_{2n+1},w_{2n}), \\ \frac{\left[\; S_b(A(a,b),A(a,b),z_{2n}) S_b(z_{2n+1},z_{2n+1},\alpha) \;\right]}{1+S_b(\alpha,\alpha,z_{2n})}, \\ \frac{\left[\; S_b(A(b,a),A(b,a),w_{2n}) S_b(w_{2n+1},w_{2n+1},\beta) \;\right]}{1+S_b(\beta,\beta,w_{2n})} \end{array} \right\} \right) \\
& = \phi\left(\max\left\{\; 0,0,S_b(A(a,b),A(a,b),\alpha),S_b(A(b,a),A(b,a),\beta),0,0,0,0 \;\right\}\right) \\
& = \phi\left(\max\left\{\; S_b(A(a,b),A(a,b),\alpha),S_b(A(b,a),A(b,a),\beta) \;\right\}\right).
\end{aligned}
$$

Similarly

$$
\frac{1}{2b} S_b(A(b,a),A(b,a),\beta) \leqslant \phi\left(\max \left\{ \begin{array}{c} S_b(A(a,b),A(a,b),\alpha), \\ S_b(A(b,a),A(b,a),\beta) \end{array} \right\} \right).
$$

Thus

$$
\tfrac{1}{2b} \max \left\{ \begin{array}{c} S_b(A(a,b),A(a,b),\alpha), \\ S_b(A(b,a),A(b,a),\beta) \end{array} \right\} \leqslant \phi\left(\max \left\{ \begin{array}{c} S_b(A(a,b),A(a,b),\alpha), \\ S_b(A(b,a),A(b,a),\beta) \end{array} \right\} \right).
$$

By the definition of ϕ, it follows that $A(a,b) = \alpha = Pa$ and $A(b,a) = \beta = Pb$. Since (A,P) is w-compatible pair, we have $A(\alpha,\beta) = P\alpha$ and $A(\beta,\alpha) = P\beta$. From the definition of S_b-metric it is clear that

$$
\begin{aligned}
& \tfrac{1}{8b^3} \min \left\{ \begin{array}{c} S_b(A(\alpha,\beta),A(\alpha,\beta),P\alpha), S_b(A(\beta,\alpha),A(\beta,\alpha),P\beta) \\ S_b(B(x_{2n+1},y_{2n+1}),B(x_{2n+1},y_{2n+1}),Qx_{2n+1}), \\ S_b(B(y_{2n+1},x_{2n+1}),B(y_{2n+1},x_{2n+1}),Qy_{2n+1}) \end{array} \right\} \\
& \quad = 0 \leqslant \max\left\{\; S_b(P\alpha,P\alpha,Qx_{2n+1}),S_b(P\beta,P\beta,Qy_{2n+1}) \;\right\}.
\end{aligned}
$$

From (2.1.4) and Lemma (1.11), we have

$$\begin{aligned}
\tfrac{1}{2b} \quad & S_b(A(\alpha,\beta),A(\alpha,\beta),\alpha) \\
& \leqslant \lim_{n\to\infty} \sup \ S_b(A(\alpha,\beta),A(\alpha,\beta),B(x_{2n+1},y_{2n+1})) \\
& \leqslant \lim_{n\to\infty} \sup 2\ b^5 \ S_b(A(\alpha,\beta),A(\alpha,\beta),B(x_{2n+1},y_{2n+1})) \\
& \leqslant \lim_{n\to\infty} \sup \ \phi \left(\max \left\{ \begin{array}{c} S_b(A(\alpha,\beta),A(\alpha,\beta),z_{2n}), S_b(A(\beta,\alpha),A(\beta,\alpha),w_{2n}), \\ 0,0,S_b(z_{2n+1},z_{2n+1},z_{2n}), S_b(w_{2n+1},w_{2n+1},w_{2n}), \\ \frac{S_b(A(\alpha,\beta),A(\alpha,\beta),z_{2n})\ S_b(z_{2n+1},z_{2n+1},A(\alpha,\beta))}{1+S_b(A(\alpha,\beta),A(\alpha,\beta),z_{2n})}, \\ \frac{S_b(A(\beta,\alpha),A(\beta,\alpha),w_{2n})\ S_b(w_{2n+1},w_{2n+1},A(\beta,\alpha))}{1+S_b(A(\beta,\alpha),A(\beta,\alpha),w_{2n})} \end{array} \right\} \right) \\
& \leqslant \lim_{n\to\infty} \sup \ \phi \left(\max \left\{ \begin{array}{c} S_b(A(\alpha,\beta),A(\alpha,\beta),z_{2n}), S_b(A(\beta,\alpha),A(\beta,\alpha),w_{2n}), \\ S_b(z_{2n+1},z_{2n+1},z_{2n}), S_b(w_{2n+1},w_{2n+1},w_{2n}), \\ S_b(z_{2n+1},z_{2n+1},A(\alpha,\beta)), S_b(w_{2n+1},w_{2n+1},A(\beta,\alpha)), \end{array} \right\} \right) \\
& \leqslant \phi \left(\max \left\{ \begin{array}{c} 2bS_b(A(\alpha,\beta),A(\alpha,\beta),\alpha), 2bS_b(A(\beta,\alpha),A(\beta,\alpha),\beta), \\ 0,0,b^2 S_b(\alpha,\alpha,A(\alpha,\beta)), b^2 S_b(\beta,\beta,A(\beta,\alpha)), \end{array} \right\} \right) \\
& \leqslant \phi \left(2b^2 \max \left\{ \ S_b(A(\alpha,\beta),A(\alpha,\beta),\alpha), S_b(A(\beta,\alpha),A(\beta,\alpha),\beta) \ \right\}\right).
\end{aligned}$$

Similarly

$$\frac{1}{2b} S_b(A(\beta,\alpha),A(\beta,\alpha),\beta) \leqslant \phi \left(2b^2 \max \left\{ \begin{array}{c} S_(A(\alpha,\beta),A(\alpha,\beta),\alpha), \\ S_b(A(\beta,\alpha),A(\beta,\alpha),\beta) \end{array} \right\} \right).$$

Thus

$$\begin{aligned}
& \tfrac{1}{2b} \max \left\{ \begin{array}{c} S_b(A(\alpha,\beta),A(\alpha,\beta),\alpha), \\ S_b(A(\beta,\alpha),A(\beta,\alpha),\beta) \end{array} \right\} \\
& \leqslant \phi \left(2b^2 \max \left\{ \begin{array}{c} S_b(A(\alpha,\beta),A(\alpha,\beta),\alpha), \\ S_b(A(\beta,\alpha),A(\beta,\alpha),\beta) \end{array} \right\} \right).
\end{aligned}$$

By the definition of ϕ, it follows that $A(\alpha,\beta) = \alpha = P\alpha$ and $A(\beta,\alpha) = \beta = P\beta$. Therefore (α,β) is a common coupled fixed point of A and P.

Since $A(X\times X) \subseteq Q(X)$ there exist x and y in X such that $A(\alpha,\beta) = \alpha = Qx$ and $A(\beta,\alpha) = \beta = Qy$. Since we have that

$$\begin{aligned}
\tfrac{1}{8b^3} \quad & \min \left\{ \begin{array}{c} S_b(A(\alpha,\beta),A(\alpha,\beta),P\alpha), S_b(A(\beta,\alpha),A(\beta,\alpha),P\beta) \\ S_b(B(x,y),B(x,y),Qx), S_b(B(y,x),B(y,x),Qy) \end{array} \right\} \\
& = 0 \leqslant \max \left\{ \begin{array}{c} S_b(P\alpha,P\alpha,Qx), \\ S_b(P\beta,P\beta,Qy) \end{array} \right\}.
\end{aligned}$$

From (2.1.4) we have

$2\ b^5\ S_b(A(\alpha,\beta),A(\alpha,\beta),B(x,y))$

$$\leqslant \phi\left(\max\left\{\begin{array}{c} S_b(P\alpha,P\alpha,Qx),S_b(P\beta,P\beta,Qy),\\ S_b(A(\alpha,\beta),A(\alpha,\beta),P\alpha),S_b(A(\beta,\alpha),A(\beta,\alpha),P\beta),\\ S_b(B(x,y),B(x,y),Qx),S(B(y,x),B(y,x),Qy),\\ \frac{S_b(A(\alpha,\beta),A(\alpha,\beta),Qx)\ S_b(B(x,y),B(x,y),P\alpha)}{1+S_b(P\alpha,P\alpha,Qx)},\\ \frac{S_b(A(\beta,\alpha),A(\beta,\alpha),Qy)S_b(B(y,x),B(y,x),P\beta)}{1+S_b(P\beta,P\beta,Qy)} \end{array}\right\}\right)$$

$$= \phi\left(\max\left\{\ 0,0,0,0,S_b(B(x,y),B(x,y),\alpha),S_b(B(y,x),B(y,x),\beta),0,0\ \right\}\right)$$
$$\leqslant \phi\left(b\max\left\{\ S_b(\alpha,\alpha,B(x,y)),S_b(\beta,\beta,B(y,x))\ \right\}\right).$$

Similarly

$$2\ b^5 S_b(\beta,\beta,B(y,x)) \leqslant \phi\left(b\max\left\{\ S_b(\alpha,\alpha,B(x,y)),S_b(\beta,\beta,B(y,x))\ \right\}\right).$$

Thus

$$2\ b^5\max\left\{\begin{array}{c} S_b(\alpha,\alpha,B(x,y)),\\ S_b(\beta,\beta,B(y,x))\end{array}\right\} \leqslant \phi\left(b\max\left\{\begin{array}{c} S_b(\alpha,\alpha,B(x,y)),\\ S_b(\beta,\beta,B(y,x))\end{array}\right\}\right).$$

It follows that $B(x,y)=\alpha=Qx$ and $B(y,x)=\beta=Qy$.

Since (B,Q) is w-compatible pair, we have $B(\alpha,\beta)=Q\alpha$, and $B(\beta,\alpha,)=Q\beta$. Since we have that

$$\frac{1}{8b^3}\quad \min\left\{\begin{array}{c} S_b(A(\alpha,\beta),A(\alpha,\beta),P\alpha),S_b(A(\beta,\alpha),A(\beta,\alpha),P\beta)\\ S_b(B(\alpha,\beta),B(\alpha,\beta),Q\alpha),S_b(B(\beta,\alpha),B(\beta,\alpha),Q\beta)\end{array}\right\}$$
$$=0\leqslant \max\left\{\begin{array}{c} S_b(P\alpha,P\alpha,Q\alpha),\\ S_b(P\beta,P\beta,Q\beta)\end{array}\right\}.$$

From (2.1.4) we have

$$2\ b^5 \quad S_b(A(\alpha,\beta),A(\alpha,\beta),B(\alpha,\beta))$$
$$\leqslant \phi\left(\max\left\{\begin{array}{c} S_b(P\alpha,P\alpha,Q\alpha),S_b(P\beta,P\beta,Q\beta),\\ S_b(A(\alpha,\beta),A(\alpha,\beta),P\alpha),S_b(A(\beta,\alpha),A(\beta,\alpha),P\beta),\\ S_b(B(\alpha,\beta),B(\alpha,\beta),Q\alpha),S_b(B(\beta,\alpha),B(\beta,\alpha),Q\beta),\\ \frac{S_b(A(\alpha,\beta),A(\alpha,\beta),Q\alpha)S_b(B(\alpha,\beta),B(\alpha,\beta),P\alpha)}{1+S_b(P\alpha,P\alpha,Q\alpha)},\\ \frac{S_b(A(\beta,\alpha),A(\beta,\alpha),Q\beta)S_b(B(\beta,\alpha),B(\beta,\alpha),P\beta)}{1+S_b(P\beta,P\beta,Q\beta)}\end{array}\right\}\right)$$

$$=\phi\left(\max\left\{\begin{array}{c} S_b(\alpha,\alpha,B(\alpha,\beta)),S_b(\beta,\beta,B(\beta,\alpha)),\\ S_b(B(\alpha,\beta),B(\alpha,\beta),\alpha),S_b(B(\beta,\alpha),B(\beta,\alpha),\beta)\end{array}\right\}\right)$$
$$\leqslant \phi\left(b\max\left\{\ S_b(\alpha,\alpha,B(\alpha,\beta)),S_b(\beta,\beta,B(\beta,\alpha))\ \right\}\right).$$

Similarly

$$2\ b^5 S_b(\beta,\beta,B(\beta,\alpha)) \leqslant \phi\left(b\max\left\{\ S_b(\alpha,\alpha,B(\alpha,\beta)),S_b(\beta,\beta,B(\beta,\alpha))\ \right\}\right).$$

Thus

$$\max\left\{\begin{array}{c} S_b(\alpha,\alpha,B(\alpha,\beta)),\\ S_b(\beta,\beta,B(\beta,\alpha))\end{array}\right\} \leqslant \phi\left(b\max\left\{\begin{array}{c} S_b(\alpha,\alpha,B(\alpha,\beta)),\\ S_b(\beta,\beta,B(\beta,\alpha))\end{array}\right\}\right).$$

It follows that $B(\alpha,\beta)=\alpha=Q\alpha$ and $B(\beta,\alpha)=\beta=Q\beta$. Thus (α,β) is a common coupled fixed point of A,B,P and Q.

To prove uniqueness let us take (α^1, β^1) as another common coupled fixed point of A, B, P and Q. Since it is clear that

$$\frac{1}{8b^3}\min\left\{\begin{array}{c} S_b(A(\alpha,\beta), A(\alpha,\beta), P\alpha),\\ S_b(A(\beta,\alpha), A(\beta,\alpha), P\beta),\\ S_b(B(\alpha^1,\beta^1), B(\alpha^1,\beta^1), Q\alpha^1),\\ S_b(B(\beta^1,\alpha^1), B(\beta^1,\alpha^1), Q\beta^1) \end{array}\right\} = 0$$

$$\leqslant \max\left\{\begin{array}{c} S_b(P\alpha, P\alpha, Q\alpha^1),\\ S_b(P\beta, P\beta, Q\beta^1) \end{array}\right\}.$$

From (2.1.4) we have

$$\begin{aligned} 2\, b^5\ S_b(\alpha,\alpha,\alpha^1) &= 2\, b^5\ S_b(A(\alpha,\beta), A(\alpha,\beta), B(\alpha^1,\beta^1))\\ &\leqslant \phi\left(\max\left\{\begin{array}{c} S_b(\alpha,\alpha,\alpha^1), S(\beta,\beta,\beta^1), S_b(\alpha,\alpha,\alpha),\\ S_b(\beta,\beta,\beta), S_b(\alpha^1,\alpha^1,\alpha^1), S_b(\beta^1,\beta^1,\beta^1),\\ \frac{S_b(\alpha,\alpha,\alpha^1)S_b(\alpha^1,\alpha^1,\alpha)}{1+S_b(\alpha,\alpha,\alpha^1)}, \frac{S_b(\beta,\beta,\beta^1)S_b(\beta^1,\beta^1,\beta)}{1+S_b(\beta,\beta,\beta^1)} \end{array}\right\}\right).\\ &\leqslant \phi(b\max\{S_b(\alpha,\alpha,\alpha^1), S_b(\beta,\beta,\beta^1)\}) \end{aligned}$$

Similarly

$$2\, b^5\ S(\beta,\beta,\beta^1) \leqslant \phi(\max\{bS(\alpha,\alpha,\alpha^1), bS_b(\beta,\beta,\beta^1)\}).$$

Thus

$$\begin{aligned} &2\, b^5\ \max\left\{\ S_b(\alpha,\alpha,\alpha^1), S_b(\beta,\beta,\beta^1)\ \right\}\\ &\leqslant \phi\left(b\max\left\{\ S_b(\alpha,\alpha,\alpha^1), S_b(\beta,\beta,\beta^1)\ \right\}\right). \end{aligned}$$

It follows that $\alpha = \alpha^1$ and $\beta = \beta^1$. Hence (α,β) is unique common coupled fixed point of A, B, P and Q.

Sub case(b).

$$\frac{1}{8b^3}\min\left\{\begin{array}{c} S_b(z_{2n}, z_{2n}, z_{2n-1}),\\ S_b(w_{2n}, w_{2n}, w_{2n-1}) \end{array}\right\} \leqslant \max\left\{\begin{array}{c} S_b(\alpha,\alpha,z_{2n-1}),\\ S_b(\beta,\beta,w_{2n-1}) \end{array}\right\}$$

holds. By proceeding as in Sub case(a), we can prove that (α,β) is unique common coupled fixed point of A, B, P and Q.

Similarly the theorem holds when $Q(X)$ is a S_b-complete subspace of (X, S_b). □

Now we give an example to illustrate the Theorem 2.1.

EXAMPLE 2.1. Let $X = [0,1]$ and $S : X \times X \times X \to \mathcal{R}^+$ by

$$S_b(x,y,z) = (|y+z-2x| + |y-z|)^2,$$

then (X, S_b) is a S_b-metric space with $b = 4$. Define $A, B : X \times X \to X$ and $P, Q : X \to X$ by

$$A(x,y) = \frac{x^2+y^2}{4^8},\ \ B = \frac{x^2+y^2}{4^9},\ \ P(x) = \frac{x^2}{4}$$

and $Q(x) = \frac{x^2}{16}$. Let $\phi : \mathcal{R}^+ \to \mathcal{R}^+$ be defined by $\phi(t) = \frac{t}{4^6}$. Consider

$$\begin{aligned}
2b^5 \quad & S_b(A(x,y), A(x,y), B(u,v)) \\
& = 2\left(4^6\right)(|A(x,y) - B(u,v)|)^2 \\
& = 2(4^6)\left|\tfrac{x^2+y^2}{4^8} - \tfrac{u^2+v^2}{4^9}\right|^2 \\
& = 2(4^6)\left|\tfrac{4x^2-u^2}{4^9} + \tfrac{4y^2-v^2}{4^9}\right|^2 \\
& \leqslant 2(4^6)\left\{\left|\tfrac{4x^2-u^2}{4^9}\right| + \left|\tfrac{4y^2-v^2}{4^9}\right|\right\}^2 \\
& \leqslant \tfrac{2(4^6)}{(4^7)^2}\left(2 \ \max\left\{\left|\tfrac{4x^2-u^2}{16}\right|, \left|\tfrac{4y^2-v^2}{16}\right|\right\}\right)^2 \\
& = \tfrac{1}{2(4^6)} \max\left\{\left|\tfrac{x^2}{4} - \tfrac{u^2}{16}\right|^2, \left|\tfrac{y^2}{4} - \tfrac{v^2}{16}\right|^2\right\} \\
& = \tfrac{1}{2(4^7)} \max\left\{S(Px,Px,Qu), S(Py,Py,Qv)\right\} \\
& \leqslant \phi\left(\max\left\{\begin{array}{c} S(Px,Px,Qu), S(Py,Py,Qv), \\ S(A(x,y),A(x,y),Px), S(A(y,x),A(y,x),Py), \\ S(B(u,v),B(u,v),Qu), S(B(v,u),B(v,u),Qv), \\ \frac{S(A(x,y),A(x,y),Qu)\ S(B(u,v),B(u,v),Px)}{1+S(Px,Px,Qu)}, \\ \frac{S(A(y,x),A(y,x),Qv)\ S(B(v,u),B(v,u),Py)}{1+S(Py,Py,Qv)} \end{array}\right\}\right).
\end{aligned}$$

Thus the condition (2.1.4) is satisfied.One can easily verify the remaining conditions of Theorem 2.1. In this example $(0,0)$ is the unique common coupled fixed point of A, B, P and Q.

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