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TWO UNIQUE FIXED POINT RESULTS OF p-CYCLIC PROBABILISTIC C-CONTRACTIONS USING DIFFERENT TYPES OF T-NORMS

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Abstract. In this paper we introduce two different types p-cyclic c-contractions through p number of subsets of a 2-probabilistic metric spaces and establish some fixed point results for such contractions. Here we introduce 3rd order Hadzic type t-norm and 3rd order Lukasiewicz t-norm. The results depend on the t-norms. Control functions have been utilized in our second theorem. These results generalized some existing fixed point results. The results are illustrated with some examples. There are also some corollaries.

1. Introduction and Mathematical preliminaries

The probabilistic fixed point theory began with the celebrated work of Sehgal and Bharucha-Reid [**35**] in which they extended the famous Banach's contraction mapping principle to probabilistic metric spaces which was introduced as early as in 1942 in the work of K. Menger [**27**]. The theory of such spaces was further developed by several authors amonst whom the names of Schweizer and Sklar are worth mentioning. Such development is described vividly in their book [**34**]. Fixed point theory developed in several directions in probabilistic spaces parallel to the development in ordinary metric spaces. A comprehensive account of development of probabilistic fixed point theory is described in the book of Hadzic and Pap [**20**]. Some more recent references are [**5, 6, 7, 15, 16**] and [**28**]. The intrinsic flexibility of probabilistic metric spaces allow us to extend the ideas in metric

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spaces in more than one inequivalent ways. An instance is the C-contraction introduced by Hicks in 1983 [**21**] which is an extension of the Banach's contraction but is not equivalent to Sehgal's probabilistic contraction which is also known as B-contraction. The idea of C-contraction has been taken up in a good number of papers in which new probabilistic contraction have appeared. In which some are generalization of the Hick's results and some other are contraction results different from probabilistic generalization of Banach contraction. Some instances of these works are [**2, 10, 13, 16, 41**]. Meanwhile a new class of contraction called cyclic contraction where defined in metric spaces and where also extended to probabilistic metric spaces. These contractions are non-self mappings and have been used in proximity point problems which are extensions of fixed point problems. Also the idea of probabilistic metric spaces has been extended to 2-probabilistic metric spaces [**3, 8, 19**]. This is parallel to the introduction of 2-Metric spaces by Gähler [**17**] which is as early as in 1963. There are quite a good number of papers on fixed point study in 2-metric and 2-probabilistic metric spaces in works where instances are [**12, 14, 29, 30, 32**].

DEFINITION 1.1. ***2-metric space*** [**17, 18**]
Let X be a non empty set. A real valued function d on $X \times X \times X$ is said to be a 2-metric on X if

(i) given distinct elements x, y of X, there exists an element z of X such that $d(x, y, z) \neq 0$;

(ii) $d(x, y, z) = 0$ when at least two of x, y, z are equal;

(iii) $d(x, y, z) = d(x, z, y) = d(y, z, x)$ for all x, y, z in X and

(iv) $d(x, y, z) \leqslant d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all x, y, z, w in X.

When d is a 2-metric on X, the ordered pair (X, d) is called a 2-metric space.

Several results of metric fixed point theory has been extended to these spaces.

Another generalization of the metric space is the probabilistic metric space which is a probabilistic generalization in which, instead of a nonnegative real number, every pair of elements is assigned to a distribution function.

DEFINITION 1.2. [**20, 34**] *A mapping $F : R \to R^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t \in R} F(t) = 0$ and $\sup_{t \in R} F(t) = 1$, where R is the set of real numbers and R^+ denotes the set of non-negative real numbers.*

An interpretation of $F_{x,y}(t)$ is that it is the probability of the event that the distance between the points x and y is less than t. A metric space becomes a Menger space if we write $F_{x,y}(t) = H(t - d(x, y))$ where H is the Heaviside function given by

$$H(t) = 1, \text{ if } t > 0, \\ \quad\quad = 0, \text{ if } t \leqslant 0.$$

Shi, Ren and Wang give the following definition of n-th order t-norm.

DEFINITION 1.3. ***n-th order t-norm*** [**36**]
A mapping $T : \Pi_{i=1}^{n}[0,1] \to [0,1]$ is called a n-th order t-norm if the following conditions are satisfied :

(i) $T(0,0,....,0) = 0$, $T(a,1,1,...,1) = a$ *for all* $a \in [0,1]$,

(ii) $T(a_1, a_{2,}, a_3, ..., a_n) = T(a_2, a_1, a_3, ..., a_n) = T(a_2, a_3, a_1, ..., a_n)$
$= .... = T(a_2, a_3, a_4, ..., a_n, a_1)$,

(iii) $a_i \geqslant b_i$, *i=1,2,3,....,n implies* $T(a_1, a_2, a_3, ..., a_n) \geqslant T(b_1, b_2, b_3, ..., b_n)$,

(iv) $T(T(a_1, a_2, a_3, ..., a_n), b_2, b_3, ...b_n) = T(a_1, T(a_2, a_3, ..., a_n, b_2), b_3, ..., b_n)$
$= T(a_1, a_2, T(a_3, a_4...., a_n, b_2, b_3), b_4, ..., b_n)$
$= ....$
$= T(a_1, a_2, ..., a_{n-1}, T(a_n, b_2, b_3, ..., b_n))$.

When $n = 2$, we have a binary t-norm, which is commonly known as t-norm. Here we extend the Hadzic type t-norm into 3rd order Hadzic type t-norm.

The following is the definition of 3rd order Hadzic type t-norm.

DEFINITION 1.4. ***3rd order Hadzic type t-norm***
A t-norm Δ is said to be 3rd order Hadzic type t-norm if the family $\{\Delta^p\}_{p\in N}$ of its iterates, defined for each $s \in (0,1)$ as
$\Delta^0(s) = 1$, $\Delta^1(s) = s$, $\Delta^2(s) = \Delta(\Delta^0(s), \Delta^1(s), s)$, $\Delta^{p+1}(s) = \Delta(\Delta^p(s), \Delta^{p-1}(s), s)$ *p3pt] for all integer $p \geqslant 2$, is equi-continuous at $s = 1$, that is, given $\lambda > 0$ there exits $\eta(\lambda) \in (0,1)$ such that*

$$1 \geqslant s > \eta(\lambda) \Rightarrow \Delta^p(s) > 1 - \lambda \text{ for all integer } p \geqslant 0.$$

Here we also extend the 2nd order Lukasiewicz t-norm into the 3rd order Lukasiewicz t-norm. The following is the definition of 3rd order Lukasiewicz t-norm.

DEFINITION 1.5. ***3rd order Lukasiewicz t-norm***
A mapping $T : \Pi_{i=1}^{3}[0,1] \to [0,1]$ is called a 3-rd order Lukasiewicz t-norm if T satisfies the following conditions.

$\Delta = T_L$, *defined by* $T_L(a,b,c) = \max\{a+b+c-2, 0\}$,
for all $a, b, c \in [0,1]$, which is the weakest t-norm.

The following are two examples of 3rd order t-norm:

(i) The minimum t-norm, $\Delta = T_m$, defined by $T_m(a,b,c) = \min\{a,b,c\}$.

(ii) The product t-norm, $\Delta = T_p$, defined by $T_p(a,b,c) = a.b.c$.

DEFINITION 1.6. ***Menger space*** [**20, 34**]
A Menger space is a triplet (X, F, Δ), where X is a non empty set, F is a function defined on $X \times X$ to the set of distribution functions and Δ is a t-norm, such that the following are satisfied:

(i) $F_{x,y}(0) = 0$ *for all* $x, y \in X$,

(ii) $F_{x,y}(s) = 1$ *for all* $s > 0$ *if and only if* $x = y$,

(iii) $F_{x,y}(s) = F_{y,x}(s)$ *for all* $x, y \in X$, $s > 0$ *and*

(iv) $F_{x,y}(u + v) \geqslant \Delta(F_{x,z}(u), F_{z,y}(v))$ *for all* $u, v \geqslant 0$ *and* $x, y, z \in X$.

EXAMPLE 1.1. *Let* $X = \{x_1, x_2, x_3\}$, $\Delta(a, b) = min(a, b)$ *and* $F_{x,\,y}(t)$ *be defined as:*

$$F_{x_1,\,x_2}(t) = \begin{cases} 0, & if \quad t \leqslant 0, \\ 0.75, & if \quad 0 < t \leqslant 2, \\ 1, & if \quad t > 2. \end{cases}$$

$$F_{x_1,\,x_3}(t) = \begin{cases} 0, & if \quad t \leqslant 0, \\ 0.92, & if \quad 0 < t \leqslant 1, \\ 1, & if \quad t > 1. \end{cases}$$

$$F_{x_2,\,x_3}(t) = \begin{cases} 0, & if \quad t \leqslant 0, \\ 0.75, & if \quad 0 < t \leqslant 2, \\ 1, & if \quad t > 2. \end{cases}$$

Then (X, F, Δ) *be a Menger space.*

The theory of these spaces is an important part of stochastic analysis.

2-Probabilistic metric space is the probabilistic generalization of 2-metric space.

DEFINITION 1.7. ***2-probabilistic metric space*** [**38**]
A probabilistic 2-metric space is an order pair (X, F) *where* X *is an arbitrary set and* F *is a mapping from* X^3 *into the set of distribution functions such that the following conditions are satisfied.*

(i) $F_{x,y,z}(t) = 0$ *for all* $t \leqslant 0$ *for all* $x, y, z \in X$,

(ii) $F_{x,y,z}(t) = 1$ *for all* $t > 0$ *iff at least two of the three points* x, y, z *are equal,*

(iii) for distinct points $x, y \in X$ *there exists a point* $z \in X$ *such that* $F_{x,y,z}(t) \neq 1$ *for* $t > 0$,

(iv) $F_{x,y,z}(t) = F_{x,z,y}(t) = F_{z,y,x}(t)$ *for all* $x, y, z \in X$ *and* $t > 0$,

(v) $F_{x,y,w}(t_1) = 1$, $F_{x,w,z}(t_2) = 1$ *and* $F_{w,y,z}(t_3) = 1$ *then* $F_{x,y,z}(t_1 + t_2 + t_3) = 1$, *for all* $x, y, z, w \in X$ *and* $t_1, t_2, t_3 > 0$.

A special case of the above definition is the following.

DEFINITION 1.8. ***2-Menger space*** [**19**]
Let X *be a nonempty set. A triplet* (X, F, Δ) *is said to be a 2-Menger space if* F *is a mapping from* X^3 *into the set of distribution functions satisfying the following conditions:*

(i) $F_{x,y,z}(0) = 0$,

(ii) $F_{x,y,z}(t) = 1$ *for all* $t > 0$ *if and only if at least two of* $x, y, z \in X$ *are equal,*

(iii) for distinct points $x, y \in X$ *there exists a point* $z \in X$ *such that* $F_{x,y,z}(t) \neq 1$ *for* $t > 0$,

(iv) $F_{x,y,z}(t) = F_{x,z,y}(t) = F_{z,y,x}(t)$*, for all* $x, y, z \in X$ *and* $t > 0$*,*

(v) $F_{x,y,z}(t) \geqslant \Delta(F_{x,y,w}(t_1), F_{x,w,z}(t_2), F_{w,y,z}(t_3))$

where $t_1, t_2, t_3 > 0$*,* $t_1 + t_2 + t_3 = t$*,* $x, y, z, w \in X$ *and* Δ *is the 3rd order t norm.*

Definition 1.9. [**19**] *A sequence* $\{x_n\}$ *in a 2-Menger space* (X, F, Δ) *is said to be converge to a limit* x *if given* $\epsilon > 0, 0 < \lambda < 1$ *there exists a positive integer* $N_{\epsilon,\lambda}$ *such that*

$$F_{x_n,x,a}(\epsilon) > 1 - \lambda \tag{1.1}$$

for all $n > N_{\epsilon,\lambda}$ *and for every* $a \in X$*.*

Definition 1.10. [**19**] *A sequence* $\{x_n\}$ *in a 2-Menger space* (X, F, Δ) *is said to be a Cauchy sequence in* X *if given* $\epsilon > 0, 0 < \lambda < 1$ *there exists a positive integer* $N_{\epsilon,\lambda}$ *such that*

$$F_{x_n,x_m,a}(\epsilon) > 1 - \lambda \tag{1.2}$$

for all $m, n > N_{\epsilon,\lambda}$ *and for every* $a \in X$*.*

Definition (1.9) and (1.10) can be equivalently written by replacing '$>$' with '$\geqslant$' in (1.1) and (1.2). More often than not, they are written in that way. We have given them in the present form for our convenience in the proofs of our theorems.

Definition 1.11. [**19**] *A 2-Menger space* (X, F, Δ) *is said to be complete if every Cauchy sequence is convergent in* X*.*

In [**23**] Khan, Swaleh and Sessa introduced a new category of contractive fixed point problems in metric spaces. They introduced the concept of "altering distance function", which is a control function that alters the distance between two points in a metric space. This concept was further generalized in a number of works. There are several works in fixed point theory involving altering distance function, some of these are noted in [**31**] and [**33**].

Recently Choudhury and Das had extended the concept of altering distance function to the context of Menger spaces in [**5**]. The definition is as follows:

Definition 1.12. Φ***-function*** [**5**]
A function $\phi : R^+ \to R^+$ *is said to be a* Φ*-function if it satisfies the following conditions:*

(i) $\phi(t) = 0$ *if and only if* $t = 0$*,*

(ii) $\phi(t)$ *is strictly monotone increasing and* $\phi(t) \to \infty$ *as* $t \to \infty$*,*

(iii) ϕ *is left continuous in* $(0, \infty)$*,*

(iv) ϕ *is continuous at* 0*.*

In [**5**] Choudhury and Das introduced a new type of contraction mapping in Menger spaces which is known as ϕ-contraction.

Definition 1.13. [**5**] *Let* (X, F, Δ) *be a Menger space. A self map* $T : X \to X$ *is said to be* ϕ*-contractive if*

$$F_{Tx,Ty}(\phi(t)) \geqslant F_{x,y}(\phi(\frac{t}{c})) \tag{1.3}$$

where $0 < c < 1,\ x, y \in X$ *and* $t > 0$*, the function* ϕ *is a* Φ*-function.*

The idea of control function has opened new possibilities of proving more fixed point results in Menger spaces. This concept has also applied to a coincidence point problems. Some recent results using Φ-function are noted in [**6, 7, 9, 13, 15, 25**] and [**28**]. Particularly in [**16**], the C-contraction was extended utilizing the Φ-function.

We will make use of the following function in our theorem.

Definition 1.14. ***Ψ-function***
A function $\psi : (0, 1) \to (0, 1)$ *is said to be a* Ψ*-function if it satisfies the following conditions:*

(i) ψ *is strictly monotone increasing,*

(ii) $\psi^n(s) \to 0$ *as* $n \to \infty$ *for all* $s > 0$*,*

(iii) $\psi(s) < s$*,*

(iv) ψ *is continuous.*

Recently Kirk, Srinivasan and Veeramani initiated the concept of cyclic contraction and cyclic contractive type mapping in their paper [**24**] in metric spaces. After that many papers have appeared on this concept.

Kirk, Srinivasan and Veeramani established the following results.

Theorem 1.1. [**24**] *Let* A *and* B *be two non-empty closed subsets of a complete metric space* X*, and suppose* $f : A \bigcup B \to A \bigcup B$ *satisfies:*

(1) $fA \subseteq B$ *and* $fB \subseteq A$*,*

(2) $d(fx, fy) \leqslant kd(x, y)$ *forall* $x \in A$ *and* $y \in B$ *where* $k \in (0, 1)$*.*

Then f *has a unique fixed point in* $A \bigcap B$*.*

The problems of cyclic contractions have been strongly associated with proximity point problems. Some other results dealing with cyclic contractions and proximity point problems are noted in [**1, 11, 26, 39**] and [**40**].

A generalization of cyclic mapping is p-cyclic mapping. The definition is the following:

Definition 1.15. *Let* $\{A_i\}_{i=1}^{p}$ *be non-empty sets. A p-cyclic mapping is a mapping* $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ *which satisfies the following conditions :*

$$(i) \quad TA_i \subseteq A_{i+1} for 1 \leqslant i < p, TA_p \subseteq A_1. \tag{1.4}$$

In this case where $p = 2$, this reduces to cyclic mapping. Some fixed point results of p-cyclic maps have been obtained in [**4, 14, 22, 37**].

In our present works we establish two different types definitions. The definitions are given below.

Definition 1.16. ***Type-I p-cyclic C-contraction on 2-Menger space***
A p-cyclic mapping T *is called a Type-I p-cyclic* C*-contraction on 2-Menger space*

if for any $r > 0$ *and* $0 < \lambda < 1$,

$$F_{x,y,a}(r) > 1 - \lambda \quad \textit{implies} \quad F_{Tx,Ty,a}(\psi_1(r)) > 1 - \psi_2(\lambda) \tag{1.5}$$

whenever $x \in A_i$, $y \in A_{i+1}$, $1 \leqslant i < p$, *for all* $a \in X$, *where* ψ_1, ψ_2 *are* Ψ*-function.*

Here we introduce another type definition which is known as type-II p-cyclic C-contraction on 2-Menger space. The definition is given below.

DEFINITION 1.17. ***Type-II p-cyclic C-contraction on 2-Menger space*** *A p-cyclic mapping T is called a Type-II p-cyclic C-contraction on 2-Menger space if for any* $r > 0$ *and* $0 < \lambda < 1$,

$$F_{x,y,a}(\phi(r)) > 1 - \lambda \quad \textit{implies} \quad F_{Tx,Ty,a}(\psi_3(r)) > 1 - \psi_2(\lambda) \tag{1.6}$$

whenever $x \in A_i$, $y \in A_j$, $1 \leqslant i,j \leqslant p$, $i \neq j$, *for all* $a \in X$, *where* ϕ *is a* Φ*-function,* ψ_2 *and* ψ_3 *are* Ψ*-function.*

The purpose of the paper is to establish fixed point results for both the above types of cyclic mappings. Illustrative examples and corollaries are also given.

2. Main Results

In our main results, we establish a lemma, two unique fixed points theorems using definitions (1.16) and (1.17). Two corollaries and two examples are also given in this part.

LEMMA 2.1. *Let* (X, F, Δ) *be a 2-Menger space. T be a p-cyclic mapping (Definition 1.15) on X satisfies the following conditions for all* $a \in X$:

$$F_{x_{n+1},x_{n+2},a}(t) \geqslant F_{x_n,x_{n+1},a}(\frac{t}{k}) \tag{2.1}$$

whenever $x_n \in A_{n+1}$, $x_{n+1} \in A_{n+2}$, $n \geqslant 0$, $k \in (0,1)$, $t > 0$, *then for* $i \geqslant 1$, $a \in X$,

$$F_{x_{n+i},x_{n+i+1},a}(t) \geqslant F_{x_n,x_{n+1},a}(\frac{t}{k^i}). \tag{2.2}$$

PROOF. Let x_0 be any arbitrary point in A_1. Now, we define the sequence $\{x_n\}_{n=0}^{\infty}$ in X by $x_n = Tx_{n-1}, n \in N$ where N is the set of natural numbers.
By (1.4), we have $x_o \in A_1$, $x_1 \in A_2$, $x_2 \in A_3$,......,$x_{r-1} \in A_r$ and in general

$$x_{nr} \in A_1, x_{nr+1} \in A_2, ..., x_{nr+(r-1)} \in A_r \tag{2.3}$$

for all $n \geqslant 0$.

For any $n \geqslant 1$ and for all $a \in X$, $t > 0$, we have

$$F_{x_n,x_{n+1},a}(t) = F_{Tx_{n-1},Tx_n,a}(t) \geqslant F_{x_{n-1},x_n,a}(\frac{t}{k})(x_{n-1} \in A_n, x_n \in A_{n+1}). \tag{2.4}$$

Again, by repeated applications of (2.4), it follows that for all $a \in X$, $t > 0$ and $n \geqslant 0$ and each $i \geqslant 1$,

$$F_{x_{n+i},x_{n+i+1},a}(t) \geqslant F_{x_n,x_{n+1},a}(\frac{t}{k^i}). \tag{2.5}$$

□

THEOREM 2.1. *Let (X, F, Δ) be a complete 2-Menger space with 3rd order Hadzic type t-norm Δ and T be a Type-I p-cyclic C-contraction on 2-Menger space. Then T has a unique fixed point in $\bigcap_{i=1}^{p} A_i$.*

PROOF. Let x_0 be any arbitrary point in A_1. Now we define the sequence $\{x_n\}_{n=0}^{\infty}$ in X by $x_n = Tx_{n-1}, n \in N$ where N is the set of natural numbers. By (1.4), we have $x_o \in A_1$, $x_1 \in A_2$, $x_2 \in A_3$,......,$x_{p-1} \in A_p$ and in general

$$x_{np} \in A_1, x_{np+1} \in A_2, ..., x_{np+(p-1)} \in A_p \tag{2.6}$$

for all $n \geqslant 0$.

Let $0 < \eta < 1$ be given. For fixed $a \in X$, we can find $r > 0$ such that

$$F_{x_0,x_1,a}(r) > 1 - \eta.$$

Then, by an application of (1.5), we get

$$F_{Tx_0,Tx_1,a}(\psi_1(r)) > 1 - \psi_2(\eta), \text{ (where } x_0 \in A_1 \text{ and } x_1 \in A_2)$$

that is,

$$F_{x_1,x_2,a}(\psi_1(r)) > 1 - \psi_2(\eta).$$

Continuing this process, we obtain

$$F_{x_2,x_3,a}(\psi_1^2(r)) = F_{Tx_1,Tx_2,a}(\psi_1^2(r)) > 1 - \psi_2^2(\eta), \text{ (where } x_1 \in A_2 \text{ and } x_2 \in A_3)$$

and, in general, for all $n \in N$, we obtain

$$F_{x_n,x_{n+1},a}(\psi_1^n(r)) > 1 - \psi_2^n(\eta). \text{ (where } x_n \in A_{n+1} \text{ and } x_{n+1} \in A_{n+2})$$

Let $\epsilon > 0$ be arbitrary. By the properties of Ψ-function we can find a positive integer N such that for all integer $n > N$,

$$\psi_1^n(r) < \epsilon.$$

Consequently, for all $n > N$, we get

$$F_{x_n,x_{n+1},a}(\epsilon) \geqslant F_{x_n,x_{n+1},a}(\psi_1^n(r)) > 1 - \psi_2^n(\eta) \to 1 \tag{2.7}$$

as $n \to \infty$. Thus, for arbitrary $\epsilon > 0$, we get

$$F_{x_n,x_{n+1},a}(\epsilon) \to 1 \tag{2.8}$$

as $n \to \infty$.

We next prove that $\{x_n\}$ is a Cauchy sequence (Definition 1.10), that is, we prove that for arbitrary $\epsilon > 0$ and $0 < \lambda < 1$, there exists $N(\epsilon, \lambda)$ such that

$$\text{for all } a \in X,\ F_{x_n,x_m,a}(\epsilon) > 1 - \lambda \text{ for all } m, n \geqslant N(\epsilon, \lambda).$$

Without loss of generality we can assume that $m > n$.

Now,

$$\epsilon = \epsilon\frac{1-k}{1-k} > \epsilon(1-k)(1 + k + k^2 + ... + k^{m-n-1})$$

Then, by the monotone increasing property of F, and for all $a \in X$, we have

$$F_{x_n,x_m,a}(\epsilon) \geqslant F_{x_n,x_m,a}(\epsilon(1-k)(1 + k + k^2 + ... + k^{m-n-1})),$$

that is,

$$F_{x_n,x_m,a}(\epsilon) \geqslant \Delta(F_{x_n,x_{n+1},a}(\epsilon(1-k)), \Delta(F_{x_{n+1},x_{n+2},a}(\epsilon k(1-k)), \Delta(.........,$$
$$\Delta(F_{x_{m-2},x_{m-1},a}(\epsilon k^{m-n-2}(1-k)), F_{x_{m-1},x_m,a}(\epsilon k^{m-n-1}(1-k)))......))). \tag{2.9}$$

Putting $t = (1-k)\epsilon k^i$ in (2.5), for all $a \in X$, we get

$$F_{x_{n+i},x_{n+i+1},a}((1-k)\epsilon k^i) \geqslant F_{x_n,x_{n+1},a}((1-k)\epsilon).$$

Then, by (2.9), for all $a \in X$, we have

$$F_{x_n,x_m,a}(\epsilon) \geqslant \Delta(F_{x_n,x_{n+1},a}(\epsilon(1-k)), \Delta(F_{x_n,x_{n+1},a}(\epsilon(1-k)), \Delta(.........,$$
$$\Delta(F_{x_n,x_{n+1},a}(\epsilon(1-k)), F_{x_n,x_{n+1},a}(\epsilon(1-k)))......))),$$

that is,

$$F_{x_n,x_m,a}(\epsilon) \geqslant \Delta^{(m-n)} F_{x_n,x_{n+1},a}(\epsilon(1-k)). \tag{2.10}$$

Since the t-norm Δ is a Hadzic type t-norm, the family $\{\Delta^p\}$ of its iterates is equicontinuous at the point $s = 1$, that is, there exists $\eta(\lambda) \in (0,1)$ such that for all $m > n$,

$$\Delta^{(m-n)}(s) > 1 - \lambda \tag{2.11}$$

whenever $\eta(\lambda) < s \leqslant 1$.

Since, $F_{x_0,x_1,a}(t) \to 1$ as $t \to \infty$ and $0 < k < 1$, there exists an positive integer $N(\epsilon, \lambda)$ such that for all $a \in X$,

$$F_{x_0,x_1,a}(\frac{(1-k)\epsilon}{k^n}) > \eta(\lambda) \tag{2.12}$$

for all $n \geqslant N(\epsilon, \lambda)$.

From (2.12) and (2.5), with $n = 0$, $i = n$ and $t = (1-k)\epsilon$, for all $a \in X$, we get

$$F_{x_n,x_{n+1},a}(\epsilon(1-k)) \geqslant F_{x_0,x_1,a}(\frac{(1-k)\epsilon}{k^n}) > \eta(\lambda)$$

for all $n \geqslant N(\epsilon, \lambda)$. Then, from (2.11) with $s = F_{x_n,x_{n+1},a}(\epsilon(1-k))$, we have

$$\Delta^{(m-n)}(F_{x_n,x_{n+1},a}(\epsilon(1-k))) > 1 - \lambda.$$

It then follows from (2.10) that for all $a \in X$,

$$F_{x_n,x_m,a}(\epsilon) > 1 - \lambda$$

for all $m, n \geqslant N(\epsilon, \lambda)$.

Thus $\{x_n\}$ is a Cauchy sequence.

By the completeness of X, there exists $z \in X$ such that

$$x_n \to z \tag{2.13}$$

as $n \to \infty$.

By the construction of the sequence $\{x_n\}$, we have $x_p \in A_1$, $x_{2p} \in A_1$, ... $x_{np} \in A_1$. Therefore the subsequence $\{x_{np}\}$ of $\{x_n\}$ which belongs to A_1 also converges to z in A_1, since A_1 is closed. Similarly subsequence $\{x_{np+1}\}$ belongs to

A_2 also converges to z in A_2. Since A_3, A_4, ..., A_p are closed sets, similarly we get $z \in A_3$, A_4, ..., A_p. Therefore $z \in A_1 \bigcap A_2 \bigcap A_3 ...... \bigcap A_p$.

Now, we prove that $Tz = z$.

By (2.13), for all $t > 0$, we have

$$F_{x_n,z,a}(\psi_1^{-1}(t)) \to 1$$

as $n \to \infty$, that is, for arbitrary $0 < \lambda < 1$, we can find $N_1 > 0$ such that for all $n > N_1$, we have

$$F_{x_n,z,a}(\psi_1^{-1}(t)) > 1 - \lambda. \tag{2.14}$$

By virtue of (1.5), we get from (2.14),

$$F_{Tx_n,Tz,a}(\psi_1(\psi_1^{-1}(t))) > 1 - \psi_2(\lambda) > 1 - \lambda,$$

(since $\psi_2(\lambda) < \lambda$) which implies that,

$$F_{x_{n+1},Tz,a}(t) > 1 - \lambda. \tag{2.15}$$

Now, taking limit as $n \to \infty$ on both sides of (2.15), for all $t > 0$, we have

$$F_{z,Tz,a}(t) \geqslant 1 - \lambda.$$

Since λ is arbitrary, and $a \in X$ is any element, for $t > 0$ we obtain

$$F_{z,Tz,a}(t) = 1, \text{ that is, } z = Tz.$$

To prove the uniqueness of the fixed point, let v be another fixed point of T, that is, $Tv = v$.

Let $a \in X$ be any element different from z and v. We can get $\epsilon_1 > 0$ such that

$$F_{z,v,a}(\epsilon_1) > 1 - \lambda. \text{ (where } 0 < \lambda < 1)$$

Then, by the inequality (1.5), we have

$$F_{Tz,Tv,a}(\psi_1(\epsilon_1)) > 1 - \psi_2(\lambda),$$

that is,

$$F_{z,v,a}(\psi_1(\epsilon_1)) > 1 - \psi_2(\lambda).$$

Continuing this process n times, we obtain

$$F_{z,v,a}(\psi_1^n(\epsilon_1)) > 1 - \psi_2^n(\lambda). \tag{2.16}$$

For arbitrary $\mu > 0$, by virtue of properties of ψ-function it is possible to find $N > 0$ such that

$$\psi_1^n(\epsilon_1) < \mu \tag{2.17}$$

for all $n > N$.

Combining (2.16) and (2.17), we have

$$F_{z,v,a}(\mu) \geqslant F_{z,v,a}(\psi_1^n(\epsilon_1)) > 1 - \psi_2^n(\lambda),$$

for all $n > N$.

Taking $n \to \infty$ both sides of the above inequality, and for all $\mu > 0$, we have $F_{z,v,a}(\mu) = 1$, that is, $z = v$.

Hence T has a unique fixed point in $A_1 \bigcap A_2 \bigcap A_3 ...... \bigcap A_p$. □

Taking $p = 3$, we get the following example to validate the Theorem 2.1.

EXAMPLE 2.1. *Let* $X = \{\alpha, \beta, \gamma, \delta\}$, $A = \{\alpha, \gamma, \delta\}$, $B = \{\alpha, \beta\}$, $C = \{\alpha, \gamma\}$ *the t-norm* Δ *is a 3rd order minimum t-norm and* F *be defined as*

$$F_{\alpha,\beta,\gamma}(t) = F_{\alpha,\beta,\delta}(t) = \begin{cases} 0, & \text{if } \ t \leqslant 0, \\ 0.40, & \text{if } \ 0 < t < 7, \\ 1, & \text{if } \ t \geqslant 7, \end{cases}$$

$$F_{\alpha,\gamma,\delta}(t) = F_{\beta,\gamma,\delta}(t) = \begin{cases} 0, & \text{if } \ t \leqslant 0, \\ 0.95, & \text{if } \ 0 < t < 1, \\ 1, & \text{if } \ t \geqslant 1, \end{cases}$$

then (X, F, Δ) *is a complete 2-Menger space. If we define* $T : X \to X$ *as follows:* $T\alpha = \alpha, T\beta = \alpha, T\gamma = \alpha, T\delta = \alpha$ *then the mapping* T *satisfies all the conditions of the Theorem 2.1 where* $\psi_1(t) = \frac{t}{2}$, $\psi_2(t) = \frac{t}{3}$ *and* α *is the unique fixed point of* T *in* $A \bigcap B \bigcap C$.

We use the control function ϕ (Definition 1.12) in our next theorem. Here we also use the 3rd order Lukasiewicz t-norm Δ (Defined as $\Delta(a, b, c) = \max\{a + b + c - 2, 0\}$ for all $a, b, c \in [0, 1]$). We also prove our second theorem by different arguments from the first theorem.

THEOREM 2.2. *Let* (X, F, Δ) *be a complete 2-Menger space with the 3rd order Lukasiewicz t-norm* Δ *(Defined as* $\Delta(a, b, c) = \max\{a+b+c-2, 0\}$ *for all* $a, b, c \in [0, 1]$*, which is the weakest t-norm.) and* T *be a Type-II p-cyclic* C*-contraction on 2-Menger space on* X *with* ϕ, ψ_2, ψ_3 *satisfying* $\phi(t) > \psi_3(\psi_2^{-1}(t))$ *for all* t *and* $(\phi^{-1} o \psi_3)^n(s) \to 0$ *as* $n \to \infty$ *for all* $s > 0$ *. Then* T *has a unique fixed point in* $\bigcap_{i=1}^{p} A_i$.

PROOF. Let x_0 be any arbitrary point in A_1. Now we define the sequence $\{x_n\}_{n=0}^{\infty}$ in X by $x_n = Tx_{n-1}, n \in N$ where N is the set of natural numbers. By (1.4), we have $x_o \in A_1$, $x_1 \in A_2$, $x_2 \in A_3$,..., $x_{p-1} \in A_p$ and in general

$$x_{np} \in A_1, x_{np+1} \in A_2, ..., x_{np+(p-1)} \in A_p \tag{2.18}$$

for all $n \geqslant 0$.

Let $0 < \eta < 1$ be given. By the property that $\phi(t) \to \infty$ as $t \to \infty$, for fixed $a \in X$, we can find $r > 0$ such that

$$F_{x_0,x_1,a}(\phi(r)) > 1 - \eta.$$

Then, by an application of (1.6), for all $a \in X$, we get

$$F_{Tx_0,Tx_1,a}(\psi_3(r)) > 1 - \psi_2(\eta). \text{ (where } x_0 \in A_1 \text{ and } x_1 \in A_2)$$

Now, by the properties of ϕ-function and Ψ-function, for all $a \in X$, we have

$$F_{Tx_0,Tx_1,a}(\phi(\phi^{-1}(\psi_3(r)))) > 1 - \psi_2(\eta),$$

that is,

$$F_{x_1,x_2,a}(\phi(p_1(r))) > 1 - \psi_2(\eta), \text{ where } \phi^{-1}(\psi_3(r)) = p_1(r).$$

Now, using the inequality (1.6), for all $a \in X$, we have

$$F_{Tx_1,Tx_2,a}(\psi_3(p_1(r))) > 1 - \psi_2^2(\eta), \text{ (where } x_1 \in A_2 \text{ and } x_2 \in A_3\text{)},$$

that is,

$$F_{x_2,x_3,a}(\psi_3(p_1(r))) > 1 - \psi_2^2(\eta),$$

that is,

$$F_{x_2,x_3,a}(\phi(\phi^{-1}(\psi_3(p_1(r))))) > 1 - \psi_2^2(\eta).$$

Now, by the properties of ϕ-function and Ψ-function, for all $a \in X$, we have

$$F_{x_2,x_3,a}(\phi(p_2(r))) > 1 - \psi_2^2(\eta),$$

where

$$\phi^{-1}(\psi_3(p_1(r))) = p_2(r).$$

Now, using the inequality (1.6), for all $a \in X$, we have

$$F_{Tx_2,Tx_3,a}(\psi_3(p_2(r))) > 1 - \psi_2^3(\eta),$$

that is,

$$F_{x_3,x_4,a}(\psi_3(p_2(r))) > 1 - \psi_2^3(\eta), \text{ (where } x_3 \in A_4 \text{ and } x_4 \in A_5\text{)}.$$

Continuing this process, we obtain in general, for all $n \in N$ and $a \in X$,

$$F_{x_n,x_{n+1},a}(\psi_3(p_{n-1}(r))) > 1 - \psi_2^n(\eta),$$

(where $x_n \in A_{n+1}$, $x_{n+1} \in A_{n+2}$ and $\phi^{-1}(\psi_3(p_{n-2}(r))) = p_{n-1}(r)$.

Let $\epsilon > 0$ be arbitrary. By the properties of Φ-function and Ψ-function we can find a positive integer N such that for all integer $n > N$, $\psi_3(p_{n-1}(r)) < \epsilon$. (By our assumption $(\phi^{-1}o\psi_3)^n(s) \to 0$ as $n \to \infty$ for all $s > 0$.)

Consequently, for all $n > N$, we get

$$F_{x_n,x_{n+1},a}(\epsilon) \geqslant F_{x_n,x_{n+1},a}(\psi_3(p_{n-1}(r))) > 1 - \psi_2^n(\eta) \to 1 \tag{2.19}$$

as $n \to \infty$. Thus, for arbitrary $\epsilon > 0$, we get

$$F_{x_n,x_{n+1},a}(\epsilon) \to 1 \tag{2.20}$$

as $n \to \infty$.

We next prove that $\{x_n\}$ is a Cauchy sequence. If possible, let $\{x_n\}$ be not a Cauchy sequence. Then, there exist $\epsilon > 0$ and $0 < \lambda < 1$ for which we can find some $a \in X$ and subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that

$$F_{x_{m(k)},x_{n(k)},a}(\epsilon) \leqslant 1 - \lambda. \tag{2.21}$$

We take $n(k)$ corresponding to $m(k)$ to be the smallest integer satisfying (2.21), so that

$$F_{x_{m(k)},x_{n(k)-1},a}(\epsilon) > 1 - \lambda. \tag{2.22}$$

If $\epsilon_1 < \epsilon$ then, we have

$$F_{x_{m(k)},x_{n(k)},a}(\epsilon_1) \leqslant F_{x_{m(k)},x_{n(k)},a}(\epsilon).$$

From the above, we conclude that it is possible to construct $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ with $n(k) > m(k) > k$ and satisfying (2.21), (2.22) whenever ϵ is replaced by a smaller positive value. As ϕ is continuous at 0 and strictly monotone increasing with $\phi(0) = 0$, it is possible to obtain $\epsilon_2 > 0$ such that $\phi(\epsilon_2) < \epsilon$.

Then, by the above argument, it is possible to obtain an increasing sequence of integers $\{m(k)\}$ and $\{n(k)\}$ with $n(k) > m(k) > k$ such that

$$F_{x_{m(k)},x_{n(k)},a}(\phi(\epsilon_2)) \leqslant 1-\lambda \tag{2.23}$$

and

$$F_{x_{m(k)},x_{n(k)-1},a}(\phi(\epsilon_2)) > 1-\lambda. \tag{2.24}$$

From the definition of ψ-function ψ_2^{-1} is continuous at ϵ_2 and $\epsilon_2 < \psi_2^{-1}(\epsilon_2)$. So there exists $\xi_1 > 0$ such that

$$\psi_2^{-1}(\epsilon_2 - \xi_1) > \epsilon_2. \tag{2.25}$$

We take $\eta_1, \eta_2 > 0$ such that

$$0 < \eta_1 + \eta_2 = \phi(\epsilon_2) - \psi_3(\psi_2^{-1}(\epsilon_2 - \xi_1)). \tag{2.26}$$

(by our assumption $\phi(t) > \psi_3(\psi_2^{-1}(t))$ for all t).

Then, from (2.23) we have

$$1-\lambda \geqslant F_{x_{m(k)},x_{n(k)},a}(\phi(\epsilon_2)) \tag{2.27}$$

$$\geqslant \Delta(F_{x_{m(k)},x_{n(k)},x_{m(k)+1}}(\eta_1), F_{x_{m(k)},x_{m(k)+1},a}(\eta_2), F_{x_{m(k)+1},x_{n(k)},a}(\psi_3(\psi_2^{-1}(\epsilon_2-\xi_1)))).$$

Now, using (2.24) and (2.25), we have

$$F_{x_{m(k)},x_{n(k)-1},a}(\phi(\psi_2^{-1}(\epsilon_2 - \xi_1))) \geqslant F_{x_{m(k)},x_{n(k)-1},a}(\phi(\epsilon_2)) > 1-\lambda.$$

Then, using (1.6), we have

$$F_{x_{m(k)+1},x_{n(k)},a}(\psi_3(\psi_2^{-1}(\epsilon_2 - \xi_1))) > 1-\psi_2(\lambda). \tag{2.28}$$

$$(x_{m(k)} \in A_{m(k)+1}, x_{n(k)-1} \in A_{n(k)}, m(k)+1 \neq n(k))$$

Then, from (2.27), we have

$$1-\lambda \geqslant \Delta(F_{x_{m(k)},x_{n(k)},x_{m(k)+1}}(\eta_1), F_{x_{m(k)},x_{m(k)+1},a}(\eta_2), 1-\psi_2(\lambda)). \tag{2.29}$$

Taking $k \to \infty$ in (2.29), and using (2.20), and the continuity of Δ, we have

$$1-\lambda \geqslant \Delta(1,1,1-\psi_2(\lambda)) = \max\{1+1+1-\psi_2(\lambda)-2, 0\} = 1-\psi_2(\lambda),$$

(since Δ is a 3rd order Lukasiewicz t-norm) which implies $\lambda \leqslant \psi_2(\lambda)$, which contradicts the fact that $\psi_2(\lambda) < \lambda$.

Thus $\{x_n\}$ is a Cauchy sequence.

By the completeness of X, there exists $z \in X$ such that

$$x_n \to z \tag{2.30}$$

as $n \to \infty$.

By the construction of the sequence $\{x_n\}$, we have $x_p \in A_1$, $x_{2p} \in A_1$, ..., $x_{np} \in A_1$. Therefore the subsequence $\{x_{np}\}$ of $\{x_n\}$ which belongs to A_1 also converges to z in A_1, since A_1 is closed. Similarly subsequence $\{x_{np+1}\}$ belongs to A_2 also converges to z in A_2. Since A_3, A_4, ..., A_p are closed sets, similarly we get $z \in A_3$, A_4, ..., A_p. Therefore $z \in A_1 \bigcap A_2 \bigcap A_3 ...... \bigcap A_p$.

Now, we prove that $Tz = z$.

By (2.30), for all $t > 0$, we have

$$F_{x_n,z,a}(\phi(\psi_2^{-1}(t))) \to 1 \text{ as } n \to \infty,$$

that is, for arbitrary $0 < \lambda < 1$, we can find $N_1 > 0$ such that for all $n > N_1$, we have

$$F_{x_n,z,a}(\phi(\psi_2^{-1}(t))) > 1 - \lambda. \tag{2.31}$$

By virtue of (1.6), we get from (2.31),

$$F_{Tx_n,Tz,a}(\psi_3(\psi_2^{-1}(t))) > 1 - \psi_2(\lambda) > 1 - \lambda, \text{ (since } \psi_2(\lambda) < \lambda)$$
$$(x_n \in A_{n+1}, z \in A_j, n+1 \neq j)$$

which implies that,

$$F_{x_{n+1},Tz,a}(\psi_3(\psi_2^{-1}(t))) > 1 - \lambda. \tag{2.32}$$

Now, taking limit as $n \to \infty$ on both sides of (2.32), for all $t > 0$, $a \in X$, we have

$$F_{z,Tz,a}(\psi_3(\psi_2^{-1}(t))) \geqslant 1 - \lambda.$$

Since λ is arbitrary, and $a \in X$ is any element, for $t > 0$, we obtain

$$F_{z,Tz,a}(\psi_3(\psi_2^{-1}(t))) = 1.$$

But the properties of ψ-function for given $s > 0$ we can find $t > 0$ such that $\psi_2^{-1}(t) < s$. Then it follows that for all $a \in X$, $s > 0$,

$$F_{z,Tz,a}(\psi_3(s)) \geqslant F_{z,Tz,a}(\psi_3(\psi_2^{-1}(t))) = 1,$$

that is, $z = Tz$.

To prove the uniqueness of the fixed point, let v be another fixed point of T, that is, $Tv = v$. Let $a \in X$ be any element different from z and v. By the properties of ϕ-function, we can get $\epsilon_3 > 0$ such that

$$F_{z,v,a}(\phi(\epsilon_3)) > 1 - \lambda. \text{ (where } 0 < \lambda < 1)$$

Then, by the inequality (1.6), we have

$$F_{Tz,Tv,a}(\psi_3(\epsilon_3)) > 1 - \psi_2(\lambda),$$

that is,

$$F_{z,v,a}(\psi_3(\epsilon_3)) > 1 - \psi_2(\lambda),$$

that is,

$$F_{z,v,a}(\phi(\phi^{-1}(\psi_3(\epsilon_3)))) > 1 - \psi_2(\lambda),$$

that is,

$$F_{z,v,a}(\phi(Q_1)) > 1 - \psi_2(\lambda), \quad \text{where } Q_1 = \phi^{-1}(\psi_3(\epsilon_3)).$$

Now, by the inequality (1.6), we have

$$F_{Tz,Tv,a}(\psi_3(Q_1)) > 1 - \psi_2^2(\lambda),$$

that is,

$$F_{z,v,a}(\psi_3(Q_1)) > 1 - \psi_2^2(\lambda),$$

that is,

$$F_{z,v,a}(\phi(\phi^{-1}(\psi_3(Q_1)))) > 1 - \psi_2^2(\lambda),$$

that is,

$$F_{z,v,a}(\phi(Q_2)) > 1 - \psi_2^2(\lambda), \text{ where } Q_2 = \phi^{-1}(\psi_3(Q_1)).$$

Continuing this process n times, we obtain

$$F_{z,v,a}(\phi(Q_n)) > 1 - \psi_2^n(\lambda) \tag{2.33}$$

where $Q_n = \phi^{-1}(\psi_3(Q_{n-1}))$.

For arbitrary $\mu > 0$, by virtue of properties of ϕ-function and ψ-function it is possible to find $N > 0$ such that

$$\phi(Q_n) < \mu \tag{2.34}$$

for all $n > N$. (By our assumption $(\phi^{-1}o\psi_3)^n(s) \to 0$ as $n \to \infty$ for all $s > 0$.)

Combining (2.33) and (2.34), we have

$$F_{z,v,a}(\mu) \geqslant F_{z,v,a}(\phi(Q_n)) > 1 - \psi_2^n(\lambda)$$

, for all $n > N$.

Taking $n \to \infty$ both sides of the above inequality, and for all $\mu > 0$, we have

$$F_{z,v,a}(\mu) = 1,$$

that is, $z = v$.

Hence T has a unique fixed point in $A_1 \bigcap A_2 \bigcap A_3 ...... \bigcap A_p$. □

Taking $p = 2$, we get the following corollary.

Corollary 2.1. *Let (X, F, Δ) be a complete 2-Menger space, where Δ is the minimum t-norm. Let A and B be two non-empty closed subsets of X. Let a mapping $T : A \bigcup B \to A \bigcup B$ satisfies the following conditions:*

(i) $TA \subseteq B$ and $TB \subseteq A$,

(ii) $F_{x,y,a}(t) > 1 - t$ implies $F_{Tx,Ty,a}(kt) > 1 - kt$

where $x \in A$ and $y \in B$, for all $a \in X$, $0 < k < 1$ and $t > 0$. Then $A \bigcap B$ is non empty and T has a unique fixed point in $A \bigcap B$.

If we take $\phi(t) = t$ and $\psi_2(t) = \psi_3(t) = kt$, in Theorem 2.2 where $0 < k < 1$, Δ is a minimum t-norm, we obtain the following corollary.

Corollary 2.2. *Let (X, F, Δ) be a complete 2-Menger space with 3rd order minimum t-norm Δ and $0 < k < 1$. Let $T : X \to X$ be a p-cyclic mapping such that for all $r > 0$, $0 < \lambda < 1$ and for all $x \in A_i$, $y \in A_j$, $1 \leqslant i, j \leqslant p$, $i \neq j$, for all $a \in X$*

$$F_{x,y,a}(r) > 1 - \lambda \Longrightarrow F_{Tx,Ty,a}(kr) > 1 - k\lambda.$$

Then T has a unique fixed point in $A_1 \bigcap A_2 \bigcap A_3 ... \bigcap A_p$.

The above corollary is actually the extension of C-contraction on 2-Menger spaces.

Taking $p = 2$ we get the following example.

Example 2.2. *Let $X = \{\alpha, \beta, \gamma, \delta\}$, $A = \{\alpha, \beta, \gamma\}$, $B = \{\gamma, \delta\}$, the t-norm Δ is a 3rd order Lukasiewicz t-norm and F be defined as*

$$F_{\alpha,\beta,\gamma}(t) = F_{\alpha,\beta,\delta}(t) = \begin{cases} 0, & \text{if} \quad t \leqslant 0, \\ 0.40, & \text{if} \quad 0 < t < 4, \\ 1, & \text{if} \quad t \geqslant 4, \end{cases}$$

$$F_{\alpha,\gamma,\delta}(t) = F_{\beta,\gamma,\delta}(t) = \begin{cases} 0, & \text{if} \quad t \leqslant 0, \\ 1, & \text{if} \quad t > 0. \end{cases}$$

Then (X, F, Δ) is a complete 2-Menger space. If we define $T : X \to X$ as follows: $T\alpha = \delta, T\beta = \gamma, T\gamma = \gamma, T\delta = \gamma$ then the mapping T satisfies all the conditions of the Theorem 2.2 where $\phi(t) = 2t$, $\psi_2(t) = \psi_3(t) = kt$, where $0 < k < 1$ and γ is the unique fixed point of T in $A \bigcap B$.

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