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THE ZAGREB INDICES OF GRAPHS BASED ON NEW OPERATIONS RELATED TO THE JOIN OF GRAPHS

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Abstract. The first and second Zagreb indices were introduced more than forty years ago and are defined as the sum of squares of the degree of the vertices and the sum of the product of the degrees of the pairs of adjacent vertices. In this work, we study the first and second Zagreb indices of new operations of different subdivisions graphs related to joining of graphs.

1. Introduction

Let G be a simple graph without directed edges and the vertex and edge sets of G are represented by V(G) and E(G) respectively. The degree of a vertex v in G is the number of edges incident to v and denoted by $d_G(v)$. Throughout this paper, we consider only simple and connected graphs. Let $\sum$ denotes the collection of all graphs. A mapping $T : \sum \rightarrow \mathbb{R}$ is called a topological index, if for every graph H isomorphic to G, $T(G) = T(H)$. In chemical graph theory, different topological indices have different applications in isomer discrimination, QSAR/QSPR investigation, pharmaceutical drug design and many more. There are various important classes of topological indices that are extensively studied by a number of researchers. Among these topological indices, the first and second Zagreb indices were most studied and have good applications in molecular graph theory. These indices were introduced by Gutman and Trinajestić [**1**] in a paper in 1972 to study the structure-dependency of the total π-electron energy(ϵ) and are respectively defined as

$$M_1(G) = \sum_{v \in V(G)} d_G(v)^2 = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]$$

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and

$$M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v).$$

We refer the reader [**2, 3, 4, 5, 6, 7**] for different mathematical properties and recent study on Zagreb indices.

Inspired by the work of X. Liu et al. [**8**] and P. Liu et al. [**9**], in this work we propose two types of four new operations on graphs based on join of graphs related to various subdivision graphs S(G), R(G), Q(G), T(G) and study their first and second Zagreb indices. For this purpose we first define some related graph operations and terminologies.

DEFINITION 1.1. *The join of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is the union $G_1 \cup G_2$ together with all the edges joining $V(G_1)$ and $V(G_2)$, so that*

$$d_{G_1 \vee G_2}(v) = \begin{cases} d_{G_1}(v) + n_2, \ if \ v \in V(G_1) \\ d_{G_2}(v) + n_1, \ if \ v \in V(G_2). \end{cases}$$

The line graph L(G) is the graph whose vertices correspond to the edges of G with two vertices being adjacent if and only if the corresponding edges in G have a vertex in common.

For a connected graph G, there are four subdivision related graphs, denoted by S(G), R(G), Q(G) and T(G), which is defined as follows [**10, 11, 12**]:

S(G) is the graph which is obtained from G by adding an extra vertex into each edge of G. In other words replaced each edge of G by a path of length 2.

The graph R(G) is obtained from G by inserting an additional vertex into each edge of G and joining each additional vertex to the end vertices of the corresponding edge of G.

Q(G) is a graph derived from G by adding a new vertex to each edge of G, then joining with edges those pairs of new vertices on adjacent edges of G.

The total graph T(G) is derive from G by inserting an new vertex to each edge of G, then joining each new vertex to the end vertices of the corresponding edge and joining with edges those pairs of new vertices on adjacent edges of G.

Let $F = \{S, R, Q, T\}$ and I(G) denotes the set of vertices F(G) which are inserted into each edge of G, so that $V(F(G)) = V(G) \cup I(G)$. Based on Cartesian product of graphs, Eliasi and Taeri [**13**], introduced F-sum graphs of two connected graphs G_1 and G_2. Deng et al. in [**14**], studied these F-sum graphs for the first and second Zagreb indices. De in [**15**], studied the these four operations for F-index of graphs. Sarala et al. in [**10**], introduced another four new operations related to lexicographic product and found their first and second Zagreb indices. Here we now introduce two types of four new operations on these graphs based on the join of two connected graphs G_1 and G_2, which are defined as follows

DEFINITION 1.2. *The vertex F-join graph of G_1 and G_2 is a graph with vertex set $V(F(G_1)) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup [uv : u \in V(G_1), v \in V(G_2)]$ that is $G_1 \dot{\vee}_F G_2$ is obtained from $F(G_1)$ and G_2 by joining each vertex of G_1 with every vertex of G_2.*

DEFINITION 1.3. *The edge F-join graph of G_1 and G_2 is a graph with vertex set $V(F(G_1)) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup [uv : u \in I(G_1), v \in V(G_2)]$ that is $G_1 \underline{\vee}_F G_2$ is obtained from $F(G_1)$ and G_2 by joining each vertex of $I(G_1)$ with every vertex of G_2.*

The figure of vertex F-join and edge F-join of graphs are shown in figure 1 and figure 2. In this work, we will study the first and second Zagreb indices of vertex and edge F-join of graphs respectively.

2. Main Results

In this work, we use the following topological indices to express first and second Zagreb indices of different vertex and edge F-join of graphs and hence we first define them. The Zagreb indices of a line graph is called the reformulated Zagreb indices, which are introduced by Miličević et al. [**16**] in 2004 and are defined as

$$EM_1(G) = \sum_{e \in E(G)} d(e)^2 = \sum_{uv \in E(G)} [d_G(u) + d_G(v) - 2]^2,$$

$$EM_2(G) = \sum_{e \sim f \in E(G)} d(e)d(f)$$

where $e \sim f$ means that the edges e and f share a common vertex in G. Different mathematical properties and applications of reformulated Zagreb indices have been studied in [**17, 18, 19, 20, 21**].

The F-index or the "forgotten topological index" was introduced in [**1**] and Furtula and Gutman investigated this index again in [**22**]. There are some other recent study of this index also [**23, 24, 25, 26**]. The F-index of a graph G is defined as

$$F(G) = \sum_{v \in V(G)} d_G(v)^3 = \sum_{uv \in E(G)} [d_G(u)^2 + d_G(v)^2].$$

Another version of Zagreb index, named as hyper Zagreb index, was introduced by Shirrdel et al. in [**27**] and is defined as

$$HM(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]^2.$$

We refer the reader to [**28, 29, 30, 31, 32**], for some recent study and application of hyper Zagreb index.

Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two connected graphs so that $|V(G_1)| = n_1$, $|V(G_2)| = n_2$ and $|E(G_1)| = m_1$, $|E(G_2)| = m_2$ respectively. In the following first we derive the first and second Zagreb indices of vertex F-join of graphs and then edge F-join of graphs respectively where $F = \{S, R, Q, T\}$. Note that, as usual, P_n denotes a path with n vertices and $(n-1)$ edges whereas C_n $(n \geqslant 3)$ denotes a cycle graph with n vertices.

2.1. Vertex F-join of Graphs. In this section, we derive the explicit expression of F-join of graphs where $F = \{S, R, Q, T\}$ respectively. At first, we consider vertex S-join of graphs.

Definition 2.1. *The vertex S-join of two vertex disjoint graphs G_1 and G_2 denoted by $G_1 \dot{\vee}_S G_2$ and is obtained from $S(G_1)$ and G_2 by joining each vertex of $V(G_1)$ with every vertex of G_2*

The degree of the vertices of vertex S-join graph are

$$d_{G_1 \dot{\vee}_S G_2}(v) = \begin{cases} d_{G_1}(v) + n_2, \ if \ v \in V(G_1) \\ d_{G_2}(v) + n_1, \ if \ v \in V(G_2) \\ 2, \ if \ v \in I(G_1). \end{cases}$$

Theorem 2.1. *If G_1 and G_2 be two connected graph then*

$$M_1(G_1 \dot{\vee}_S G_2) = M_1(G_1) + M_1(G_2) + 4n_2m_1 + 4n_1m_2 + n_1^2n_2 + n_1n_2^2 + 4m_1.$$

Proof. From definition of first Zagreb index, we have

$$\begin{aligned} M_1(G_1 \dot{\vee}_S G_2) &= \sum_{v \in V(G_1 \dot{\vee}_S G_2)} d_{G_1 \dot{\vee}_S G_2}(v)^2 \\ &= \sum_{v \in V(G_1)} d_{G_1 \dot{\vee}_S G_2}(v)^2 + \sum_{v \in V(G_2)} d_{G_1 \dot{\vee}_S G_2}(v)^2 \\ &\quad + \sum_{v \in I(G_1)} d_{G_1 \dot{\vee}_S G_2}(v)^2 \\ &= \sum_{v \in V(G_1)} (d_{G_1}(v) + n_2)^2 + \sum_{v \in V(G_2)} (d_{G_2}(v) + n_1)^2 + \sum_{v \in I(G_1)} 2^2 \\ &= \sum_{v \in V(G_1)} [d_{G_1}(v)^2 + 2n_2 d_{G_1}(v) + n_2^2] \\ &\quad + \sum_{v \in V(G_2)} [d_{G_2}(v)^2 + 2n_1 d_{G_2}(v) + n_1^2] + \sum_{v \in I(G_1)} 2^2 \\ &= \sum_{v \in V(G_1)} d_{G_1}(v)^2 + 2n_2 \sum_{v \in V(G_1)} d_{G_1}(v) + n_1n_2^2 + \sum_{v \in V(G_2)} d_{G_2}(v)^2 \\ &\quad + 2n_1 \sum_{v \in V(G_2)} d_{G_2}(v) + n_2n_1^2 + 4m_1 \\ &= M_1(G_1) + M_1(G_2) + 4n_2m_1 + 4n_1m_2 + n_1^2n_2 \\ &\quad + n_1n_2^2 + 4m_1. \end{aligned}$$

Which is the desired result. $\square$

EXAMPLE 2.1. Using theorem 2.1, we get

$$\begin{aligned}
(i)\ M_1(P_n\dot{\vee}_s P_m) &= 4n+mn(m+n+8)-16,\\
(ii)\ M_1(P_n\dot{\vee}_s C_m) &= 8n(m+1)+mn(m+n)-10,\\
(iii)\ M_1(C_n\dot{\vee}_s C_m) &= 8n+4m+mn(m+n+8),\\
(iv)\ M_1(C_n\dot{\vee}_s P_m) &= 4n+4m+mn(m+n+8)-6.
\end{aligned}$$

THEOREM 2.2. *If G_1 and G_2 be two connected graph then*

$$\begin{aligned}
M_2(G_1\dot{\vee}_S G_2) &= 2M_1(G_1)+n_1M_1(G_2)+M_2(G_2)+4m_1(m_2+n_2)\\
&\quad +2n_1n_2(m_1+m_2)+n_1^2(m_2+n_2^2).
\end{aligned}$$

PROOF. By definition of second Zagreb index, we have

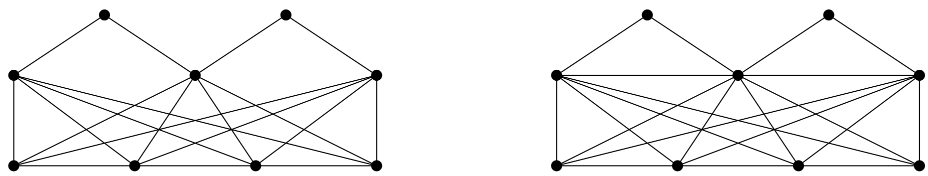
$$\begin{aligned}
M_2(G_1\dot{\vee}_S G_2) &= \sum_{uv\in E(G_1\dot{\vee}_S G_2)} d_{G_1\dot{\vee}_S G_2}(u)d_{G_1\dot{\vee}_S G_2}(v)\\
&= \sum_{uv\in E(S(G_1))} d_{G_1\dot{\vee}_S G_2}(u)d_{G_1\dot{\vee}_S G_2}(v)\\
&\quad + \sum_{uv\in E(G_2)} d_{G_1\dot{\vee}_S G_2}(u)d_{G_1\dot{\vee}_S G_2}(v)\\
&\quad + \sum_{u\in V(G_1)}\sum_{v\in V(G_2)} d_{G_1\dot{\vee}_S G_2}(u)d_{G_1\dot{\vee}_S G_2}(v)\\
&= \sum_{uv\in E(S(G_1))} (d_{G_1}(u)+n_2).2\\
&\quad + \sum_{uv\in E(G_2)} (d_{G_2}(u)+n_1)(d_{G_2}(v)+n_1)\\
&\quad + \sum_{u\in V(G_1)}\sum_{v\in V(G_2)} (d_{G_1}(u)+n_2)(d_{G_2}(v)+n_1)\\
&= \sum_{v\in V(G_1)} (d_{G_1}(v)+n_2).2d_{G_1}(v)\\
&\quad + \sum_{uv\in E(G_2)} [d_{G_2}(u)d_{G_2}(v)+n_1(d_{G_2}(u)+d_{G_2}(v))+n_1^2]\\
&\quad + \sum_{u\in V(G_1)}\sum_{v\in V(G_2)} [d_{G_1}(u)d_{G_2}(v)+n_1d_{G_1}(u)+n_2d_{G_2}(v)\\
&\quad +n_1n_2]
\end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{v\in V(G_1)} d_{G_1}(v)^2 + 2n_2 \sum_{v\in V(G_1)} d_{G_1}(v) \\
&\quad + \sum_{uv\in E(G_2)} d_{G_2}(u)d_{G_2}(v) + n_1 \sum_{uv\in E(G_2)} (d_{G_2}(u)+d_{G_2}(v)) \\
&\quad + n_1^2 m_2 + \sum_{u\in V(G_1)} d_{G_1}(u) \sum_{v\in V(G_2)} d_{G_2}(v) \\
&\quad + n_1 \sum_{v\in V(G_2)} \sum_{u\in V(G_1)} d_{G_1}(u) \\
&\quad + n_2 \sum_{v\in V(G_1)} \sum_{u\in V(G_2)} d_{G_2}(u) + n_1^2 n_2^2 \\
&= 2M_1(G_1) + n_1 M_1(G_2) + M_2(G_2) + 4m_1(m_2+n_2) \\
&\quad + 2n_1n_2(m_1+m_2) + n_1^2(m_2+n_2^2).
\end{aligned}
$$

Which is the required result. □

EXAMPLE 2.2. From theorem 2.2, we get

$$
\begin{aligned}
(i)\ M_2(P_n\dot{\vee}_S P_m) &= 3mn^2 + 2m^2n + m^2n^2 - n^2 + 8mn - 4m - 2n - 20, \\
(ii)\ M_2(P_n\dot{\vee}_s C_m) &= m^2n^2 + mn(3n+2m) + 10mn - 4m + 4n, \\
(iv)\ M_2(C_n\dot{\vee}_s C_m) &= m^2n^2 + mn(2m+3n) + 12mn + 4m + 8n, \\
(v)\ M_2(C_n\dot{\vee}_s P_m) &= 3mn^2 + m^2n(n+2) - n^2 + 10mn + 4m - 2n - 8.
\end{aligned}
$$



$P_3\dot{\vee}_S P_4$ $P_3\dot{\vee}_R P_4$

FIGURE 1. The example of $P_3\dot{\vee}_S P_4$ and $P_3\dot{\vee}_R P_4$ graphs.

DEFINITION 2.2. *The vertex R-join of two vertex disjoint graphs G_1 and G_2 denoted by $G_1\dot{\vee}_R G_2$ and is obtained from $R(G_1)$ and G_2 by joining each vertex of $V(G_1)$ with every vertex of G_2.*

The degree of the vertices of vertex R-join graph are

$$
d_{G_1\dot{\vee}_R G_2}(v) = \begin{cases} 2d_{G_1}(v) + n_2, v \in V(G_1) \\ d_{G_2}(v) + n_1, v \in V(G_2) \\ 2, v \in I(G_1). \end{cases}
$$

THEOREM 2.3. *If G_1 and G_2 be two connected graph then*

$$M_1(G_1\dot{\vee}_R G_2) = 4M_1(G_1) + M_1(G_2) + 8m_1n_2 + 4m_2n_1 + {n_1}^2n_2 + n_1{n_2}^2 + 4m_1.$$

PROOF. From definition of first Zagreb index, we have

$$\begin{aligned}
M_1(G_1\dot{\vee}_R G_2) &= \sum_{v\in V(G_1\dot{\vee}_R G_2)} d_{G_1\dot{\vee}_R G_2}(v)^2 \\
&= \sum_{v\in V(G_1)} d_{G_1\dot{\vee}_R}G_2(v)^2 + \sum_{v\in I(G_1)} d_{G_1\dot{\vee}_R G_2}(v)^2 \\
&\quad + \sum_{v\in V(G_2)} d_{G_1\dot{\vee}_R G_2}(v)^2 \\
&= \sum_{v\in V(G_1)} (2d_{G_1}(v)+n_2)^2 + \sum_{v\in I(G_1)} 2^2 + \sum_{v\in V(G_2)} (d_{G_2}(v)+n_1)^2 \\
&= \sum_{v\in V(G_1)} \{4d_{G_1}(v)^2 + 4n_2d_{G_1}(v) + {n_2}^2\} + \sum_{v\in I(G_1)} 2^2 \\
&\quad + \sum_{v\in V(G_2)} \{d_{G_2}(v)^2 + 2n_1d_{G_2}(v) + n_1^2\} \\
&= 4\sum_{v\in V(G_1)} d_{G_1}(v)^2 + 4n_2\sum_{v\in V(G_1)} d_{G_1}(v) + n_1{n_2}^2 + \sum_{v\in I(G_1)} 2^2 \\
&\quad + \sum_{v\in V(G_2)} d_{G_2}(v)^2 + 2n_1\sum_{v\in V(G_2)} d_{G_2}(v) + n_1^2n_2 \\
&= 4M_1(G_1) + M_1(G_2) + 8m_1n_2 + 4m_2n_1 + {n_1}^2n_2 \\
&\quad + n_1{n_2}^2 + 4m_1.
\end{aligned}$$

Which is the desired result. □

EXAMPLE 2.3. From theorem 2.3, we get

$$\begin{aligned}
(i)\ M_1(P_n\dot{\vee}_R P_m) &= mn(m+n) + 12mn - 4m + 16n - 34, n \geqslant 2, \\
(ii)\ M_1(P_n\dot{\vee}_R C_m) &= mn(m+n-8) - 4m + 20n - 28, n \geqslant 2, m \geqslant 3, \\
(iii)\ M_1(C_n\dot{\vee}_R C_m) &= mn(m+n+12) + 4m + 20n, n, m \geqslant 3, \\
(iv)\ M_1(C_n\dot{\vee}_R P_m) &= mn(m+n+12) + 4m + 16n - 6, n \geqslant 3, m \geqslant 2.
\end{aligned}$$

THEOREM 2.4. *If G_1 and G_2 be two connected graph then*

$$\begin{aligned}
M_2(G_1\dot{\vee}_R G_2) &= 4M_2(G_1) + M_2(G_2) + (2n_2+4)M_1(G_1) + n_1M_1(G_2) \\
&\quad + n_2^2m_1 + n_1^2m_2 + 2n_1n_2(2m_1+m_2) + 8m_1m_2 \\
&\quad + n_1^2n_2^2 + 4m_1n_2.
\end{aligned}$$

PROOF. Using definition of second Zagreb index, we have

$$\begin{aligned}
M_2(G_1\dot{\vee}_R G_2) &= \sum_{uv\in E(G_1\dot{\vee}G_2)} d_{G_1\dot{\vee}_R G_2}(u)d_{G_1\dot{\vee}_R G_2}(v)\\
&= \sum_{uv\in E(G_1)} d_{G_1\dot{\vee}_R G_2}(u)d_{G_1\dot{\vee}_R G_2}(v)\\
&\quad+\sum_{u\in I(G_1),v\in V(G_1)} d_{G_1\dot{\vee}_R G_2}(u)d_{G_1\dot{\vee}_R G_2}(v)\\
&\quad+\sum_{uv\in E(G_2)} d_{G_1\dot{\vee}_R G_2}(u)d_{G_1\dot{\vee}_R G_2}(v)\\
&\quad+\sum_{u\in V(G_1)}\sum_{v\in V(G_2)} d_{G_1\dot{\vee}_R G_2}(u)d_{G_1\dot{\vee}_R G_2}(v)\\
&= \sum_{uv\in E(G_1)} (2d_{G_1}(u)+n_2)(2d_{G_1}(v)+n_2)\\
&\quad+\sum_{u\in I(G_1),v\in V(G_1)} (2d_{G_1}(v)+n_2).d_{I(G_1)}(u)\\
&\quad+\sum_{uv\in E(G_2)} (d_{G_2}(u)+n_1)(d_{G_2}(v)+n_1)\\
&\quad+\sum_{u\in V(G_1)}\sum_{v\in V(G_2)} (2d_{G_1}(u)+n_2)(d_{G_2}(v)+n_1)\\
&= \sum_{uv\in E(G_1)} (2d_{G_1}(u)+n_2)(2d_{G_1}(v)+n_2)\\
&\quad+\sum_{v\in V(G_1)} (2d_{G_1}(v)+n_2)2d_{G_1}(v)\\
&\quad+\sum_{uv\in E(G_2)} (d_{G_2}(u)+n_1)(d_{G_2}(v)+n_1)\\
&\quad+\sum_{u\in V(G_1)}\sum_{v\in V(G_2)} (2d_{G_1}(u)+n_2)(d_{G_2}(v)+n_1)\\
&= 4\sum_{uv\in E(G_1)} d_{G_1}(u)d_{G_1}(v)+2n_2\sum_{uv\in E(G_1)}(d_{G_1}(u)+d_{G_1}(v))\\
&\quad+n_2^2m_1+4\sum_{v\in V(G_1)} d_{G_1}(v)^2+2n_2\sum_{v\in V(G_1)} d_{G_1}(v)\\
&\quad+\sum_{uv\in E(G_2)} d_{G_2}(u)d_{G_2}(v)+n_1\sum_{uv\in E(G_2)}(d_{G_2}(u)+d_{G_2}(v))\\
&\quad+n_1^2m_2+2\sum_{u\in V(G_1)} d_{G_1}(u)\sum_{v\in V(G_2)} d_{G_2}(v)\\
&\quad+2n_1\sum_{v\in V(G_2)}\sum_{u\in V(G_1)} d_{G_1}(u)\\
&\quad+n_2\sum_{u\in V(G_1)}\sum_{v\in V(G_2)} d_{G_2}(v)+n_1^2n_2^2\\
&= 4M_2(G_1)+M_2(G_2)+(2n_2+4)M_1(G_1)+n_1M_1(G_2)+n_2^2m_1\\
&\quad+n_1^2m_2+2n_1n_2(2m_1+m_2)+8m_1m_2+n_1^2n_2^2+4m_1n_2.
\end{aligned}$$

Which is the required expression. $\square$

EXAMPLE 2.4. Using theorem 2.4, we get

$$\begin{aligned}
(i)\ M_2(P_n\dot{\vee}_R P_m) &= m^2n^2+3m^2n+5mn^2-(m^2+n^2)+18mn-20m \\
&\quad +18n-56,\ m,n\geqslant 3,\\
(ii)\ M_2(P_n\dot{\vee}_R C_m) &= m^2n^2+m^3+3m^2n+5mn^2-m^2+20mn-24m \\
&\quad +32n-56,\ m,n\geqslant 3,\\
(iii)\ M_2(C_n\dot{\vee}_R C_m) &= m^2n^2+3m^2n+5mn^2+m^3+4n^3+24mn-16n,\\
&\quad m,n\geqslant 3,\\
(iv)\ M_2(C_n\dot{\vee}_R P_m) &= m^2n^2+3m^2n+5mn^2+3n^2+18mn+4m+18n-8,\\
&\quad m,n\geqslant 3.
\end{aligned}$$

DEFINITION 2.3. *The vertex Q-join of two vertex disjoint graphs G_1 and G_2 denoted by $G_1\dot{\vee}_Q G_2$ and is obtained from $Q(G_1)$ and G_2 by joining each vertex of $V(G_1)$ with every vertex of G_2.*

The degree of the vertices of vertex Q-join graph are

$$d_{G_1\dot{\vee}_Q G_2}(v)=\begin{cases} d_{G_1}(v)+n_2,\ if\ v\in V(G_1)\\ d_{G_2}(v)+n_1,\ if\ v\in V(G_2)\\ d_{G_1}(w)+d_{G_1}(k),\ if\ e=(w,k),e\in I(G_1).\end{cases}$$

THEOREM 2.5. *If G_1 and G_2 be two connected graph then*

$$M_1(G_1\dot{\vee}_Q G_2)=M_1(G_1)+M_1(G_2)+HM(G_1)+4n_2m_1+4n_1m_2+n_1n_2^2+n_1^2n_2.$$

PROOF. By definition of first Zagreb index, we have

$$\begin{aligned}
M_1(G_1\dot{\vee}_Q G_2) &= \sum_{v\in V(G_1\dot{\vee}_Q G_2)} d_{G_1\dot{\vee}_Q G_2}(v)^2\\
&= \sum_{v\in V(G_1)} d_{G_1\dot{\vee}_Q G_2}(v)^2+\sum_{v\in V(G_2)} d_{G_1\dot{\vee}_Q G_2}(v)^2\\
&\quad +\sum_{v\in I(G_1)} d_{G_1\dot{\vee}_Q G_2}(v)^2\\
&= \sum_{v\in V(G_1)} (d_{G_1}(v)+n_2)^2+\sum_{v\in V(G_2)} (d_{G_2}(v)+n_1)^2\\
&\quad +\sum_{v\in I(G_1)} d_{I(G_1)}(v)^2
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{v\in V(G_1)} [d_{G_1}(v)^2 + 2n_2 d_{G_1}(v) + n_2^2] \\
&\quad + \sum_{v\in V(G_2)} [d_{G_2}(v)^2 + 2n_1 d_{G_2}(v) + n_1^2] \\
&\quad + \sum_{uv\in E(G_1)} (d_{G_1}(u) + d_{G_1}(v))^2 \\
&= \sum_{v\in V(G_1)} d_{G_1}(v)^2 + 2n_2 \sum_{v\in V(G_1)} d_{G_1}(v) + n_1 n_2^2 + \sum_{v\in V(G_2)} d_{G_2}(v)^2 \\
&\quad +2n_1 \sum_{v\in V(G_2)} d_{G_2}(v) + n_1^2 n_2 + \sum_{uv\in E(G_1)} (d_{G_1}(u) + d_{G_1}(v))^2 \\
&= M_1(G_1) + M_1(G_2) + HM(G_1) + 4n_2 m_1 + 4n_1 m_2 \\
&\quad +n_1 n_2^2 + n_1^2 n_2.
\end{aligned}
$$

Which is the desired result. □

EXAMPLE 2.5. By theorem 2.5, we get

$$
\begin{aligned}
(i)\ M_1(P_n \dot{\vee}_Q P_m) &= mn(m+n) + 4[n(m-1) + m(n-1)] + 4m \\
&\quad +20n - 42,\ m \geqslant 2, n \geqslant 3, \\
(ii)\ M_1(P_n \dot{\vee}_Q C_m) &= mn(m+n) + 4m(2n-1) + 4m + 20n - 36, \\
&\quad m, n \geqslant 3, \\
(iii)\ M_1(C_n \dot{\vee}_Q C_m) &= mn(m+n+8) + 4m + 20n,\ m, n \geqslant 3, \\
(iv)\ M_1(C_n \dot{\vee}_Q P_m) &= m^2 n + mn^2 + 8mn + 4m + 16n - 6,\ m, n \geqslant 3.
\end{aligned}
$$

THEOREM 2.6. *If G_1 and G_2 be two connected graph then*

$$
\begin{aligned}
M_2(G_1 \dot{\vee}_Q G_2) &= EM_2(G_1) + 2EM_1(G_1) + 2M_1(G_1) + HM(G_1) \\
&\quad +2n_2 M_1(G_1) + M_2(G_2) + n_1 M_1(G_2) + n_1^2 m_2 \\
&\quad +4m_1 m_2 + 2m_1 n_1 n_2 + 2m_2 n_1 n_2 + n_1^2 n_2^2 - 4m_1.
\end{aligned}
$$

PROOF. From definition of second Zagreb index, we have

$$
M_2(G_1 \dot{\vee}_Q G_2) = \sum_{uv\in E(G_1 \dot{\vee}_Q G_2)} d_{G_1 \dot{\vee}_Q G_2}(u) d_{G_1 \dot{\vee}_Q G_2}(v)
$$

$$
\begin{aligned}
&= \sum_{u,v\in I(G_1)} d_{G_1\dot{\vee}_Q G_2}(u)d_{G_1\dot{\vee}_Q G_2}(v) \\
&\quad + \sum_{u\in V(G_1),v\in I(G_1)} d_{G_1\dot{\vee}_Q G_2}(u)d_{G_1\dot{\vee}_Q G_2}(v) \\
&\quad + \sum_{uv\in E(G_2)} d_{G_1\dot{\vee}_Q G_2}(u)d_{G_1\dot{\vee}_Q G_2}(v) \\
&\quad + \sum_{u\in V(G_1)}\sum_{v\in V(G_2)} d_{G_1\dot{\vee}_Q G_2}(u)d_{G_1\dot{\vee}_Q G_2}(v) \\
&= \sum_{u,v\in I(G_1)} d_{I(G_1)}(u)d_{I(G_1)}(v) \\
&\quad + \sum_{u\in V(G_1),v\in I(G_1)} d_{G_1}(u)d_{I(G_1)}(v) \\
&\quad + \sum_{uv\in E(G_2)} (d_{G_2}(u)+n_1)(d_{G_2}(v)+n_1) \\
&\quad + \sum_{u\in V(G_1)}\sum_{v\in V(G_2)} (d_{G_2}(v)+n_1)(d_{G_1}(u)+n_2) \\
&= \sum_{uv,vw\in E(G_1)} (d_{G_1}(u)+d_{G_1}(v))(d_{G_1}(v)+d_{G_1}(w)) \\
&\quad + \sum_{uv\in E(G_1)} (d_{G_1}(u)+n_2+d_{G_1}(v)+n_2)(d_{G_1}(u)+d_{G_1}(v)) \\
&\quad + \sum_{uv\in E(G_2)} (d_{G_2}(u)+n_1)(d_{G_2}(v)+n_1) \\
&\quad + \sum_{u\in V(G_1)}\sum_{v\in V(G_2)} (d_{G_2}(v)+n_1)(d_{G_1}(u)+n_2) \\
&= P_1+P_2+P_3+P_4.
\end{aligned}
$$

Now,

$$
\begin{aligned}
P_1 &= \sum_{uv,vw\in E(G)} (d_{G_1}(u)+d_{G_1}(v))(d_{G_1}(v)+d_{G_1}(w)) \\
&= \sum_{e,f\in L(G_1),e=uv,f=vw} (d_{L(G_1)}(e)+2)(d_{L(G_1)}(f)+2) \\
&= \sum_{e,f\in E(L(G_1)} [d_{L(G_1)}(e)d_{L(G_1)}(f)+2(d_{L(G_1)}(e)+d_{L(G_1)}(f))+4] \\
&= M_2(L(G_1))+2M_1(L(G_1))+4\,|E(L(G_1)| \\
&= M_2(L(G_1))+2M_1(L(G_1))+4[\frac{M_1(G_1)}{2}-m_1]
\end{aligned}
$$

$$= \quad EM_2(G_1) + 2EM_1(G_1) + 2M_1(G_1) - 4m_1.$$

$$\begin{aligned}
P_2 &= \sum_{uv\in E(G_1)} (d_{G_1}(u) + n_2 + d_{G_1}(v) + n_2)(d_{G_1}(u) + d_{G_1}(v)) \\
&= \sum_{uv\in E(G_1)} [(d_{G_1}(u) + d_{G_1}(v))^2 + 2n_2(d_{G_1}(u) + d_{G_1}(v)) \\
&= \sum_{uv\in E(G_1)} (d_{G_1}(u) + d_{G_1}(v))^2 + 2n_2 \sum_{uv\in E(G_1)} (d_{G_1}(u) + d_{G_1}(v)) \\
&= HM(G_1) + 2n_2M_1(G_1).
\end{aligned}$$

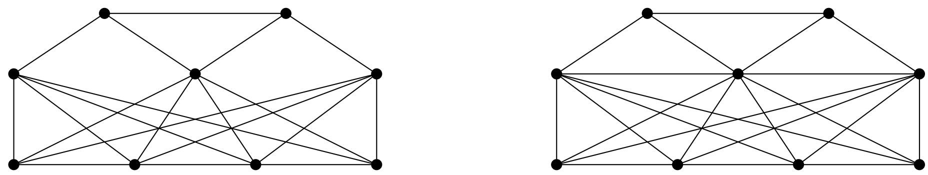
$$\begin{aligned}
P_3 &= \sum_{uv\in E(G_2)} (d_{G_2}(v)+n_1)(d_{G_2}(u) + n_1) \\
&= \sum_{uv\in E(G_2)} d_{G_2}(v)d_{G_2}(u) + n_1 \sum_{uv\in E(G_2)} (d_{G_2}(v)+d_{G_2}(u)) + n_1^2m_2 \\
&= M_2(G_2) + n_1M_1(G_2) + n_1^2m_2.
\end{aligned}$$

$$\begin{aligned}
P_4 &= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} (d_{G_2}(v) + n_1)(d_{G_1}(u) + n_2) \\
&= \sum_{u\in V(G_1)} d_{G_1}(u) \sum_{v\in V(G_2)} d_{G_2}(v) + n_1 \sum_{v\in V(G_2)} \sum_{u\in V(G_1)} d_{G_1}(u) \\
&\quad +n_2 \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} d_{G_2}(v) + n_1^2n_2^2 \\
&= 4m_1m_2 + 2n_1n_2m_1 + 2n_1n_2m_2 + n_1^2n_2^2.
\end{aligned}$$

Combining the above contributions we get the desired result. □

Example 2.6. Applying theorem 2.6, we get

$$\begin{aligned}
(i)\ M_2(P_n\dot{\vee}_QP_m) &= m^2n^2 + 2m^2n + 3mn^2 - n^2 + 12mn - 12m \\
&\quad +22n - 74,\ m \geqslant 3, n \geqslant 4, \\
(ii)\ M_2(P_n\dot{\vee}_QC_m) &= m^2n^2 + 2m^2n + 3mn^2 + 14mn - 12m \\
&\quad +32n - 70,\ m \geqslant 3, n \geqslant 4, \\
(iii)\ M_2(C_n\dot{\vee}_QC_m) &= m^2n^2 + 2m^2n + 3mn^2 + 16mn + 4m + 32n, \\
&\quad m, n \geqslant 3, \\
(iv)\ M_2(C_n\dot{\vee}_QP_m) &= m^2n^2 + 2m^2n + 3mn^2 + 3n^2 + 10mn + 4m + 26n, \\
&\quad m, n \geqslant 3.
\end{aligned}$$



$P_3\dot{\vee}_Q P_4$ $\qquad$ $P_3\dot{\vee}_T P_4$

FIGURE 2. The example of $P_3\dot{\vee}_Q P_4$ and $P_3\dot{\vee}_T P_4$ graphs.

DEFINITION 2.4. *The vertex T-join of two vertex disjoint graphs G_1 and G_2 denoted by $G_1\dot{\vee}_T G_2$ and is obtained from $T(G_1)$ and G_2 by joining each vertex of $V(G_1)$ with every vertex of G_2.*

The degree of the vertices of vertex T-join graph are

$$d_{G_1\dot{\vee}_T G_2}(v) = \begin{cases} 2d_{G_1}(v) + n_2, \ if \ v \in V(G_1) \\ d_{G_2}(v) + n_1, \ if \ v \in V(G_2) \\ d_{G_1}(w) + d_{G_1}(k), \ if \ e = (w,k), e \in I(G_1). \end{cases}$$

THEOREM 2.7. *If G_1 and G_2 be two connected graph then*

$$M_1(G_1\dot{\vee}_T G_2) = HM(G_1) + 4M_1(G_1) + M_1(G_2) + n_1 n_2(n_1 + n_2) + 8n_2 m_1 + 4m_2 n_1.$$

PROOF. From definition of first Zagreb index, we have

$$\begin{aligned} M_1(G_1\dot{\vee}_T G_2) &= \sum_{v\in V(G_1\dot{\vee}_T G_2)} d_{G_1\dot{\vee}_T G_2}(v)^2 \\ &= \sum_{v\in V(G_1)} d_{G_1\dot{\vee}_T G_2}(v)^2 + \sum_{v\in V(G_2)} d_{G_1\dot{\vee}_T G_2}(v)^2 \\ &\quad + \sum_{v\in I(G_1)} d_{G_1\dot{\vee}_T G_2}(v)^2 \\ &= \sum_{v\in V(G_1)} (2d_{G_1}(v) + n_2)^2 + \sum_{v\in V(G_2)} (d_{G_2}(v) + n_1)^2 \\ &\quad + \sum_{u\in I(G_1)} d_{I(G_1)}(u)^2 \end{aligned}$$

$$\begin{aligned}
&= \sum_{v\in V(G_1)} [4d_{G_1}(v)^2 + 4n_2 d_{G_1}(v) + n_2^2] \\
&\quad + \sum_{v\in V(G_2)} [d_{G_2}(v)^2 + 2n_1 d_{G_2}(v) + n_1^2] \\
&\quad + \sum_{uv\in E(G_1)} [d_{G_1}(u) + d_{G_1}(v)]^2 \\
&= 4\sum_{v\in V(G_1)} d_{G_1}(v)^2 + 4n_2 \sum_{v\in V(G_1)} d_{G_1}(v) + n_1n_2^2 + \sum_{v\in V(G_2)} d_{G_2}(v)^2 \\
&\quad +2n_1 \sum_{v\in V(G_2)} d_{G_2}(v) + n_1^2n_2 + \sum_{uv\in E(G_1)} [d_{G_1}(u) + d_{G_1}(v)]^2 \\
&= HM(G_1) + 4M_1(G_1) + M_1(G_2) + n_1n_2(n_1+n_2) + 8n_2m_1 \\
&\quad +4m_2n_1.
\end{aligned}$$

Which is the required expression. □

Example 2.7. Using theorem 2.7, we get

$$\begin{aligned}
(i)\ M_1(P_n\dot{\vee}_T P_m) &= mn(m+n) + 12mn - 4m + 28n - 60, n \geqslant 3, m \geqslant 2, \\
(ii)\ M_1(P_n\dot{\vee}_T C_m) &= mn(m+n) + 12mn - 4m + 32n - 54,\ n \geqslant 3, m \geqslant 2, \\
(iii)\ M_1(C_n\dot{\vee}_T C_m) &= mn(m+n) + 12mn + 4m + 32n,\ n, m \geqslant 3, \\
(iv)\ M_1(C_n\dot{\vee}_T P_m) &= mn(m+n) + 12mn + 4m + 24n - 6,\ n \geqslant 3, m \geqslant 3.
\end{aligned}$$

Theorem 2.8. *If G_1 and G_2 be two connected graph then*

$$\begin{aligned}
M_2(G_1\dot{\vee}_T G_2) &= EM_2(G_1) + 2EM_1(G_1) + 2HM(G_1) + (4n_2+2)M_1(G_1) \\
&\quad +4M_2(G_1) + n_1M_1(G_2) + M_2(G_2) + n_1^2m_2 \\
&\quad +n_2^2m_1 + 8m_1m_2 + 4m_1n_1n_2 + 2m_2n_1n_2 + n_1^2n_2^2 - 4m_1.
\end{aligned}$$

Proof. From definition of second Zagreb index, we have

$$\begin{aligned}
M_2(G_1\dot{\vee}_T G_2) &= \sum_{uv\in E(G_1\dot{\vee}_T G_2)} d_{G_1\dot{\vee}_T G_2}(u) d_{G_1\dot{\vee}_T G_2}(v) \\
&= \sum_{u,v\in I(G_1)} d_{G_1\dot{\vee}_T G_2}(u) d_{G_1\dot{\vee}_T G_2}(v) \\
&\quad + \sum_{u\in V(G_1), v\in I(G_1)} d_{G_1\dot{\vee}_T G_2}(u) d_{G_1\dot{\vee}_T G_2}(v) \\
&\quad + \sum_{uv\in E(G_2)} d_{G_1\dot{\vee}_T G_2}(u) d_{G_1\dot{\vee}_T G_2}(v) \\
&\quad + \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} d_{G_1\dot{\vee}_T G_2}(u) d_{G_1\dot{\vee}_T G_2}(v) \\
&\quad + \sum_{uv\in E(G_1)} d_{G_1\dot{\vee}_T G_2}(u) d_{G_1\dot{\vee}_T G_2}(v)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{u,v\in I(G_1)} d_{I(G_1)}(u)d_{I(G_1)}(v) \\
&\quad + \sum_{u\in V(G_1),v\in I(G_1)} d_{G_1}(u)d_{I(G_1)}(v) \\
&\quad + \sum_{uv\in E(G_2)} (d_{G_2}(u)+n_1)(d_{G_2}(v)+n_1) \\
&\quad + \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} (d_{G_2}(v)+n_1)(2d_{G_1}(u)+n_2) \\
&\quad + \sum_{uv\in E(G_1)} (2d_{G_1}(u)+n_2)(2d_{G_1}(v)+n_2) \\
&= D_1+D_2+D_3+D_4+D_5.
\end{aligned}$$

Now,

$$\begin{aligned}
D_1 &= \sum_{u,v\in I(G_1)} d_{I(G_1)}(u)d_{I(G_1)}(v) \\
&= \sum_{e,f\in L(G_1),e=uv,f=vw} (d_{L(G_1)}(e)+2)(d_{L(G_1)}(f)+2) \\
&= \sum_{e,f\in L(G_1)} d_{L(G_1)}(e)d_{L(G_1)}(f) + 2\sum_{e,f\in L(G_1)} (d_{L(G_1)}(e)+d_{L(G_1)}(f)) \\
&\quad +4\,|E(L(G_1))| \\
&= M_2(L(G_1)) + 2M_1(L(G_1)) + 4\,|E(L(G_1)| \\
&= M_2(L(G_1)) + 2M_1(L(G_1)) + 4[\frac{M_1(G_1)}{2} - m_1] \\
&= EM_2(G_1) + 2EM_1(G_1) + 2M_1(G_1) - 4m_1.
\end{aligned}$$

$$\begin{aligned}
D_2 &= \sum_{u\in V(G_1),v\in I(G_1)} d_{G_1}(u)d_{I(G_1)}(v) \\
&= \sum_{uv\in E(G_1)} (2d_{G_1}(u)+n_2+2d_{G_1}(v)+n_2)(d_{G_1}(u)+d_{G_1}(v)) \\
&= \sum_{uv\in E(G_1)} [2(d_{G_1}(u)+d_{G_1}(v))^2 + 2n_2(d_{G_1}(u)+d_{G_1}(v)) \\
&= 2\sum_{uv\in E(G_1)} (d_{G_1}(u)+d_{G_1}(v))^2 + 2n_2\sum_{uv\in E(G_1)} (d_{G_1}(u)+d_{G_1}(v)) \\
&= 2HM(G_1) + 2n_2M_1(G_1).
\end{aligned}$$

$$\begin{aligned}
D_3 &= \sum_{uv\in E(G_2)} (d_{G_2}(u)+n_1)(d_{G_2}(v)+n_1)\\
&= \sum_{uv\in E(G_2)} d_{G_2}(u)d_{G_2}(v) + n_1 \sum_{uv\in E(G_2)} (d_{G_2}(u)+d_{G_2}(v)) + n_1^2 m_2\\
&= M_2(G_2) + n_1 M_1(G_2) + n_1^2 m_2.\\
D_4 &= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} (d_{G_2}(v)+n_1)(2d_{G_1}(u)+n_2)\\
&= \sum_{u\in V(G_1)} 2d_{G_1}(u) \sum_{v\in V(G_2)} d_{G_2}(v) + n_1 \sum_{v\in V(G_2)} \sum_{u\in V(G_1)} 2d_{G_1}(u)\\
&\quad + n_2 \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} d_{G_2}(v) + n_1^2 n_2^2\\
&= 8m_1 m_2 + 4n_1 n_2 m_1 + 2n_1 n_2 m_2 + n_1^2 n_2^2.\\
D_5 &= \sum_{uv\in E(G_1)} (2d_{G_1}(u)+n_2)(2d_{G_1}(v)+n_2)\\
&= 4 \sum_{uv\in E(G_1)} d_{G_1}(u)d_{G_1}(v) + 2n_2 \sum_{uv\in E(G_1)} (d_{G_1}(u)+d_{G_1}(v)) + n_2^2 m_1\\
&= 4M_2(G_1) + 2n_2 M_1(G_1) + n_2^2 m_1.
\end{aligned}$$

Combining the above contributions we get the required result as in theorem 2.8 □

Example 2.8. Applying theorem 2.8, we get

$$\begin{aligned}
(i)\ M_2(P_n \dot{\vee}_T P_m) &= m^2n^2 + m^2(n-1) + n^2(m-1) + 2mn(2n+m-3)\\
&\quad +28mn - 28m + 50n - 132,\ n \geqslant 4, m \geqslant 3,\\
(ii)\ M_2(P_n \dot{\vee}_T C_m) &= m^2n^2 + m^2(n-1) + n^2m + 2mn(2n+m-2) + 28mn\\
&\quad -28m + 64n - 132,\ n \geqslant 4, m \geqslant 3,\\
(iii)\ M_2(C_n \dot{\vee}_T C_m) &= m^2n^2 + 3m^2n + 5n^2m + 28mn + 4m + 64n,\ n, m \geqslant 3,\\
(iv) M_2(C_n \dot{\vee}_T P_m) &= n^2(m^2-1) + 3m^2n + 5n^2m + 26mn + 4m\\
&\quad +50n - 8,\ m \geqslant 3, n \geqslant 3.
\end{aligned}$$

2.2. Edge F-join of Graphs. In the following, we calculate explicit expressions of edge F-join of graphs where $F = \{S, R, Q, T\}$ respectively. First, we consider edge S-join of graphs.

Definition 2.5. *The edge S-join of two vertex disjoint graphs G_1 and G_2 denoted by $G_1 \veebar_S G_2$ and is obtained from $S(G_1)$ and G_2 by joining each vertex of $I(G_1)$ with every vertex of G_2.*

The degree of the vertices of edge S-join graph are

$$d_{G_1 \veebar_S G_2}(v) = \begin{cases} d_{G_1}(v),\ if\ v \in V(G_1)\\ d_{G_2}(v) + m_1,\ if\ v \in V(G_2)\\ 2 + n_2,\ if\ v \in I(G_1). \end{cases}$$

THEOREM 2.9. *If G_1 and G_2 be two connected graph then*

$$M_1(G_1 \underline{\vee}_S G_2) = M_1(G_1) + M_1(G_2) + m_1\{(n_2+2)^2 + 4m_2 + n_2m_1\}.$$

PROOF. From definition of first Zagreb index, we have

$$\begin{aligned}
M_1(G_1 \underline{\vee}_S G_2) &= \sum_{v \in V(G_1 \underline{\vee}_S G_2)} d_{G_1 \underline{\vee}_S G_2}(v)^2 \\
&= \sum_{v \in V(G_1)} d_{G_1 \underline{\vee}_S G_2}(v)^2 + \sum_{v \in I(G_1)} d_{G_1 \underline{\vee}_S G_2}(v)^2 \\
&\quad + \sum_{v \in V(G_2)} d_{G_1 \underline{\vee}_S G_2}(v)^2 \\
&= \sum_{v \in V(G_1)} d_{G_1}(v)^2 + \sum_{v \in I(G_1)} (2+n_2)^2 + \sum_{v \in V(G_2)} (d_{G_2}(v) + m_1)^2 \\
&= \sum_{v \in V(G_1)} d_{G_1}(v)^2 + \sum_{v \in I(G_1)} (2+n_2)^2 + \sum_{v \in V(G_2)} d_{G_2}(v)^2 \\
&\quad + 2m_1 \sum_{v \in V(G_2)} d_{G_2}(v) + m_1^2 n_2 \\
&= M_1(G_1) + m_1(n_2+2)^2 + M_1(G_2) + 4m_1m_2 + n_2m_1^2 \\
&= M_1(G_1) + M_1(G_2) + m_1\{(n_2+2)^2 + 4m_2 + n_2m_1\}.
\end{aligned}$$

Which is the desired result. □

EXAMPLE 2.9. Applying theorem 2.9, we get

$$\begin{aligned}
(i)\ M_1(P_n \underline{\vee}_S P_m) &= mn(m+n+6) - m^2 - 3m + 4n - 12, \\
(ii)\ M_1(P_n \underline{\vee}_S C_m) &= mn(m+n+6) - m^2 - 3m + 8n - 10, \\
(iii)\ M_1(C_n \underline{\vee}_S C_m) &= n(m^2 + 8m + mn + 4) + 4m + 4n, \\
(iv)\ M_1(C_n \underline{\vee}_S P_m) &= n\{(m+2)^2 + 4(m-1) + mn\} + 4m + 4n - 6.
\end{aligned}$$

THEOREM 2.10. *If G_1 and G_2 be two connected graph then*

$$\begin{aligned}
M_2(G_1 \underline{\vee}_S G_2) &= (n_2+2)M_1(G_1) + m_1M_1(G_2) + M_2(G_2) + m_1\{m_1m_2 \\
&\quad + (2m_2 + m_1n_2)(2+n_2)\}.
\end{aligned}$$

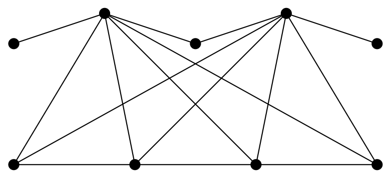
PROOF. By definition of second Zagreb index, we have

$$\begin{aligned}
M_2(G_1\underline{\vee}_S G_2) &= \sum_{uv\in E(G_1\underline{\vee}_S G_2)} d_{G_1\underline{\vee}_S G_2}(u)d_{G_1\underline{\vee}_S G_2}(v)\\
&= \sum_{uv\in E(S(G_1)} d_{G_1\underline{\vee}_S G_2}(u)d_{G_1\underline{\vee}_S G_2}(v)\\
&\quad + \sum_{uv\in E(G_2)} d_{G_1\underline{\vee}_S G_2}(u)d_{G_1\underline{\vee}_S G_2}(v)\\
&\quad + \sum_{u\in I(G_1),v\in V(G_2)} d_{G_1\underline{\vee}_S G_2}(u)d_{G_1\underline{\vee}_S G_2}(v)\\
&= \sum_{v\in V(G_1)} d_{G_1}(v)\{d_{G_1}(v).(n_2+2)\}\\
&\quad + \sum_{uv\in E(G_2)} (d_{G_2}(u)+m_1)(d_{G_2}(v)+m_1)\\
&\quad + \sum_{u\in I(G_1),v\in V(G_2)} (2+n_2)(d_{G_2}(v)+m_1)\\
&= (2+n_2)\sum_{v\in V(G_1)} d_{G_1}(v)^2 + m_1(2+n_2)\sum_{v\in V(G_2)}(d_{G_2}(v)+m_1)\\
&\quad + \sum_{uv\in E(G_2)}\{d_{G_2}(u)d_{G_2}(v)+m_1(d_{G_2}(u)+d_{G_2}(v))+m_1^2\}\\
&= (2+n_2)\sum_{v\in V(G_1)} d_{G_1}(v)^2 + \sum_{uv\in E(G_2)} d_{G_2}(u)d_{G_2}(v)+m_2m_1^2\\
&\quad +m_1\sum_{uv\in E(G_2)}(d_{G_2}(u)+d_{G_2}(v))+m_1(2+n_2)\sum_{v\in V(G_2)} d_{G_2}(v)\\
&\quad +{m_1}^2n_2(2+n_2)\\
&= (n_2+2)M_1(G_1)+m_1M_1(G_2)+M_2(G_2)\\
&\quad +m_1\{m_1m_2+(2m_2+m_1n_2)(2+n_2)\}.
\end{aligned}$$

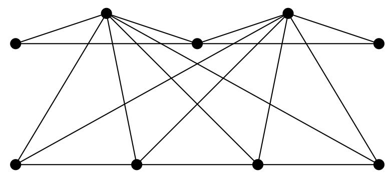
Which is the required result. □

EXAMPLE 2.10. Using theorem 2.10, we get

$$\begin{aligned}
(i)\ M_2(P_n\underline{\vee}_S P_m) &= (m-1)(n-1)^2+(m+2)(n-1)\{2(m-1)+m(n-1)\}\\
&\quad +8mn-6m+2n-14,\\
(ii)\ M_2(P_n\underline{\vee}_S C_m) &= m(m+2)(n-1)(n+1)+m(n-1)^2\\
&\quad +8mn-6m+8n-12,\\
(iii)\ M_2(C_n\underline{\vee}_S C_m) &= m^2n^2+2m^2n+3mn^2+12mn+4m+8n,\\
(iv)\ M_2(C_n\underline{\vee}_S P_m) &= m^2n^2+2m^2n+3mn^2-n^2+10mn+4m-2n-8.
\end{aligned}$$



$P_3 \underline{\vee}_S P_4$



$P_3 \underline{\vee}_R P_4$

FIGURE 3. The example of $P_3 \underline{\vee}_S P_4$ and $P_3 \underline{\vee}_R P_4$ graphs.

DEFINITION 2.6. *The edge R-join of two vertex disjoint graphs G_1 and G_2 denoted by $G_1 \underline{\vee}_R G_2$ and is obtained from $R(G_1)$ and G_2 by joining each vertex of $I(G_1)$ with every vertex of G_2.*

The degree of the vertices of edge R-join graph are

$$d_{G_1 \underline{\vee}_R G_2}(v) = \begin{cases} 2d_{G_1}(v), \ if \ v \in V(G_1) \\ d_{G_2}(v) + m_1, \ if \ v \in V(G_2) \\ 2 + n_2, \ if \ v \in I(G_1). \end{cases}$$

THEOREM 2.11. *If G_1 and G_2 be two connected graph*

$$M_1(G_1 \underline{\vee}_R G_2) = 4M_1(G_1) + M_1(G_2) + 4m_1m_2 + {m_1}^2 n_2 + (n_2+2)^2 m_1.$$

PROOF. From definition of first Zagreb index, we have

$$\begin{aligned} M_1(G_1 \underline{\vee}_R G_2) &= \sum_{v \in V(G_1 \underline{\vee}_R G_2)} d_{G_1 \underline{\vee}_R G_2}(v)^2 \\ &= \sum_{v \in V(G_1)} (2d_{G_1}(v))^2 + \sum_{v \in V(G_2)} (d_{G_2}(v) + m_1)^2 \\ &\quad + \sum_{v \in I(G_1)} (n_2+2)^2 \\ &= \sum_{v \in V(G_1)} 4d_{G_1}(v)^2 + \sum_{v \in V(G_2)} [d_{G_2}(v)^2 + 2m_1 d_{G_2}(v) + {m_1}^2] \\ &\quad + m_1(n_2+2)^2 \\ &= 4 \sum_{v \in V(G_1)} d_{G_1}(v)^2 + \sum_{v \in V(G_2)} d_{G_2}(v)^2 + 2m_1 \sum_{v \in V(G_2)} d_{G_2}(v) \\ &\quad + n_2 {m_1}^2 + m_1(n_2+2)^2 \\ &= 4M_1(G_1) + M_1(G_2) + 4m_1m_2 + {m_1}^2 n_2 + (n_2+2)^2 m_1. \end{aligned}$$

This is the required expression. $\square$

EXAMPLE 2.11. Applying theorem 2.11, we get

$$\begin{aligned}
(i)\ M_1(P_n\underline{\vee}_R P_m) &= (n-1)\{(m+2)^2+m(n-1)\}+4mn+12n-26,\ m,n\geqslant 2,\\
(ii)\ M_1(P_n\underline{\vee}_R C_m) &= (n-1)\{(m+2)^2+m(n-1)\}+4mn+16n-24,\\
&\quad m\geqslant 2, n\geqslant 3,\\
(iii)\ M_1(C_n\underline{\vee}_R C_m) &= n\{(m+2)^2+mn\}+4mn+4m+16n,\ m\geqslant 3, n\geqslant 3,\\
(iv)\ M_1(C_n\underline{\vee}_R P_m) &= n(m+2)^2+mn^2+4mn+4m+12n-6,\ m\geqslant 2, n\geqslant 3.
\end{aligned}$$

THEOREM 2.12. *If G_1 and G_2 be two connected graph then*

$$\begin{aligned}
M_2(G_1\underline{\vee}_R G_2) &= 4M_2(G_1)+M_2(G_2)+2(n_2+2)M_1(G_1)+m_1M_1(G_2)\\
&\quad +m_1(2+n_2)(m_1n_2+2m_2)+m_1^2m_2.
\end{aligned}$$

PROOF. From definition of second Zagreb index, we have

$$\begin{aligned}
M_2(G_1\underline{\vee}_R G_2) &= \sum_{uv\in E(G_1\underline{\vee}_R G_2)} d_{G_1\underline{\vee}_R G_2}(u)\, d_{G_1\underline{\vee}_R G_2}(v)\\
&= \sum_{uv\in E(G_1)} d_{G_1\underline{\vee}_R G_2}(u) d_{G_1\underline{\vee}_R G_2}(v)\\
&\quad + \sum_{u\in I(G_1), v\in V(G_1)} d_{G_1\underline{\vee}_R G_2}(u) d_{G_1\underline{\vee}_R G_2}(v)\\
&\quad + \sum_{u\in I(G_1), v\in V(G_2)} d_{G_1\underline{\vee}_R G_2}(u) d_{G_1\underline{\vee}_R G_2}(v)\\
&\quad + \sum_{uv\in E(G_2)} d_{G_1\underline{\vee}_R G_2}(u) d_{G_1\underline{\vee}_R G_2}(v)\\
&= \sum_{uv\in E(G_1)} 2d_{G_1}(u).2d_{G_1}(v) + \sum_{u\in I(G_1), v\in V(G_1)} d_{I(G_1)}(u)2d_{G_1}(v)\\
&\quad + \sum_{v\in V(G_2)} m_1(2+n_2)(d_{G_2}(v)+m_1)\\
&\quad + \sum_{uv\in E(G_2)} (d_{G_2}(u)+m_1)(d_{G_2}(v)+m_1)\\
&= 4\sum_{uv\in E(G_1)} d_{G_1}(u).d_{G_1}(v) + 2(2+n_2)\sum_{v\in V(G_1)} d_{G_1}(v).d_{G_1}(v)\\
&\quad +m_1(2+n_2)\sum_{v\in V(G_2)} d_{G_2}(v) + {m_1}^2 n_2(2+n_2) + m_1^2m_2\\
&\quad + \sum_{uv\in E(G_2)} d_{G_2}(u)d_{G_2}(v) + m_1\sum_{uv\in E(G_2)} (d_{G_2}(v)+d_{G_2}(u))\\
&= 4M_2(G_1)+M_2(G_2)+2(n_2+2)M_1(G_1)+m_1M_1(G_2)\\
&\quad +m_1(2+n_2)(m_1n_2+2m_2)+m_1^2m_2.
\end{aligned}$$

Which is the required result. $\square$

EXAMPLE 2.12. Using theorem 2.12, we get

$$\begin{aligned}
(i)\ M_2(P_n \underline{\vee}_R P_m) &= (n-1)(m+2)(mn+m-2)+(n-1)^2(m-1)+12mn \\
&\quad -12m+26n-58,\ m,n \geqslant 3, \\
(ii)\ M_2(P_n \underline{\vee}_R C_m) &= (n-1)(m+2)(mn+m)+(n-1)^2 m+12mn-12m \\
&\quad +32n-56,\ m,n \geqslant 3, \\
(iii)\ M_2(C_n \underline{\vee}_R C_m) &= mn(m+2)(n+2)+mn^2+12mn+4m+32n, \\
&\quad m,n \geqslant 3, \\
(iv)\ M_2(P_n \underline{\vee}_R C_m) &= n(m+2)(mn+2m-2)+n^2(m-1)+12mn \\
&\quad +4m+26n-8,\ m,n \geqslant 3.
\end{aligned}$$

DEFINITION 2.7. *The edge Q-join of two vertex disjoint graphs G_1 and G_2 denoted by $G_1 \underline{\vee}_Q G_2$ and is obtained from $Q(G_1)$ and G_2 by joining each vertex of $I(G_1)$ with every vertex of G_2.*

The degree of the vertices of edge Q-join graph are

$$d_{G_1 \underline{\vee}_Q G_2}(v) = \begin{cases} d_{G_1}(v),\ if\ v \in V(G_1) \\ d_{G_2}(v)+m_1,\ if\ v \in V(G_2) \\ d_{G_1}(w)+d_{G_1}(k)+n_2,\ if\ e=(w,k), e \in I(G_1). \end{cases}$$

THEOREM 2.13. *If G_1 and G_2 be two connected graph then*

$$M_1(G_1 \underline{\vee}_Q G_2) = M_1(G_1)+M_1(G_2)+HM(G_1)+2n_2M_1(G_1)+4m_1m_2+{m_1}^2n_2+n_2^2m_1.$$

PROOF. From definition of first Zagreb index, we have

$$\begin{aligned}
M_1(G_1 \underline{\vee}_Q G_2) &= \sum_{v \in V(G_1 \underline{\vee}_Q G_2)} d_{G_1 \underline{\vee}_Q G_2}(v)^2 \\
&= \sum_{v \in V(G_1)} d_{G_1 \underline{\vee}_Q G_2}(v)^2 + \sum_{v \in V(G_2)} d_{G_1 \underline{\vee}_Q G_2}(v)^2 \\
&\quad + \sum_{v \in I(G_1)} d_{G_1 \underline{\vee}_Q G_2}(v)^2 \\
&= \sum_{v \in V(G_1)} d_{G_1}(v)^2 + \sum_{v \in V(G_2)} (d_{G_2}(v)+m_1)^2 \\
&\quad + \sum_{uv \in E(G_1)} (d_{G_1}(u)+d_{G_1}(v)+n_2)^2 \\
&= \sum_{v \in V(G_1)} d_{G_1}(v)^2 + \sum_{v \in V(G_2)} [d_{G_2}(v)^2+2m_1d_{G_2}(v)+{m_1}^2] \\
&\quad + \sum_{uv \in E(G_1)} [(d_{G_1}(u)+d_{G_1}(v))^2+2n_2(d_{G_1}(u)+d_{G_1}(v))+n_2^2]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{v\in V(G_1)} d_{G_1}(v)^2 + \sum_{v\in V(G_2)} d_{G_2}(v)^2 + 2m_1 \sum_{v\in V(G_2)} d_{G_2}(v) \\
&\quad + n_2{m_1}^2 + \sum_{uv\in E(G_1)} (d_{G_1}(u)+d_{G_1}(v))^2 \\
&\quad + 2n_2 \sum_{uv\in E(G_1)} (d_{G_1}(u)+d_{G_1}(v)) + m_1 n_2^2 \\
&= M_1(G_1)+M_1(G_2)+HM(G_1)+2n_2M_1(G_1)+4m_1m_2 \\
&\quad + {m_1}^2 n_2 + n_2^2 m_1.
\end{aligned}$$

Which is the desired result. □

EXAMPLE 2.13. By theorem 2.13, we get

$$\begin{aligned}
(i)\ M_1(P_n \underline{\vee}_Q P_m) &= m(n-1)(m+n-1)+12mn-12m+16n-38,\ m,n\geqslant 2,\\
(ii)\ M_1(P_n \underline{\vee}_Q C_m) &= m(n-1)(m+n-1)+12mn-12m+20n-36,\\
&\quad\ m\geqslant 3, n\geqslant 2,\\
(iii)\ M_1(C_n \underline{\vee}_Q C_m) &= mn(m+n)+12mn+4m+20n,\ m,n\geqslant 3,\\
(iv)\ M_1(C_n \underline{\vee}_Q P_m) &= mn(m+n)+12mn+4m+16n-6,\ m,n\geqslant 3.
\end{aligned}$$

THEOREM 2.14. *If G_1 and G_2 be two connected graph then*

$$\begin{aligned}
M_2(G_1 \underline{\vee}_Q G_2) &= EM_2(G_1)+(2+n_2)EM_1(G_1)+(2+n_2)^2\frac{M_1(G_1)}{2}\\
&\quad + HM(G_1)+n_2M_1(G_1)+2m_2M_1(G_1)+m_1n_2M_1(G_1)\\
&\quad + M_2(G_2)+m_1M_1(G_2)+2n_2m_2m_1+m_1^2n_2^2\\
&\quad + m_1^2m_2 - m_1(2+n_2)^2.
\end{aligned}$$

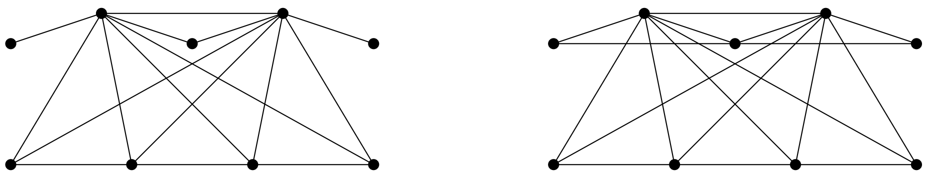
PROOF. From definition of second Zagreb index, we have

$$\begin{aligned}
M_2(G_1 \underline{\vee}_Q G_2) &= \sum_{uv\in E(G_1 \underline{\vee}_Q G_2)} d_{G_1 \underline{\vee}_Q G_2}(u)\, d_{G_1 \underline{\vee}_Q G_2}(v)\\
&= \sum_{uv\in E(I(G_1))} d_{G_1 \underline{\vee}_Q G_2}(u)\, d_{G_1 \underline{\vee}_Q G_2}(v)\\
&\quad + \sum_{u\in V(G_1), v\in I(G_1)} d_{G_1 \underline{\vee}_Q G_2}(u)\, d_{G_1 \underline{\vee}_Q G_2}(v)\\
&\quad + \sum_{v\in V(G_2)} \sum_{u\in I(G_1)} d_{G_1 \underline{\vee}_Q G_2}(u)\, d_{G_1 \underline{\vee}_Q G_2}(v)\\
&\quad + \sum_{uv\in E(G_2)} d_{G_1 \underline{\vee}_Q G_2}(u)\, d_{G_1 \underline{\vee}_Q G_2}(v)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{uv,vw\in E(G_1)} (d_{G_1}(u)+d_{G_1}(v)+n_2)(d_{G_1}(v)+d_{G_1}(w)+n_2)\\
&\quad + \sum_{uv\in E(G_1)} (d_{G_1}(u)+d_{G_1}(v))(d_{G_1}(u)+d_{G_1}(v)+n_2)\\
&\quad + \sum_{v\in V(G_2)} \sum_{u\in I(G_1)} (d_{I(G_1)}(u)+n_2)(d_{G_2}(v)+m_1)\\
&\quad + \sum_{uv\in E(G_2)} (d_{G_2}(u)+m_1)(d_{G_2}(v)+m_1)\\
&= A_1+A_2+A_3+A_4.
\end{aligned}
$$

Now,

$$
\begin{aligned}
A_1 &= \sum_{uv,vw\in E(G_1)} (d_{G_1}(u)+d_{G_1}(v)+n_2)(d_{G_1}(v)+d_{G_1}(w)+n_2)\\
&= \sum_{e,f\in L(G_1),e=uv,f=vw} (d_{L(G_1)}(e)+2+n_2)(d_{L(G_1)}(f)+2+n_2)\\
&= \sum_{e,f\in E(L(G_1))} [d_{L(G_1)}(e)d_{L(G_1)}(f)+(2+n_2)(d_{L(G_1)}(e)+d_{L(G_1)}(f))\\
&\quad +(2+n_2)^2]\\
&= M_2(L(G_1))+(2+n_2)M_1(L(G_1))+(2+n_2)^2\,|E(L(G_1)|\\
&= EM_2(G_1)+(2+n_2)EM_1(G_1)+(2+n_2)^2[M_1(G_1)/2-m_1].\\
A_2 &= \sum_{uv\in E(G_1)} (d_{G_1}(u)+d_{G_1}(v))(d_{G_1}(u)+d_{G_1}(v)+n_2)\\
&= \sum_{uv\in E(G_1)} (d_{G_1}(u)+d_{G_1}(v))^2+n_2\sum_{uv\in E(G_1)} (d_{G_1}(u)+d_{G_1}(v))\\
&= HM(G_1)+n_2M_1(G_1).\\
A_3 &= \sum_{v\in V(G_2)} \sum_{u\in I(G_1)} (d_{I(G_1)}(u)+n_2)(d_{G_2}(v)+m_1)\\
&= \sum_{uv\in E(G_1)} \sum_{w\in V(G_2)} (d_{G_1}(u)+d_{G_1}(v)+n_2)(d_{G_2}(w)+m_1)\\
&= \sum_{uv\in E(G_1)} (d_{G_1}(u)+d_{G_1}(v)) \sum_{w\in V(G_2)} d_{G_2}(w)+m_1n_2\sum_{w\in V(G_2)} d_{G_2}(w)\\
&\quad +m_1n_2\sum_{uv\in E(G_1)} (d_{G_1}(u)+d_{G_1}(v))+n_2^2m_1^2\\
&= 2m_2M_1(G_1)+2n_2m_2m_1+m_1n_2M_1(G_1)+m_1^2n_2^2.
\end{aligned}
$$



$P_3 \underline{\vee}_Q P_4$ $\quad$ $P_3 \underline{\vee}_T P_4$

FIGURE 4. The example of $P_3 \underline{\vee}_Q P_4$ and $P_3 \underline{\vee}_T P_4$ graphs.

$$\begin{aligned} A_4 &= \sum_{uv \in E(G_2)} (d_{G_2}(u) + m_1)(d_{G_2}(v) + m_1) \\ &= \sum_{uv \in E(G_2)} d_{G_2}(v) d_{G_2}(u) + m_1 \sum_{uv \in E(G_2)} (d_{G_2}(v) + d_{G_2}(u)) + m_1^2 m_2 \\ &= M_2(G_2) + m_1 M_1(G_2) + m_1^2 m_2. \end{aligned}$$

Combining the above contributions we get the desired expression as follows in theorem 2.14. □

EXAMPLE 2.14. From theorem 14, we get

$$\begin{aligned} (i)\ M_2(P_n \underline{\vee}_Q P_m) &= m^2n^2 + m^2n + 5mn^2 - 3m^2 - n^2 + 10mn - 27m \\ &\quad +20n - 61,\ n \geqslant 4, m \geqslant 3. \\ (ii)\ M_2(P_n \underline{\vee}_Q C_m) &= m^2n^2 + m^2n + 5mn^2 - 3m^2 + 12mn - 29m \\ &\quad +32n - 70,\ n \geqslant 4, m \geqslant 3. \\ (iii)\ M_2(C_n \underline{\vee}_Q C_m) &= m^2n^2 + 3m^2n + 5mn^2 + 24mn \\ &\quad +4m + 32n,\ n \geqslant 4, m \geqslant 3. \\ (iv)\ M_2(C_n \underline{\vee}_Q P_m) &= m^2n^2 + 3m^2n + 5mn^2 - n^2 + 22mn \\ &\quad +4m - 18n - 8,\ n \geqslant 4, m \geqslant 3. \end{aligned}$$

DEFINITION 2.8. *The edge T-join of two vertex disjoint graphs G_1 and G_2 denoted by $G_1 \underline{\vee}_T G_2$ and is obtained from $T(G_1)$ and G_2 by joining each vertex of $I(G_1)$ with every vertex of G_2.*

The degree of the vertices of edge T-join graph are

$$d_{G_1 \underline{\vee}_T G_2}(v) = \begin{cases} 2d_{G_1}(v),\ if\ v \in V(G_1) \\ d_{G_2}(v) + m_1,\ if\ v \in V(G_2) \\ d_{G_1}(w) + d_{G_1}(k) + n_2,\ if\ e = (w,k), e \in I(G_1). \end{cases}$$

THEOREM 2.15. *If G_1 and G_2 be two connected graph then*

$$M_1(G_1 \underline{\vee}_T G_2) = (2n_2 + 4)M_1(G_1) + HM(G_1) + M_1(G_2) + m_1^2 n_2 + n_2^2 m_1 + 4m_1 m_2.$$

PROOF. From definition of first Zagreb index, we have

$$
\begin{aligned}
M_1(G_1 \veebar_T G_2) &= \sum_{v \in (G_1 \veebar_T G_2)} d_{G_1 \veebar_T G_2}(v)^2 \\
&= \sum_{v \in V(G_1)} d_{G_1 \veebar_T G_2}(v)^2 + \sum_{v \in V(G_2)} d_{G_1 \veebar_T G_2}(v)^2 \\
&\quad + \sum_{v \in I(G_1)} d_{G_1 \veebar_T G_2}(v)^2 \\
&= \sum_{v \in V(G_1)} (2d_{G_1}(v))^2 + \sum_{v \in V(G_2)} (d_{G_2}(v) + m_1)^2 \\
&\quad + \sum_{uv \in E(G_1)} (d_{G_1}(u) + d_{G_1}(v) + n_2)^2 \\
&= \sum_{v \in V(G_1)} (2d_{G_1}(v))^2 + \sum_{v \in V(G_2)} [d_{G_2}(v)^2 + 2m_1 d_{G_2}(v) + {m_1}^2] \\
&\quad + \sum_{uv \in E(G_1)} [(d_{G_1}(u) + d_{G_1}(v))^2 + 2n_2(d_{G_1}(u) + d_{G_1}(v)) + n_2^2] \\
&= 4 \sum_{v \in V(G_1)} d_{G_1}(v)^2 + \sum_{v \in V(G_2)} d_{G_2}(v)^2 + 2m_1 \sum_{v \in V(G_2)} d_{G_2}(v) \\
&\quad + n_2 {m_1}^2 + \sum_{uv \in E(G_1)} [d_{G_1}(u) + d_{G_1}(v)]^2 \\
&\quad + 2n_2 \sum_{uv \in E(G_1)} (d_{G_1}(u) + d_{G_1}(v)) + {n_2}^2 m_1 \\
&= (2n_2 + 4)M_1(G_1) + HM(G_1) + M_1(G_2) + m_1^2 n_2 \\
&\quad + n_2^2 m_1 + 4m_1 m_2.
\end{aligned}
$$

Which is the desired result. □

EXAMPLE 2.15. Using theorem 2.15, we get

$$
\begin{aligned}
(i)\ M_1(P_n \veebar_T P_m) &= m^2(n-1) + m(n-1)^2 + 12mn - 12m \\
&\quad + 28n - 56,\ n \geqslant 3, m \geqslant 2, \\
(ii)\ M_1(P_n \veebar_T C_m) &= m^2(n-1) + m(n-1)^2 + 12mn - 12m \\
&\quad + 32n - 54,\ n \geqslant 3, m \geqslant 3, \\
(iv)\ M_1(C_n \veebar_T C_m) &= m^2 n + mn^2 + 12mn + 4m + 32n,\ m, n \geqslant 3, \\
(iv)\ M_1(C_n \veebar_T P_m) &= m^2 n + mn^2 + 12mn + 4m + 28n - 6,\ n \geqslant 3, m \geqslant 2.
\end{aligned}
$$

THEOREM 2.16. *If G_1 and G_2 be two connected graph then*

$$\begin{aligned} M_2(G_1\underline{\vee}_T G_2) &= EM_2(G_1)+(2+n_2)EM_1(G_1)+2HM(G_1) \\ &+\{4m_2+4n_2+2m_1n_2+(2+n_2)^2\}\frac{M_1(G_1)}{2}+4M_2(G_1) \\ &+M_2(G_2)+m_1M_1(G_2)+m_1^2(m_2+n_2^2) \\ &+2m_1m_2n_2-m_1(2+n_2)^2. \end{aligned}$$

PROOF. By definition of second Zagreb index, we have

$$\begin{aligned} M_2(G_1\underline{\vee}_T G_2) &= \sum_{uv\in E(G_1\underline{\vee}_T G_2)} d_{G_1\underline{\vee}_T G_2}(u)d_{G_1\underline{\vee}_T G_2}(v) \\ &= \sum_{uv\in E(I(G_1))} d_{G_1\underline{\vee}_T G_2}(u)d_{G_1\underline{\vee}_T G_2}(v) \\ &+ \sum_{u\in V(G_1),v\in I(G_1)} d_{G_1\underline{\vee}_T G_2}(u)d_{G_1\underline{\vee}_T G_2}(v) \\ &+ \sum_{uv\in E(G_1)} d_{G_1\underline{\vee}_T G_2}(u)d_{G_1\underline{\vee}_T G_2}(v) \\ &+ \sum_{v\in V(G_2)}\sum_{u\in I(G_1)} d_{G_1\underline{\vee}_T G_2}(u)d_{G_1\underline{\vee}_T G_2}(v) \\ &+ \sum_{uv\in E(G_2)} d_{G_1\underline{\vee}_T G_2}(u)d_{G_1\underline{\vee}_T G_2}(v) \\ &= \sum_{uv\in E(I(G_1))} d_{I(G_1)}(u)d_{I(G_1)}(v) + \sum_{u\in V(G_1),v\in I(G_1)} d_{G_1}(u)d_{I(G_1)}(v) \\ &+ \sum_{uv\in E(G_1)} 2d_{G_1}(u)2d_{G_1}(v) \\ &+ \sum_{v\in V(G_2)}\sum_{u\in I(G_1)} (d_{I(G_1)}(u)+n_2)(d_{G_2}(v)+m_1) \\ &+ \sum_{uv\in E(G_2)} (d_{G_2}(u)+m_1)(d_{G_2}(v)+m_1) \\ &= Q_1+Q_2+Q_3+Q_4+Q_5. \end{aligned}$$

Now,

$$\begin{aligned} Q_1 &= \sum_{uv\in E(I(G_1))} d_{I(G_1)}(u)d_{I(G_1)}(v) \\ &= \sum_{uv,vw\in E(G_1)} (d_{G_1}(u)+d_{G_1}(v)+n_2)(d_{G_1}(v)+d_{G_1}(w)+n_2) \end{aligned}$$

$$\begin{aligned}
&= \sum_{e,f\in E(L(G_1)),e=uv,f=vw} (d_{L(G_1)}(e)+2+n_2)(d_{L(G_1)}(f)+2+n_2) \\
&= \sum_{e,f\in E(L(G_1))} [d_{L(G_1)}(e)d_{L(G_1)}(f)+(2+n_2)(d_{L(G_1)}(e)+d_{L(G_1)}(f)) \\
&\quad +(2+n_2)^2] \\
&= M_2(L(G_1))+(2+n_2)M_1(L(G_1))+(2+n_2)^2\,|E(L(G_1)| \\
&= EM_2(G_1)+(2+n_2)EM_1(G_1)+(2+n_2)^2[M_1(G_1)/2-m_1].
\end{aligned}$$

$$\begin{aligned}
Q_2 &= \sum_{u\in V(G_1),v\in I(G_1)} d_{G_1}(u)d_{I(G_1)}(v) \\
&= \sum_{uv\in E(G_1)} (2d_{G_1}(u)+2d_{G_1}(v))(d_{G_1}(u)+d_{G_1}(v)+n_2) \\
&= 2\sum_{uv\in E(G_1)} (d_{G_1}(u)+d_{G_1}(v))^2+2n_2\sum_{uv\in E(G_1)} (d_{G_1}(u)+d_{G_1}(v)) \\
&= 2HM(G_1)+2n_2M_1(G_1).
\end{aligned}$$

$$\begin{aligned}
Q_3 &= \sum_{v\in V(G_2)}\sum_{u\in I(G_1)} (d_{I(G_1)}(u)+n_2)(d_{G_2}(v)+m_1) \\
&= \sum_{uv\in E(G_1)}\sum_{v\in V(G_2)} (d_{G_1}(u)+d_{G_1}(v)+n_2)(d_{G_2}(v)+m_1) \\
&= \sum_{v\in V(G_2)} d_{G_2}(v)\sum_{uv\in E(G_1)} (d_{G_1}(u)+d_{G_1}(v)) \\
&\quad +m_1n_2\sum_{uv\in E(G_1)} (d_{G_1}(u)+(d_{G_1}(v)) \\
&\quad +m_1n_2\sum_{v\in V(G_2)} d_{G_2}(v)+m_1^2n_2^2 \\
&= 2m_2M_1(G_1)+2n_2m_2m_1+m_1n_2M_1(G_1)+m_1^2n_2^2.
\end{aligned}$$

$$\begin{aligned}
Q_4 &= \sum_{uv\in E(G_2)} (d_{G_2}(u)+m_1)(d_{G_2}(v)+m_1) \\
&= \sum_{uv\in E(G_2)} d_{G_2}(u)d_{G_2}(v)+m_1\sum_{uv\in E(G_2)} (d_{G_2}(u)+d_{G_2}(v))+m_1^2m_2 \\
&= M_2(G_2)+m_1M_1(G_2)+m_1^2m_2.
\end{aligned}$$

$$\begin{aligned}
Q_5 &= \sum_{uv\in E(G_1)} 2d_{G_1}(u)2d_{G_1}(v) \\
&= 4M_2(G_1).
\end{aligned}$$

Combining the above contributions we get the desired result. □

EXAMPLE 2.16. From theorem 2.16, we get

$$\begin{aligned}
(i)\ M_2(P_n \veebar_T P_m) &= (m+2)^2(2n-3)+(n-1)^2(m^2+m-1) \\
&\quad +2m(m-1)(n-1)+4mn^2-(n-1)(2+m)^2+14mn \\
&\quad -28m+46n-114,\ n \geqslant 4, m \geqslant 3, \\
(ii)\ M_2(P_n \veebar_T C_m) &= (m+2)^2(2n-3)+m(n-1)^2(m+1)+2m^2(n-1) \\
&\quad -(m+2)^2(n-1)+2m(n-1)(2n-3)+24mn-34m \\
&\quad +60n-124,\ n \geqslant 4, m \geqslant 3, \\
(iii)\ M_2(C_n \veebar_T C_m) &= mn^2(m+1)+n(m+2)^2+2mn(m+2n) \\
&\quad +24mn+4m+60n,\ m,n \geqslant 3, \\
(iv)\ M_2(C_n \veebar_T P_m) &= n^2(m^2+m-1)+n(m+2)^2+2mn(m-1+2n) \\
&\quad +24mn+4m+46n-8,\ m \geqslant 4, n \geqslant 3.
\end{aligned}$$

3. Conclusions

In this paper, we derived some closed formula of the first and second Zagreb index of graphs based on new operations related to the vertex and edge F-join of graphs in terms of different topological indices of their factor graphs. Also, we apply our results to compute the first and second Zagreb index for some important classes of graphs. For further study, some other topological indices for this graph operations can be computed.

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