JOURNAL OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4866, ISSN (o) 2303-4947
www.imvibl.org /JOURNALS / BULLETIN
Vol. 7(2017), 17-35

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354–5792 (o), ISSN 1986–521X (p)

LACEABILITY IN THE MODIFIED DISTANCE GRAPH OF GRID GRAPHS

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Abstract. A connected graph G is Hamiltonian-t^*-laceable if there exists in it a Hamiltonian path between at least one pair of distinct vertices u and v with the property $d(u,v) = t,\ 1 \leqslant t \leqslant \operatorname{diam} G$. G is termed t^*-connected if it is Hamiltonian-t^*-laceable for all t. In this paper, we show that the modified distance graph of the grid graph $M_{gr}(m,n)$ for $n = m$, $4 < m < 11$ and for $n = 2m$, $2 < m < 8$ is t^*-connected.

1. Introduction

All graphs considered here are finite, simple, connected and undirected. Let u and v be two vertices in G. The distance between u and v denoted by $d(u,v)$ is the length of a shortest u–v path in G. G is Hamiltonian-t -laceable (Hamiltonian-t^*-laceable) if there exists in it a Hamiltonian path between every pair (at least one pair) of distinct vertices u and v with the property $d(u,v) = t,\ 1 \leqslant t \leqslant \operatorname{diam} G$. G is termed t^*-connected if it is Hamiltonian-t^*-laceable for all t such that $1 \leqslant t \leqslant \operatorname{diam} G$.

Let D be the set of all distances between every pair of vertices in a graph G and let S be a subset of D. The distance graph associated with G denoted by $D(G,S)$ is the graph having the same vertex set as that of G with two vertices x and y being adjacent in $D(G,S)$ whenever $d(x,y) \in S$. The concept of distance in graphs has been explained in detail in [1] and in [3], Thimmaraju and Murali have studied the Laceability properties in some distance graphs. Laceability properties in the distance graphs of paths P_{2n}, P_{2n+1} and P_{3n} with distance sets $\{1, 2k\}, \{1, 2k+1\}$ and $\{1, 3k\}$ have been studied by Murali and Harinath in [2] and by Leena Shenoy and Murali in [4]. In this paper we prove that the modified distance graph of the grid graph $M_{gr}(m,n)$ is t^*-connected for $n = m$, $4 < m < 11$ and for $n = 2m$, $2 < m < 8$.

2000 *Mathematics Subject Classification.* 05C45; 05C99.

Key words and phrases. Hamiltonian-t^*-laceable graph, Grid graph.

Definition 1.1 (The Modified Distance graph of Grid graph $M_{gr}(m,n)$). This graph is obtained as follows:

- We consider the grid graph $P_m \times P_n$ with vertex set:
  $V = \{a_{11}, a_{12} \ldots, a_{1n}\} \cup \{a_{21}, a_{22} \ldots, a_{2n}\} \cup \cdots \cup \{a_{m1}, a_{m2} \ldots, a_{mn}\}$
  and edge set
  $E = \{a_{ij}a_{i(j+1)}, a_{i(j+1)}a_{i(j+2)}, a_{i(j+2)}a_{i(j+3)}, \ldots, a_{i(j+(n-2))}a_{i(j+(n-1))},$
  $j = 1, i = 1, 2, 3, \ldots, n\} \cup \{a_{ij}a_{(i+1)j}, a_{(i+1)j}a_{(i+2)j}, \ldots, a_{(i+(n-2))j}a_{(i+(n-1))j},$
  $i = 1, j = 1, 2, \ldots, n\}$.
- We consider the distance set $S = \{2\}$ and construct the graph $D(P_m \times P_n, S)$.
- Next we join the vertices a_{12}, a_{13} and a_{22}, a_{23} by an edge each.
- The resulting graph is the graph $M_{gr}(m,n)$.

In Fig 1, the grid graph $P_4 \times P_5$ and the graph $M_{gr}(4,5)$ are shown.

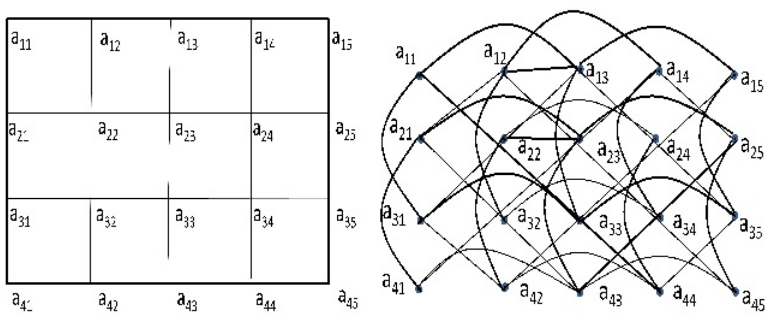

Figure 1. The grid graph $P_4 \times P_5$ and the modified grid graph $M_{gr}(4,5)$.

Definition 1.2. Let P be a path from the verticesa_i to a_j in a graph G and let P' be a path froma_j to a_k. Then the path $P \cup P'$ is the path obtained by extending the path P from a_i to a_j to a_k through the common vertex a_j.

2. Results

In this section, we shall show that the graph $M_{gr}(m,n)$ is t^*-connected for $n = m$, $4 < m < 11$ and for $n = 2m$, $2 < m < 8$. First, we need to introduce the following terminologies which will be applied to prove t^*-connectedness.

$$a_{ij}I^m[n] = a_{ij}, a_{(i+m)j}a_{(i+2m)j}, a_{(i+3m)j}, \ldots, a_{nj},$$
$$a_{ij}J^x[y] = a_{ij}, a_{i(j+x)}a_{i(j+2x)}, a_{i(j+3x)}, \ldots, a_{iy},$$
$$a_{ij}I^m[n]J^x[y] = a_{ij}, a_{(i+m)(j+x)}, \ldots, a_{ny}.$$

Theorem 2.1. *The graph $M_{gr}(m,n)$ for $n = m$ and $4 < m < 11$ is t^*-connected.*

Proof. Let $G = M_{gr}(m,n)$. Let

$$V = \{a_{11}, a_{12} \ldots, a_{1n}\} \cup \{a_{21}, a_{22} \ldots, a_{2n}\} \cup \cdots \cup \{a_{m1}, a_{m2} \ldots, a_{mn}\} \text{ and}$$
$$\cup \{a_{ij}a_{(i+1)(j-1)}, a_{(i+1)(j-1)}a_{(i+2)(j-2)}, \ldots, a_{(i+(n-(n-1)))(j-(n-(n-1)))}$$
$$a_{(i+(n-(n-2)))(j-(n-(n-2)))}, j = 3, i = 1\} \cup \cdots \cup \{a_{ij}a_{(i+1)(j-1)}, a_{(i+1)(j-1)}$$
$$a_{(i+2)(j-2)}, \ldots, a_{(i+(n-(n-1)))(j-(n-(n-1)))}a_{(i+(n-1))(j-(n-1))}, j = n, i = 1\}$$
$$\cup \{a_{ij}a_{(i+1)(j-1)}, a_{(i+1)(j-1)}a_{(i+2)(j-2)}, \ldots, a_{(i+(n-3))(j-(n-3))}$$
$$a_{(i+(n-2))(j-(n-2))}, i = 2, j = n\}$$
$$\cup \{a_{ij}a_{(i+1)(j-1)}, a_{(i+1)(j-1)}a_{(i+2)(j-2)}, \ldots, a_{(i+(n-4))(j-(n-4))}$$
$$a_{(i+(n-3))(j-(n-3))}, i = 3, j = n\} \cup \cdots \cup \{a_{ij}a_{(i+1)(j-1)}, a_{(i+1)(j-1)}a_{(i+2)(j-2)},$$
$$\ldots, a_{(i+(n-(n)))(j-(n-(n)))}a_{(i+(n-(n-1)))(j-(n-(n-1)))}, i = (n-1), j = n\} + e_1$$
$$= \{a_{ij}a_{i(j+1)}, i = 1, j = 2\} + e_2 = \{a_{ij}a_{i(j+1)}, i = 2, j = 2\}$$

Clearly, $d(G) = n - 1$. To establish the result, we consider the following cases.
Case (i): n is odd.
Sub case (i): $t = 1$.
In $G = M_{gr}(m,n)$, $d(a_{11}, a_{31}) = 1$ and the path

$$P : a_{11}I^1J^1P^0J^1a_{14}J^2[n-1]a_{2n}I^2[n-1]a_{n(n-1)}J^{-2}[2]a_{(n-1)1}J^2[n-2]$$
$$a_{(n-2)(n-1)}J^{-2}[2]a_{(n-3)1}J^2[n-2]a_{5(n-1)}J^{-2}[2]a_{41}J^2[n-2]$$
$$a_{3(n-1)}P^0(a_{2(n-2)}, a_{3(n-3)}) \cup (a_{3(n-3)}, a_{2(n-4)})P^0a_{34}P^0J^{-2}P^0I^{-1}J^{-1}P^0I^1J^1$$
$$P^0J^1P^0J^2[n]a_{3n}I^2[n]a_{n(n-2)}J^{-2}[1]a_{(n-2)1}J^2[n-1]a_{(n-2)(n-2)}J^{-2}[1]$$
$$a_{(n-3)2}J^2[n-1]P^0a_{51}P^0a_{42}J^2[n-1]P^0(a_{3(n-2)}, a_{2(n-1)})$$
$$\cup (a_{2(n-1)}, a_{2(n-3)}) \cup (a_{2(n-3)}, a_{3(n-3)})P^0a_{24}P^0a_{33}P^0a_{31}$$

is a Hamiltonian path between the vertices a_{11} and a_{31}.
Sub case (ii): $t = 2$.
In $G = M_{gr}(m,n)$, $d(a_{11}, a_{51}) = 2$ then the path

$$P : a_{11}I^1J^1P^0J^1a_{14}J^2[n-1]a_{2n}I^2[n-1]a_{n(n-1)}J^{-2}[2]$$
$$a_{(n-1)1}J^2[n-2]a_{(n-2)(n-1)}J^{-2}[2]a_{(n-3)1}J^2[n-2]a_{5(n-1)}J^{-2}[2]a_{41}J^2[n-2]$$
$$a_{3(n-1)}P^0(a_{2(n-2)}, a_{3(n-3)}) \cup (a_{3(n-3)}, a_{2(n-4)})P^0a_{34}P^0J^{-2}P^0I^{-1}J^{-1}P^0I^1J^1$$
$$P^0J^1P^0J^2[n]a_{3n}I^2[n]a_{(n-1)(n-1)}I^{-2}[2]a_{3(n-2)}I^2[n]a_{(n-1)(n-3)}I^{-2}[2]a_{3(n-4)}$$
$$I^2[n]a_{(n-1)(n-5)}I^{-2}[2]a_{3(n-6)}I^2[n]P^0a_{n1}P^0a_{(n-2)1}P^0a_{(n-1)2}I^{-2}[4]P^0a_{31}P^0a_{51}$$

is a Hamiltonian path between the vertices a_{11} and a_{51}. In Fig 2, Hamiltonian path from vertex a_{11} to a_{31} in the graph $M_{gr}(5,5)$ is shown.

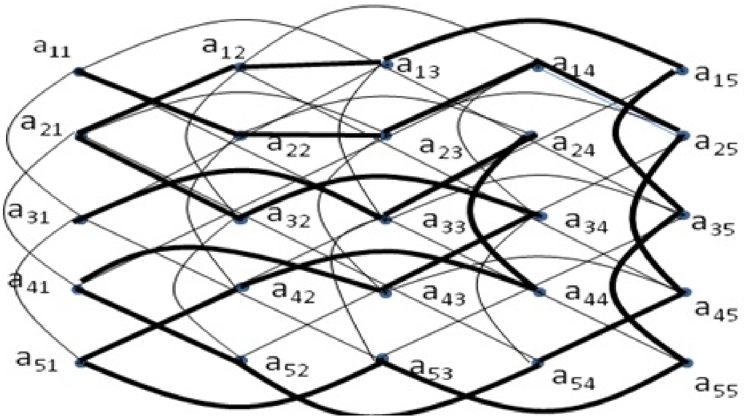

FIGURE 2. Hamiltonian path from vertex $a_{11}$to a_{31} in the graph $M_{gr}(5,5)$.

Sub case (iii): $t = 3$.

In $G = M_{gr}(m,n)$, $d(a_{11}, a_{21}) = 3$ then the path

$$P : a_{11}J^2[n]a_{3n}I^2[n]a_{n(n-2)}J^{-2}[1]a_{(n-1)2}J^2[n-1]a_{(n-2)(n-2)}J^{-2}[1]a_{(n-3)2} J^2[n-1]a_{5(n-2)}J^{-2}[1]a_{52}J^2[n-1]a_{2(n-1)}P^0(a_{3(n-2)}, a_{2(n-3)}) \cup (a_{2(n-3)}, a_{3(n-4)})P^0a_{24}P^0I^1J^{-1}P^0J^{-2}P^0I^{-1}J^1P^0J^1P^0I^{-1}J^{-1}P^0 J^2[n-1]a_{2n}I^2[n-1]a_{n(n-1)}J^{-2}[2]a_{(n-1)1}J^2[n-2]a_{(n-2)(n-1)}J^{-2}[2]a_{(n-3)1} J^2[n-2]a_{5(n-1)}J^{-2}[2]a_{41}J^2[n-2]P^0(a_{3(n-1)}, a_{2(n-2)}) \cup (a_{2(n-2)}, a_{3(n-3)}) \cup (a_{3(n-3)}, a_{2(n-4)})P^0a_{34}P^0a_{32}P^0a_{21}$$

is the Hamiltonian path between the vertices a_{11} and a_{21}.

Sub case (iv): $t = 4$.

In $G = M_{gr}(m,n)$, $d(a_{11}, a_{41}) = 4$ then the path

$$P : a_{11}J^2[n]a_{3n}I^2[n]a_{n(n-2)}J^{-2}[1]a_{(n-1)2}J^2[n-1]a_{(n-2)(n-2)}J^{-2}[1]a_{(n-3)2} J^2[n-1]a_{(n-4)(n-2)}J^{-2}[1]a_{42}J^2[n-1]a_{2(n-1)}P^0(a_{3(n-2)}, a_{2(n-3)}) \cup (a_{2(n-3)}, a_{3(n-4)})P^0a_{24}P^0I^1J^{-1}P^0J^{-2}P^0I^{-1}J^1P^0J^1P^0J^2[n-2] a_{3(n-1)}J^{-2}[2]P^0a_{21}P^0a_{12}J^2[n-1]a_{2n}I^2[n-1]a_{n(n-1)}J^{-2}[2]a_{(n-1)1} J^2[n-2]a_{7(n-1)}J^{-2}[2]a_{61}J^2[n-2]P^0(a_{5(n-1)}, a_{4(n-2)}) \cup (a_{4(n-2)}, a_{5(n-3)}) \cup (a_{5(n-3)}, a_{(n-5)(n-4)})P^0a_{54}P^0a_{43}P^0a_{52}P^0a_{41}$$

is the Hamiltonian path between the vertices a_{11} and a_{41}.

Sub case (v): $t = 5$.

In $G = M_{gr}(m,n)$, $d(a_{11}, a_{61}) = 5$ then the path

$$P : a_{11}J^2[n]a_{3n}I^2[n]a_{n(n-2)}J^{-2}[1]a_{(n-1)2}J^2[n-1]a_{(n-2)(n-2)}J^{-2}[1]a_{(n-3)2}$$

$$J^2[n-1]a_{(n-4)(n-2)}J^{-2}[1]a_{42}J^2[n-1]a_{2(n-1)}P^0(a_{3(n-2)}, a_{2(n-3)})$$
$$\cup (a_{2(n-3)}, a_{3(n-4)})P^0a_{24}P^0I^1J^{-1}P^0J^{-2}P^0I^{-1}J^1P^0J^1P^0J^2[n-2]$$
$$a_{3(n-1)}J^{-2}[2]a_{43}J^2[n-2]a_{5(n-1)}J^{-2}[2]a_{41}I^{-2}[2]a_{12}J^2[n-1]a_{2n}$$
$$I^2[n-1]a_{n(n-1)}J^{-2}[2]a_{(n-1)1}J^2[n-2]P^0(a_{(n-2)(n-1)}, a_{(n-3)(n-2)})$$
$$P^0a_{76}[I^{-1}J^{-1}P^0I^1J^{-1}]^2P^0a_{61}$$

is the Hamiltonian path between the vertices a_{11} and a_{61}. In Fig 3, Hamiltonian path from vertex a_{11} to a_{61} in the graph $M_{gr}(7,7)$ is shown.

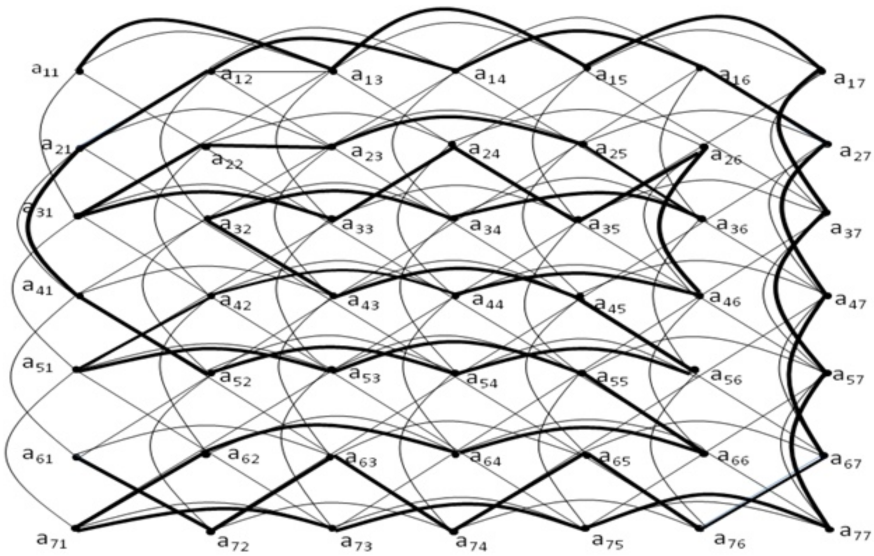

FIGURE 3. Hamiltonian path from vertex a_{11} to a_{61} in the graph $M_{gr}(7,7)$.

Sub case (vi): $t = 6$**.**

In $G = M_{gr}(m,n)$, $d(a_{11}, a_{77}) = 6$ then the path

$$P : a_{11}I^1J^1P^0J^1a_{14}J^2[n-1]a_{2n}I^2[n-1]a_{n(n-1)}J^{-2}[2]a_{(n-1)1}J^2[n-2]a_{(n-2)(n-1)}$$
$$J^{-2}[2]a_{(n-3)1}J^2[n-2]a_{(n-4)(n-1)}J^{-2}[2]a_{41}J^2[n-2]a_{3(n-1)}P^0I^{-1}[2]$$
$$J^{-1}[n-2]P^0I^1[3]J^{-1}[n-3]P^0I^{-1}[2]J^{-1}[n-4]P^0a_{34}J^{-2}[2]a_{21}P^0a_{12}P^0a_{13}$$
$$J^2[n]a_{2(n-1)}J^{-2}[4]a_{33}J^2[n]a_{4(n-1)}J^{-2}[4]a_{53}J^2[n]a_{6(n-1)}J^{-2}[2]a_{42}P^0a_{31}$$
$$I^2[n]a_{n3}J^2[n]P^0a_{(n-2)n}P^0a_{(n-1)(n-1)}J^{-2}[2]a_{(n-2)3}J^2[n-2]$$

is the Hamiltonian path between the vertices a_{11} and a_{77}.

Sub case (vii): $t = 7$**.**

In $G = M_{gr}(m,n)$, $d(a_{11}, a_{88}) = 7$ then the path

$$P : a_{11}I^1J^1P^0J^1a_{14}J^2[n]P^0I^2P^0a_{2n}I^2[n-1]a_{n(n-1)}J^{-2}[2]a_{(n-1)1}J^2[n-2]$$

$$a_{(n-2)(n-1)}J^{-2}[2]a_{(n-3)1}J^{2}[n-2]a_{(n-4)(n-1)}J^{-2}[2]a_{(n-5)1}J^{2}[n-2]$$
$$a_{(n-6)(n-1)}I^{-1}[n-7]J^{-1}[n-2]P^{0}I^{1}[n-6]J^{-1}[n-3]P^{0}$$
$$I^{-1}[n-8]J^{-1}[n-4]P^{0}I^{1}[n-6]J^{-1}[n-5]P^{0}J^{-2}[n-7]P^{0}a_{(n-7)1}I^{-1}[1]$$
$$J^{1}[2]a_{1(n-6)}J^{2}[n]a_{(n-6)n}I^{2}[n-2]a_{(n-3)(n-1)}I^{-2}[2]a_{(n-6)(n-2)}$$
$$I^{2}[n-2]a_{(n-1)(n-3)}I^{-2}[2]a_{(n-6)(n-4)}I^{2}[n-2]P^{0}a_{(n-3)(n-5)}I^{-2}[2]$$
$$a_{(n-6)(n-6)}J^{2}[n-2]a_{(n-3)2}I^{-2}[4]a_{(n-6)1}I^{2}[n]a_{(n-1)(n-7)}I^{1}[n]J^{1}[n-6]$$
$$P^{0}I^{-1}[n-1]J^{1}[n-5]P^{0}I^{1}[n]J^{1}[n-4]P^{0}I^{-1}[n-1]J^{1}[n-3]P^{0}$$
$$I^{1}[n]J^{1}[n-2]P^{0}J^{2}[n]P^{0}I^{-1}[n-1]J^{-1}[n-1]$$

is the Hamiltonian path between the vertices a_{11} and a_{88}.
Sub case (viii): $t = 8$.

In $G = M_{gr}(m, n)$, $d(a_{11}, a_{99}) = 8$ then the path

$$P : a_{11}I^{1}J^{1}P^{0}J^{1}a_{14}J^{2}[n-1]a_{2n}I^{2}[n-1]a_{n(n-1)}J^{-2}[2]a_{(n-1)1}J^{2}[n-2]$$
$$a_{(n-2)(n-1)}J^{-2}[2]a_{(n-3)1}J^{2}[n-2]a_{(n-4)(n-1)}J^{-2}[2]a_{(n-5)1}J^{2}[n-2]$$
$$a_{(n-6)(n-1)}I^{-1}J^{-1}P^{0}I^{1}J^{-1}P^{0}I^{-1}J^{-1}P^{0}I^{1}J^{-1}P^{0}J^{-1}P^{0}I^{-1}J^{-1}P^{0}I^{-1}J^{1}$$
$$P^{0}J^{1}P^{0}J^{2}[n]a_{3n}I^{2}[n-2]a_{(n-1)(n-1)}I^{-2}[2]a_{(n-6)(n-2)}I^{2}[n-2]$$
$$a_{(n-1)(n-3)}I^{-2}[2]a_{3(n-4)}I^{2}[n-2]a_{(n-1)4}$$
$$I^{-2}[2]a_{(n-6)(n-6)}I^{2}[n-2]a_{(n-1)2}I^{-2}[4]a_{(n-6)1}I^{2}[n]a_{n3}J^{2}[n]$$

is the Hamiltonian path between the vertices a_{11} and a_{99}. In Fig 4, Hamiltonian path from vertex a_{11} to a_{99} in the graph $M_{gr}(9, 9)$ is shown.

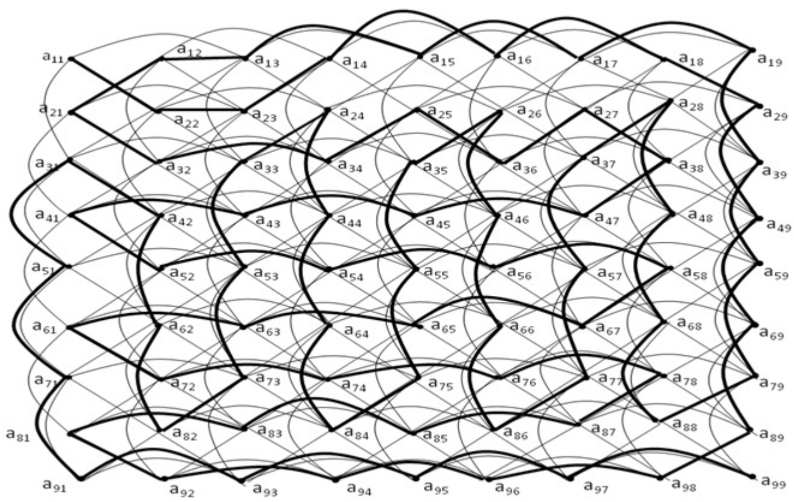

FIGURE 4. Hamiltonian path from vertex a_{11} to a_{99} in the graph $G = M_{gr}(9, 9)$.

Case (ii): n is even.
Sub case (i): $t = 1$.
In $G = M_{gr}(m,n)$, $d(a_{11}, a_{31}) = 1$ and the path

$$P : a_{11}I^1J^1P^0J^1a_{14}J^2[n]a_{3n}I^2[n-1]a_{n(n-1)}I^{-2}[2]a_{3(n-2)}I^2[n-1]a_{(n-1)(n-2)}$$
$$P^0a_{n(n-3)}I^{-2}[2]a_{3(n-4)}I^2[n-1]a_{n(n-5)}I^{-2}[2]a_{3(n-6)}I^2[n-1]a_{n(n-7)}$$
$$I^{-2}[n-6]P^0a_{63}P^0a_{43}P^0a_{32}I^2[n-1]a_{n1}I^{-2}[2]P^0a_{12}P^0a_{13}J^2[n-1]$$
$$a_{2n}I^2[n]a_{(n-1)(n-1)}I^{-2}[3]a_{2(n-2)}I^2[n]a_{(n-1)(n-3)}I^{-2}[3]a_{2(n-4)}I^2[n]a_{(n-1)(n-5)}$$
$$I^{-2}[3]a_{2(n-6)}I^2[n]a_{(n-1)(n-7)}I^{-2}[3]P^0a_{33}P^0a_{42}I^2[n]a_{(n-1)1}I^{-2}[3]$$

is a Hamiltonian path between the vertices a_{11} and a_{31}.
Sub case (ii): $t = 2$
In$G = M_{gr}(m,n)$, $d(a_{11}, a_{51}) = 2$ then the path

$$P : a_{11}I^1J^1P^0J^1a_{14}J^2[n]a_{3n}I^2[n-1]a_{n(n-1)}I^{-2}[2]a_{3(n-2)1}I^2[n-1]a_{n(n-3)}$$
$$I^{-2}[2]a_{3(n-4)}I^2[n-1]a_{n(n-5)}I^{-2}[2]a_{3(n-6)}I^2[n-1]a_{n(n-7)}I^{-2}[4]P^0a_{63}$$
$$P^0a_{43}P^0a_{32}I^2[n-1]a_{n1}I^{-2}[2]P^0a_{12}P^0a_{13}J^2[n-1]a_{2n}I^2[n]$$
$$a_{n(n-2)}J^{-2}[2]a_{(n-1)(n-9)}J^2[n-1]a_{(n-2)(n-2)}J^{-2}[2]a_{71}J^2[n-1]a_{6(n-2)}J^{-2}[2]$$
$$a_{53}J^2[n-1]P^0a_{4(n-2)}P^0(a_{3(n-1)}, a_{2(n-2)}) \cup (a_{2(n-2)}, a_{3(n-3)})$$
$$\cup (a_{3(n-3)}, a_{(n-2)(n-4)}) \cup (a_{(n-2)(n-4)}, a_{(n-6)(n-4)})P^0a_{35}P^0I^{-1}J^{-1}$$
$$P^0I^2P^0I^{-1}J^{-1}P^0I^1J^{-1}P^0I^{-1}J^{-1}P^0I^2$$

is a Hamiltonian path between the vertices a_{11} and a_{51}. In Fig 5, Hamiltonian path from vertex a_{11} to a_{21} in the graph $M_{gr}(6,6)$ is shown.

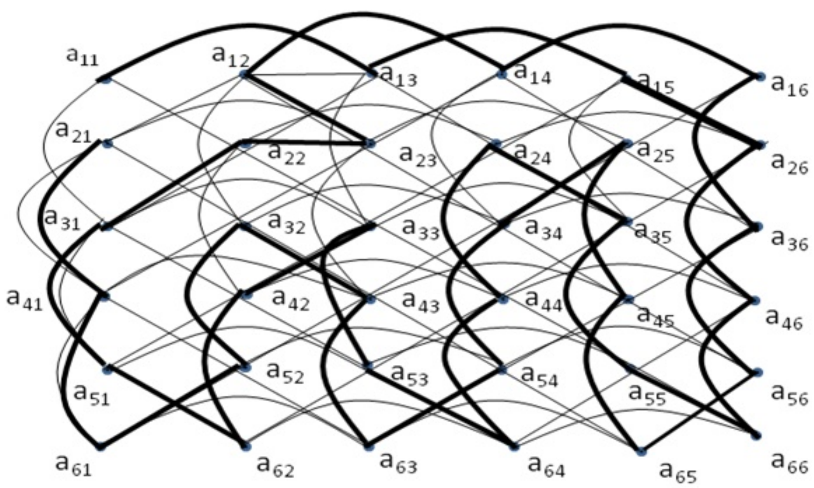

FIGURE 5. Hamiltonian path from vertex a_{11} to a_{21} in the graph $M_{gr}(6,6)$.

Sub case (iii): $t = 3$

In $G = M_{gr}(m,n)$, $d(a_{11}, a_{21}) = 3$ then the path

$$P : a_{11}J^2[n-1]a_{2n}I^2[n]a_{(n-1)(n-1)}I^{-2}[3]a_{2(n-2)}I^2[n]a_{(n-1)(n-3)}I^{-2}[3]a_{2(n-4)} I^2[n]a_{(n-1)(n-5)}I^{-2}[3]a_{24}I^2[n]a_{(n-1)3}I^{-2}[3]a_{42}I^2[n]a_{(n-1)1}I^{-2}[3] P^0a_{22}P^0a_{23}P^0a_{12}J^2[n]a_{3n}I^2[n-1]a_{n(n-1)}I^{-2}[2]a_{3(n-2)} I^{-2}[n-1]a_{n(n-3)}I^{-2}[2]a_{3(n-4)}I^2[n-1]a_{n(n-5)}I^{-2}[2]a_{34}I^2[n-1]a_{n3} I^{-2}[4]a_{32}I^2[n-1]a_{n1}I^{-2}[2]$$

is a Hamiltonian path between the vertices a_{11} and a_{21}.

Sub case (iv): $t = 4$

In $G = M_{gr}(m,n)$, $d(a_{11}, a_{41}) = 4$then the path

$$P : a_{11}J^2[n-1]a_{2n}I^2[n]a_{(n-1)(n-1)}I^{-2}[3]a_{2(n-2)}I^2[n]a_{(n-1)(n-3)}I^{-2}[3]a_{26}I^2[n] a_{(n-1)5}I^{-2}[3]a_{24}I^2[n]a_{(n-1)3}I^{-2}[3]a_{42}I^2[n]a_{(n-1)1}I^{-2}[3]P^0a_{22}P^0 a_{23}J^2[n-1]a_{3(n-2)}J^{-2}[2]P^0a_{21}P^0a_{12}J^2[n]a_{3n}I^2[n-1]a_{n(n-1)}J^{-2}[1] a_{(n-1)2}J^2[n-2]a_{(n-2)(n-1)}J^{-2}[1]a_{(n-3)2}J^2[n-2]a_{(n-4)(n-1)1}J^{-2}[1] a_{52}J^2[n-2]a_{4(n-1)}J^{-2}[1]$$

is a Hamiltonian path between the vertices a_{11} and a_{41}.

Sub case (v): $t = 5$**.**

In $G = M_{gr}(m,n)$, $d(a_{11}, a_{61}) = 5$ then the path

$$P : a_{11}J^2[n-1]a_{2n}I^2[n]a_{(n-1)(n-1)}I^{-2}[3]a_{2(n-2)}I^2[n]a_{(n-1)(n-3)}I^{-2}[3]a_{26}I^2[n] a_{(n-1)5}I^{-2}[3]a_{24}I^2[n]a_{(n-1)3}I^{-2}[3]a_{42}I^2[n]a_{(n-1)1}I^{-2}[3]P^0a_{22}P^0 a_{23}J^2[n-1]a_{3(n-2)}J^{-2}[2]a_{43}J^2[n-1]a_{5(n-2)}J^{-2}[2]a_{41}I^{-2}[2]a_{12}J^2[n]a_{3n} I^2[n-1]a_{n(n-1)}J^{-2}[1]a_{(n-1)2}J^2[n-2]a_{(n-2)(n-1)}J^{-2}[1]a_{(n-3)2} J^2[n-2]a_{6(n-1)}J^{-2}[1]$$

is a Hamiltonian path between the vertices a_{11} and a_{61}.

Sub case (vi): $t = 6$**.**

In $G = M_{gr}(m,n)$, $d(a_{11}, a_{77}) = 6$ then the path

$$P : a_{11}I^1J^1P^0J^1a_{14}J^2[n]a_{3n}I^2[n-1]a_{n(n-1)}I^{-2}[2]a_{3(n-2)}I^2[n-1]a_{n(n-3)} I^{-2}[2]a_{3(n-4)}I^2[n-1]a_{n5}I^{-2}[2]a_{34}I^2[n-1]a_{n3}I^{-2}[4]a_{32}I^2[n-1]a_{n1} I^{-2}[2]P^0a_{12}P^0a_{13}J^2[n-1]a_{2n}J^{-2}[4]a_{33}J^2[n-1]a_{4n}J^{-2}[4]a_{53} J^2[n-1]a_{6n}J^{-2}[6]a_{75}I^{-1}J^{-1}P^0I^1J^{-1}P^0I^{-1}J^{-1}P^0I^{-2}P^0 a_{31}I^2[n-1]a_{n2}J^2[n]a_{(n-1)(n-1)}P^0J^{-1}[3]a_{(n-2)2}J^2[n]a_{(n-3)(n-1)}J^{-2}[n-2]$$

is a Hamiltonian path between the vertices a_{11} and a_{77}. In Fig 6, Hamiltonian path from vertex a_{11} to a_{77} in the graph $M_{gr}(8,8)$ is shown.

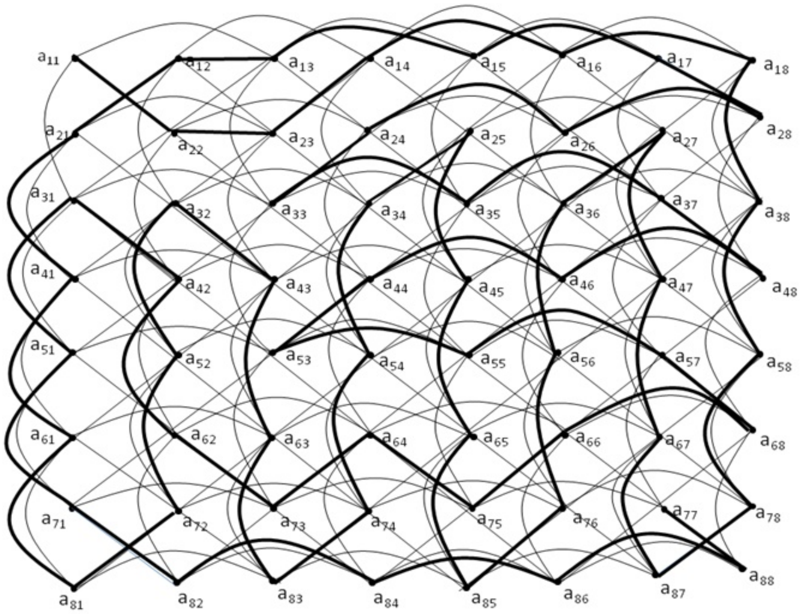

FIGURE 6. Hamiltonian path from vertex a_{11} to a_{77} in the graph $M_{gr}(8, 8)$.

Sub case (vii): $t = 7$.

In $G = M_{gr}(m, n)$, $d(a_{11}, a_{88}) = 7$ then the path

$$P : a_{11}I^1J^1P^0J^1a_{14}J^2[n]a_{3n}I^2[n-1]a_{n(n-1)}I^{-2}[2]a_{3(n-2)}I^2[n-1]a_{n(n-3)}$$
$$I^{-2}[2]a_{3(n-4)}I^2[n-1]a_{n5}I^{-2}[2]a_{(n-7)4}I^2[n-1]a_{n3}I^{-2}[4]a_{32}I^2[n-1]a_{n1}$$
$$I^{-2}[2]a_{12}J^1[3]P^0J^2[n-1]a_{2n}I^2[6]a_{7(n-1)}I^{-2}[3]a_{2(n-2)}I^2[6]a_{7(n-3)}$$
$$I^{-2}[3]a_{2(n-4)}I^2[6]a_{7(n-5)}I^{-2}[3]a_{2(n-6)}I^2[6]a_{(n-3)3}I^{-2}[3]a_{31}$$
$$P^0[I^1J^1P^0I^1J^{-1}P^0]^2a_{(n-1)1}P^0a_{n2}J^2[n]P^0I^{-2}P^0$$
$$a_{(n-1)(n-1)}J^{-2}[1]a_{(n-2)2}J^2[n-2]$$

is a Hamiltonian path between the vertices a_{11} and a_{88}.

Sub case (viii): $t = 8$.

In $G = M_{gr}(m, n)$, $d(a_{11}, a_{99}) = 8$ then the path

$$P : a_{11}I^1J^1P^0J^1a_{14}J^2[n]a_{3n}I^2[n-1]a_{n(n-1)}I^{-2}[2]a_{3(n-2)}I^2[n-1]a_{n(n-3)}$$
$$I^{-2}[2]a_{3(n-3)}I^2[n-1]a_{n(n-5)}I^{-2}[2]a_{3(n-6)}I^2[n-1]a_{n(n-7)}I^{-2}[4]a_{3(n-8)}I^2$$
$$[n-1]a_{n1}I^{-2}[2]P^0I^{-1}J^1P^0J^1P^0J^2[n-1]a_{2n}J^{-2}[n-6]a_{(n-7)(n-7)}J^2$$
$$[n-1]a_{4n}J^{-2}[n-6]a_{(n-5)3}J^2[n-1]a_{(n-4)n}J^{-2}[n-6]a_{(n-3)3}J^2[n-1]$$
$$a_{(n-2)n}J^{-2}[2]P^0I^{-2}P^0I^{-2}P^0I^{-1}J^{-1}P^0I^{-2}[n-1]P^0a_{n2}P^0[I^{-1}J^1P^0I^1$$

$J^1P^0]^3J^2P^0I^{-1}J^{-1}$

is a Hamiltonian path between the vertices a_{11} and a_{99}.

Sub case (ix): $t = 9$.

In $G = M_{gr}(m, n)$, $d(a_{11}, a_{1010}) = 9$ then the path

$$P : a_{11}I^1J^1P^0J^1a_{14}J^2[n]a_{3n}I^2[n-1]a_{n(n-1)}I^{-2}[2]a_{3(n-2)}I^2[n-1]a_{n(n-3)}$$
$$I^{-2}[2]a_{3(n-4)}I^2[n-1]a_{n(n-5)}I^{-2}[2]a_{3(n-6)}I^2[n-1]a_{n(n-7)}I^{-2}[4]a_{3(n-8)}$$
$$I^2[n-1]a_{n(n-9)}I^{-2}[2]P^0I^{-1}J^1P^0J^1P^0J^2[n-1]P^0I^1J^1P^0J^{-2}[4]$$
$$P^0I^1J^{-1}P^0J^2[n-1]P^0I^1J^1P^0J^2[4]P^0I^1J^{-1}P^0J^2[n-1]P^0$$
$$I^1J^1P^0J^{-2}[4]P^0I^1J^{-1}P^0J^2[n-1]P^0I^1J^1P^0[I^1J^{-1}P^0$$
$$I^{-1}J^{-1}P^0]^4P^0J^{-2}[4]P^0I^{-1}J^{-1}P^0I^2[n-1]P^0a_{n2}J^2[n]$$

is a Hamiltonian path between the vertices a_{11} and a_{1010}. □

THEOREM 2.2. *The graph $G = M_{gr}(m, n)$ for $n = 2m$, where $2 < m < 8$ is t^*-connected*

PROOF. Let $G = M_{gr}(m, n)$. Then

$$V(G) = \{a_{11}, a_{12}, \ldots, a_{1n}\} \cup \{a_{21}, a_{22}, \ldots, a_{2n}\} \cup \cdots \cup \{a_{m1}, a_{m2}, \ldots, a_{mn}\}$$
$$E(G) = \{a_{ij}a_{i(j+2)}, a_{i(j+2)}a_{i(j+4)}, \ldots, a_{i(j+(n-3))}a_{i(j+(n-1))},$$
$$i = 1, 2, 3, \ldots, m, j = 1\} \cup \{a_{ij}a_{i(j+2)}, a_{i(j+2)}a_{i(j+4)}, \ldots, a_{i(j+(n-4))}$$
$$a_{i(j+(n-2))}, i = 1, 2, 3, \ldots, m, j = 2\}$$
$$\cup \{a_{ij}a_{(i+2)j}, a_{(i+2)j}a_{(i+4)j}, \ldots, a_{(i+(m-3))j}a_{(i+(m-1))j}\ [m = \text{odd}]$$
$$a_{(i+(m-4))j}a_{(i+(m-2))j}\ [m = \text{even}],$$
$$i = 1, j = 1, 2, 3, \ldots, n\} \cup \{a_{ij}a_{(i+2)j}, a_{(i+2)j}a_{(i+4)j},$$
$$\ldots, a_{(i+(m-5))j}a_{(i+(m-3))j}[m = \text{odd}]$$
$$a_{(i+(m-4))j}a_{(i+(m-2))j}\ [m = \text{even}], i = 2, j = 1, 2, 3, \ldots, n\}$$
$$\cup \{a_{ij}a_{(i+1)(j+1)}, a_{(i+1)(j+1)}a_{(i+2)(j+2)}, \ldots,$$
$$a_{(i+(m-2))(j+(n-(m+2)))}a_{(i+(m-1))(j+(n-(m+1)))}, i = 1, j = 1\}$$
$$\cup \{a_{ij}a_{(i+1)(j+1)}, a_{(i+1)(j+1)}a_{(i+2)(j+2)}, \ldots,$$
$$a_{(i+(m-3))(j+(n-(m+3)))}a_{(i+(m-2))(j+(n-(m+2)))}, i = 2, j = 1\} \cup \cdots$$
$$\cup \{a_{ij}a_{(i+1)(j+1)}, a_{(i+1)(j+1)}a_{(i+2)(j+2)}, \ldots,$$
$$a_{(i+(m-m))(j+(n-n))}a_{(i+m-((m-1)))(j+(n-(n-1)))}, i = (m-1), j = 1\}$$
$$\cup \{a_{ij}a_{(i+1)(j+1)}, a_{(i+1)(j+1)}a_{(i+2)(j+2)}, \ldots,$$
$$a_{(i+(m-2))(j+(n-(m+2)))}a_{(i+(m-1))(j+(n-(m+1)))}, i = 1, j = 2\}$$
$$\{a_{ij}a_{(i+1)(j+1)}, a_{(i+1)(j+1)}a_{(i+2)(j+2)}, \ldots, a_{(i+(m-m))(j+(n-(n-n)))}$$
$$a_{(i+(m-2))(j+(n-(n-1)))}, i = 1, j = (n-1)\} \cup \{a_{ij}a_{(i+1)(j-1)}, a_{(i+1)(j-1)}$$
$$a_{(i+2)(j-2)}, \ldots, a_{(i+(m-m))(j-(n-n))}a_{(i+(m-(m-1)))(j-(n-(n-1)))},$$

$i = 1, j = 2)\} \cup \{a_{ij}a_{(i+1)(j-1)}, a_{(i+1)(j-1)}a_{(i+2)(j-2)}, \cdots,$
$a_{(i+(m-(m-1)))(j-(n-(n-1)))}a_{(i+(m-(m-2)))(j-(n-(n-2)))},$
$i = 1, j = 3)\} \cup \cdots \cup \{a_{ij}a_{(i+1)(j-1)}, a_{(i+1)(j-1)}a_{(i+2)(j-2)}, \cdots,$
$a_{(i+(m-2))(j-(n-(m+2)))}a_{(i+(m-1))(j-(n-(m+1)))},$
$i = 1, j = n\} \cup \{a_{ij}a_{(i+1)(j-1)}, a_{(i+1)(j-1)}a_{(i+2)(j-2)}, \cdots,$
$a_{(i+(m-3))(j-(n-(m+3)))}a_{(i+(m-2))(j-(n-(m+2)))},$
$i = 2, j = n\} \cup \cdots \cup \{a_{ij}a_{(i+1)(j-1)}, a_{(i+1)(j-1)}a_{(i+2)(j-2)}, \cdots$
$a_{(i+(m-(m)))(j-(n-(n)))}a_{(i+(m-(m-1)))(j-(n-(n-1)))},$
$i = (m-1), j = n\} + e_1 = \{a_{ij}a_{i(j+1)}, i = 1, j = 2\}$
$+ e_2 = \{a_{ij}a_{i(j+1)}, i = 2, j = 2\}.$

Clearly, $d(G) = \lceil \frac{m+n}{2} \rceil - 1$. To establish the result, we consider the following cases,
Case (i): n is odd.
Sub case (i): $t = 1$.
In $G = M_{gr}(m, n)$, $d(a_{11}, a_{13}) = 1$ and the path

$$\begin{aligned}P :\ & a_{11}I^2[m]a_{m3}J^2[n-1]a_{(m-1)n}I^{-2}[2]a_{1(n-1)}I^2[m-2]a_{(m-1)(n-2)}I^{-2}[2]\\ & a_{1(n-3)}I^2[m-2]a_{(m-1)(n-4)}I^{-2}[2]a_{1(n-5)}I^2[m-2]a_{(m-1)(n-6)}I^{-2}[2]a_{1m}\\ & I^2[m-2]a_{(m-1)(n-8)}I^{-2}[2]a_{15}I^2[m-2]a_{(m-1)4}I^{-2}[2]a_{(m-4)3}I^2[m]\\ & a_{(m-1)2}I^{-2}[2]a_{23}P^0a_{14}J^2[n]a_{3n}I^2[m]a_{(m-1)(n-1)}I^{-2}[2]\\ & a_{3(n-2)}I^2[m]a_{(m-1)(n-3)}I^{-2}[2]a_{3(n-4)}I^2[m]a_{(m-1)9}I^{-2}[2]a_{3(n-6)}\\ & I^2[m]a_{(m-1)(n-7)}I^{-2}[2]a_{36}I^2[m]a_{(m-1)5}I^{-2}[2]a_{34}I^2[m]a_{(m-1)3}\\ & I^{-2}[4]a_{(m-2)2}I^2[m]a_{(m-1)1}I^{-2}[4]a_{32}P^0I^{-1}J^{-1}P^0I^{-1}J^1P^0J^1\end{aligned}$$

is a Hamiltonian path between the vertices a_{11} and a_{13}.
Sub case (ii): $t = 2$.
In $G = M_{gr}(m, n)$, $d(a_{11}, a_{24}) = 2$ and the path

$$\begin{aligned}P :\ & a_{11}I^2[3]P^0I^{-1}J^1P^0J^1P^0I^{-1}J^1P^0J^2[n]a_{3n}\\ & I^2[m]a_{(m-1)(n-1)}I^{-2}[2]a_{3(n-2)}I^2[m]a_{(m-1)(n-3)}I^{-2}[2]a_{3(n-4)}\\ & I^2[m]a_{(m-1)(n-5)}I^{-2}[2]a_{3(n-6)}I^2[m]a_{(m-1)(n-7)}I^{-2}[2]a_{3(n-8)}\\ & I^2[m]a_{(m-1)5}I^{-2}[2]a_{34}I^2[m]a_{(m-1)3}I^{-2}[4]a_{52}J^2[m]P^0\\ & a_{61}P^0a_{41}P^0a_{32}P^0I^{-1}J^{-1}P^0I^{-1}J^1P^0J^1\\ & P^0J^2[n-1]a_{2n}I^2[m-1]a_{m(n-1)}J^{-2}[1]a_{(m-1)2}J^2[n-2]a_{5(n-1)}\\ & J^{-2}[1]a_{(m-3)2}J^2[n-2]a_{3(n-1)}[P^0I^{-1}J^{-1}P^0I^1J^{-1}]^3\\ & P^0a_{26}P^0I^1J^{-1}P^0J^{-2}P^0I^{-1}J^1\end{aligned}$$

is a Hamiltonian path between the vertices a_{11} and a_{24}. In Fig 7, Hamiltonian path from vertex a_{11} to a_{24} in the graph $M_{gr}(3, 6)$ is shown.

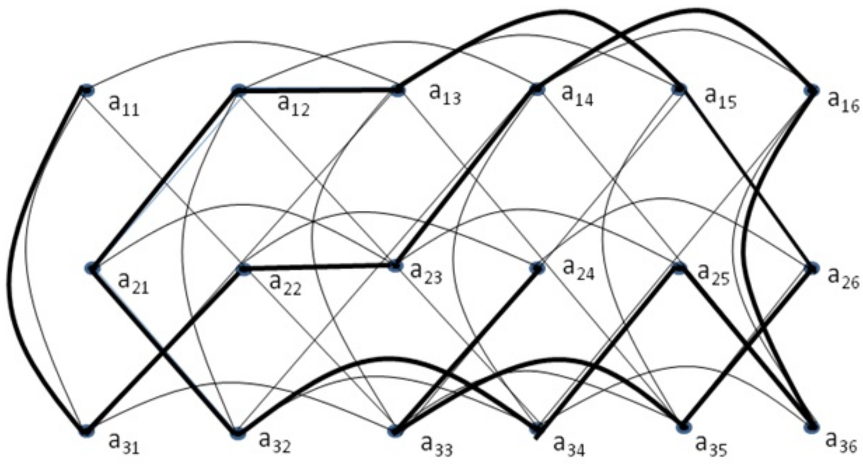

FIGURE 7. Hamiltonian path from vertex a_{11} to a_{24} in the graph $M_{gr}(3,6)$.

Sub case (iii): $t = 3$.

In $G = M_{gr}(m,n)$, $d(a_{11}, a_{14}) = 3$ and the path

$$P : a_{11}I^2[m]a_{m3}J^2[n-1]a_{(m-1)n}I^{-2}[2]a_{1(n-1)}I^2[m-2]a_{(m-1)(n-2)}I^{-2}[2]a_{1(n-3)} I^2[m-2]a_{(m-1)(n-4)}I^{-2}[2]a_{1(n-5)}I^2[m-2]a_{(m-1)(n-6)}I^{-2}[2]a_{1(n-7)}I^2[m-2] a_{(m-1)(n-8)}I^{-2}[2]a_{15}I^2[m-2]a_{(m-1)4}I^{-2}[2]a_{13}I^2[m-2]a_{(m-1)2}I^{-2}[2] P^0a_{23}P^0I^{-1}J^{-1}P^0I^1J^{-1}P^0I^1J^1P^0a_{41}I^2[m-1]a_{m2} J^2[n]a_{(m-1)(n-1)}J^{-2}[3]a_{(m-2)2}J^2[n]a_{4(n-1)}J^{-2}[3]a_{34}J^2[n]a_{2(n-1)} P^0I^{-1}J^1P^0J^{-2}P^0I^1J^{-1}[P^0I^{-1}J^{-1}P^0I^1J^{-1}]^2P^0 a_{16}P^0I^1J^{-1}P^0I^{-1}J^{-1}$$

is a Hamiltonian path between the vertices a_{11} and a_{14}.

Sub case (iv): $t = 4$.

In $G = M_{gr}(m,n)$, $d(a_{11}, a_{16}) = 4$ and the path

$$P : a_{11}I^2[m-2]a_{m1}J^2[n-1]a_{(m-1)n}I^{-2}[2]a_{1(n-1)}I^2[m-2]a_{(m-1)(n-2)}I^{-2}[2] a_{1(n-3)}I^2[m-2]a_{(m-1)(n-4)}I^{-2}[2]a_{1(n-5)}I^2[m-2]a_{(m-1)(n-6)}I^{-2}[2]a_{1(n-7)} I^2[m-2]a_{(m-1)(n-6)}I^{-2}[2]a_{15}I^2[m-2]a_{(m-1)4}I^{-2}[2]a_{13}I^2[m-2] a_{(m-1)2}I^{-2}[2]P^0J^1P^0I^{-1}J^1P^0J^{-2}P^0I^1J^{-1}P^0I^1 J^1P^0a_{41}I^2[m-1]a_{m2}J^2[n]a_{(m-1)(n-1)}J^{-2}[3]a_{(m-2)2}J^2[n]a_{4(n-1)} J^{-2}[3]a_{34}J^2[n]a_{2(n-1)}P^0I^{-1}J^1P^0J^{-2}P^0a_{2(n-3)}P^0J^{-2} P^0I^{-1}J^1P^0J^{-2}P^0I^1J^{-1}P^0a_{25}P^0a_{16}$$

is a Hamiltonian path between the vertices a_{11} and a_{16}.

Sub case (v): $t = 5$.

In $G = M_{gr}(m, n)$, $d(a_{11}, a_{18}) = 5$ and the path

$$P : a_{11}I^2[m-2]a_{m1}J^2[n-1]a_{(m-1)n}I^{-2}[2]a_{1(n-1)}I^2[m-2]a_{(m-1)(n-2)}I^{-2}[2]$$
$$a_{1(n-3)}I^2[m-2]a_{(m-1)(n-4)}I^{-2}[2]a_{1(n-5)}I^2[m-2]a_{(m-1)(n-6)}I^{-2}[2]a_{1m}$$
$$I^2[m-2]a_{(m-1)(n-8)}I^{-2}[2]a_{15}I^2[m-2]a_{(m-1)4}I^{-2}[2]a_{13}I^2[m-2]$$
$$a_{(m-1)2}I^{-2}[2]P^0J^1P^0I^{-1}J^1P^0J^{-2}P^0I^1J^{-1}P^0$$
$$I^1J^1P^0a_{41}I^2[m-1]a_{m2}J^2[n]a_{(m-1)(n-1)}J^{-2}[3]a_{(m-2)2}J^2[n]$$
$$a_{4(n-1)}J^{-2}[3]a_{34}J^2[n]P^0I^{-2}P^0a_{2(n-1)}P^0I^{-1}J^{-1}P^0$$
$$I^1J^{-1}P^0I^{-1}J^{-1}P^0a_{29}J^{-2}P^0I^{-1}J^1P^0J^2$$

is a Hamiltonian path between the vertices a_{11} and a_{18}.

Sub case (vi): $t = 6$.

In $G = M_{gr}(m, n)$, $d(a_{11}, a_{110}) = 6$ and the path

$$P : a_{11}I^2[m-2]a_{m1}J^2[n-1]a_{(m-1)n}I^{-2}[2]a_{1(n-1)}I^2[m-2]a_{(m-1)(n-2)}I^{-2}[2]$$
$$a_{1(n-3)}I^2[m-2]a_{(m-1)(n-4)}I^{-2}[2]a_{1(n-5)}I^2[m-2]a_{(m-1)(n-6)}I^{-2}[2]a_{1(n-7)}$$
$$I^2[m-2]a_{(m-1)(n-8)}I^{-2}[2]a_{15}I^2[m-2]a_{(m-1)4}I^{-2}[2]a_{13}I^2[m-2]a_{(m-1)2}$$
$$I^{-2}[2]P^0J^1P^0I^{-1}J^1P^0J^{-2}P^0I^1J^{-1}P^0I^1J^1P^0$$
$$I^1J^{-1}P^0a_{(m-1)1}P^0a_{m2}J^2[n]a_{(m-1)(n-1)}J^{-2}[3]a_{(m-2)2}J^2[n]$$
$$a_{4(n-1)}J^{-2}[3]a_{34}J^2[n]P^0a_{(m-6)n}$$
$$P^0I^1J^{-1}P^0I^{-1}J^{-1}P^0I^1J^{-1}P^0a_{29}J^{-2}[5]P^0I^{-1}J^1P^0J^2[10]$$

is a Hamiltonian path between the vertices a_{11} and a_{110}. In Fig 8, Hamiltonian path from vertex a_{11} to a_{110} in the graph $M_{gr}(5, 10)$ is shown.

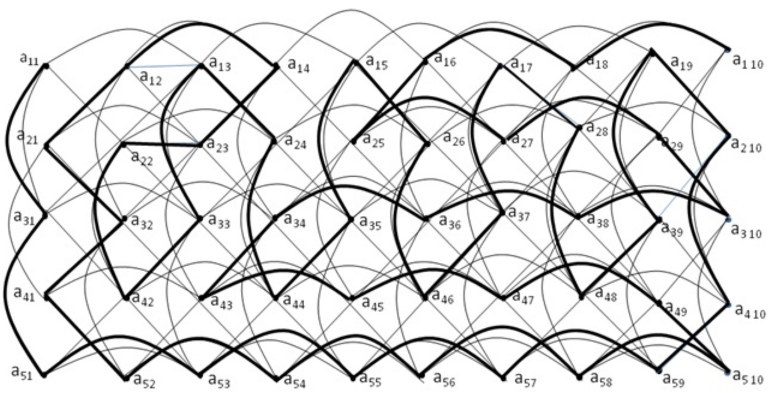

FIGURE 8. Hamiltonian path from vertex a_{11} to a_{110} in the graph $M_{gr}(5, 10)$.

Sub case (vii): $t = 7$**.**

In $G = M_{gr}(m,n)$, $d(a_{11}, a_{610}) = 7$ and the path

$$P : a_{11}P^0I^1J^1P^0J^1P^0a_{1(m-3)}J^2[n]a_{3n}I^2[m] \\ a_{(m-1)(n-1)}I^{-2}[2]a_{(m-4)(n-2)}I^2[m]a_{(m-1)(n-3)}I^{-2}[2]a_{3(n-4)} \\ I^2[m]a_{(m-1)(n-5)}I^{-2}[2]a_{3(n-6)}I^2[m]a_{(m-1)(n-7)}I^{-2}[2]a_{3(n-8)} \\ I^2[m]a_{(m-1)(n-9)}I^{-2}[2]a_{34}I^2[m]a_{(m-1)3}I^{-2}[4]a_{(m-4)2}I^2[m] \\ a_{(m-1)1}I^{-2}[2]P^0I^{-1}J^1P^0J^2[n-1]P^0a_{1(n-3)}J^2[n-1]a_{2n} \\ J^{-2}[4]a_{(m-4)3}J^2[n-1]a_{(m-3)n}J^{-2}[2]P^0I^{-1}J^{-1}P^0I^2P^0 \\ J^2[n-1]P^0I^1J^1P^0I^1J^{-1}P^0I^{-1}J^{-1}P^0I^1J^{-1}P^0 \\ J^{-2}[1]a_{(m-1)2}J^2[n-4]$$

is a Hamiltonian path between the vertices a_{11} and a_{610}.

Sub case (viii): $t = 8$**.**

In $G = M_{gr}(m,n)$, $d(a_{11}, a_{612}) = 8$ and the path

$$P : a_{11}P^0I^1J^1P^0J^1P^0a_{1(m-3)}J^2[n]a_{3n}I^2[m]a_{(m-1)(n-1)}I^{-2}[2]a_{(m-4)(n-2)} \\ I^2[m]a_{(m-1)(n-3)}I^{-2}[2]a_{3(n-4)}I^2[m]a_{(m-1)(n-5)}I^{-2}[2]a_{3(n-6)}I^2[m]a_{(m-1)(n-7)} \\ I^{-2}[2]a_{3(n-8)}I^2[m]a_{(m-1)(n-9)}I^{-2}[2]a_{34}I^2[m]a_{(m-1)3}I^{-2}[4]a_{(m-4)2} \\ I^2[m]a_{(m-1)1}I^{-2}[2]P^0I^{-1}J^1P^0J^2[n-1]P^0a_{1(n-3)}J^2[n-1]a_{2n}J^{-2}[4] \\ a_{(m-4)3}J^2[n-1]a_{(m-3)n}J^{-2}[2]P^0I^{-1}J^{-1}P^0I^2P^0J^2[n-1]P^0I^1J^1 \\ P^0I^1J^{-1}P^0J^{-2}[1]a_{(m-1)2}J^2[n-2]$$

is a Hamiltonian path between the vertices a_{11} and a_{612}.

Sub case (ix): $t = 9$**.**

In $G = M_{gr}(m,n)$, $d(a_{11}, a_{712}) = 9$ and the path

$$P : a_{11}J^2[n-1]a_{2n}I^2[m-1]a_{m(n-1)}I^{-2}[3]a_{2(n-2)}I^2[m-1]a_{m(n-3)}I^{-2}[m]a_{2(n-4)} \\ I^2[m-1]a_{m(n-5)}I^{-2}[3]a_{2(n-6)}I^2[m-1]a_{m(n-7)}I^{-2}[3]a_{2(n-8)}I^2[m-1]a_{m(n-9)} \\ I^{-2}[3]a_{2(n-10)}I^2[m-1]a_{m3}I^{-2}[m-4]a_{(m-2)2}I^2[m-1]a_{m1}I^{-2}[3]P^0I^{-1} \\ J^1P^0J^{-2}P^0I^{-1}J^1P^0J^2[n]a_{2(m-1)}J^{-2}[5]a_{(m-4)4}J^2[n]a_{(m-3)(n-1)} \\ J^{-2}[3]P^0I^{-1}J^{-1}P^0I^1J^{-1}P^0I^1J^1P^0J^2[n]a_{mn} \\ P^0I^{-1}J^{-1}P^0J^{-2}[1]a_{m2}J^2[n-2]$$

is a Hamiltonian path between the vertices a_{11} and a_{712}.

Sub case (x): $t = 10$**.**

In $G = M_{gr}(m,n)$, $d(a_{11}, a_{714}) = 10$ and the path

$$P : a_{11}J^2[n-1]a_{2n}I^2[m-1]a_{m(n-1)}I^{-2}[3]a_{2(n-2)}I^2[m-1]a_{m(n-3)}I^{-2}[m]a_{2(n-4)} \\ I^2[m-1]a_{m(n-5)}I^{-2}[3]a_{2(n-6)}I^2[m-1]a_{m(n-7)}I^{-2}[3]a_{2(n-8)}I^2[m-1]a_{m(n-9)} \\ I^{-2}[3]a_{2(n-10)}I^2[m-1]a_{m3}I^{-2}[m-4]a_{(m-2)2}I^2[m-1]a_{m1}I^{-2}[3]P^0I^{-1}$$

$J^1P^0J^{-2}P^0I^{-1}J^1P^0J^2[n]a_{2(m-1)}J^{-2}[5]a_{(m-4)4}J^2[n]a_{(m-3)(n-1)}$
$J^{-2}[3]P^0I^{-1}J^{-1}P^0I^1J^{-1}P^0I^1J^1P^0J^2[n]P^0I^1J^{-1}P^0J^{-2}[1]a_{m2}J^2[n]$

is a Hamiltonian path between the vertices a_{11} and a_{714}. In Fig 9, Hamiltonian path from vertex a_{11} to a_{714} in the graph $M_{gr}(7,14)$ is shown.

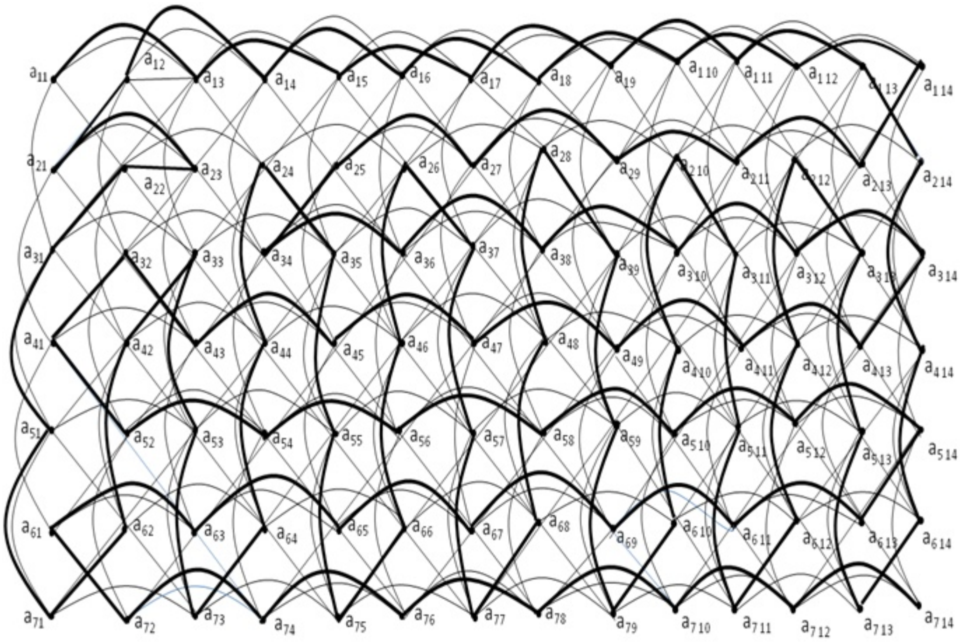

FIGURE 9. Hamiltonian path from vertex a_{11} to a_{714} in the graph $M_{gr}(7,14)$.

Case (ii): n **is even.**
Sub case (i): $t = 1$.
In $G = M_{gr}(m,n)$, $d(a_{11}, a_{13}) = 1$ and the path

$$P: a_{11}I^2[m-1]a_{m2}J^2[n]a_{(m-2)n}I^{-2}[2]a_{1(n-1)}I^2[m-1]a_{(m-2)(n-2)}I^{-2}[2]a_{1(n-3)}$$
$$I^2[m-1]a_{(m-2)(n-4)}I^{-2}[2]a_{17}I^2[m-1]a_{(m-2)(n-6)}I^{-2}[2]a_{15}I^2[m-1]a_{(m-2)4}$$
$$I^{-2}[2]a_{33}I^2[m-1]a_{(m-2)2}I^{-2}[2]a_{23}P^0a_{14}J^2[n]a_{3n}I^2[m-1]a_{m(n-1)}$$
$$I^{-2}[2]a_{3(n-2)}I^2[m-1]a_{m9}I^{-2}[2]a_{38}I^2[m-1]a_{6(n-5)}I^{-2}[2]a_{36}I^2[m-1]$$
$$a_{m5}I^{-2}[2]a_{34}I^2[m-1]a_{m3}I^{-2}[4]a_{52}I^{-2}[4]$$
$$P^0a_{m1}I^{-2}[4]a_{32}P^0I^{-1}J^{-1}P^0I^{-1}J^1P^0J^1$$

is a Hamiltonian path between the vertices a_{11} and a_{13}.
Sub case (ii): $t = 2$.

In $G = M_{gr}(m,n)$, $d(a_{11}, a_{24}) = 2$ and the path

$$\begin{aligned}P : & a_{11}I^2P^0I^{-1}J^1P^0J^1P^0I^{-1}J^1P^0J^2[n]a_{3n}I^2[m-1]a_{m(n-1)}\\ & I^{-2}[2]a_{3(n-2)}I^2[m-1]a_{m(n-3)}I^{-2}[2]a_{3(n-4)}I^2[m-1]a_{m(n-5)}\\ & I^{-2}[2]a_{3(n-6)}I^2[m-1]a_{m(n-7)}I^{-2}[2]a_{34}I^2[m-1]a_{m3}P^0a_{(m-1)(n-10)}P^0a_{m1}\\ & I^{-2}[4]P^0a_{32}P^0I^{-1}J^{-1}P^0I^{-1}J^1P^0J^1P^0J^2[n-1]a_{2n}\\ & I^2[m]a_{m(n-2)}J^{-2}[2]a_{(m-1)1}J^2[n-1]a_{4(n-2)}J^{-2}[2]a_{33}J^2[n-2]a_{2(n-2)}J^{-2}[4]\end{aligned}$$

is a Hamiltonian path between the vertices a_{11} and a_{24}.

Sub case (iii): $t = 3$.

In $G = M_{gr}(m,n)$, $d(a_{11}, a_{14}) = 3$ and the path

$$\begin{aligned}P : & a_{11}I^2[m-1]a_{m2}J^2[n]a_{4n}I^{-2}[2]a_{1(n-1)}I^2[m-1]a_{(m-2)(n-2)}I^{-2}[2]a_{1(n-3)}\\ & I^2[m-1]a_{(m-2)(n-4)}I^{-2}[2]a_{1(n-5)}I^2[m-1]a_{(m-2)(n-6)}I^{-2}[2]a_{1(n-7)}I^2[m-1]\\ & a_{(m-2)(n-8)}I^{-2}[2]a_{13}I^2[m-2]a_{(m-2)2}I^{-2}[2]P^0a_{23}P^0I^{-1}J^{-1}P^0I^1J^{-1}P^0\\ & I^1J^1P^0I^1J^{-1}P^0a_{m1}J^2[n-1]a_{5n}J^{-2}[2]a_{(m-2)3}J^2[n-1]a_{3n}J^{-2}[4]a_{25}\\ & J^2[n-1]a_{1n}J^{-2}[4]\end{aligned}$$

is a Hamiltonian path between the vertices a_{11} and a_{14}. In Fig 10, Hamiltonian path from vertex a_{11} to a_{114} in the graph $M_{gr}(4,8)$ is shown.

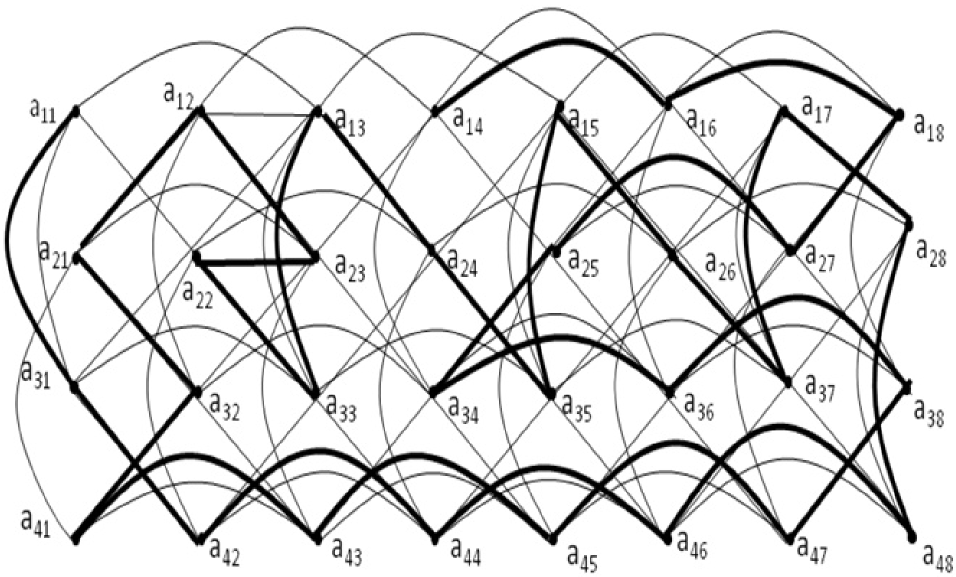

FIGURE 10. Hamiltonian path from vertex a_{11} to a_{114} in the graph $M_{gr}(4,8)$.

Sub case (iv): $t = 4$.

In $G = M_{gr}(m, n)$, $d(a_{11}, a_{16}) = 4$ and the path

$$\begin{aligned}P :\ & a_{11}I^2[m-1]a_{m2}J^2[n]a_{(m-2)n}I^{-2}[2]a_{1(n-1)}I^2[m-1]a_{(m-2)(n-2)}I^{-2}[2]a_{1(n-3)}\\ & I^2[m-1]a_{(m-2)(n-4)}I^{-2}[2]a_{1(n-5)}I^2[m-1]a_{(m-2)(n-6)}I^{-2}[2]a_{15}I^2[m-1]\\ & a_{(m-2)(n-8)}I^{-2}[2]a_{13}I^2[m-1]a_{(m-2)2}I^{-2}[2]P^0J^1P^0I^{-1}J^1P^0\\ & J^{-2}P^0I^1J^{-1}P^0I^1J^1P^0I^1J^{-1}P^0a_{m1}J^2[n-1]a_{5n}\\ & J^{-2}[2]a_{(m-2)3}J^2[n-1]a_{3n}J^{-2}[4]a_{25}J^2[n-1]a_{1n}J^{-2}[6]\end{aligned}$$

is a Hamiltonian path between the vertices a_{11} and a_{16}.

Sub case (v): $t = 5$.

In $G = M_{gr}(m, n)$, $d(a_{11}, a_{18}) = 5$ and the path

$$\begin{aligned}P :\ & a_{11}I^2[m-1]a_{m2}J^2[n]a_{(m-2)n}I^{-2}[2]a_{1(n-1)}I^2[m-1]a_{(m-2)(n-2)}\\ & I^{-2}[2]a_{1(n-3)}I^2[m-1]a_{(m-2)(n-4)}I^{-2}[2]a_{1(n-5)}I^2[m-1]a_{(m-2)(n-6)}\\ & I^{-2}[2]a_{15}I^2[m-1]a_{(m-2)4}I^{-2}[2]a_{13}I^2[m-1]a_{(m-2)2}I^{-2}[2]P^0J^1\\ & P^0I^{-1}J^1P^0J^{-2}P^0I^1J^{-1}P^0I^1J^1P^0I^1J^{-1}P^0a_{m1}J^2[n-1]a_{5n}\\ & J^{-2}[2]a_{(m-2)3}J^2[n-1]a_{3n}J^{-2}[4]a_{25}P^0a_{16}P^0a_{27}J^2[n-1]a_{1n}J^{-2}[8]\end{aligned}$$

is a Hamiltonian path between the vertices a_{11} and a_{18}.

Sub case (vi): $t = 6$.

In $G = M_{gr}(m, n)$, $d(a_{11}, a_{110}) = 6$ and the path

$$\begin{aligned}P :\ & a_{11}I^2[m-1]a_{m2}J^2[n]a_{(m-2)n}I^{-2}[2]a_{1(n-1)}I^2[m-1]a_{(m-2)(n-2)}I^{-2}[2]a_{1(n-3)}\\ & I^2[m-1]a_{(m-2)(n-4)}I^{-2}[2]a_{1(n-5)}I^2[m-1]a_{(m-2)(n-6)}I^{-2}[2]a_{1(n-7)}I^2[m-1]\\ & a_{(m-2)(n-8)}I^{-2}[2]a_{1(n-9)}I^2[m-1]a_{(m-2)2}I^{-2}[2]P^0J^1P^0I^{-1}J^1P^0J^{-2}P^0I^1\\ & J^{-1}P^0I^1J^1P^0I^1J^{-1}P^0a_{m1}J^2[n-1]a_{(m-1)n}J^{-2}[2]a_{(m-2)3}J^2[n-1]\\ & a_{3n}J^{-2}[4]a_{2(n-7)}P^0I^{-1}J^1P^0I^1J^1P^0I^{-1}J^1P^0a_{2(n-3)}\\ & J^2[n-1]a_{1n}J^{-2}[n-2]\end{aligned}$$

is a Hamiltonian path between the vertices a_{11} and a_{110}.

Sub case (vii): $t = 7$.

In $G = M_{gr}(m, n)$, $d(a_{11}, a_{610}) = 7$ and the path

$$\begin{aligned}P :\ & a_{11}P^0I^1J^1P^0J^1P^0a_{1(m-2)}J^2[n]a_{(m-3)n}I^2[m-1]a_{m(n-1)}\\ & I^{-2}[2]a_{(m-3)(n-2)}I^2[m-1]a_{m(n-3)}I^{-2}[2]a_{(m-3)(n-4)}I^2[m-1]a_{m(n-5)}\\ & I^{-2}[2]a_{(m-3)m}I^2[m-1]a_{m(n-7)}I^{-2}[2]a_{(m-3)(n-8)}I^2[m-1]a_{m(n-9)}I^{-2}[m-2]\\ & a_{(m-3)2}I^2[m-1]a_{m1}I^{-2}[2]a_{12}P^0J^1P^0J^2[n-1]a_{2n}J^{-2}[4]a_{(m-3)3}\\ & J^2[n-1]a_{(m-2)n}J^{-2}[2]a_{(m-3)1}I^2[m-1]a_{m2}[P^0I^{-1}J^1P^0I^1J^1]^3\\ & P^0I^{-1}J^1P^0J^2P^0a_{mn}J^{-2}[n-2]\end{aligned}$$

is a Hamiltonian path between the vertices a_{11} and a_{610}. In Fig 11, Hamiltonian path from vertex a_{11} to a_{610} in the graph $M_{gr}(6,12)$ is shown.

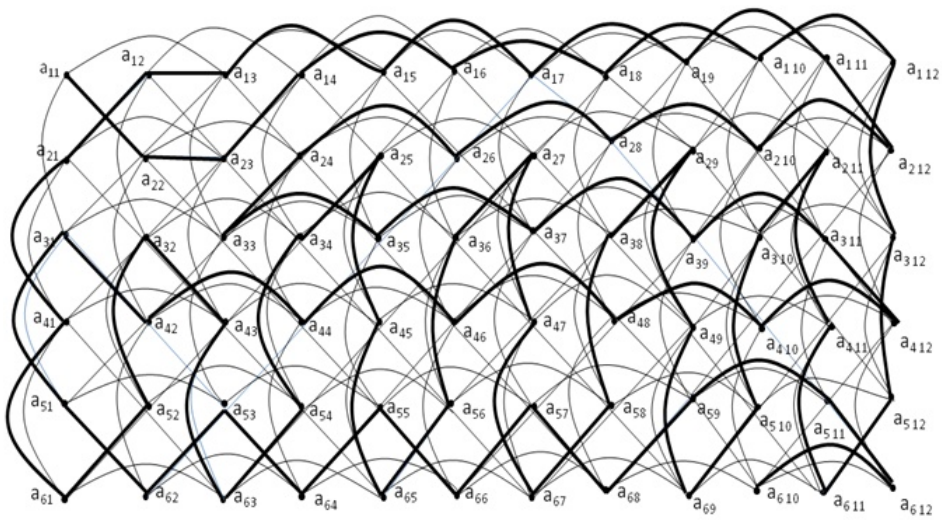

FIGURE 11. Hamiltonian path from vertex a_{11} to a_{610} in the graph $M_{gr}(6,12)$.

Sub case (viii): $t = 8$.

In $G = M_{gr}(m,n)$, $d(a_{11}, a_{612}) = 8$ and the path

$$\begin{aligned} P : {} & a_{11}P^0I^1J^1P^0J^1P^0a_{1(m-2)}J^2[n]a_{(m-3)n}I^2[m-1]a_{m(n-1)}I^{-2}[2]a_{(m-3)(n-2)} \\ & I^2[m-1]a_{m(n-3)}I^{-2}[2]a_{(m-3)(n-4)}I^2[m-1]a_{m(n-5)}I^{-2}[2]a_{(m-3)m} \\ & I^2[m-1]a_{m(n-7)}I^{-2}[2]a_{(m-3)(n-8)}I^2[m-1]a_{m(n-9)}I^{-2}[m-2]a_{(m-3)2} \\ & I^2[m-1]a_{m1}I^{-2}[2]a_{12}P^0J^1P^0J^2[n-1]a_{.2n}J^{-2}[4]a_{(m-3)3} \\ & J^2[n-1]a_{(m-2)n}J^{-2}[2]a_{(m-3)1}I^2[m-1]a_{m2}[P^0I^{-1}J^1P^0I^1J^1]^5 \end{aligned}$$

is a Hamiltonian path between the vertices a_{11} and a_{612}. □

3. Conclusion

In this paper we have explored the Hamiltonian-t^*-laceability properties of modified distance graph of grid graphs $M_{gr}(m,n)$ for $n = m$, $4 < m < 11$ and for $n = 2m$, $2 < m < 8$. Hamiltonian laceability properties of distance graphs and product graphs respectively can be used to design the network topology. Hamiltonian laceability in single connected graph can be used to overcome network failures if they are connected in the form of hybrid topology.

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Received by editors 01.04.2016; Available online 12.12.2016.

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