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COMMON FIXED POINTS OF GENERALIZED TAC-RATIONAL CONTRACTIVE MAPPINGS IN $\alpha\beta$-COMPLETE METRIC SPACES

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Abstract. In this paper, we introduce generalized TAC-rational contractive condition for four selfmaps in metric spaces and prove some new fixed point results for this class of mappings in $\alpha\beta$-complete metric spaces. We provide an example in support of our results.

1. Introduction and Preliminaries

Banach contraction principle plays a vital role in fixed point theory and many authors used contractive type conditions to generalize or extend this principle. In 1997, Alber and Guerre-Delabriere [**1**] introduced weakly contractive maps as a generalization of contraction maps and established fixed point results in the setting of Hilbert spaces and subsequently Rhoades [**17**] extended and improved this concept to metric spaces. In 2008, Dutta and Choudhury [**8**] introduced (ψ,φ)- weakly contractive maps and proved the existence of fixed points in complete metric spaces. In continuation to the extensions of contraction maps, Doric [**7**] extended (ψ,φ)-weakly contractive maps and proved the existence of fixed points in complete metric spaces.

Throughout this paper, $\mathbf{R}$ denotes the set of all reals.

Theorem 1.1. [**7**] *Let (X,d) be a complete metric space and let $f,\ g : X \to X$ be two selfmaps such that for all $x,y \in X$*

$$\psi(d(fx,gy)) \leqslant \psi(M(x,y)) - \varphi(M(x,y)))$$

where $M(x,y) = \max\{d(x,y), d(x,fx), d(y,gy), \frac{d(x,gy)+d(y,fx)}{2}\}$,
$\psi : [0,\infty) \to [0,\infty)$ *is a continuous and nondecreasing function with* $\psi(t) = 0$ *if*

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and only if $t = 0$ *and* $\varphi\ : [0, \infty) \to [0, \infty)$ *is lower semi-continuous function with* $\varphi(t) = 0$ *if and only if* $t = 0$.

Then there exists a unique $u \in X$ *such that* $u = fu = gu$.

Definition 1.1. [**3**] Let X be a nonempty set, f be a selfmap of X and $\alpha, \beta : X \to [0, \infty)$ be two mappings. We say that f is a cyclic (α, β) - admissible mapping if (i) for any $x \in X$ with $\alpha(x) \geqslant 1 \implies \beta(fx) \geqslant 1$, and
(ii) for any $y \in X$ with $\beta(y) \geqslant 1 \implies \alpha(fy) \geqslant 1$.

Definition 1.2. [**2**] A function $f : [0, \infty)^2 \to \mathbf{R}$ is called C - class function if it is continuous and satisfies the following axioms:
(i) $f(s, t) \leqslant s$, and
(ii) for any $s, t \in [0, \infty)$ with $f(s, t) = s$ implies that either $s = 0$ or $t = 0$.

Throughout this paper, we denote the set of all C- class functions by $\mathscr{C}$,
$\Psi = \{\psi : [0, \infty) \to [0, \infty)|\ \psi$ is continuous, nondecreasing and $\psi^{-1}(\{0\}) = 0\ \}$, and
$\Phi = \{\varphi : [0, \infty) \to [0, \infty)|\ \lim_{n\to\infty} \varphi(t_n) = 0 \implies \lim_{n\to\infty} t_n = 0\}$.

Remark 1.1. $f(0, 0) = 0$.

Remark 1.2. *If* $\varphi \in \Phi$ *then* $\varphi(t) = 0 \implies t = 0$.

Example 1.1. The following functions $h : [0, \infty)^2 \to \mathbf{R}$ are elements of $\mathscr{C}$.
For $s, t \in [0, \infty)$, (i) $h(s, t) = s - t$, (ii) $h(s, t) = \frac{s-t}{1+t}$, (iii) $h(s, t) = \frac{s}{1+t}$, and (iv) $h(s, t) = \frac{s}{1+ts}$.

Definition 1.3. [**12**] Let f and g be selfmaps of a metric space (X, d). A point $x \in X$ is said be a coincidence point of f and g if $fx = gx$. we denote the set of all coincidence points of f and g by $C(f, g)$.

Definition 1.4. [**13**] Let f and g be selfmaps of a metric space (X, d). The pair (f, g) is said to be weakly compatible if they commute at their coincidence points, i.e., $fgx = gfx$ whenever $gx = fx, x \in X$.

Very recently, Chandok, Tas and Ansari [**6**] introduced the concept of TAC-contractive mappings and proved some fixed point results in the setting of complete metric spaces as follows:

Definition 1.5. [**6**] Let (X, d) be a metric space and let $\alpha, \beta : X \to [0, \infty)$ be two given mappings. We say that $T : X \to X$ is a TAC- contractive mapping if

$$x, y \in X \text{ with } \alpha(x)\beta(y) \geqslant 1 \implies \psi(d(Tx, Ty)) \leqslant f(\psi(d(x, y)), \varphi(d(x, y)))$$

where $\psi \in \Psi$ and $\varphi : [0, \infty) \to [0, \infty)$ is continuous with
$\lim_{n\to\infty} \varphi(t_n) = 0 \implies \lim_{n\to\infty} t_n = 0$ and $h\ \in \mathscr{C}$

Theorem 1.2. [**6**] *Let (X, d) be a complete metric space,* $\alpha,\ \beta : X \to [0, \infty)$ *be two mappings and let* $T : X \to X$ *is cyclic* (α, β)*-admissible mapping. Assume that* T *be a* TAC*-contractive mapping. Suppose that there exists* $x_0 \in X$ *such that* $\alpha(x_0) \geqslant 1$ *and* $\beta(x_0) \geqslant 1$ *and either of the following conditions hold:*
(a) T *is continuous*
(b) If $\{x_n\}$ *is a sequence in* X *such that* $x_n \to z$ *and* $\beta(x_n) \geqslant 1$ *for all* n*, then*

$\beta(z) \geqslant 1$.

Then T has a fixed point.

Moreover, if $\alpha(x) \geqslant 1$ and $\beta(y) \geqslant 1$ for all $x, y \in$ Fix (T) where Fix (T) is the set of all fixed points of T, then T has a unique fixed point.

Definition 1.6. [**10**] Let $f, g, S, T : X \to X$ be selfmaps on X and $\alpha, \beta : X \to [0, \infty)$ be two maps. We say that (f, g) is a cyclic (α, β)- admissible mapping with respect to (S, T) if
(i) for any $x \in X$ with $\alpha(Sx) \geqslant 1$ implies $\beta(fx) \geqslant 1$,
(ii) for any $x \in X$ with $\beta(Tx) \geqslant 1$ implies $\alpha(gx) \geqslant 1$.

Motivated by the works on α-complete metric spaces of Hussain, Kutbi and Salimi [**11**] and Pansuwon, Sintunavarat,Parvaneh and Cho [**16**], we introduce $\alpha\beta$-complete metric spaces as follows.

Definition 1.7. Let (X, d) be a metric space and let $\alpha, \beta : X \to [0, \infty)$ be two maps. The metric space X is said to be $\alpha\beta$-complete if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n) \geqslant 1$ and $\beta(x_n) \geqslant 1$ for all $n \in \mathbf{N} \cup \{0\}$ converges in X.

Remark 1.3. *If X is a complete metric space, then X is also an $\alpha\beta$-complete metric space, but its converse need not be true due to the following example.*

Example 1.2. Let $X = (-100, 100)$ with the usual metric. Define mappings $\alpha, \beta : X \to [0, \infty)$ by

$$\alpha(x) = \begin{cases} \frac{2}{|x|+1} \text{ if } x \in (-1, 1) \\ 0 \text{ otherwise,} \end{cases} \quad \text{and} \quad \beta(x) = \begin{cases} 2 \text{ if } x \in \ (-1, 1) \\ 0 \text{ otherwise.} \end{cases}$$

With these mappings α and β, we have (X, d) is an $\alpha\beta$-complete metric space. Indeed, if $\{x_n\}$ is a Cauchy sequence in X such that $\alpha(x_n) \geqslant 1$ and $\beta(x_n) \geqslant 1$ for all n, then $\{x_n\} \subseteq (-1, 1) \subset [-1, 1]$, since $[-1, 1]$ is closed subset of $\mathbf{R}$, it follows that $([-1, 1], d)$ is a complete metric space and so that there exists $z \in [-1, 1]$ such that $x_n \to z$ as $n \to \infty$. Hence X is $\alpha\beta$-complete. But X is not a complete metric space.

Definition 1.8. Let (X, d) be a metric space and let $\alpha, \beta : X \to [0, \infty)$ be two maps. A set $A \subset X$ is said to be $\alpha\beta$-closed if for any sequence $\{x_n\} \subset A$ with $\alpha(x_n) \geqslant 1$ and $\beta(x_n) \geqslant 1$ for all $n \in \mathbf{N}$ such that $\{x_n\}$ converges $z \in X$ then $z \in A$.

Motivated by the works of Chandok, Tas and Ansari [**6**] and Doric [**7**], we introduce generalized $TAC-(S, T)$-rational contractive mappings in metric spaces.

Definition 1.9. Let f, g, S and T be selfmaps of a metric space (X, d) and let $\alpha, \beta : X \to [0, \infty)$ be two given mappings. If there exist $\psi \in \Psi$, $\varphi \in \Phi$ and $h \in \mathscr{C}$ such that

$$\text{for all } x, y \in X \ \text{ with } \alpha(Sx)\beta(Ty) \geqslant 1 \implies \psi(d(fx, gy)) \leqslant h(\psi(M(x, y)),\ \varphi(M(x, y))), \tag{1.1}$$

where

$$(1.2)\quad \begin{aligned} M(x,y) = \max\{d(Sx,Ty), d(Sx,fx), d(Ty,gy), \frac{d(Sx,gy)+d(Ty,fx)}{2}, \\ \frac{d(Ty,gy)[1+d(Sx,fx)]}{1+d(Sx,Ty)}, \frac{d(fx,Ty)[1+d(Sx,gy)]}{1+d(Sx,Ty)}\}, \end{aligned}$$

then we say that the pair (f,g) is a generalized $TAC-(S,T)$ rational contractive map.

If we take $T=S=I,\ I$ is identity map of X in Definition 1.9, we have the following:

Definition 1.10. Let f and g be selfmaps of a metric space (X,d) and let $\alpha,\beta : X \to [0,\infty)$ be two given mappings. If there exist $\psi \in \Psi$, $\varphi \in \Phi$ and $h \in \mathscr{C}$ such that

$$(1.3)\quad \begin{aligned} &\text{for all } x,y \in X \text{ with } \alpha(x)\beta(y) \geqslant 1 \\ &\implies \psi(d(fx,gy)) \leqslant h(\psi(M(x,y)),\ \varphi(M(x,y))), \end{aligned}$$

where

$$M(x,y) = \max\{d(x,y), d(x,fx), d(y,gy), \frac{d(x,gy)+d(y,fx)}{2}, \frac{d(y,gy)[1+d(x,fx)]}{1+d(x,y)}, \frac{d(fx,y)[1+d(x,gy)]}{1+d(x,y)}\},$$

then we say that the pair (f,g) is a generalized TAC-rational contractive map.

If we take $f=g$ and $T=S=I,\ I$ is identity map of X in above definition, we have

Definition 1.11. Let f be selfmap of a metric space (X,d) and let $\alpha,\beta : X \to [0,\infty)$ be two given mappings. If there exist $\psi \in \Psi$, $\varphi \in \Phi$ and $h \in \mathscr{C}$ such that

$$(1.4)\quad \begin{aligned} &\text{for all } x,y \in X \text{ with } \alpha(x)\beta(y) \geqslant 1 \\ &\implies \psi(d(fx,fy)) \leqslant h(\psi(M(x,y)),\ \varphi(M(x,y))) \end{aligned}$$

where

$$M(x,y) = \max\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}, \frac{d(y,fy)[1+d(x,fx)]}{1+d(x,y)}, \frac{d(fx,y)[1+d(x,fy)]}{1+d(x,y)}\},$$

then we say that f is a generalized TAC-rational contractive map.

In Section 2, we prove our main results in which we study the existence of common fixed points of generalized $TAC-(S,T)$-rational contractive mappings in $\alpha\beta$-complete metric spaces. We provide corollaries and an example in support of our results in Section 3.

The following lemma is useful in our subsequent discussion.

Lemma 1.1. [**5**] *Suppose (X,d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n,x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exists an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k \geqslant k$*

such that $d(m_k, n_k) \geqslant \epsilon$. For each $k > 0$, corresponding to m_k, we can choose n_k to be the smallest positive integer such that $d(x_{m_k}, x_{n_k}) \geqslant \epsilon$, $d(x_{m_k}, x_{n_k-1}) < \epsilon$ and

$$(i) \lim_{k\to\infty} d(x_{n_k-1}, x_{m_k+1}) = \epsilon, \quad (ii) \lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \epsilon$$

$$\text{(iii)} \lim_{k\to\infty} d(x_{m_k-1}, x_{n_k}) = \epsilon \text{ and (iv) } \lim_{k\to\infty} d(x_{n_k}, x_{m_k+1}) = \epsilon.$$

2. Main results

Let (X, d) be an $\alpha\beta$-complete metric space. Let f, g, S and T be selfmaps of X. We assume that
(i) $fX \subset TX$, $gX \subset SX$ and
(ii) there exists x_0 in X such that $\alpha(Sx_0) \geqslant 1$ and $\beta(Tx_0) \geqslant 1$.

We define sequence $\{y_n\}$ in X, under the assumptions (i) and (ii) above, as follows.

Let $x_0 \in X$ be as in (ii). By (i), we can choose a point x_1 in X such that $fx_0 = Tx_1 = y_0$ (say), and again since $gX \subset SX$, corresponding to x_1, we can choose x_2 in X such that $gx_1 = Sx_2 = y_1$(say). On continuing this process, it follows that there exists a sequence $\{x_n\}$ in X such that

$$(2.1) \qquad \begin{cases} \quad fx_{2n} = Tx_{2n+1} = y_{2n} \text{ (say)}, \\ \text{and} \\ \quad gx_{2n+1} = Sx_{2n+2} = y_{2n+1} \text{ (say) } n = 0, 1, 2, \dots . \end{cases}$$

Theorem 2.1. *Let (X, d) be an $\alpha\beta$-complete metric space. Let f, g, S and T be selfmaps of X and let (f, g) be a pair generalized $TAC-(S, T)$ rational contractive mappings. Assume that (i) and (ii) hold. Further assume that:*
(iii) (f, g) is a cyclic (α, β)-admissible pair with respect to (S, T);
(iv) if $\{x_n\}$ is a sequence in X such that $x_n \to x$, and $\alpha(x_n) \geqslant 1$ and $\beta(x_n) \geqslant 1$ for all n, then $\alpha(x) \geqslant 1$ and $\beta(x) \geqslant 1$, and
(v) one of the ranges fX, TX, gX, SX is $\alpha\beta$-closed.

Then the sequence $\{y_n\}$ defined by (2.1) is Cauchy in X. Also, $C(f, S) \neq \emptyset$ and $C(g, T) \neq \emptyset$. Let $\lim_{n\to\infty} y_n = z$ (say), $z \in X$.

In fact, $fu = Su = gv = Tv = z$ for some $u \in C(f, S)$ and $v \in C(g, T)$.

Proof. Since $\alpha(Sx_0) \geqslant 1$ and (f, g) is cyclic (α, β)-admissible with respect to (S, T), we have $\beta(fx_0) \geqslant 1$ i.e., $\beta(Tx_1) \geqslant 1$ which further implies that $\alpha(gx_1) \geqslant 1$ i.e.,$(Sx_2) \geqslant 1$. Continuing this way, we obtain that

$$(2.2) \qquad \alpha(Sx_{2n}) \geqslant 1 \text{ and } \beta(Tx_{2n+1}) \geqslant 1 \text{ for all } n \in \mathbf{N} \cup \{0\}.$$

Similarly, by $\beta(Tx_0) \geqslant 1$, we have

$$(2.3) \qquad \beta(Tx_{2n}) \geqslant 1 \text{ and } \alpha(Sx_{2n+1}) \geqslant 1 \text{ for all } n \in \mathbf{N} \cup \{0\}.$$

Thus from (2.2) and (2.3), we have

$$(2.4) \qquad \alpha(Sx_n) \geqslant 1 \text{ and } \beta(Tx_n) \geqslant 1 \text{ for all } n \in \mathbf{N} \cup \{0\}.$$

Now, we show that $\{y_n\}$ is a Cauchy sequence.

Suppose that $y_{2n} = y_{2n+1}$ for some $n \in \mathbf{N}$. Now, we have

$$\begin{aligned} M(x_{2n+2}, x_{2n+1}) = \max\{ & d(Sx_{2n+2}, Tx_{2n+1}), d(Sx_{2n+2}, fx_{2n+2}), d(Tx_{2n+1}, gx_{2n+1}), \\ & \frac{d(Sx_{2n+2}, gx_{2n+1}) + d(fx_{2n+2}, Tx_{2n+1})}{2}, \\ & \frac{d(Tx_{2n+1}, gx_{2n+1})[1 + d(Sx_{2n+2}, fx_{2n+2})]}{1 + d(Sx_{2n+2}, Tx_{2n+1})}, \\ & \frac{d(fx_{2n+2}, Tx_{2n+1})[1 + d(Sx_{2n+2}, gx_{2n+1})]}{1 + d(Sx_{2n+2}, Tx_{2n+1})}\} \\ = \max\{ & d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}), \\ & \frac{d(y_{2n+1}, y_{2n+1}) + d(y_{2n+2}, y_{2n})}{2}, \\ & \frac{d(y_{2n}, y_{2n+1})[1 + d(y_{2n+1}, y_{2n+2})]}{1 + d(y_{2n+1}, y_{2n})}, \\ & \frac{d(y_{2n+2}, y_{2n})[1 + d(y_{2n+1}, y_{2n+1})]}{1 + d(y_{2n+1}, y_{2n})}\}. \end{aligned}$$

Hence

$$M(x_{2n+2}, x_{2n+1}) = d(y_{2n+2}, y_{2n+1}). \tag{2.5}$$

Now, from (1.1) and using (2.5), we have

$$\begin{aligned} \psi(d(y_{2n+2}, y_{2n+1})) &= \psi(d(fx_{2n+2}, gx_{2n+1})) \\ &\leqslant h(\psi(M(x_{2n+2}, x_{2n+1})),\ \varphi(M(x_{2n+2}, x_{2n+1}))) \\ &= h(\psi(d(y_{2n+2}, y_{2n+1})),\ \varphi(d(y_{2n+2}, y_{2n+1})) \\ &\leqslant \psi(d(y_{2n+2}, y_{2n+1})) \end{aligned}$$

Hence we have
$h(\psi(d(y_{2n+2}, y_{2n+1})),\ \varphi(d(y_{2n+2}, y_{2n+1}))) = \psi(d(y_{2n+2}, y_{2n+1}))$, which implies that $\psi(d(y_{2n+2}, y_{2n+1})) = 0$ or $\varphi(d(y_{2n+2}, y_{2n+1})) = 0$, in any case we have $d(y_{2n+2}, y_{2n+1}) = 0$, this implies $y_{2n+1} = y_{2n+2}$. Therefore $y_{2n} = y_{2n+1} = y_{2n+2}$.
In a similar way it is easy to see that $y_{2n} = y_{2n+1} = y_{2n+2} = y_{2n+3}$.
Now, by applying induction it is easy to show that $y_{2n} = y_{2n+k}$ for all $k = 0, 1, 2, ...$
Therefore, $\{y_m\}$ is a constant sequence for $m \geqslant 2n$, hence $\{y_n\}$ is Cauchy in X.

Hence, with out loss of generality, we assume that $y_{2n} \neq y_{2n+1}$ for all $n \in \mathbf{N}$. First we show that $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$. By (2.4) we have $\alpha(Sx_{2n}) \geqslant 1$ and $\beta(Tx_{2n+1}) \geqslant 1$ for all $n \in \mathbf{N} \cup \{0\}$ which implies that $\alpha(Sx_{2n})\beta(Tx_{2n+1}) \geqslant 1$, and hence by putting $x = x_{2n}$ and $y = x_{2n+1}$ in (1.1), we have

$$\psi(d(y_{2n}, y_{2n+1})) = \psi(d(fx_{2n}, gx_{2n+1})) \leqslant h(\psi(M(x_{2n}, x_{2n+1})),\ \varphi(M(x_{2n}, x_{2n+1}))), \tag{2.6}$$

where

$$\begin{aligned}
M(x_{2n}, x_{2n+1}) &= \max\{d(Sx_{2n}, Tx_{2n+1}), d(Sx_{2n}, fx_{2n}), d(Tx_{2n+1}, gx_{2n+1}),\\
&\quad \frac{d(Sx_{2n}, gx_{2n+1}) + d(fx_{2n}, Tx_{2n+1})}{2},\\
&\quad \frac{d(Tx_{2n+1}, gx_{2n+1})[1 + d(Sx_{2n}, fx_{2n})]}{1 + d(Sx_{2n}, Tx_{2n+1})},\\
&\quad \frac{d(fx_{2n}, Tx_{2n+1})[1 + d(Sx_{2n}, gx_{2n+1})]}{1 + d(Sx_{2n}, Tx_{2n+1})}\}\\
&= \max\{d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}),\\
&\quad \frac{d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})}{2},\\
&\quad \frac{d(y_{2n}, y_{2n+1})[1 + d(y_{2n-1}, y_{2n})]}{1 + d(y_{2n-1}, y_{2n})},\\
&\quad \frac{d(y_{2n}, y_{2n})[1 + d(y_{2n-1}, y_{2n+1})]}{1 + d(y_{2n-1}, y_{2n})}\},\\
&= \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\}.
\end{aligned}$$

Suppose that $d(y_{2n-1}, y_{2n}) < d(y_{2n}, y_{2n+1})$ for some $n \in \mathbb{N} \cup \{0\}$. From (2.6) we have

$$\begin{aligned}
\psi(d(y_{2n}, y_{2n+1})) &= \psi(d(fx_{2n}, gx_{2n+1})) \leqslant h(\psi(M(x_{2n}, x_{2n+1})),\ \varphi(M(x_{2n}, x_{2n+1})))\\
&= h(\psi(d(y_{2n},\ y_{2n+1})),\ \varphi(d(y_{2n}, y_{2n+1})) \leqslant \psi(d(y_{2n}, y_{2n+1})).
\end{aligned}$$

Hence we have $h(\psi(d(y_{2n},\ y_{2n+1}),\ \varphi(d(y_{2n},\ y_{2n+1}))) = \psi(d(y_{2n},\ y_{2n+1}))$, and from property (ii) of h, we have either $\psi(d(y_{2n},\ y_{2n+1}) = 0$ or $\varphi(d(y_{2n},\ y_{2n+1})) = 0$. In either case we have $d(y_{2n}, y_{2n+1}) = 0$, a contradiction, since $y_{2n} \neq y_{2n+1}$.

Hence $d(y_{2n-1}, y_{2n}) \geqslant d(y_{2n}, y_{2n+1})$ for all $n \in \mathbf{N} \cup \{0\}$.
Similarly it can be shown that $d(y_{2n+1}, y_{2n}) \geqslant d(y_{2n+1}, y_{2n+2})$ for all $n \in \mathbf{N} \cup \{0\}$. Hence it follows that $\{d(y_n, y_{n+1})\}$ is a decreasing sequence of reals, which is bounded from below. Thus there exists $r \geqslant 0$ such that $\lim_{n\to\infty} d(y_n, x_{n+1}) = r$. Suppose that $r > 0$. We consider

$$\begin{aligned}
\psi(d(y_{2n}, y_{2n+1})) &= \psi(d(fx_{2n}, gx_{2n+1}))\\
&\leqslant h(\psi(M(x_{2n}, x_{2n+1})),\ \varphi(M(x_{2n}, x_{2n+1}))\\
&= h(\psi(d(y_{2n-1}, y_{2n})),\ \varphi(d(y_{2n-1}, y_{2n}))\\
&\leqslant \psi(d(y_{2n-1}, y_{2n})).
\end{aligned} \tag{2.7}$$

On letting $n \to \infty$ in (2.7) and using the continuity of ψ and f, we have $\psi(r) \leqslant f(\psi(r),\ \lim_{n\to\infty} \varphi(d(y_{2n-1}, y_{2n}))) \leqslant \psi(r)$, so that $f(\psi(r),\ \lim_{n\to\infty} \varphi(d(y_{2n-1}, y_{2n}))) = \psi(r)$. Now, by using property (ii) of h, we have either $\psi(r) = 0$ or $\lim_{n\to\infty} \varphi(d(y_{2n-1}, y_{2n})) = 0$ which implies that $r = 0$ Hence $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.

We now prove that $\{y_n\}$ is a Cauchy sequence. To prove it, it is sufficient to show that $\{y_{2n}\}$ is a Cauchy sequence in X. Suppose $\{y_{2n}\}$ is not a Cauchy sequence. Then by Lemma 1.1 there exist $\epsilon > 0$ and sequence of positive integers $\{2n_k\}$ and $\{2m_k\}$ such that $2n_k > 2m_k \geqslant k$ satisfying

$$d(y_{2m_k}, y_{2n_k}) \geqslant \epsilon. \tag{2.8}$$

Let us choose the smallest $2n_k$ satisfying (2.8). Then we have $2n_k > 2m_k \geqslant k$ with $d(y_{2m_k}, y_{2n_k}) \geqslant \epsilon$ and $d(y_{2m_k}, y_{2n_k-2}) < \epsilon$. Also from (i)- (iv) of Lemma 1.1 we have (i) $\lim_{k\to\infty} d(y_{2n_k-1}, y_{2m_k+1}) = \epsilon$ (ii) $\lim_{k\to\infty} d(y_{2n_k}, y_{2m_k}) = \epsilon$
(iii) $\lim_{k\to\infty} d(y_{2m_k-1}, y_{2n_k}) = \epsilon$ and (iv) $\lim_{k\to\infty} d(y_{2n_k}, y_{2m_k+1}) = \epsilon$.
Since $\alpha(Sx_{2m_k}) \geqslant 1$ and $\beta(Tx_{2n_k+1}) \geqslant 1$, we have $\alpha(Sx_{2m_k})\beta(Tx_{2n_k+1}) \geqslant 1$, and by substituting $x = x_{2n_k}$ and $y = x_{2m_k+1}$ in (1.1) we have

$$\begin{aligned}\psi(d(y_{2n_k}, y_{2m_k+1})) &= \psi(d(fx_{2n_k}, gx_{2m_K+1}))\\ &\leqslant h(\psi(M(x_{2n_k}, x_{2m_k+1})),\ \varphi(M(x_{2n_k}, x_{2m_k+1}))),\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}M(x_{2n_k}, x_{2m_k+1}) = \max\{&d(Sx_{2n_k}, Tx_{2m_k+1}), d(Sx_{2n_k}, fx_{2n_k}), d(Tx_{2m_k+1}, gx_{2m_k+1}),\\ &\frac{d(Sx_{2n_k}, gx_{2m_k+1}) + d(fx_{2n_k}, Tx_{2m_k+1})}{2},\\ &\frac{d(Tx_{2m_k+1}, gx_{2m_k+1})[1 + d(Sx_{2n_k}, fx_{2n_k})]}{1 + d(Sx_{2n_k}, Tx_{2m_k+1})},\\ &\frac{d(fx_{2n_k}, Tx_{2m_k+1})[1 + d(Sx_{2n_k}, gx_{2m_k+1})]}{1 + d(Sx_{2n_k}, Tx_{2m_k+1})}\}\end{aligned}$$

On taking limits as $k \to \infty$, we have
$\lim\limits_{k\to\infty} M(x_{2n_k}, x_{2m_k+1}) = \max\{\epsilon, 0, 0, \frac{\epsilon+\epsilon}{2}, 0, \frac{\epsilon[1+\epsilon]}{1+\epsilon}\} = \epsilon,$

From (2.9), we have

$$\begin{aligned}\psi(d(y_{2n_k}, y_{2m_K+1})) &= \psi(d(fx_{2n_k}, gx_{2m_K+1}))\\ &\leqslant h(\psi(M(x_{2n_k}, x_{2m_k+1})),\ \varphi(M(x_{2n_k}, x_{2m_k+1}))),\\ &\leqslant \psi(M(x_{2n_k}, x_{2m_k+1})).\end{aligned} \tag{2.10}$$

on taking limits as $k \to \infty$ in (2.10) we have
$\psi(\epsilon) \leqslant h(\psi(\epsilon), \lim\limits_{k\to\infty} \varphi(M(x_{2n_k}, x_{2m_k+1}))) \leqslant \psi(\epsilon)$, which implies that $h(\psi(\epsilon), \lim\limits_{k\to\infty} \varphi(M(x_{2n_k}, x_{2m_k+1}))) = \psi(\epsilon)$. Now, from the property (ii) of h we have, $\psi(\epsilon) = 0$ or $\lim\limits_{k\to\infty} \varphi(M(x_{2n_k}, x_{2m_k+1})) = 0$, which implies that $\epsilon = 0$ or $\lim\limits_{k\to\infty} M(x_{2n_k}, x_{2m_k+1}) = \epsilon = 0$, in both cases $\epsilon = 0$, a contradiction.
Therefore $\{y_{2n}\}$ is a Cauchy sequence, hence $\{y_n\}$ is a Cauchy sequence in X.
Since $\alpha(y_n) \geqslant 1$ and $\beta(y_n) \geqslant 1$ for all n and X is $\alpha\beta$-complete, there exists $z \in X$ such that $\lim\limits_{n\to\infty} y_n = z$. Hence from (2.1) we have
$\lim\limits_{n\to\infty} fx_{2n} = \lim\limits_{n\to\infty} Tx_{2n+1} = \lim\limits_{n\to\infty} gx_{2n+1} = \lim\limits_{n\to\infty} Sx_{2n+2} = \lim\limits_{n\to\infty} y_{2n} = z$
Case (i): Suppose that SX is $\alpha\beta$-closed.
In this case $z \in SX$ and hence we can choose $u \in X$ such that $z = Su$.

Now, we show that $fu = z$. From (ii), we have $\alpha(Su) \geqslant 1$ and $\beta(Tx_{2n+1}) \geqslant 1$. Now, by substituting $x = u$ and $y = x_{2n+1}$ in (1.1), we have

$$\psi(d(fu, gx_{2n+1})) \leqslant h(\psi(M(u, x_{2n+1})),\ \varphi(M(u, x_{2n+1}))), \tag{2.11}$$

where
$$M(u, x_{2n+1}) = \max\{d(Su, Tx_{2n+1}), d(Su, fu), d(Tx_{2n+1}, gx_{2n+1}), \frac{d(Su, gx_{2n+1}) + d(fu, Tx_{2m_k+1})}{2}, \frac{d(Tx_{2n+1}, gx_{2n+1})[1 + d(Su, fu)]}{1 + d(Su, Tx_{2n+1})}, \frac{d(fu, Tx_{2n+1})[1 + d(Su, gx_{2n+1})]}{1 + d(Su, Tx_{2n+1})}\}.$$

On taking limits as $n \to \infty$ we have

$$\lim_{n\to\infty} M(u, x_{2n+1}) = d(fu,\ z). \tag{2.12}$$

On letting $n \to \infty$ in (2.11) and using (2.12), we have
$\psi(d((fu, z)) \leqslant h(\psi(d(fu, z)),\ \lim_{k\to\infty} \varphi(M(u, x_{2n+1}))) \leqslant \psi(d(fu, z))$, and hence $h(\psi(d(fu, z)),\ \lim_{k\to\infty} \varphi(M(u, x_{2n+1}))) = \psi(d(fu, z))$, which implies that $\psi(d(fu, z)) = 0$ or $\lim_{k\to\infty} \varphi(M(u, x_{2n+1})) = 0$ which further implies $d(fu, z) = 0$ or $\lim_{k\to\infty} M(u, x_{2n+1}) = d(fu, z) = 0$.
Therefore

$$fu = z, \text{i.e., } z = fu = su. \tag{2.13}$$

Hence $Su = fu$, and u is a coincidence point of f and S.

Since $z = fu \in fX$ and $fX \subseteq TX$, we have $z \in TX$ and hence there exists $v \in X$ such that $Tv = z$. By (2.4), we have $\alpha(Sx_{2n}) \geqslant 1$ and by (iv), $\beta(Tv) \geqslant 1$. Now, by substituting $x = x_{2n}$ and $y = v$ in (1.1), we have

$$\psi(d(fx_{2n}, gv)) \leqslant h(\psi(M(x_{2n}, v)),\ \varphi(M(x_{2n}, v))), \tag{2.14}$$

where

$$M(x_{2n}, v) = \max\{d(Sx_{2n}, Tv), d(Sx_{2n}, fx_{2n}), d(Tv, gv), \frac{d(Sx_{2n}, gv) + d(fx_{2n}, Tv)}{2}, \frac{d(Tv, gv)[1 + d(Sx_{2n}, fx_{2n})]}{1 + d(Sx_{2n}, Tv)}, \frac{d(fx_{2n}, Tv)[1 + d(Sx_{2n}, gv)]}{1 + d(Sx_{2n}, Tv)}\}. \tag{2.15}$$

On letting limit as $n \to \infty$ in (2.15), we have

$$\lim_{n\to\infty} M(x_{2n}, v) = d(z, gv). \tag{2.16}$$

On letting $n \to \infty$ in (2.14) and using (2.16), we have
$\psi(d((z, gv)) \leqslant h(\psi(d(z, gv)),\ \lim_{k\to\infty} \varphi(M(x_{2n}, v))) \leqslant \psi(d(z, gv))$,
hence $h(\psi(d(z, gv)),\ \lim_{k\to\infty} \varphi(M(x_{2n}, v))) = \psi(d(z, gv))$, which implies that $\psi(d(z, gv)) = 0$ or $\lim_{k\to\infty} \varphi(M(x_{2n}, v)) = 0$ in both cases $d(z, gv) = 0$. Since

$\lim_{k\to\infty} M(x_{2n}, v) = d(z, gv) = 0$, which implies that $gv = z$.
Therefore

$$z = Tv = gv, \tag{2.17}$$

and v is a coincidence point of T and g. From (2.13) and (2.17), we have
$z = fu = Su = gv = Tv$.

Hence the pairs $C(f, S) \neq \emptyset$ and $C(g, T) \neq \emptyset$.

Case (ii): Suppose that gX is $\alpha\beta$-closed.
In this case $z \in gX$, since $gX \subseteq SX$, we have $z \in SX$ and hence we can choose $u \in X$ such that $z = Su$. Hence the proof follows as in case (i).

For the cases TX is $\alpha\beta$-closed and fX is $\alpha\beta$-closed, we follow the arguments similar to the cases of SX is $\alpha\beta$-closed and gX is $\alpha\beta$-closed receptively. □

THEOREM 2.2. *In addition to the hypotheses of Theorem 2.1, if*
(i) (f, S) *and* (g, T) *are weakly compatible and,*
(ii) $\alpha(Su) \geqslant 1$ *and* $\beta(Tv) \geqslant 1$ *whenever* u *and* v *are coincidence points of* (f, S) *and* (T, g) *respectively.*

Then f, g, T *and* S *have a unique common fixed point.*

PROOF. By Theorem 2.1, we have $z = fu = Su = Tv = gv$. Since (f, S) is weakly compatible we have $fz = fSu = Sfu = Sz$, so that z is also a coincidence point of (f, S). By hypotheses, we have $\alpha(Sz) \geqslant 1$ and $\beta(Tv) \geqslant 1$ which implies that $\alpha(Sz)\beta(Tv) \geqslant 1$.
Now, substituting $x = z$ and $y = v$ in (1.1) we have

$$\psi(d(fz, gv)) \leqslant h(\psi(M(z, v)),\ \varphi(M(z, v))) \tag{2.18}$$

where

$$\begin{aligned} M(z, v) = {} & \max\{d(Sz, Tv), d(Sz, fz), d(Tv, gv), \frac{d(Sz, gv) + d(fz, Tv)}{2}, \\ & \frac{d(Tv, gv)[1 + d(Sz, fz)]}{1 + d(Sz, Tv)}, \frac{d(fz, Tv)[1 + d(Sz, gv)]}{1 + d(Sz, Tv)}\} \\ = {} & d(Sz, Tv) = d(fz, gv). \end{aligned} \tag{2.19}$$

From (2.18) and (2.19), we have

$$\psi(d(fz, gv)) \leqslant h(\psi(d(fz, gv)),\ \varphi(d(fz, gv))) \leqslant \psi(d(fz, gv)), \tag{2.20}$$

which implies that $h(\psi(d(fz, gv)),\ \varphi(d(fz, gv))) = \psi(d(fz, gv))$ which further implies that $\psi(d(fz, gv)) = 0$ or $\varphi(d(fz, gv)) = 0$ in either case we have $d(fz, gv) = 0$. Therefore $fz = gv$, hence $fz = Sz = gv = z$ so that z is a common fixed point of f and S.

Similarly we can show that $z = gz = Tz$. Hence $z = fz = Sz = gz = Tz$.

We now show that f, g, S and T have unique fixed point. Suppose that $u = fu = gu = Su = Tu$ and $z = fz = gz = Sz = Tz$. By hypothesis (ii), we have $\alpha(Su)\beta(Tz) \geqslant 1$ and from (1.1) we have

$$\psi(d(fu, gz)) \leqslant h(\psi(M(u, z))), \varphi(M(u, z)), \tag{2.21}$$

where

$$
\begin{aligned}
M(u,z) = \ & \max\{d(Su,Tz), d(Su,fu), d(Tz,gz), \frac{d(Su,gz)+d(fu,Tz)}{2}, \\
& \frac{d(Tz,gz)[1+d(Su,fu)]}{1+d(Su,Tz)}, \frac{d(fu,Tz)[1+d(Su,gz)]}{1+d(Su,Tz)}\} \\
= \ & d(u,z).
\end{aligned} \tag{2.22}
$$

By using (2.22) in (2.21), we have

$$
\begin{aligned}
\psi(d(fu,gz)) = \psi(d(fu,gz)) &\leqslant h((M(u,z)), \varphi(M(u,z))) \\
&= h(\psi(d(u,z)),\ \varphi(d(u,z)) \leqslant\ \psi(d(u,z)).
\end{aligned} \tag{2.23}
$$

Hence $h(\psi(d(u,z)),\ \varphi(d(u,z))) = \psi(d(u,z))$ which implies that $\psi(d(u,z)) = 0$ or $\lim_{k\to\infty} \varphi(d(u,z)) = \varphi(d(u,z)) = 0$. In either case we have $d(u,z) = 0$. Hence $u = z$. Therefore f, S, g and T have a unique common fixed point in X. □

THEOREM 2.3. *Let A and B be two closed subsets of a complete metric space (X,d) such that $A \cap B \neq \emptyset$. Let $f, g : A \cup B \to A \cup B$ be mappings with $fA \subset B$ and $gB \subset A$. Assume that there exist $\psi \in \Psi$, $\varphi \in \Phi$ and $h \in \mathscr{C}$ such that*

$$
\psi(d(fx,gy)) \leqslant h(\psi(M(x,y)),\ \varphi(M(x,y))) \text{ for all } x\ \in A \text{ and } y \in B, \tag{2.24}
$$

where $M(x,y)$ is defined as in (1.3).

Then f and g have a unique common fixed point $u \in A \cap B$.

PROOF. Let us define $\alpha, \beta : A \cup B \to A \cup B$ by

$$
\alpha(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ otherwise,} \end{cases} \quad \text{and} \quad \beta(x) = \begin{cases} 1 \text{ if } x \in\ B \\ 0 \text{ otherwise.} \end{cases}
$$

We have, for any $x, y \in A \cup B$ with $\alpha(x)\beta(y) \geqslant 1$ if and only if $\alpha(x) \geqslant 1$ and $\beta(y) \geqslant 1$ if and only if $x \in A$ and $y \in B$. Hence from (2.24), we have

$$
\psi(d(fx,gy)) \leqslant h(\psi(M(x,y)),\ \varphi(M(x,y))) \text{ for all } x\ \in A \text{ and } y \in B.
$$

Therefore (f,g) is a generalized TAC- rational contractive map.

Suppose $x \in A \cup B$ with $\alpha(x) \geqslant 1$. Then $x \in A$ and hence $fx \in fA \subset B$ so that $\beta(fx) \geqslant 1$. And suppose $y \in A \cup B$ with $\beta(y) \geqslant 1$. Then $y \in B$ and hence $gx \in gB \subset A$ so that $\alpha(gy) \geqslant 1$. Therefore, (f,g) is a cyclic (α,β)-admissible mapping.

Since $A \cap B \neq \emptyset$ there exists $x_0 \in A \cap B$ so that $\alpha(x_0) \geqslant 1$ and $\beta(x_0) \geqslant 1$.

If $\{x_n\}$ is a sequence in $A \cup B$ such that $x_n \to x$, and $\alpha(x_n) \geqslant 1$ and $\beta(x_n) \geqslant 1$ for all n then $x_n \in A$ and $x_n \in B$. Since A and B are closed we have $x \in A$ and $x \in B$, which implies that $\alpha(x) \geqslant 1$ and $\beta(x) \geqslant 1$.

Therefore, by choosing $S = T = I, I$ is identity mapping on X, it follows that f and g satisfy all hypotheses of Theorem 2.1. Hence there exist $z, u, v \in A \cup B$ such that $z = fu = u, z = gv = v$. If $z \in A$ implies that $\alpha(u) = \alpha(fu) = \alpha(z) \geqslant 1$. If $z \in B$ implies that $\beta(v) = \beta(gv) = \beta(z) \geqslant 1$.

Hence the pair (f,g) satisfies all the hypotheses of Theorem 2.2 with $S = T = I$. and therefore f and g have a unique common fixed point.

Suppose $w = fw = gw$, if $w \in A$ then $w = fw \in fA \subset B$, hence $w \in B$, and also if $w \in B$, we have $w = gw \in gB \subset A$, hence $w \in A$. Therefore the fixed point $w \in A \cap B$. □

3. Corollaries and an example

COROLLARY 3.1. *Let $\alpha, \beta : X \to [0, \infty)$ be two mappings. Let f, g, S and T be selfmaps of an $\alpha\beta$-complete metric space (X, d) with $fX \subset TX$, $gX \subset SX$ and (f, g) be a pair of cyclic (α, β)-admissible mapping with respect to (S, T). Assume that there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that*

$$\text{for all } x, y \in X \text{ with } \alpha(Sx)\beta(Ty) \geqslant 1 \implies \psi(d(fx, gy)) \leqslant \psi(M(x, y)) - \varphi(M(x, y)),$$

where $M(x, y)$ is defined as in (1.2).

Further, suppose that the following conditions hold:
(i) there exists $x_0 \in X$ such that $\alpha(Sx_0) \geqslant 1$ and $\beta(Tx_0) \geqslant 1$,
(ii) if $\{x_n\}$ is a sequence in X such that $x_n \to z$, and $\alpha(x_n) \geqslant 1$ and $\beta(x_n) \geqslant 1$ for all n, then $\alpha(z) \geqslant 1$ and $\beta(z) \geqslant 1$
(iii) one of the ranges fX, TX, gX, SX is $\alpha\beta$-closed.
(iv) (f, S) and (g, T) are weakly compatible, and
(v) $\alpha(Su) \geqslant 1$ and $\beta(Tv) \geqslant 1$ whenever u and v are coincidence point of (f, S) and (T, g) respectively.

Then f, g, T and S. have a unique common fixed point.

PROOF. Follows from Theorem 2.2 by taking $h(s, t) = s - t$. □

COROLLARY 3.2. *Let $\alpha, \beta : X \to [0, \infty)$ be two mappings. Let f, g, S and T be selfmaps of an $\alpha\beta$-complete metric space (X, d) with $fX \subset TX$, $gX \subset SX$ and (f, g) be a pair of cyclic (α, β)-admissible mapping with respect to (S, T). Assume that there exist $\psi \in \Psi$, $\varphi \in \Phi$ and $h \in \mathscr{C}$ such that*

$$\alpha(Sx)\beta(Ty)\psi(d(fx, gy)) \leqslant h(\psi(M(x, y)), \varphi(M(x, y))), \text{ for all } x, y \in X \tag{3.1}$$

where $M(x, y)$ is defined as in (1.2).

Further, suppose that the following conditions hold:
(i) there exists $x_0 \in X$ such that $\alpha(Sx_0) \geqslant 1$ and $\beta(Tx_0) \geqslant 1$,
(ii) if $\{x_n\}$ is a sequence in X such that $x_n \to z$, and $\alpha(x_n) \geqslant 1$ and $\beta(x_n) \geqslant 1$ for all n, then $\alpha(z) \geqslant 1$ and $\beta(z) \geqslant 1$
(iii) one of the ranges fX, TX, gX, SX is $\alpha\beta$-closed.
(iv) (f, S) and (g, T) are weakly compatible, and
(v) $\alpha(Su) \geqslant 1$ and $\beta(Tv) \geqslant 1$ whenever u and v are coincidence point of (f, S) and (T, g) respectively.

Then f, g, T and S. have a unique common fixed point.

PROOF. Let $x, y \in X$ with $\alpha(Sx)\beta(Ty) \geqslant 1$. Then we have
$\psi(d(fx, gy)) \leqslant \alpha(Sx)\beta(Ty)\psi(d(fx, gy)) \leqslant h(\psi(M(x, y)), \phi(M(x, y)))$. Hence the inequality (3.1) implies the inequality (1.1). Now by applying Theorem 2.2, it follows that f, g, T and S. have a unique common fixed point. □

COROLLARY 3.3. Let f, g, S and T be selfmaps of a complete metric space (X, d) with $fX \subset TX$, $gX \subset SX$ and one of the ranges fX, TX, $gX$and SX is closed. Assume that there exist $\psi \in \Psi$, $\varphi \in \Phi$ *and* $h \in \mathscr{C}$ *such that*

$$\psi(d(fx, gy)) \leqslant h(\psi(M(x, y)), \varphi(M(x, y))), \text{for all } x, y \in X \tag{3.2}$$

where $M(x, y)$ is defined as in (1.2).

Then f, g, T *and* S. *have a unique common fixed point.*

PROOF. The result follows from Theorem 2.2 by taking $\alpha(x) = \beta(x) = 1$ for all $x \in X$. □

COROLLARY 3.4. Let f, g, S and T be selfmaps of a complete metric space (X, d) with $fX \subset TX$ and $gX \subset SX$ Assume that there exist $\psi \in \Psi$ *and* $\varphi \in \Phi$ *such that*

$$\psi(d(fx, gy)) \leqslant \psi(M(x, y)) - \varphi(M(x, y)), \tag{3.3}$$

where $M(x, y)$ is defined as in (1.2). *Then* f, g, T *and* S. *have a common fixed point.*

PROOF. The result follows from Corollary 3.3 by taking $h(s, t) = s - t$. □

COROLLARY 3.5. Let $f, g : X \to X$ be selfmappings on complete metric space (X, d). Assume that there exist $\psi \in \Psi$, $\varphi \in \Phi$ and $h \in \mathscr{C}$ such that

$$\psi(d(fx, gy)) \leqslant h(\psi(M(x, y)), \varphi(M(x, y))), \tag{3.4}$$

where
$M(x, y) = \max\{d(x, y), d(x, fx), d(y, gy), \frac{d(x,gy)+d(y,fx)}{2}, \frac{d(y,gy)[1+d(x,fx)]}{1+d(x,y)}, \frac{d(fx,y)[1+d(x,gy)]}{1+d(x,y)}\}$.

Then f *and* g *have a unique fixed point.*

PROOF. Follows from Corollary 3.3 by taking $T = S = I, I$ identity map of$X$. □

COROLLARY 3.6. *Let* α, $\beta : X \to [0, \infty)$ *be two mappings and* (X, d) *be an* $\alpha\beta$*-complete metric space. Let* $f : X \to X$ *is cyclic* (α, β)*-admissible mapping. Assume that there exist* $\psi \in \Psi$, $\varphi \in \Phi$ *and* $h \in \mathscr{C}$ *such that*

$$\begin{aligned} &\text{for all } x, y \in X \text{ with } \alpha(x)\beta(y) \geqslant 1 \\ &\qquad \Longrightarrow \psi(d(fx, fy)) \leqslant h(\psi(M(x, y)), \varphi(M(x, y))), \end{aligned} \tag{3.5}$$

where
$M(x, y) = \max\{d(x, y), d(x, fx), d(y, fy), \frac{d(x,fy)+d(y,fx)}{2}, \frac{d(y,fy)[1+d(x,fx)]}{1+d(x,y)}, \frac{d(fx,y)[1+d(x,y)]}{1+d(x,y)}\}$.

Further, suppose that the following conditions hold:

(i) there exists $x_0 \in X$ *such that* $\alpha(x_0) \geqslant 1$ *and* $\beta(x_0) \geqslant 1$,

(ii) if $\{x_n\}$ *is a sequence in* X *such that* $x_n \to z$, *and* $\alpha(x_n) \geqslant 1$ *and* $\beta(x_n) \geqslant 1$ *for all* n, *then* $\alpha(z) \geqslant 1$ *and* $\beta(z) \geqslant 1$

Moreover, if $\alpha(x) \geqslant 1$ *and* $\beta(y) \geqslant 1$ *for all* $x, y \in$ *Fix* (f) *where Fix* (f) *is the set of all fixed points of* f, *then* f *has a unique fixed point.*

Proof. Follows from Theorem 2.2 by taking $f = g$ and $T = S = I$, where I identity map of X. $\square$

Remark 3.1. *Corollary 3.6 is an extension of Theorem 1.2 to $\alpha\beta$-complete metric spaces.*

Corollary 3.7. Let A and B be two closed subsets of a complete metric space (X, d) such that $A \cap B \neq \emptyset$. Let $f, g : A \cup B \to A \cup B$ be mappings with $fA \subset B$ and $gB \subset A$. Assume that there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that
$\psi(d(fx, gy)) \leqslant \psi(M(x, y)) - \varphi(M(x, y))$ for all $x \in A$ and $y \in B$
where $M(x, y)$ is defined as in (1.3).
Then f and g have a unique common fixed point in $A \cap B$.

Proof. Follows from Theorem 2.3 by taking $h(s, t) = s - t$. $\square$

Example 3.1. Let $X = [0, 10)$ with the usual metric d. We define f, g, S and T on X by

$$f(x) = \begin{cases} \frac{x}{8} \text{ if } x \in [0, 1] \\ x - \frac{1}{3} \text{ if } x \in (1, 10), \end{cases} \qquad g(x) = \begin{cases} \frac{x}{4} \text{ if } x \in [0, 1] \\ x - 1 \text{ if } x \in (1, 10), \end{cases}$$

$$S(x) = \begin{cases} \frac{x}{2} \text{ if } x \in [0, 1] \\ x - \frac{1}{2} \text{ if } x \in [1, 10), \end{cases} \quad \text{and } \ T(x) = x.$$

Now, we have $fX = [0, \frac{1}{8}] \cup (\frac{2}{3}, \frac{29}{3}) \subseteq [0, 10) = TX$ and
$gX = [0, \frac{1}{4}] \cup (0, 9) = [0, 9) \subseteq [0, \frac{19}{2}) = SX$. It is clear that the pairs (f, S) and (g, T) are weakly compatible.

we now define $\alpha, \beta : X \to [0, \infty)$ by

$$\alpha(x) = \begin{cases} 1 \text{ if } x \in [0, \frac{1}{2}] \\ 0 \text{ otherwise} \end{cases} \quad \text{and } \ \beta(x) = \begin{cases} e \text{ if } x \in \ [0, 1] \\ 0 \text{ otherwise.} \end{cases}$$

Since for any $x \in X$ with $\alpha(Sx) \geqslant 1 \Leftrightarrow Sx \in [0, \frac{1}{2}] \Leftrightarrow x \in [0, 1]$, we have $fx \in [0, \frac{1}{8}]$ which implies that $\beta(fx) = e \geqslant 1$. Also, for any $x \in X$ with
$\beta(Tx) \geqslant 1 \Leftrightarrow Tx \in [0, 1]$ which implies that $x \in [0, 1]$, and hence $gx \in [0, \frac{1}{4}]$, so that $\alpha(gx) = 1$. Therefore (f, g) is a cyclic (α, β)-admissible mapping with respect to (S, T). Moreover at $x_0 = 0$, $\alpha(Sx_0) = \alpha(0) \geqslant 1$ and $\beta(Tx_0) = \beta(0) \geqslant 1$.

If $\{x_n\}$ is any sequence in X such that $\alpha(x_n) \geqslant 1$ and $\beta(x_n) \geqslant 1$ for all $n \in \mathbf{N} \cup \{0\}$ and $x_n \to x$, by the definition of α and β, we have $x_n \in [0, \frac{1}{2}]$. Therefore $x \in [0, \frac{1}{2}]$. Hence $\alpha(x) \geqslant 1$ and $\beta(x) \geqslant 1$.

Now, we show that (f, g) is a generalized $TAC - (S, T)$ rational contractive mapping. For this purpose, we choose $\psi, \ \varphi : [0, \infty) \to [0, \infty)$ by $\psi(t) = t, \ t \geqslant 0$,

$$\varphi(t) = \begin{cases} 1 \text{ if } t = 0 \\ \frac{t}{2} \text{ if } t \in (0, \infty), \end{cases} \quad \text{and } h : [0, \infty)^2 \to \mathbf{R} \text{ by } \ h(s, t) = \frac{s}{1 + t}, \ \forall s, t \in [0, \infty).$$

Then clearly $\psi \in \Psi$ and $\varphi \in \Phi$. We now verify the inequality (1.1).

It is easy to see that $\alpha(Sx)\beta(Ty) \geqslant 1$ if and only if $x \in [0, 1]$ and $y \in [0, 1]$. Hence we verify inequality (1.1) for $x, y \in [0, 1]$.

Now, for $x, y \in [0,1]$, we have

$$\psi(d(fx,gy)) = d(fx,gy) = |fx-gy| = |\frac{x}{8}-\frac{y}{4}| = \frac{1}{8}|x-2y| = \frac{1}{4}.\frac{1}{2}|x-2y| \tag{3.6}$$
$$= \frac{1}{4}d(Sx,Ty) \leqslant \frac{d(Sx,Ty)}{1+\frac{d(Sx,Ty)}{2}},$$

since $d(Sx,Ty) = |Sx-Ty| = |\frac{x}{2}-y| \leqslant 1$.

Since the function $c:[0,\infty) \to \mathbf{R}$ defined by $c(t) = \frac{t}{1+\frac{t}{2}},\ t \geqslant 0$ monotonically increasing on $[0,4]$, it follows from (3.6) that

$$\psi(d(fx,gy)) \leqslant \frac{d(Sx,Ty)}{1+\frac{d(Sx,Ty)}{2}} \leqslant \frac{M(x,y)}{1+\frac{M(x,y)}{2}}$$
$$= \frac{\psi(M(x,y))}{1+\varphi(M(x,y))} = h(\psi(M(x,y)),\varphi(M(x,y))),$$

so that the inequality (1.1) holds.

Clearly $X=[0,10)$ is $\alpha\beta$-complete and $TX = X = [0,1)$ is $\alpha\beta$-closed. Hence f, g, S and T satisfy all the hypotheses of Theorem 2.2 and $\{0\}$ is the unique common fixed point of f, g, S and T.

Here we observe that (X,d) is not complete, φ is not continuous and $\phi(0) \neq 0$. Hence Theorem 1.1 is not applicable.

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