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FIXED POINT THEOREMS FOR A PAIR OF MAPPINGS SATISFYING ϕ - IMPLICIT RELATIONS IN WEAK PARTIAL METRIC SPACES

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Abstract. In this paper we prove two general fixed point theorems for a pair of mappings satisfying ϕ - implicit relations in weak partial metric spaces, generalizing some known results.

1. Introduction

In 1994, Matthews [**14**] introduced the concept of partial metric spaces as a part of the study of denotional semantics of dataflow networks and proved the Banach contraction principle in such spaces. The partial metric spaces play an important role in constructing models in the theory of computation. Many authors studied the fixed points for mappings satisfying contractive conditions in [**3**], [**6**], [**12**] and in other papers. In [**12**] some fixed point theorems for particular pairs of mappings are proved, which generalize some results from [**3**], [**6**] and from other papers.

In 1999, Heckmann [**10**] introduced the concept of weak partial metric spaces, which is a generalization of partial metric space.

Some results of self mappings on weak partial metric spaces are recently obtained in [**2**] and [**7**].

Several classical fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit relation in [**15**], [**16**] and in other papers. Recently, this method is used in the study of fixed points in metric spaces, symmetric spaces, quasi - metric spaces, b - metric spaces, Hilbert spaces, ultra - metric spaces, convex metric spaces, compact metric spaces, in two

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and three metric spaces, for single - valued mappings, hybrid pairs of mappings and set - valued mappings.

Quite recently, the method is used in the study of fixed points for mappings satisfying contractive / extensive conditions of integral type, in fuzzy metric spaces, probabilistic metric spaces, intuitionistic metric spaces, G - metric spaces and G_p - metric spaces.

With this method the proofs of some fixed point theorems are more simple. Also, the method allows the study of local and global properties of fixed point structures.

The study of fixed points of self mappings in complete partial metric spaces using implicit relations is introduced in [**8**], [**9**], [**17**] - [**19**].

The notion of φ - implicit relation is introduced in [**4**], generalizing the results from [**15**], [**16**].

The notion of weakly compatible mappings is introduced in [**11**]. This notion is used often in proofing the existence of common fixed points.

The purpose of this paper is to prove two general fixed point theorems for a pair of self mappings satisfying a φ - contractive condition in weak partial metric spaces, generalizing Theorem 3.1 [**2**] and Theorems 3.5 and 3.9 [**12**].

2. Preliminaries

Definition 2.1 ([**14**]). A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}_+$ such that for all $x, y, z \in X$:

$(P_1) : x = y$ if and only if $p(x,x) = p(x,y) = p(y,y)$,

$(P_2) : p(x,x) \leqslant p(x,y)$,

$(P_3) : p(x,y) = p(y,x)$,

$(P_4) : p(x,z) \leqslant p(x,y) + p(y,z) - p(y,y)$.

The pair (X,p) is called a partial metric space.

If $p(x,y) = 0$, then $x = y$, but the converse does not always hold.

Each partial metric p on X generates a T_0 - topology τ_p which has as base the family of p - open balls $\{B_p(x,\varepsilon) : x \in X,\ \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) \leqslant p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

A sequence $\{x_n\}$ of a partial metric space (X,p) converges with respect to τ_p to a point $x \in X$ if and only if

$$p\,(x,x) = \lim_{n\to\infty} p\,(x,x_n)\,.$$

If p is a partial metric on X, then the function

$$d_w(x,y) = p(x,y) - \min\{p(x,x), p(y,y)\}$$

is an ordinary metric on X.

Remark 2.1. Let $\{x_n\}$ be a sequence in a partial metric space (X,p) and $x \in X$, then $\lim_{n\to\infty} d_w(x,y) = 0$ if and only if

$$p\,(x,x) = \lim_{n\to\infty} p\,(x_n,x) = \lim_{n,m\to\infty} p\,(x_n,x_m)\,. \tag{2.1}$$

Definition 2.2 ([**14**]).

1) A sequence $\{x_n\}$ in a partial metric space (X,p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.

2) A partial metric (X,p) is said to be complete if every Cauchy sequence in X converges with respect to τ_p to a point $x \in X$ such that

$$p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m).$$

Theorem 2.1 ([**12**]). *Let (X,p) be a partial metric space and $f,g : X \to X$ be two self mappings. If there exists nonnegative constants A,B,C,D,E with $A+B+C+2D+E<1$ and $A+B+C+D+2E<1$ such that*

$$p(fx,gy) \leqslant Ap(x,y) + Bp(x,fx) + Cp(y,gy) + Dp(x,gy) + Ep(y,fx)$$

for all $x,y \in X$, then f and g have a unique common fixed point z such that $p(z,z)=0$.

Theorem 2.2 ([**12**]). *Let (X,p) be a complete partial metric space and $f : X \to X$ be a self map. If there exists $k \in \left[0,\frac{1}{2}\right)$ such that*

$$p(fx,fy) \leqslant k \max\{p(x,y), p(x,fx), p(y,fy), p(x,fy), p(y,fx)\}$$

for all $x,y \in X$, then f has a unique fixed point.

Definition 2.3 ([**10**]). A weak partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}_+$ such that for all $x,y,z \in X$:
$(wP_1) : x = y$ if and only if $p(x,x) = p(x,y) = p(y,y)$,
$(wP_2) : p(x,y) = p(y,x)$,
$(wP_3) : p(x,z) \leqslant p(x,y) + p(y,z) - p(y,y)$.
The pair (X,p) is called a weak partial metric space.

Obviously, every partial metric space is a weak partial metric space, but the converse may not be true.

If $X = [0,\infty)$ and $p(x,y) = \frac{x+y}{2}$, then (X,p) is a weak partial metric space but (X,p) is not a partial metric space.

Theorem 2.3 ([**2**]). *Let (X,p) be a weak partial metric space. Then $d_w(x,y) : X \times X \to \mathbb{R}_+$ is a metric on X.*

Remark 2.2. In a weak partial metric space, the convergence of sequences, Cauchy sequences and completeness are defined as in partial metric space.

Theorem 2.4 ([**2**]). *Let (X,p) be a weak partial metric space.*
a) $\{x_n\}$ is a Cauchy sequence in (X,p) if and only if $\{x_n\}$ is a Cauchy sequence in metric space (X,d_w).
b) (X,p) is complete if and only if (X,d_w) is complete.

Lemma 2.1. *Let (X,p) be a weak partial metric space and $\{x_n\}$ is a sequence in X. If $\lim_{n\to\infty} x_n = x$ and $p(x,x)=0$, then*

$$\lim_{n\to\infty} p(x_n,y) = p(x,y), \forall y \in X.$$

Proof. By (wP_3) we have

$$p(x,y) \leqslant p(x,x_n) + p(x_n,y).$$

Hence

$$p(x,y) - p(x,x_n) \leqslant p(x_n,y) \leqslant p(x_n,x) + p(x,y).$$

Letting n tends to infinity we obtain

$$\lim_{n\to\infty} p(x_n,y) = p(x,y).$$

□

Theorem 2.5 ([**2**]). *Let (X,p) be a complete weak partial metric space and $f : X \to X$ be a map such that*

$$p(fx,fy) \leqslant ap(x,y) + bp(x,fx) + cp(y,fy) + dp(x,fy) + ep(y,fx),$$

for all $x,y \in X$, $a,b,c,d,e \geqslant 0$ and if $d > e$, then $a+b+c+2d < 1$, and if $a < e$, then $a+b+c+2e < 1$. Then f has a unique fixed point.

Remark 2.3. Remark 2.1 is still true for weak partial metric space.

Let $\varphi : [0,\infty) \to [0,\infty)$ be a function. In connection with the function φ consider the following properties [**5**, pp. 41].

(i) φ is nondecreasing;
(ii) $\varphi(t) < t$ for all $t > 0$,
(iii) $\varphi(0) = 0$,
(iv) φ is continuous,
(v) $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t \geqslant 0$,
(vi) $\sum\limits_{n=1}^{\infty} \varphi^n(t)$ is convergent for all $t \geqslant 0$.

The next lemma shows some relationships existing between the above conditions.

Lemma 2.2 (Lemma 2.5 [**5**, pp. 41]).

(1) (i) and (ii) imply (iii),
(2) (ii) and (iv) imply (iii),
(3) (i) and (v) imply (ii).

Definition 2.4 ([**5**]).

1) A function φ satisfying (i) and (v) is said to be a comparison function.
2) A function φ satisfying (i) and (vi) is said to be a (C) - comparison function.

Lemma 2.3 (Lemma 2.2 [**5**]).

1) Any (C) - comparison function is a comparison function.
2) Any comparison function satisfies (iii).
3) If φ is a (C) - comparison function, then the function $s : \mathbb{R}_+ \to \mathbb{R}_+$, $s(t) = \sum\limits_{k=0}^{\infty} \varphi^k(t), t \in \mathbb{R}_+$ satisfies (i) and (iii).

Definition 2.5. Let X be a nonempty set and let $f, g : X \to X$ be two mappings. If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g and w is called a point of coincidence of f and g.

The set of all coincidence points of f and g is denoted by $\mathcal{C}(f, g)$.

Definition 2.6 ([**11**]). Two self mappings f and g on a nonempty set X are weakly compatible if $fgx = gfx$ for all $x \in \mathcal{C}(f, g)$.

Theorem 2.6 ([**1**]). *Let f and g be self mappings of a nonempty set X. If f and g are weakly compatible and f and g have a unique point of coincidence $w = fx = gx$, for some $x \equiv X$, then w is the unique fixed point of f and g.*

3. φ - implicit relations

Similarly, as in [**4**], we introduce a new type of implicit relation, named φ - implicit relation.

Definition 3.1. Let $\mathcal{F}_{6w}$ be the set of all lower semi - continuous functions $F : \mathbb{R}_+^6 \to \mathbb{R}$ such that:

(F_1) : F is nonincreasing in variables t_5, t_6;

(F_2) : there exists a (C) - comparison function φ such that for all $u, v \geqslant 0$:

(F_{2a}) : $F(u, v, v, u, u + v, 2v) \leqslant 0$,

and

(F_{2b}) : $F(u, v, u, v, 2v, u + v) \leqslant 0$

implies $u \leqslant \varphi(v)$;

(F_3) : $F(t, t, 0, 0, t, t) > 0, \forall t > 0$.

In the following example, the proofs of property (F_1) are easy.

Example 3.1.

$$F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6,$$

where $a, b, c, d, e \geqslant 0$ and $a + b + c + 2d + 2e < 1$.

(F_2) : Let $u, v \geqslant 0$ be such that

$$F(u, v, v, u, u + v, 2v) = u - av - bv - cu - d(u + v) - e \cdot 2v \leqslant 0.$$

Hence

$$u \leqslant av + bv + cu + d(u + v) + 2ev.$$

If

$$u > v,$$

then

$$u[1 - (a + b + c + 2d + 2e)] \leqslant 0,$$

a contradiction. Hence

$$u \leqslant v,$$

which implies

$$u \leqslant (a + b + c + 2d + 2e)v.$$

Therefore

$$u \leqslant \varphi(v),$$

where

$$\varphi(t) = (a + b + c + 2d + 2e)t.$$

Similarly, $F(u, v, u, v, 2v, u + v) \leqslant 0$ implies $u \leqslant \varphi(v)$.

$(F_3) : F(t, t, 0, 0, t, t) = t[1 - (a + d + e)] > 0, \forall t > 0$.

Example 3.2.

$$F(t_1, ..., t_6) = t_1 - k \max\{t_2, t_3, t_4, t_5, t_6\},$$

where $k \in [0, \frac{1}{2})$.

(F_2) : Let $u, v \geqslant 0$ be such that

$$F(u, v, v, u, u + v, 2v) = u - k \max\{u, v, u + v, 2v\} \leqslant 0$$

which implies

$$u \leqslant k \max\{u + v, 2v\}.$$

If

$$u > v,$$

then

$$u(1 - 2k) \leqslant 0,$$

a contradiction. Hence

$$u \leqslant v,$$

which implies

$$u \leqslant kv.$$

Therefore

$$u \leqslant \varphi(v),$$

where

$$\varphi(t) = kt, \ k \in \left[0, \frac{1}{2}\right).$$

Similarly, $F(u, v, u, v, 2v, u + v) \leqslant 0$ implies $u \leqslant \varphi(v)$.

$(F_3) : F(t, t, 0, 0, t, t) = t(1 - k) > 0, \forall t > 0$.

Example 3.3.

$$F(t_1, ..., t_6) = t_1 - \max\{at_2, b(t_3 + 2t_4), b(t_5 + t_6)\},$$

where $a \in (0, 1)$ and $b \in \left[0, \frac{1}{4}\right)$.

(F_2) : Let $u, v \geqslant 0$ be such that

$$F(u, v, v, u, u + v, 2v) = u - \max\{av, b(v + 2u), b(3v + u)\} \leqslant 0,$$

which implies

$$u \leqslant \max\{av, b(3v + u)\}.$$

If

$$u > v,$$

then

$$u(1-\max\{a,4b\})\leqslant 0,$$

a contradiction. Hence

$$u\leqslant v,$$

which implies

$$u\leqslant \max\{a,4b\}v.$$

Therefore

$$u\leqslant \varphi\left(v\right),$$

where

$$\varphi\left(t\right)=\max\{a,4b\}t,$$

with $\max\{a,4b\}<1$.

Similarly, $F\left(u,v,u,v,2v,u+v\right)\leqslant 0$ implies $u\leqslant \varphi\left(v\right)$.

(F_3) : $F\left(t,t,0,0,t,t\right)=t(1-\max\{a,4b\})>0,\forall t>0$.

Example 3.4.

$$F(t_1,...,t_6)=t_1^2-a\max\{t_2^2,t_3^2,t_4^2\}-bt_5t_6,$$

where $a,b\geqslant 0$ and $a+4b<1$.

(F_2) : Let $u,v\geqslant 0$ be such that

$$F(u,v,v,u,u+v,2v)=u^2-a\max\{u^2,v^2\}-b\left(u+v\right)\cdot 2v\leqslant 0,$$

which implies

$$u^2\leqslant a\max\{u^2,v^2\}+2bv\left(u+v\right).$$

If

$$u>v,$$

then

$$u^2[1-(a+4b)]\leqslant 0,$$

a contradiction. Hence

$$u\leqslant v,$$

which implies

$$u\leqslant \sqrt{a+4b}v.$$

Therefore

$$u\leqslant \varphi\left(v\right),$$

where

$$\varphi\left(t\right)=\sqrt{a+4b}t,$$

with $\sqrt{a+4b}<1$.

Similarly, $F\left(u,v,u,v,2v,u+v\right)\leqslant 0$ implies $u\leqslant \varphi\left(v\right)$.

(F_3) : $F\left(t,t,0,0,t,t\right)=t^2[1-(a+b)]>0,\forall t>0$.

Example 3.5.

$$F(t_1, ..., t_6) = t_1 - \varphi\left(\max\left\{t_2, t_3, t_4, \frac{t_5}{2}, \frac{t_6}{2}\right\}\right).$$

(F_2) : Let $u, v \geqslant 0$ be such that

$$F(u, v, v, u, u+v, 2v) = u - \varphi\left(\max\left\{u, v, \frac{u+v}{2}\right\}\right) \leqslant 0,$$

which implies

$$u \leqslant \varphi\left(\max\left\{u, v, \frac{u+v}{2}\right\}\right).$$

If

$$u > v,$$

then

$$u \leqslant \varphi(u) < u,$$

a contradiction. Hence

$$u \leqslant v,$$

which implies

$$u \leqslant \varphi(v).$$

Similarly, $F(u, v, u, v, 2v, u+v) \leqslant 0$ implies $u \leqslant \varphi(v)$.

(F_3) : $F(t, t, 0, 0, t, t) = t - \varphi(t) > 0, \forall t > 0$.

Example 3.6.

$$F(t_1, ..., t_6) = t_1 - \varphi(at_2 + bt_3 + c\max\{4t_4, t_5 + t_6\}),$$

where $a, b, c \geqslant 0$ and $a + b + 4c < 1$.

(F_2) : Let $u, v \geqslant 0$ be such that

$$F(u, v, v, u, u+v, 2v) = u - \varphi(av + bv + c\max\{4v, u+3v\}) \leqslant 0,$$

which implies

$$u \leqslant \varphi(av + bv + c\max\{4v, u+3v\}).$$

If

$$u > v,$$

then

$$u \leqslant \varphi((a+b+4c)u) \leqslant \varphi(u) < u,$$

a contradiction. Hence

$$u \leqslant v,$$

which implies

$$u \leqslant \varphi(v).$$

Similarly, $F(u, v, u, v, 2v, u+v) \leqslant 0$ implies $u \leqslant \varphi(v)$.

(F_3) : $F(t, t, 0, 0, t, t) = t - \varphi((a+2c)t) \geqslant t - \varphi(t) > 0, \forall t > 0$.

Example 3.7.
$$F(t_1, ..., t_6) = t_1 + \frac{t_1}{t_5 + t_6} - \varphi\left(at_2 + bt_3 + ct_4\right),$$
where $a, b, c \geqslant 0$ and $a + b + c < 1$.

(F_2) : Let $u, v \geqslant 0$ be such that
$$F(u, v, v, u, u + v, 2v) = u + \frac{u}{u + 3v} - \varphi\left(av + bv + cu\right) \leqslant 0.$$
This implies
$$u - \varphi\left(av + bv + cu\right) \leqslant 0.$$
If
$$u > v,$$
then
$$u \leqslant \varphi\left(\left(a + b + c\right) u\right) \leqslant \varphi\left(u\right) < u,$$
a contradiction. Hence
$$u \leqslant v,$$
which implies
$$u \leqslant \varphi\left(\left(a + b + c\right) v\right) \leqslant \varphi\left(v\right).$$
Similarly, $F\left(u, v, u, v, 2v, u + v\right) \leqslant 0$ implies $u \leqslant \varphi\left(v\right)$.

(F_3) : $F\left(t, t, 0, 0, t, t\right) = t + \dfrac{1}{2} - \varphi\left(at\right) \geqslant t - \varphi\left(t\right) + \dfrac{1}{2} > 0, \forall t > 0.$

4. Main results

4.1. Fixed point theorems for non weakly compatible mappings.

Theorem 4.1. *Let (X, p) be a weak partial metric space and $f, g : X \to X$ such that for all $x, y \in X$*
$$F\left(\begin{array}{c} p\left(fx, gy\right), p\left(x, y\right), p\left(x, fx\right), \\ p\left(y, gy\right), p\left(x, gy\right), p\left(y, fx\right) \end{array}\right) \leqslant 0 \tag{4.1}$$
for some $F \in \mathcal{F}_{6w}$. If z is a common fixed point of f and g, then $p\left(z, z\right) = 0$.

Proof. By (4.1) for $x = y = z$ we obtain
$$F\left(\begin{array}{c} p\left(fz, gz\right), p\left(z, z\right), p\left(z, fz\right), \\ p\left(z, gz\right), p\left(z, gz\right), p\left(z, fz\right) \end{array}\right) \leqslant 0,$$
$$F\left(p\left(z, z\right), p\left(z, z\right), p\left(z, z\right), p\left(z, z\right), p\left(z, z\right), p\left(z, z\right)\right) \leqslant 0.$$
By (F_1) we have
$$F\left(p\left(z, z\right), p\left(z, z\right), p\left(z, z\right), p\left(z, z\right), 2p\left(z, z\right), 2p\left(z, z\right)\right) \leqslant 0.$$
By (F_2) we have
$$p\left(z, z\right) \leqslant \phi\left(p\left(z, z\right)\right) < p\left(z, z\right) \text{ if } p\left(z, z\right) > 0.$$
Hence, $p\left(z, z\right) = 0$. □

Theorem 4.2. *Let (X, p) be a weak partial metric space, $f, g : X \to X$ and some $F \in \mathcal{F}_{6w}$. Then f and g have at most a common fixed point.*

Proof. Suppose that f and g have two common fixed points u and v. By (4.1) we have

$$F\left(\begin{array}{c} p(fu,gv),p(u,v),p(u,fu), \\ p(v,gv),p(u,gv),p(v,fu) \end{array}\right)\leqslant 0,$$

$$F\left(\begin{array}{c} p(u,v),p(u,v),p(u,u), \\ p(v,v),p(u,v),p(u,v) \end{array}\right)\leqslant 0.$$

By Theorem 4.1,

$$p(u,u)=p(v,v)=0.$$

Hence,

$$F(p(u,v),p(u,v),0,0,p(u,v),p(u,v))\leqslant 0,$$

a contradiction of (F_3) if $p(u,v)>0$. Hence, $p(u,v)=0$, so $u=v$. □

Theorem 4.3. *Let (X,p) be a complete weak partial metric space satisfying inequality (4.1) for all $x,y\in X$ and some $F\in\mathcal{F}_{6w}$. Then f and g have a unique common fixed point z such that $p(z,z)=0$.*

Proof. Let x_0 be an arbitrary point of (X,p). Then we define the sequence $\{x_n\}$ in X such that

$$x_{2n+1}=fx_{2n} \text{ and } x_{2n+2}=gx_{2n+1}, \text{ for } n\in\mathbb{N}. \tag{4.2}$$

$$F\left(\begin{array}{c} p(fx_{2n},gx_{2n+1}),p(x_{2n},x_{2n+1}),p(x_{2n},fx_{2n}), \\ p(x_{2n+1},gx_{2n+1}),p(x_{2n},gx_{2n+1}),p(x_{2n+1},fx_{2n}) \end{array}\right)\leqslant 0.$$

By (4.2) we obtain

$$F\left(\begin{array}{c} p(x_{2n+1},x_{2n+2}),p(x_{2n},x_{2n+1}),p(x_{2n},x_{2n+1}), \\ p(x_{2n+1},x_{2n+2}),p(x_{2n},x_{2n+2}),p(x_{2n+1},x_{2n+1}) \end{array}\right)\leqslant 0. \tag{4.3}$$

By (wP_3),

$$\begin{aligned} p(x_{2n},x_{2n+2}) &\leqslant p(x_{2n},x_{2n+1})+p(x_{2n+1},x_{2n+2})- \\ &\quad -p(x_{2n+1},x_{2n+1}) \\ &\leqslant p(x_{2n},x_{2n+1})+p(x_{2n+1},x_{2n+2}), \end{aligned}$$

$$p(x_{2n+1},x_{2n+1})\leqslant p(x_{2n+1},x_{2n})+p(x_{2n},x_{2n+1})=2p(x_{2n+1},x_{2n}).$$

By (F_1) and (4.3) we obtain

$$F\left(\begin{array}{c} p(x_{2n+1},x_{2n+2}),p(x_{2n},x_{2n+1}), \\ p(x_{2n},x_{2n+1}),p(x_{2n+1},x_{2n+2}), \\ p(x_{2n},x_{2n+1})+p(x_{2n+1},x_{2n+2}), \\ 2p(x_{2n},x_{2n+1}) \end{array}\right)\leqslant 0.$$

By (F_{2a}) we obtain

$$p(x_{2n+1},x_{2n+2})\leqslant\phi(p(x_{2n},x_{2n+1})).$$

Similarly, by (4.1) for $x=x_{2n}$ and $y=x_{2n-1}$ we obtain

$$F\left(\begin{array}{c} p(fx_{2n},gx_{2n-1}),p(x_{2n},x_{2n-1}),p(x_{2n},fx_{2n}), \\ p(x_{2n-1},gx_{2n-1}),p(x_{2n},gx_{2n-1}),p(x_{2n-1},fx_{2n}) \end{array}\right)\leqslant 0.$$

By (4.2) we obtain

$$F\left(\begin{array}{c} p(x_{2n+1},x_{2n}),p(x_{2n},x_{2n-1}),p(x_{2n},x_{2n+1}), \\ p(x_{2n-1},x_{2n}),p(x_{2n},x_{2n}),p(x_{2n-1},x_{2n+1}) \end{array}\right) \leqslant 0. \tag{4.4}$$

By (wP_3),

$$\begin{aligned} p(x_{2n-1},x_{2n+1}) &\leqslant p(x_{2n-1},x_{2n})+p(x_{2n},x_{2n+1}), \\ p(x_{2n},x_{2n}) &\leqslant 2p(x_{2n},x_{2n-1}). \end{aligned}$$

By (4.4) and (F_1) we obtain

$$F\left(\begin{array}{c} p(x_{2n+1},x_{2n}),p(x_{2n},x_{2n-1}), \\ p(x_{2n},x_{2n+1}),p(x_{2n},x_{2n-1}), \\ 2p(x_{2n},x_{2n-1}), \\ p(x_{2n-1},x_{2n})+p(x_{2n},x_{2n+1}) \end{array}\right) \leqslant 0. \tag{4.5}$$

By (4.5) and (F_{2b}) we obtain

$$p(x_{2n},x_{2n+1}) \leqslant \phi(p(x_{2n-1},x_{2n})).$$

Hence,

$$p(x_n,x_m) \leqslant \phi(p(x_{n-1},x_n)) \leqslant \ldots \leqslant \phi^n(p(x_0,x_1)).$$

For $n,m \in \mathbb{N}$, $m>n$, we have from repeated use of (wP_3) that

$$\begin{aligned} p(x_n,x_m) &\leqslant p(x_n,x_{n+1})+p(x_{n+1},x_{n+2})+\ldots+p(x_{m-1},x_m) \\ &\leqslant \sum_{k=n}^{m-1}\phi^k(p(x_0,x_1)). \end{aligned}$$

Since $\sum\limits_{k=0}^{\infty}\phi^k(p(x_0,x_1))<+\infty$, then $\lim_{n,m\to\infty}\sum\limits_{k=n}^{m-1}\phi^k(p(x_0,x_1))=0$.

Since $d_w(x_n,x_m) \leqslant p(x_n,x_m)$, then this implies that $\{x_n\}$ is a Cauchy sequence in (X,d_w). Since (X,p) is complete, then by Theorem 2.4, (X,d_w) is complete and the sequence $\{x_n\}$ converges to a point z and $\lim_{n\to\infty} d_w(x_{2n},z)=0$. Again, by Theorem 2.5,

$$p(z,z)=\lim_{n\to\infty}p(x_n,z)=\lim_{n,m\to\infty}p(x_n,x_m)=0. \tag{4.6}$$

By (4.1) for $x=z$ and $y=x_{2n+1}$ we obtain

$$F\left(\begin{array}{c} p(fz,gx_{2n+1}),p(z,x_{2n+1}),p(z,fz), \\ p(x_{2n+1},gx_{2n+1}),p(z,gx_{2n+1}),p(x_{2n+1},fz) \end{array}\right) \leqslant 0.$$

By (4.2) we obtain

$$F\left(\begin{array}{c} p(fz,x_{2n+2}),p(z,x_{2n+1}),p(z,fz), \\ p(x_{2n+1},x_{2n+2}),p(z,x_{2n+2}),p(x_{2n+1},fz) \end{array}\right) \leqslant 0.$$

Letting n tends to infinity, by Lemma 2.1 we obtain

$$F(p(z,fz),0,p(z,fz),0,0,p(z,fz)) \leqslant 0,$$

which implies by (F_{2b}) that

$$p(z,fz) \leqslant \phi(p(z,fz)) < p(z,fz),$$

a contradiction if $p(z, fz) > 0$. Hence $p(z, fz) \leqslant \phi(0) = 0$. Therefore $z = fz$ and z is a fixed point of f.

Similarly, for $x = x_{2n}$ and $y = z$ we obtain

$$F\left(\begin{array}{c} p(fx_{2n}, gz), p(x_{2n}, z), p(x_{2n}, fx_{2n}), \\ p(z, gz), p(x_{2n}, gz), p(z, fx_{2n}) \end{array}\right) \leqslant 0.$$

By (4.2) we obtain

$$F\left(\begin{array}{c} p(x_{2n+1}, gz), p(x_{2n}, z), p(x_{2n}, x_{2n+1}), \\ p(z, gz), p(x_{2n}, gz), p(z, x_{2n+1}) \end{array}\right) \leqslant 0.$$

Letting n tends to infinity we obtain

$$F(p(z, gz), 0, 0, p(z, gz), p(z, gz), 0) \leqslant 0.$$

By (F_{2a}) we have

$$p(z, gz) \leqslant \phi(0) = 0.$$

Hence $z = gz$ and z is a fixed point of g. Therefore, z is a coincidence point of f and g. By Theorem 4.1, $p(z, z) = 0$ and by Theorem 4.2, z is the unique common fixed point of f and g. □

Example 4.1. Let $X = [0, 1]$ and $p(x, y) = \frac{x+y}{2}$. Then $d_w(x, y) = \frac{1}{2}|x - y|$. Therefore, (X, d_w) is a complete metric space. By Theorem 2.4, (X, p) is a complete weakly partial metric space. Suppose $fx = \frac{x}{2}$ and $gx = 0$. Then $p(fx, gy) = \frac{x}{4}$ and $p(x, fx) = \frac{x+\frac{x}{2}}{2} = \frac{3x}{4}$. Then

$$p(fx, gy) \leqslant kp(x, fx),$$

where $k \in \left[\frac{1}{3}, \frac{1}{2}\right)$. Hence

$$\begin{aligned} p(fx, gy) \quad \leqslant \quad & k \max\{p(x, y), p(x, fx), \\ & p(y, gy), p(x, gy), p(y, fx)\}. \end{aligned}$$

By Theorem 4.3 and Example 3.2, for $k \in \left[\frac{1}{3}, \frac{1}{2}\right)$, f and g have a unique common fixed point.

By Theorem 4.3, for $f = g$ we obtain

Theorem 4.4. *Let (X, p) be a complete weak partial metric space and $f : X \to X$ such that for all $x, y \in X$*

$$F\left(\begin{array}{c} p(fx, fy), p(x, y), p(x, fx), \\ p(y, fy), p(x, fy), p(y, fx) \end{array}\right) \leqslant 0$$

and some F satisfying $(F_1), (F_{2a}), (F_3)$. Then f has a unique common fixed point z.

Remark 4.1.

1. By Theorem 4.3 and Example 3.1 we obtain a generalization of Theorem 2.1.
2. By Theorem 4.4 and Example 3.2 we obtain a generalization of Theorem 2.2.
3. By Theorem 4.4 and Example 3.1 we obtain a generalization of Theorem 2.5.

4.2. Fixed points for weakly compatible mappings.

THEOREM 4.5. *Let (X,p) be a weak partial metric space and f,g be two self mappings on X such that*

$$F\left(\begin{array}{c} p(fx,fy), p(gx,gy), p(fx,gx), \\ p(fy,gy), p(fx,gy), p(fy,gx) \end{array}\right) \leqslant 0 \tag{4.7}$$

for all $x,y \in X$, $\mathcal{C}(f,g) \neq \emptyset$ and some F satisfying property (F_{2a}). If z is a point of coincidence of f and g, then $p(z,z) = 0$.

PROOF. Since z is a point of coincidence of f and g, there exists $x \in \mathcal{C}(f,g) \neq \emptyset$ such that $z = fx = gx$.

By (4.7) for $x = y$ we obtain

$$F\left(\begin{array}{c} p(fx,fx), p(gx,gx), p(fx,gx), \\ p(fx,gx), p(fx,gx), p(fx,gx) \end{array}\right) \leqslant 0,$$

$$F(p(z,z), p(z,z), p(z,z), p(z,z), p(z,z), p(z,z)) \leqslant 0.$$

By (F_1) and (4.7) we obtain

$$F(p(z,z), p(z,z), p(z,z), p(z,z), 2p(z,z), 2p(z,z)) \leqslant 0.$$

By (F_{2a}) we obtain

$$p(z,z) \leqslant \phi(p(z,z)) < p(z,z),$$

a contradiction if $p(z,z) > 0$. Hence, $p(z,z) = 0$. □

THEOREM 4.6. *Let f,g be self mappings of a weak partial metric space satisfying inequality (4.7) for all $x,y \in X$, $\mathcal{C}(f,g) \neq \emptyset$ and some F satisfying property (F_3). Then f and g have a unique point of coincidence.*

PROOF. Suppose that f and g have two common fixed points $z = fx = gx$ and $t = fy = gy$, for some $x,y \in \mathcal{C}(f,g)$. By (4.7) and Theorem 4.5 we obtain

$$F(p(z,t), p(z,t), 0, 0, p(z,t), p(z,t)) \leqslant 0,$$

a contradiction of (F_3) if $p(z,t) > 0$. Hence, $p(z,t) = 0$, which implies $z = t$. □

THEOREM 4.7. *Let f,g be self mappings of a weak partial metric space satisfying inequality (4.7) for all $x,y \in X$ and some F satisfying properties $(F_1), (F_{2a}), (F_3)$. If*

1) $f(X) \subset g(X)$,

2) $f(X)$ or $g(X)$ is a complete subspace of X,

then $\mathcal{C}(f,g) \neq \emptyset$.

PROOF. Let x_0 be an arbitrary point of (X,p). Then, since $f(X) \subset g(X)$, there exists a sequence $\{x_n\}$ such that

$$y_0 = fx_0 = gx_1, y_1 = fx_1 = gx_2, ..., y_n = fx_n = gx_{n+1}, n \in \mathbb{N}. \tag{4.8}$$

Then by (4.7) we obtain

$$F\left(\begin{array}{c} p(fx_n, fx_{n+1}), p(gx_n, gx_{n+1}), p(fx_n, gx_n), \\ p(fx_{n+1}, gx_{n+1}), p(fx_n, gx_{n+1}), p(fx_{n+1}, gx_n) \end{array}\right) \leqslant 0.$$

By (4.8) we obtain

$$F\left(\begin{array}{c} p\left(y_n,y_{n+1}\right),p\left(y_{n-1},y_n\right),p\left(y_{n-1},y_n\right), \\ p\left(y_n,y_{n+1}\right),p\left(y_n,y_n\right),p\left(y_{n-1},y_{n+1}\right) \end{array}\right)\leqslant 0. \tag{4.9}$$

By (wP_3).

$$\begin{aligned} p\left(y_n,y_n\right) &\leqslant p\left(y_n,y_{n-1}\right)+p\left(y_{n-1},y_n\right) \\ &= 2p\left(y_{n-1},y_n\right), \end{aligned}$$

$$p\left(y_{n-1},y_{n+1}\right)\leqslant p\left(y_{n-1},y_n\right)+p\left(y_n,y_{n+1}\right).$$

By (4.9) and (F_3) we obtain

$$F\left(\begin{array}{c} p\left(y_n,y_{n+1}\right),p\left(y_{n-1},y_n\right), \\ p\left(y_{n-1},y_n\right),p\left(y_n,y_{n+1}\right), \\ 2p\left(y_n,y_{n-1}\right), \\ p\left(y_{n-1},y_n\right)+p\left(y_n,y_{n+1}\right) \end{array}\right)\leqslant 0.$$

By (F_{2a}) we obtain

$$p\left(y_n,y_{n+1}\right)\leqslant \phi\left(p\left(y_{n-1},y_n\right)\right)\leqslant ...\leqslant \phi^n\left(p\left(y_0,y_1\right)\right).$$

For $n,m\in\mathbb{N}$, $m>n$, we obtain from repeated use of (wP_3) that

$$\begin{aligned} p\left(y_n,y_m\right) &\leqslant p\left(y_n,y_{n+1}\right)+p\left(y_{n+1},y_{n+2}\right)+...+p\left(y_{m-1},y_m\right) \\ &\leqslant \sum_{k=n}^{m-1}\phi^k\left(p\left(y_0,y_1\right)\right). \end{aligned}$$

Since

$$\sum_{k=0}^{\infty}\phi^k\left(p\left(y_0,y_1\right)\right)<\infty,$$

then

$$\lim_{n,m\to\infty}\sum_{k=n}^{m-1}\phi^k\left(p\left(y_0,y_1\right)\right)=0$$

and

$$\lim_{n,m\to\infty}p\left(y_n,y_m\right)=0.$$

Since $d_w\left(y_n,y_m\right)\leqslant p\left(y_n,y_m\right)\to 0$ as $n,m\to\infty$, then $\{y_n\}$ is a Cauchy sequence in X. Hence, $\{x_n\}$ converges at a point $z\in X$, and by Theorem 2.3,

$$z=\lim_{n\to\infty}f\left(z\right)=\lim_{n\to\infty}gx_{n+1}=gu,\ \text{for some } u\in X \tag{4.10}$$

and $\lim_{n\to\infty}d_w\left(y_n,z\right)=0$. By Theorem 2.4 we obtain

$$p\left(z,z\right)=\lim_{n\to\infty}p\left(y_n,z\right)=\lim_{n,m\to\infty}p\left(y_n,y_m\right). \tag{4.11}$$

We prove that $fu=gu$. By (4.7) we have

$$F\left(\begin{array}{c} p\left(fx_n,fu\right),p\left(gx_n,gu\right),p\left(fx_n,gx_n\right), \\ p\left(fu,gu\right),p\left(fx_n,gu\right),p\left(fu,gx_n\right) \end{array}\right)\leqslant 0,$$

$$F\left(\begin{array}{c} p\left(y_n,fu\right),p\left(y_{n-1},gu\right),p\left(y_{n-1},y_n\right), \\ p\left(fu,gu\right),p\left(y_n,gu\right),p\left(fu,y_{n-1}\right) \end{array}\right)\leqslant 0.$$

By (4.10) and Lemma 2.1, letting n tends to infinity we obtain

$$F\left(p\left(z,fu\right),0,0,p\left(z,fu\right),0,p\left(z,fu\right)\right)\leqslant 0.$$

By (F_{2a}) we obtain

$$p\left(z,fu\right)\leqslant \phi\left(0\right)=0.$$

Hence $p\left(z,fu\right)=0$, which implies $z=fu=gu$ and z is a point of coincidence. By Theorem 4.6, z is the unique point of coincidence.

Moreover, if f and g are weakly compatible, by Theorem 2.6, z is the unique common fixed point. □

Example 4.2. Let $X=\left[0,\frac{1}{3}\right]$ and $p\left(x,y\right)=\frac{x+y}{2}$. Then $d_w\left(x,y\right)=\frac{1}{2}\left|x-y\right|$. Therefore, (X,d_w) is a complete metric space. By Theorem 2.4, (X,p) is a complete weakly partial metric space. Suppose $fx=x^2$ and $gx=3x$. Then

$$p\left(fx,fy\right)=\frac{x^2+y^2}{2}\leqslant\frac{x+y}{2}$$

and

$$p\left(gx,gy\right)=\frac{3\left(x+y\right)}{2}.$$

Hence

$$p\left(fx,fy\right)\leqslant 3k\cdot\frac{x+y}{2}=kp\left(gx,gy\right),$$

where $k\in\left[\frac{1}{3},\frac{1}{2}\right)$. Therefore

$$\begin{aligned} p\left(fx,fy\right) \quad\leqslant\quad & k\max\{p\left(gx,gy\right),p\left(fx,gx\right), \\ & p\left(fy,gy\right),p\left(fx,gy\right),p\left(fy,gx\right)\}, \end{aligned}$$

where $k\in\left[\frac{1}{3},\frac{1}{2}\right)$. Since $\mathcal{C}\left(f,g\right)=\{0\}$, then $fg0=gf0=0$.

Hence f and g are weakly compatible. By Theorem 4.7 and Example 3.2, for $k\in\left[\frac{1}{3},\frac{1}{2}\right)$, f and g have a unique common fixed point $z=0$ with $p\left(z,z\right)=0$.

Remark 4.2. If $gx=x$, we obtain Theorem 4.4.

References

[1] M. Abbas and B. E. Rhoades. Common fixed point results for noncommuting mappings without continuity in generalized metric spaces., *Appl. Math. Comput.*, **215**(1)(2009), 262 – 269.

[2] I. Altun and G. Durmaz. Weak partial metric spaces and some fixed point results. *Appl. Gen. Topol.*, **13** (2) (2012), 179 – 191.

[3] I. Altun, F. Sola and H. Simsek. Generalized contractions on partial metric spaces. *Topology Appl.*, **157** (18) (2010), 2778 – 2785.

[4] I. Altun and D. Turkoglu. Some fixed point theorems for weakly compatible mappings satisfying an implicit relation. *Taiwanese J. Math.*, **13** (4) (2008), 1291 – 1304.

[5] V. Berinde. *Iterative Approximation of Fixed Points*, Springer - Verlag Berlin Heidelberg, 2007.

[6] R. P. Chi, E. Karapinar and T. D. Thanh. A generalized contraction principle in partial metric spaces. *Math. Comput. Modelling*, **55** (5 – 6) (2012), 1673 – 1681.

[7] G. Durmaz, Ö. Acar and I. Altun. Some fixed point results on weak partial metric spaces. *Filomat*, **27** (2) (2013), 317 – 326.

[8] S. Güliaz and E. Karapinar. A coupled fixed point result in partially ordered partial metric spaces through implicit function. *Hacet. J. Math. Stat.*, **42** (4) (2013), 347 – 357.

[9] S. Güliaz, E. Karapinar and I. S. Yüce. A coupled coincidence point theorem in partially ordered metric spaces with an implicit relation. *Fixed Point Theory Appl.*, **2013:38** (2013).

[10] R. Heckmann. Approximation of metric spaces by partial metric spaces. *Appl. Categ. Structures*, **7** (1999), 71 – 83.

[11] G. Jungck. Common fixed points for noncontinuous nonself maps on non-metric spaces. *Far East J. Math. Sci.*, **4** (2) (1996), 199 – 215.

[12] Z. Kadelburg, H. K. Nashine and S. Radenović. Fixed point results under various contractive conditions in partial metric spaces. *Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat.*, RACSAM, **107** (2) (2013), 241 – 256.

[13] E. Karapinar and U. Yüksel. Some fixed point theorems in partial metric spaces. *J. Appl. Math.*, **2011** (2011), Article ID 263621, 16 pages.

[14] S. G. Matthews. Partial metric topology. *Proc.* 8^{th} *Summer Conference on General Topology and Applications*, Ann. New York Acad. Sci. **728** (1994), 183 – 197.

[15] V. Popa. Fixed point theorems for implicit contractive mappings. *Stud. Cerc. Ştiinţ., Ser. Mat., Univ. Bacău* , **7** (1997), 127 – 134.

[16] V. Popa. Some fixed point theorems for compatible mappings satisfying an implicit relation. *Demonstr. Math.*, **32** (1) (1999), 157 – 163.

[17] V. Popa and A.-M. Patriciu. A general fixed point theorem for a pair of self mappings with common limit range property in partial metric spaces. *Bul. Inst. Politeh. Iaşi, Secţ. I, Mat. Mec. Teor. Fiz.*, **61** (65) (2015), 85 – 99.

[18] V. Popa and A.-M. Patriciu. A general fixed point theorem for a pair of mappings in partial metric spaces. *Acta Univ. Apulensis, Math. Inform.*, **43** (2015), 93 – 103.

[19] C. Vetro and F. Vetro. Common fixed points of mappings satisfying implicit relations in partial metric spaces. *J. Nonlinear Sci. Appl.*, **6** (2013), 152 – 161.

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