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Eigenvalue Intervals for the Existence of Positive Solutions to System of Multi-Point Fractional Order Boundary Value Problems

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Abstract. We determine eigenvalue intervals of λ_1 and λ_2 for the existence of at least one positive solution for a coupled system of Riemann–Liouville type multi-point fractional order boundary value problems by utilizing a fixed point theorem on a cone under suitable conditions.

1. Introduction

Differential equations with fractional order have recently proved to be strong tools in modeling of many physical phenomena [**16, 12, 8**]. It owes to the intensive development of the theory of fractional calculus as well as its applications in various fields of science and technology such as aerodynamics, biology, Bode's analysis of feedback amplifiers, capacitor theory, chemistry, control theory, physics, polymer rheology, electrical circuits. In consequence, the study of fractional order differential equations is attained much importance and attention due to their wide applicability.

Much interest has been created in establishing positive solutions and multiple positive solutions for two-point, multi-point boundary value problems associated with odinary and fractional order differential equations. To mention the related papers along these lines, we refer to Bai and Fang [**1**], Gupta, Ntouyas and Tsamatos [**6, 7**], Ma [**11, 10**] for ordinary differential equations and Bai and Lü [**2**], Benchohra, Henderson, Ntoyuas and Ouahab [**3**], Su and Zhang [**17**], Prasad and Krushna [**13, 15**] for fractional order differential equations.

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Gupta [**5**] proved the existence of positive solutions for more general multi-point boundary value problems

$$x''(t) = g\big(t, x(t), x'(t)\big) + e(t),\ a.\ e.\ t \in (0,1),$$
$$x(0) = \sum_{i=1}^{m-2} h_i x\big(\tau_i\big),\ x'(1) = \sum_{i=1}^{m-2} k_i x'\big(\xi_i\big).$$

In 2007, Yao [**19**] obtained the existence of n solutions and/or positive solutions to the following semipositone elastic beam equation boundary value problem

$$u^{(4)}(t) = f\big(t, u(t), u'(t), u''(t)\big),\ t \in (0,1),$$
$$u(0) = u'(0) = u''(0) = u''(1) = 0.$$

In 2008, Sun and Zhang [**18**] considered the third order m-point boundary value problem

$$u'''(t) + f(t, u(t), u'(t), u''(t)) = 0,\ t \in (0,1),$$
$$u(0) = u'(0) = 0, u''(1) = \sum_{i=1}^{m-2} k_i u''\big(\xi_i\big),$$

where $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is L_p-Caratheodory, $1 < p < \infty$, $0 = \xi_0 < \xi_2 < \cdots < \xi_{m-1} = 1, k_i \in \mathbb{R}(i = 1, 2, \cdots, m-2)$ and $\sum_{i=1}^{m-2} k_i \neq 1$. Some criteria for the existence of at least one solution are established by using the well-known Leray–Schauder continuation principle.

Recently Prasad and Krushna [**14**] determined the eigenvalue intervals for which there exist at least one positive solution to the fractional order boundary value problem

$$D_{a^+}^{q_1} y_i(t) + \lambda_i p_i(t) f_i(y_{i+1}(t)) = 0,\ 1 \leqslant i \leqslant n,\ t \in [a,b],$$
$$y_{n+1}(t) = y_1(t),$$
$$\alpha y_i(a) - \beta D_{a^+}^{q_2} y_i(a) = 0,\ \gamma y_i(b) + \delta D_{a^+}^{q_3} y_i(b) = 0,$$

where $D_{a^+}^{q_i}, i = 1, 2, 3$ are the standard Riemann–Liouville fractional order derivatives, $1 < q_1 \leqslant 2$, $0 < q_2, q_3 \leqslant 1$, by an application of standard fixed point theorem.

Inspired by the papers mentioned above, in this paper, we determine eigenvalue intervals of λ_1 and λ_2 for which there exist positive solutions to a coupled system of fractional order differential equations

$$D_{0^+}^{\alpha_1} y_1(t) + \lambda_1 p(t) f\big(y_1(t), y_2(t)\big) = 0,\ t \in (0,1), \tag{1.1}$$

$$D_{0^+}^{\alpha_2} y_2(t) + \lambda_2 q(t) g\big(y_1(t), y_2(t)\big) = 0,\ t \in (0,1), \tag{1.2}$$

satisfying the multi-point boundary conditions

$$y_1(0) = y_1'(0) = y_1''(0) = 0, y_1''(1) = \sum_{k=2}^{m-1} \vartheta_k y_1''\big(\xi_k\big),\ m \geqslant 3, \tag{1.3}$$

$$y_2(0) = y_2'(0) = y_2''(0) = 0, y_2''(1) = \sum_{k=2}^{n-1} \zeta_k y_2''\big(\eta_k\big),\ n \geqslant 3, \tag{1.4}$$

where $0 < \xi_2 < \cdots < \xi_{m-1} < 1,\ 0 < \eta_2 < \cdots < \eta_{n-1} < 1,\ \lambda_1, \lambda_2 > 0,\ \alpha_i \in (3, 4]$ and $D_{0^+}^{\alpha_i}$, for $i = 1, 2$ are the standard Riemann–Liouville fractional order derivatives. We shall give sufficient conditions on λ_i for $i = 1, 2$ and f, g such that the fractional order boundary value problem (1.1)-(1.4) has positive solutions. By a positive solution of the system of fractional order boundary value problem (1.1)-(1.4), we mean $\Big(y_1(t), y_2(t)\Big) \in \Big(C^{\alpha_1}[0, 1] \times C^{\alpha_2}[0, 1]\Big)$ satisfying (1.1)-(1.4) with $y_1(t) \geqslant 0$, $y_2(t) \geqslant 0$, for all $t \in [0, 1]$ and $(y_1, y_2) \neq (0, 0)$.

We assume the following conditions hold throughout the paper:

(A1) the functions $f, g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ are continuous,

(A2) the functions $p, q : [0, 1] \to \mathbb{R}^+$ are continuous and p, q do not vanish identically on any closed subinterval of $[0, 1]$,

(A3) $\vartheta_1, \vartheta_2, \vartheta_3, \cdots, \vartheta_{m-1}$ and $\zeta_1, \zeta_2, \zeta_3, \cdots, \zeta_{n-1}$ are positive constants such that

$$\sum_{k=2}^{m-1} \vartheta_k \xi_k^{\alpha_1 - 3} < 1 \text{ and } \sum_{k=2}^{n-1} \zeta_k \eta_k^{\alpha_2 - 3} < 1,$$

(A4) each of $f_0^s, f_0^i, f_\infty^s, f_\infty^i, g_0^s, g_0^i, g_\infty^s$ and g_∞^i, by

$$f_0^s = \lim_{(y_1, y_2) \to (0^+, 0^+)} \sup \frac{f(y_1, y_2)}{\sum_{i=1}^{2} y_i}, f_0^i = \lim_{(y_1, y_2) \to (0^+, 0^+)} \inf \frac{f(y_1, y_2)}{\sum_{i=1}^{2} y_i},$$

$$f_\infty^s = \lim_{(y_1, y_2) \to (\infty, \infty)} \sup \frac{f(y_1, y_2)}{\sum_{i=1}^{2} y_i}, f_\infty^i = \lim_{(y_1, y_2) \to (\infty, \infty)} \inf \frac{f(y_1, y_2)}{\sum_{i=1}^{2} y_i},$$

$$g_0^s = \lim_{(y_1, y_2) \to (0^+, 0^+)} \sup \frac{g(y_1, y_2)}{\sum_{i=1}^{2} y_i}, g_0^i = \lim_{(y_1, y_2) \to (0^+, 0^+)} \inf \frac{g(y_1, y_2)}{\sum_{i=1}^{2} y_i},$$

$$g_\infty^s = \lim_{(y_1, y_2) \to (\infty, \infty)} \sup \frac{g(y_1, y_2)}{\sum_{i=1}^{2} y_i}, g_\infty^i = \lim_{(y_1, y_2) \to (\infty, \infty)} \inf \frac{g(y_1, y_2)}{\sum_{i=1}^{2} y_i},$$

exist as positive real numbers.

The organization of this paper is as follows: In §2, we compute the Green functions for the fractional order boundary value problems and estimate the bounds for these Green functions. In §3, we establish criteria for the existence of positive solutions to fractional order boundary value problem (1.1)-(1.4) by utilizing Guo–Krasnosel'skii fixed point theorem. In §4, as an application, we demonstrate our results with an example.

2. Green functions and bounds

In this section the Green functions for the fractional order boundary value problems are constructed and the bounds for these Green functions are estimated, which are essential to establish the main results.

LEMMA 2.1. *Let* $d_1 = 1 - \vartheta_k \xi_k^{\alpha_1 - 3} \neq 0$. *If* $h_1(t) \in C[0,1]$, *then the fractional order differential equations*

$$D_{0^+}^{\alpha_1} y_1(t) + h_1(t) = 0, \ t \in (0,1), \tag{2.1}$$

satisfying (1.3) *has a unique solution*

$$y_1(t) = \int_0^1 G_1(t,s) h_1(s) ds,$$

where

$$G_1(t,s) = G_1^*(t,s) + \frac{t^{\alpha_1 - 1}}{d_1} \sum_{k=2}^{m-1} \vartheta_k G_1^{**}\big(\xi_k, s\big), \tag{2.2}$$

$$G_1^*(t,s) = \begin{cases} \dfrac{t^{\alpha_1-1}(1-s)^{\alpha_1-3}}{\Gamma(\alpha_1)}, & 0 \leqslant t \leqslant s \leqslant 1, \\ \dfrac{t^{\alpha_1-1}(1-s)^{\alpha_1-3} - (t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}, & 0 \leqslant s \leqslant t \leqslant 1, \end{cases}$$

$$G_1^{**}\big(\xi_k, s\big) = \begin{cases} \dfrac{\big[\xi_k(1-s)\big]^{\alpha_1-3}}{\Gamma(\alpha_1)}, & 0 \leqslant \xi_k \leqslant s \leqslant 1, \\ \dfrac{\big[\xi_k(1-s)\big]^{\alpha_1-3} - (\xi_k - s)^{\alpha_1-3}}{\Gamma(\alpha_1)}, & 0 \leqslant s \leqslant \xi_k \leqslant 1. \end{cases}$$

PROOF. Let $y_1 \in C^{\alpha_1}[0,1]$ be the solution of fractional order boundary value problem given by (2.1) and (1.3). An equivalent integral equation for (2.1) is given by

$$y_1(t) = \frac{-1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} h_1(s) ds + c_1 t^{\alpha_1-1} + c_2 t^{\alpha_1-2} + c_3 t^{\alpha_1-3} + c_4 t^{\alpha_1-4}.$$

Using the boundary conditions (1.3), one can determine $c_4 = c_3 = c_2 = 0$ and

$$c_1 = \frac{1}{d_1}\left[\int_0^1 \frac{(1-s)^{\alpha_1-3}}{\Gamma(\alpha_1)} h_1(s) ds - \sum_{k=2}^{m-1} \int_0^{\xi_k} \frac{\vartheta_k(\xi_k - s)^{\alpha_1-3}}{\Gamma(\alpha_1)} h_1(s) ds\right].$$

Hence the unique solution of the fractional order boundary value problem given by (2.1) and (1.3) is

$$y_1(t) = \frac{t^{\alpha_1-1}}{d_1}\left[\int_0^1 \frac{(1-s)^{\alpha_1-3}}{\Gamma(\alpha_1)}h_1(s)ds - \sum_{k=2}^{m-1}\int_0^{\xi_k}\frac{\vartheta_k(\xi_k-s)^{\alpha_1-3}}{\Gamma(\alpha_1)}h_1(s)ds\right]$$
$$- \int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}h_1(s)ds$$
$$= \int_0^t \frac{t^{\alpha_1-1}(1-s)^{\alpha_1-3}}{\Gamma(\alpha_1)}h_1(s)ds - \int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}h_1(s)ds + \frac{t^{\alpha_1-1}}{d_1}\times$$
$$\left[\int_0^1 \frac{\vartheta_k\xi_k^{\alpha_1-3}(1-s)^{\alpha_1-3}}{\Gamma(\alpha_1)}h_1(s)ds - \sum_{k=2}^{m-1}\int_0^{\xi_k}\frac{\vartheta_k(\xi_k-s)^{\alpha_1-3}}{\Gamma(\alpha_1)}h_1(s)ds\right]$$
$$= \int_0^1 \left\{G_1^*(t,s) + \frac{t^{\alpha_1-1}}{d_1}\sum_{k=2}^{m-1}\vartheta_k G_1^{**}\big(\xi_k,s\big)\right\}h_1(s)ds$$
$$= \int_0^1 G_1(t,s)h_1(s)ds.$$

□

LEMMA 2.2. *Assume that the condition (A3) is satisfied. Then the Green's function $G_1(t,s)$ given in (2.2) is nonnegative, for all $t,s \in [0,1]$.*

PROOF. Consider the Green's function $G_1(t,s)$ given by (2.2). Let $0 \leqslant t \leqslant s \leqslant 1$. Then, we have

$$G_1^*(t,s) = \frac{t^{\alpha_1-1}(1-s)^{\alpha_1-3}}{\Gamma(\alpha_1)} \geqslant 0.$$

Let $0 \leqslant s \leqslant t \leqslant 1$. Then, we have

$$G_1^*(t,s) = \frac{t^{\alpha_1-1}(1-s)^{\alpha_1-3} - (t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}$$
$$\geqslant \frac{t^{\alpha_1-1}\big[(1-s)^{\alpha_1-3} - (1-s)^{\alpha_1-1}\big]}{\Gamma(\alpha_1)} \geqslant 0.$$

Let $0 \leqslant \xi_k \leqslant s \leqslant 1$. Then

$$G_1^{**}\big(\xi_k,s\big) = \frac{\big[\xi_k(1-s)\big]^{\alpha_1-3}}{\Gamma(\alpha_1)} \geqslant 0.$$

Let $0 \leqslant s \leqslant \xi_k \leqslant 1$. Then

$$G_1^{**}\big(\xi_k,s\big) = \frac{\big[\xi_k(1-s)\big]^{\alpha_1-3} - (\xi_k-s)^{\alpha_1-3}}{\Gamma(\alpha_1)}$$
$$\geqslant \frac{\big[\xi_k(1-s)\big]^{\alpha_1-3} - \big[\xi_k(1-s)\big]^{\alpha_1-3}}{\Gamma(\alpha_1)} = 0.$$

□

Lemma 2.3. *Assume that the condition (A3) is satisfied. Then the Green's function $G_1(t,s)$ given in (2.2) satisfies the following inequality:*

$$m_1(t)G_1(1,s) \leqslant G_1(t,s) \leqslant G_1(1,s), \ for\ all\ t,s \in [0,1], \tag{2.3}$$

where $m_1(t) = t^{\alpha_1-1}$.

Proof. Consider the Green's function $G_1(t,s)$ given by (2.2). Let $0 \leqslant t \leqslant s \leqslant 1$ and $0 \leqslant \xi_k \leqslant s \leqslant 1$. Then, we have

$$\begin{aligned} G_1(t,s) =& G_1^*(t,s) + \frac{t^{\alpha_1-1}}{d_1}\sum_{k=2}^{m-1}\vartheta_k G_1^{**}\big(\xi_k,s\big) \\ \leqslant& G_1^*(1,s) + \frac{1}{d_1}\sum_{k=2}^{m-1}\vartheta_k G_1^{**}\big(1,s\big) \\ \leqslant& G_1(1,s). \end{aligned}$$

Similarly $G_1(t,s) \leqslant G_1(1,s)$ for $0 \leqslant s \leqslant t \leqslant 1$ and $0 \leqslant s \leqslant \xi_k \leqslant 1$.

Let $0 \leqslant t \leqslant s \leqslant 1$ and $0 \leqslant \xi_k \leqslant s \leqslant 1$. Then, we have

$$\begin{aligned} G_1(t,s) =& \frac{t^{\alpha_1-1}(1-s)^{\alpha_1-3}}{\Gamma(\alpha_1)} + \frac{t^{\alpha_1-1}}{d_1}\sum_{k=2}^{m-1}\vartheta_k G_1^{**}\big(\xi_k,s\big) \\ \geqslant& t^{\alpha_1-1}\left[G_1^*(1,s) + \frac{1}{d_1}\sum_{k=2}^{m-1}\vartheta_k G_1^{**}\big(1,s\big)\right] \\ \geqslant& t^{\alpha_1-1}G_1(1,s). \end{aligned}$$

Similarly $G_1(t,s) \geqslant t^{\alpha_1-1}G_1(1,s)$ for $0 \leqslant s \leqslant t \leqslant 1$ and $0 \leqslant s \leqslant \xi_k \leqslant 1$. □

Lemma 2.4. *Assume that the condition (A3) is satisfied and $s \in [0,1]$. Then the Green's function $G_1(t,s)$ given in (2.2) satisfies*

$$\min_{t\in[\vartheta_{m-1},1]} G_1(t,s) \geqslant k_1 G_1(1,s), \tag{2.4}$$

where $k_1 = \vartheta_k \xi_k^{\alpha_1-3} < 1$.

Proof. By Lemma 2.3, we can easily establish the result. □

Lemma 2.5. *Let $d_2 = 1 - \zeta_k \eta_k^{\alpha_2-3} \neq 0$. If $h_2(t) \in C[0,1]$, then the fractional order differential equations*

$$D_{0^+}^{\alpha_2} y_2(t) + h_2(t) = 0, \ t \in (0,1), \tag{2.5}$$

satisfying (1.3) has a unique solution,

$$y_2(t) = \int_0^1 G_2(t,s)h_2(s)ds,$$

where

$$G_2(t,s) = G_2^*(t,s) + \frac{t^{\alpha_2-1}}{d_2}\sum_{k=2}^{n-1}\zeta_k G_2^{**}\big(\eta_k,s\big), \tag{2.6}$$

$$G_2^*(t,s) = \begin{cases} \dfrac{t^{\alpha_2-1}(1-s)^{\alpha_2-3}}{\Gamma(\alpha_2)}, & 0 \leqslant t \leqslant s \leqslant 1, \\ \dfrac{t^{\alpha_2-1}(1-s)^{\alpha_2-3} - (t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}, & 0 \leqslant s \leqslant t \leqslant 1, \end{cases}$$

$$G_2^{**}\big(\eta_k,s\big) = \begin{cases} \dfrac{\big[\eta_k(1-s)\big]^{\alpha_2-3}}{\Gamma(\alpha_2)}, & 0 \leqslant \eta_k \leqslant s \leqslant 1, \\ \dfrac{\big[\eta_k(1-s)\big]^{\alpha_2-3} - (\eta_k-s)^{\alpha_2-3}}{\Gamma(\alpha_2)}, & 0 \leqslant s \leqslant \eta_k \leqslant 1. \end{cases}$$

Proof. Proof is similar to Lemma 2.1. □

Lemma 2.6. *Assume that the condition* ($A3$) *is satisfied. Then the Green's function* $G_2(t,s)$ *given in* (2.6) *is nonnegative, for all* $t,s \in [0,1]$.

Proof. Proof is similar to Lemma 2.2. □

Lemma 2.7. *Assume that the condition* ($A3$) *is satisfied. Then the Green's function* $G_2(t,s)$ *given in* (2.6) *satisfies the following inequality:*

$$m_2(t)G_2(1,s) \leqslant G_2(t,s) \leqslant G_2(1,s), \ \ for\ all\ t,s \in [0,1], \tag{2.7}$$

where $m_2(t) = t^{\alpha_2-1}$.

Proof. Proof is similar to Lemma 2.3. □

Lemma 2.8. *Assume that the condition* ($A3$) *is satisfied and* $s \in [0,1]$. *Then the Green's function* $G_2(t,s)$ *given in* (2.6) *satisfies*

$$\min_{t\in[\zeta_{n-1},1]} G_2(t,s) \geqslant k_2 G_2(1,s), \tag{2.8}$$

where $k_2 = \zeta_k \eta_k^{\alpha_2-3} < 1$.

Proof. By Lemma 2.7, we can easily establish the result. □

Theorem 2.1. [**4, 9**] *Let* X *be a Banach Space,* $P \subseteq X$ *be a cone and suppose that* Ω_1, Ω_2 *are open subsets of* X *with* $0 \in \Omega_1$ *and* $\overline{\Omega}_1 \subset \Omega_2$. *Suppose further that* $T : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ *is completely continuous operator such that either*

(i) $\| Tu \| \leqslant \| u \|,\ u \in P \cap \partial\Omega_1$ *and* $\| Tu \| \geqslant \| u \|,\ u \in P \cap \partial\Omega_2$, *or*

(ii) $\| Tu \| \geqslant \| u \|,\ u \in P \cap \partial\Omega_1$ *and* $\| Tu \| \leqslant \| u \|,\ u \in P \cap \partial\Omega_2$ *holds.*

Then T *has a fixed point in* $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Existence of at least one positive solution

In this section we establish the existence of at least one positive solution for a coupled system of fractional order boundary value problem (1.1)-(1.4).

Consider the Banach space $\mathcal{B} = \mathcal{E} \times \mathcal{E}$, where $\mathcal{E} = \big\{ y_1 : y_1 \in C[0,1] \big\}$ equipped with the norm $\|(y_1, y_2)\| = \sum_{i=1}^{2} \|y_i\|$ for $(y_1, y_2) \in \mathcal{B}$ and the norm is defined as

$$\|y_1\| = \max_{t\in[0,1]} |y_1(t)|.$$

Define a cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \Big\{ (y_1, y_2) \in \mathcal{B} : y_1(t), y_2(t) \geqslant 0 \text{ on } [0,1] \text{ and } \min_{t\in[r,1]} \sum_{i=1}^{2} y_i(t) \geqslant k\|(y_1, y_2)\| \Big\},$$

where

$$r = \max\big\{\vartheta_{m-1}, \zeta_{n-1}\big\} \text{ and } k = \min\big\{k_1, k_2\big\}. \tag{3.1}$$

Let $T_1, T_2 : \mathcal{P} \to \mathcal{E}$ and $T : \mathcal{P} \to \mathcal{B}$ be the operators defined by

$$\begin{aligned} T_1(y_1, y_2)(t) =& \lambda_1 \int_0^1 G_1(t,s) p(s) f\big(y_1(s), y_2(s)\big) ds, \\ T_2(y_1, y_2)(t) =& \lambda_2 \int_0^1 G_2(t,s) q(s) g\big(y_1(s), y_2(s)\big) ds \end{aligned}$$

and

$$T(y_1, y_2)(t) = \Big(T_1(y_1, y_2)(t), T_2(y_1, y_2)(t) \Big), \text{ for } (y_1, y_2) \in \mathcal{B}. \tag{3.2}$$

Lemma 3.1. *The operator T defined by* (3.2) *is a self map on $\mathcal{P}$.*

Proof. Let $(y_1, y_2) \in \mathcal{P}$. Clearly $T_1(y_1, y_2)(t)$ and $T_2(y_1, y_2)(t)$ are nonnegative for $t \in [0,1]$. Also for $(y_1, y_2) \in \mathcal{P}$,

$$\begin{aligned} \|T_1(y_1, y_2)\| \leqslant& \lambda_1 \int_0^1 G_1(1,s) p(s) f\big(y_1(s), y_2(s)\big) ds, \\ \|T_2(y_1, y_2)\| \leqslant& \lambda_2 \int_0^1 G_2(1,s) q(s) g\big(y_1(s), y_2(s)\big) ds, \end{aligned}$$

and

$$\begin{aligned} \min_{t\in[r,1]} T_1(y_1, y_2)(t) =& \min_{t\in[r,1]} \lambda_1 \int_0^1 G_1(t,s) p(s) f\big(y_1(s), y_2(s)\big) ds \\ \geqslant& k_1 \lambda_1 \int_0^1 G_1(1,s) p(s) f\big(y_1(s), y_2(s)\big) ds \\ \geqslant& k_1 \|T_1(y_1, y_2)\|_0. \end{aligned}$$

Similarly $\min_{t\in[r,1]} T_2(y_1,y_2)(t) \geqslant k_2\|T_2(y_1,y_2)\|_0$. Therefore,

$$\begin{aligned}\min_{t\in[r,1]} \sum_{i=1}^{2} T_i(y_1,y_2)(t) &\geqslant k_1\|T_1(y_1,y_2)\|_0 + k_2\|T_2(y_1,y_2)\|_0\\ &\geqslant k\sum_{i=1}^{2}\big\|T_i(y_1,y_2)\big\|_0\\ &= k\big\|\big(T_1(y_1,y_2),T_2(y_1,y_2)\big)\big\|\\ &= k\big\|T(y_1,y_2)\big\|.\end{aligned}$$

Hence $T(y_1,y_2)\in\mathcal{P}$ and so $T:\mathcal{P}\to\mathcal{P}$. Standard arguments involving the Arzela–Ascoli theorem shows that T is completely continuous. □

Let

$$\begin{aligned}\Lambda_1 =&\gamma_1\left[kk_1f_\infty^i\int_r^1 G_1(1,s)p(s)ds\right]^{-1}, \Lambda_2=\gamma_1\left[f_0^s\int_0^1 G_1(1,s)p(s)ds\right]^{-1},\\ \Lambda_3 =&\gamma_2\left[kk_2g_\infty^i\int_r^1 G_2(1,s)q(s)ds\right]^{-1}, \Lambda_4=\gamma_2\left[g_0^s\int_0^1 G_2(1,s)q(s)ds\right]^{-1},\end{aligned}$$

where γ_1,γ_2 are two positive real numbers such that $\gamma_1+\gamma_2=1$.

THEOREM 3.1. *Assume that the conditions (A1)-(A4) hold.*

- (H1) *If* $f_0^s, g_0^s, f_\infty^i, g_\infty^i \in (0,\infty), \Lambda_1 < \Lambda_2$ *and* $\Lambda_3 < \Lambda_4$, *then for each* $\lambda_1 \in (\Lambda_1,\Lambda_2)$ *and* $\lambda_2\in(\Lambda_3,\Lambda_4)$ *there exists a positive solution* (y_1,y_2) *for the problem* (1.1)-(1.4).
- (H2) *If* $f_0^s = g_0^s = 0, f_\infty^i, g_\infty^i \in (0,\infty)$, *then for each* $\lambda_1\in(\Lambda_1,\infty)$ *and* $\lambda_2\in(\Lambda_3,\infty)$ *there exists a positive solution* (y_1,y_2) *for the problem* (1.1)-(1.4).
- (H3) *If* $f_0^s, g_0^s\in(0,\infty), f_\infty^i = g_\infty^i=\infty$, *then for each* $\lambda_1\in(0,\Lambda_2)$ *and* $\lambda_2\in(0,\Lambda_4)$ *there exists a positive solution* (y_1,y_2) *for the problem* (1.1)-(1.4).
- (H4) *If* $f_0^s=g_0^s=0, f_\infty^i=g_\infty^i=\infty$, *then for each* $\lambda_1\in(0,\infty)$ *and* $\lambda_2\in(0,\infty)$ *there exists a positive solution* (y_1,y_2) *for the problem* (1.1)-(1.4).

PROOF. Let $\lambda_1 \in (\Lambda_1, \Lambda_2)$, $\lambda_2 \in (\Lambda_3, \Lambda_4)$ and let $\epsilon > 0$ be a real number such that $\epsilon < f^i_\infty$, $\epsilon < g^i_\infty$ and

$$\gamma_1\left[kk_1\big(f^i_\infty - \epsilon\big)\int_r^1 G_1(1,s)p(s)ds\right]^{-1} \leqslant \lambda_1,$$

$$\gamma_1\left[\big(f^s_0 + \epsilon\big)\int_0^1 G_1(1,s)p(s)ds\right]^{-1} \geqslant \lambda_1,$$

$$\gamma_2\left[kk_2\big(g^i_\infty - \epsilon\big)\int_r^1 G_2(1,s)q(s)ds\right]^{-1} \leqslant \lambda_2,$$

$$\gamma_2\left[\big(g^s_0 + \epsilon\big)\int_0^1 G_2(1,s)q(s)ds\right]^{-1} \geqslant \lambda_2.$$

From the definitions of f^s_0 and g^s_0, there exists $J_1 > 0$ such that

$$f(y_1, y_2) \leqslant \big(f^s_0 + \epsilon\big)(y_1 + y_2) \text{ and } g(y_1, y_2) \leqslant \big(g^s_0 + \epsilon\big)(y_1 + y_2), 0 < \sum_{i=1}^{2} y_i \leqslant J_1.$$

By $(A1)$, the above inequalities are also valid for $y_1 = y_2 = 0$. Let $(y_1, y_2) \in \mathcal{P}$ with $\|(y_1, y_2)\|_{\mathcal{B}} = J_1$. *i.e.*, $\|y_1\| + \|y_2\| = J_1$. Then, from Lemma 2.3, for $t \in [0,1]$, we have

$$\begin{aligned}T_1\big(y_1, y_2\big)(t) =& \lambda_1 \int_0^1 G_1(t,s)p(s)f\big(y_1(s), y_2(s)\big)ds\\ \leqslant& \lambda_1 \int_0^1 G_1(1,s)p(s)\big(f^s_0 + \epsilon\big)\big(y_1(s) + y_2(s)\big)ds\\ \leqslant& \lambda_1\big(f^s_0 + \epsilon\big)\int_0^1 G_1(1,s)p(s)\big(\|y_1\| + \|y_2\|\big)ds\\ \leqslant& \gamma_1\big(\|y_1\| + \|y_2\|\big) = \gamma_1\big\|(y_1, y_2)\big\|_{\mathcal{B}}.\end{aligned}$$

Hence $\|T_1\big(y_1, y_2\big)\| \leqslant \gamma_1\big\|(y_1, y_2)\big\|_{\mathcal{B}}$. In a similar manner we conclude that

$$\|T_2\big(y_1, y_2\big)\| \leqslant \gamma_2\big\|(y_1, y_2)\big\|_{\mathcal{B}}.$$

Therefore,

$$\begin{aligned}\big\|T_1\big(y_1, y_2\big)\big\|_{\mathcal{B}} =& \big\|\big(T_1(y_1, y_2), T_2(y_1, y_2)\big)\big\|_{\mathcal{B}}\\ =& \big\|T_1(y_1, y_2)\big\| + \big\|T_2(y_1, y_2)\big\|\\ =& \big(\gamma_1 + \gamma_2\big)\big\|(y_1, y_2)\big\|_{\mathcal{B}}\\ =& \big\|(y_1, y_2)\big\|_{\mathcal{B}}.\end{aligned}$$

Hence $\big\|T\big(y_1, y_2\big)\big\|_{\mathcal{B}} \leqslant \big\|(y_1, y_2)\big\|_{\mathcal{B}}$.

If we set $\Omega_1 = \Big\{(y_1, y_2) \in \mathcal{B} : \big\|(y_1, y_2)\big\|_{\mathcal{B}} < J_1\Big\}$, then

$$\big\|T(y_1, y_2)\big\|_{\mathcal{B}} \leqslant \big\|(y_1, y_2)\big\|_{\mathcal{B}}, \text{ for } (y_1, y_2) \in \mathcal{P} \cap \partial\Omega_1. \tag{3.3}$$

Next from the definitions of f^i_∞ and g^i_∞, there exists $\overline{J}_2 > 0$ such that

$$f(y_1, y_2) \geqslant (f^i_\infty - \epsilon)(y_1 + y_2) \text{ and } g(y_1, y_2) \geqslant (g^i_\infty - \epsilon)(y_1 + y_2),\ \sum_{i=1}^{2} y_i \geqslant \overline{J}_2.$$

Let $J_2 = \max\left\{2J_1, \dfrac{\overline{J}_2}{k}\right\}$. Choose $(y_1, y_2) \in \mathcal{P}$ with $\big\|(y_1, y_2)\big\|_{\mathcal{B}} = H_2$. Then

$$\min_{t\in[r,1]} \sum_{i=1}^{2} y_i(t) \geqslant k\big\|(y_1, y_2)\big\|_{\mathcal{B}} \geqslant \overline{J}_2.$$

From Lemma 2.4, we have

$$\begin{aligned}
T_1(y_1, y_2)(t) &= \lambda_1 \int_0^1 G(t, s)p(s)f\big(y_1(s), y_2(s)\big)ds \\
&\geqslant \lambda_1 k_1 \int_r^1 G(1, s)p(s)(f^i_\infty - \epsilon)\big(y_1(s) + y_2(s)\big)ds \\
&\geqslant \lambda_1 k_1 (f^i_\infty - \epsilon) \int_r^1 G(1, s)p(s)k\big\|\big(y_1(s), y_2(s)\big)\big\|_{\mathcal{B}} ds \\
&\geqslant \gamma_1 \big\|\big(y_1(s), y_2(s)\big)\big\|_{\mathcal{B}}.
\end{aligned}$$

Hence $\big\|T_1(y_1, y_2)\big\| \geqslant \gamma_1\big\|(y_1, y_2)\big\|_{\mathcal{B}}$. In a similar manner we conclude that

$$\big\|T_2(y_1, y_2)\big\| \geqslant \gamma_2\big\|(y_1, y_2)\big\|_{\mathcal{B}}.$$

Therefore,

$$\begin{aligned}
\big\|T_1\big(y_1, y_2\big)\big\|_{\mathcal{B}} =& \big\|\big(T_1(y_1, y_2), T_2(y_1, y_2)\big)\big\|_{\mathcal{B}} \\
=& \big\|T_1(y_1, y_2)\big\| + \big\|T_2(y_1, y_2)\big\| \\
=& \big(\gamma_1 + \gamma_2\big)\big\|(y_1, y_2)\big\|_{\mathcal{B}} \\
=& \big\|(y_1, y_2)\big\|_{\mathcal{B}}.
\end{aligned}$$

Hence $\big\|T\big(y_1, y_2\big)\big\|_{\mathcal{B}} \geqslant \big\|(y_1, y_2)\big\|_{\mathcal{B}}$.

If we set $\Omega_2 = \Big\{(y_1, y_2) \in \mathcal{B} : \big\|(y_1, y_2)\big\|_{\mathcal{B}} < J_2\Big\}$, then

$$\big\|T(y_1, y_2)\big\|_{\mathcal{B}} \geqslant \big\|(y_1, y_2)\big\|_{\mathcal{B}}, \text{ for } (y_1, y_2) \in \mathcal{P} \cap \partial\Omega_2. \tag{3.4}$$

Applying Theorem 2.1 to (3.3) and (3.4), we obtain that T has a fixed point $(y_1, y_2) \in \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$ and hence the boundary value problem (1.1)-(1.4) has a positive solution such that $J_1 \leqslant \sum_{i=1}^{2} \|y_i\| \leqslant J_2$.

$(H2)$ Let $\lambda_1 \in (\Lambda_1, \infty)$, $\lambda_2 \in (\Lambda_3, \infty)$ and let $\epsilon > 0$ be a real number such that

$\epsilon < f^i_\infty,\ \epsilon < g^i_\infty$ and

$$\gamma_1\left[kk_1\big(f^i_\infty-\epsilon\big)\int_r^1 G_1(1,s)p(s)ds\right]^{-1}\leqslant\lambda_1;\ \epsilon\leqslant\frac{\gamma_1}{\lambda_1}\left[\int_0^1 G_1(1,s)p(s)ds\right]^{-1},$$

$$\gamma_2\left[kk_2\big(g^i_\infty-\epsilon\big)\int_r^1 G_2(1,s)q(s)ds\right]^{-1}\leqslant\lambda_2;\ \epsilon\leqslant\frac{\gamma_2}{\lambda_2}\left[\int_0^1 G_2(1,s)q(s)ds\right]^{-1}.$$

From the definitions of $f^s_0=0$ and $g^s_0=0$, there exists $J_1>0$ such that

$$f(y_1,y_2)\leqslant\epsilon(y_1+y_2)\text{ and }g(y_1,y_2)\leqslant\epsilon(y_1+y_2),0\leqslant\sum_{i=1}^{2}y_i\leqslant J_1.$$

Let $(y_1,y_2)\in\mathcal{P}$ with $\|(y_1,y_2)\|_{\mathcal{B}}=J_1$. *i.e.*, $\|y_1\|+\|y_2\|=J_1$. Then, from Lemma 2.3, for $t\in[0,1]$, we have

$$\begin{aligned}T_1\big(y_1,y_2\big)(t)=&\lambda_1\int_0^1 G_1(t,s)p(s)f\big(y_1(s),y_2(s)\big)ds\\ \leqslant&\lambda_1\int_0^1 G_1(1,s)p(s)\epsilon\big(y_1(s)+y_2(s)\big)ds\\ \leqslant&\lambda_1\epsilon\int_0^1 G_1(1,s)p(s)\big(\|y_1\|+\|y_2\|\big)ds\\ \leqslant&\gamma_1\big(\|y_1\|+\|y_2\|\big)=\gamma_1\big\|(y_1,y_2)\big\|_{\mathcal{B}}.\end{aligned}$$

Hence $\|T_1\big(y_1,y_2\big)\|\leqslant\gamma_1\big\|(y_1,y_2)\big\|_{\mathcal{B}}$. In a similar manner we conclude that

$$\|T_2\big(y_1,y_2\big)\|\leqslant\gamma_2\big\|(y_1,y_2)\big\|_{\mathcal{B}}.$$

Therefore,

$$\begin{aligned}\big\|T\big(y_1,y_2\big)\big\|_{\mathcal{B}}=&\big\|\big(T_1(y_1,y_2),T_2(y_1,y_2)\big)\big\|_{\mathcal{B}}\\ =&\big\|T_1(y_1,y_2)\big\|+\big\|T_2(y_1,y_2)\big\|\\ \leqslant&\gamma_1\big\|(y_1,y_2)\big\|_{\mathcal{B}}+\gamma_2\big\|(y_1,y_2)\big\|_{\mathcal{B}}\\ =&\big(\gamma_1+\gamma_2\big)\big\|(y_1,y_2)\big\|_{\mathcal{B}}\\ =&\big\|(y_1,y_2)\big\|_{\mathcal{B}}.\end{aligned}$$

Hence $\big\|T\big(y_1,y_2\big)\big\|_{\mathcal{B}}\leqslant\big\|(y_1,y_2)\big\|_{\mathcal{B}}$.

Define the set $\Omega_1=\Big\{(y_1,y_2)\in\mathcal{B}:\big\|(y_1,y_2)\big\|_{\mathcal{B}}<J_1\Big\}$ then

$$\big\|T(y_1,y_2)\big\|_{\mathcal{B}}\leqslant\big\|(y_1,y_2)\big\|_{\mathcal{B}},\ \text{for }(y_1,y_2)\in\mathcal{P}\cap\partial\Omega_1. \tag{3.5}$$

Define the set $\Omega_2=\Big\{(y_1,y_2)\in\mathcal{B}:\big\|(y_1,y_2)\big\|_{\mathcal{B}}<J_2\Big\}$ and proceeding in a similar manner of proof $(H1)$, we get

$$\big\|T(y_1,y_2)\big\|_{\mathcal{B}}\geqslant\big\|(y_1,y_2)\big\|_{\mathcal{B}},\ \text{for }(y_1,y_2)\in\mathcal{P}\cap\partial\Omega_2. \tag{3.6}$$

Applying Theorem 2.1 to (3.5) and (3.6), we obtain that T has a fixed point $(y_1,y_2)\in\mathcal{P}\cap(\overline{\Omega}_2\backslash\Omega_1)$ and hence the boundary value problem (1.1)-(1.4) has a positive solution such that $J_1 \leqslant \sum_{i=1}^{2} \|y_i\| \leqslant J_2$. Similarly we can prove the remaining. □

Prior to our next result, we define

$$\Upsilon_1 = \gamma_3 \left[k k_1 f_0^i \int_r^1 G_1(1,s)p(s)ds \right]^{-1}, \Upsilon_2 = \gamma_3 \left[f_\infty^s \int_0^1 G_1(1,s)p(s)ds \right]^{-1},$$

$$\Upsilon_3 = \gamma_4 \left[k k_2 g_0^i \int_r^1 G_2(1,s)q(s)ds \right]^{-1}, \Upsilon_4 = \gamma_4 \left[g_\infty^s \int_0^1 G_2(1,s)q(s)ds \right]^{-1},$$

where γ_3, γ_4 are two positive real numbers such that $\gamma_3 + \gamma_4 = 1$.

Theorem 3.2. *Assume that the conditions* $(A1)$-$(A4)$ *hold.*

(H5) *If* $f_0^i, g_0^i, f_\infty^s, g_\infty^s \in (0,\infty), \Upsilon_1 < \Upsilon_2$ *and* $\Upsilon_3 < \Upsilon_4$, *then for each* $\lambda_1 \in (\Upsilon_1, \Upsilon_2)$ *and* $\lambda_2 \in (\Upsilon_3, \Upsilon_4)$ *there exists a positive solution* (y_1, y_2) *for the problem* (1.1)-(1.4).

(H6) *If* $f_\infty^s = g_\infty^s = 0, f_0^i, g_0^i \in (0,\infty)$, *then for each* $\lambda_1 \in (\Upsilon_1, \infty)$ *and* $\lambda_2 \in (\Upsilon_3, \infty)$ *there exists a positive solution* (y_1, y_2) *for the problem* (1.1)-(1.4).

(H7) *If* $f_\infty^s, g_\infty^s \in (0,\infty), f_0^i = g_0^i = \infty$, *then for each* $\lambda_1 \in (0, \Upsilon_2)$ *and* $\lambda_2 \in (0, \Upsilon_4)$ *there exists a positive solution* (y_1, y_2) *for the problem* (1.1)-(1.4).

(H8) *If* $f_\infty^s = g_\infty^s = 0, f_0^i = g_0^i = \infty$, *then for each* $\lambda_1 \in (0,\infty)$ *and* $\lambda_2 \in (0,\infty)$ *there exists a positive solution* (y_1, y_2) *for the problem* (1.1)-(1.4).

Proof. $(H5)$ Let $\lambda_1 \in (\Upsilon_1, \Upsilon_2)$, $\lambda_2 \in (\Upsilon_3, \Upsilon_4)$ and let $\epsilon > 0$ be a real number such that $\epsilon < f_0^i,\ \epsilon < g_0^i$ and

$$\gamma_3 \left[k k_1 \big(f_0^i - \epsilon\big) \int_r^1 G_1(1,s)p(s)ds \right]^{-1} \leqslant \lambda_1,$$

$$\gamma_3 \left[\big(f_\infty^s + \epsilon\big) \int_0^1 G_1(1,s)p(s)ds \right]^{-1} \geqslant \lambda_1,$$

$$\gamma_4 \left[k k_2 \big(g_0^i - \epsilon\big) \int_r^1 G_2(1,s)q(s)ds \right]^{-1} \leqslant \lambda_2,$$

$$\gamma_4 \left[\big(g_\infty^s + \epsilon\big) \int_0^1 G_2(1,s)q(s)ds \right]^{-1} \geqslant \lambda_2.$$

From the definitions of $f_0^i, g_0^i \in (0,\infty)$ there exists $J_3 > 0$ such that

$$f(y_1,y_2) \leqslant \big(f_0^i - \epsilon\big)(y_1+y_2) \text{ and } g(y_1,y_2) \leqslant \big(g_0^i - \epsilon\big)(y_1+y_2), 0 < \sum_{i=1}^{2} y_i \leqslant J_3.$$

By $(A1)$, the above inequalities are also valid for $y_1 = y_2 = 0$. Let $(y_1, y_2) \in \mathcal{P}$ with $\|(y_1,y_2)\|_{\mathcal{B}} = J_3$. *i.e.*, $\|y_1\| + \|y_2\| = J_3$. Then, from Lemma 2.4, for $t \in [0,1]$, we have

$$\begin{aligned}T_1\big(y_1, y_2\big)(t) =&\lambda_1 \int_0^1 G_1(t,s)p(s)f\big(y_1(s), y_2(s)\big)ds\\ \geqslant&\lambda_1 k_1 \int_r^1 G_1(1,s)p(s)\big(f_0^i-\epsilon\big)\big(y_1(s)+y_2(s)\big)ds\\ \geqslant&\lambda_1 k_1\big(f_0^i-\epsilon\big)\int_r^1 G_1(1,s)p(s)k\big(\|y_1\|+\|y_2\|\big)ds\\ \geqslant&\gamma_3\big(\|y_1\|+\|y_2\|\big)=\gamma_3\big\|(y_1,y_2)\big\|_{\mathcal{B}}.\end{aligned}$$

Hence $\|T_1\big(y_1,y_2\big)\| \geqslant \gamma_3\big\|(y_1,y_2)\big\|_{\mathcal{B}}$. In a similar manner we conclude that

$$\|T_2\big(y_1,y_2\big)\| \geqslant \gamma_4\big\|(y_1,y_2)\big\|_{\mathcal{B}}.$$

Therefore,

$$\big\|T\big(y_1,y_2\big)\big\|_{\mathcal{B}} = \big\|\big(T_1(y_1,y_2),T_2(y_1,y_2)\big)\big\|_{\mathcal{B}} \geqslant \big(\gamma_3+\gamma_4\big)\big\|(y_1,y_2)\big\|_{\mathcal{B}} = \big\|(y_1,y_2)\big\|_{\mathcal{B}}.$$

Hence $\big\|T\big(y_1,y_2\big)\big\|_{\mathcal{B}} \geqslant \big\|(y_1,y_2)\big\|_{\mathcal{B}}$.

If we set $\Omega_3 = \Big\{(y_1,y_2)\in\mathcal{B} : \big\|(y_1,y_2)\big\|_{\mathcal{B}} < J_3\Big\}$, then

$$\big\|T(y_1,y_2)\big\|_{\mathcal{B}} \geqslant \big\|(y_1,y_2)\big\|_{\mathcal{B}},\ \text{for } (y_1,y_2)\in\mathcal{P}\cap\partial\Omega_3. \tag{3.7}$$

Now we define the functions $f^*, g^* : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\begin{cases} f^*(x) = \max\limits_{y_1+y_2\in[0,x]} f(y_1,y_2)\\ g^*(x) = \max\limits_{y_1+y_2\in[0,x]} g(y_1,y_2),\ \text{for all } x\in\mathbb{R}^+.\ \text{Then}\end{cases}$$

$$f(y_1,y_2)\leqslant f^*(x) \text{ and } g(y_1,y_2)\leqslant g^*(x),\ \sum_{i=1}^{2} y_i \leqslant x.$$

It follows that the functions f^*, g^* are nondecreasing and satisfy the conditions

$$\begin{cases} \lim\limits_{x\to\infty}\sup \dfrac{f^*(x)}{x} = f_\infty^s,\\ \lim\limits_{x\to\infty}\sup \dfrac{g^*(x)}{x} = g_\infty^s.\end{cases}$$

Next from the definitions of $f_\infty^s, g_\infty^s \in (0,\infty)$, there exists $\overline{J}_4 > 0$ such that

$$f^*(x)\leqslant (f_\infty^s+\epsilon)x \text{ and } g^*(x)\leqslant (g_\infty^s+\epsilon)x,\ x\geqslant \overline{J}_4.$$

Let $J_4 = \max\Big\{2J_3, \overline{J}_4\Big\}$. Choose $(y_1,y_2)\in\mathcal{P}$ with $\big\|(y_1,y_2)\big\|_{\mathcal{B}} = J_4$. Then by the definitions of f^* and g^*, we have

$$\begin{cases} f\big(y_1(t),y_2(t)\big)\leqslant f^*\big(y_1(t)+y_2(t)\big)\leqslant f^*\big(\|(y_1\|+\|y_2)\|\big) = f^*\big(\|(y_1,y_2)\|_{\mathcal{B}}\big),\\ g\big(y_1(t),y_2(t)\big)\leqslant g^*\big(y_1(t)+y_2(t)\big)\leqslant g^*\big(\|(y_1\|+\|y_2)\|\big) = g^*\big(\|(y_1,y_2)\|_{\mathcal{B}}\big).\end{cases}$$

From Lemma 2.3, we have

$$\begin{aligned}T_1(y_1,y_2)(t) &= \lambda_1\int_0^1 G_1(t,s)p(s)f\big(y_1(s),y_2(s)\big)ds\\ &\leqslant \lambda_1\int_0^1 G_1(1,s)p(s)f^*\big(\|(y_1,y_2)\|_{\mathcal{B}}\big)ds\\ &\leqslant \lambda_1\int_0^1 G_1(1,s)p(s)(f_\infty^s+\epsilon)\big\|(y_1,y_2)\big\|_{\mathcal{B}}ds\\ &\leqslant \gamma_3\big\|(y_1,y_2)\big\|_{\mathcal{B}}.\end{aligned}$$

Hence $\big\|T_1(y_1,y_2)\big\| \leqslant \gamma_3\big\|(y_1,y_2)\big\|_{\mathcal{B}}$. In a similar manner, we conclude that

$$\big\|T_2(y_1,y_2)\big\| \leqslant \gamma_4\big\|(y_1,y_2)\big\|_{\mathcal{B}}.$$

Therefore,

$$\big\|T_1\big(y_1,y_2\big)\big\|_{\mathcal{B}} = \big\|\big(T_1(y_1,y_2),T_2(y_1,y_2)\big)\big\|_{\mathcal{B}} \leqslant \big(\gamma_3+\gamma_4\big)\big\|(y_1,y_2)\big\|_{\mathcal{B}} = \big\|(y_1,y_2)\big\|_{\mathcal{B}}.$$

Hence $\big\|T\big(y_1,y_2\big)\big\|_{\mathcal{B}} \leqslant \big\|(y_1,y_2)\big\|_{\mathcal{B}}$. If we set

$$\Omega_4 = \Big\{(y_1,y_2)\in\mathcal{B} : \big\|(y_1,y_2)\big\|_{\mathcal{B}} < J_4\Big\},$$

then

$$\big\|T(y_1,y_2)\big\|_{\mathcal{B}} \leqslant \big\|(y_1,y_2)\big\|_{\mathcal{B}},\ \text{for } (y_1,y_2)\in\mathcal{P}\cap\partial\Omega_4. \tag{3.8}$$

Applying Theorem 2.1 to (3.7) and (3.8), we obtain that T has a fixed point $(y_1,y_2)\in\mathcal{P}\cap(\overline{\Omega}_4\backslash\Omega_3)$ and hence the boundary value problem (1.1)-(1.4) has a positive solution such that $J_3 \leqslant \sum_{i=1}^{2}\|y_i\| \leqslant J_4$.

The proofs of the remaining cases $(H5)$-$(H8)$ are similar that of $(H1)$ and we shall omit them. □

4. Example

In this section we give an example to illustrate the utility of our main results.

Consider the fractional order three-point boundary value problem

$$D_{0^+}^{3.6}y_1(t)+\lambda_1 f\big(y_1,y_2\big)=0,\ t\in(0,1), \tag{4.1}$$

$$D_{0^+}^{3.7}y_2(t)+\lambda_2 g\big(y_1,y_2\big)=0,\ t\in(0,1), \tag{4.2}$$

$$y_1(0)=y_1'(0)=y_1''(0)=0, y_1''(1)=\frac{3}{2}y_1''\Big(\frac{1}{2}\Big), \tag{4.3}$$

$$y_2(0)=y_2'(0)=y_2''(0)=0, y_2''(1)=2y_2''\Big(\frac{1}{3}\Big), \tag{4.4}$$

where

$$\begin{cases} f(y_1, y_2) = \dfrac{(Siny_2 + 3)(y_1 + y_2)\Big[1100(y_1 + y_2) + 1\Big]}{y_1 + y_2 + 1}, \\ g(y_1, y_2) = \dfrac{\Big[500(y_1 + y_2) + 1\Big](Cosy_1 + 7)(y_1 + y_2)}{y_1 + y_2 + 1}, \end{cases}$$

$p(t) = q(t) = 1, f_0^s = 4, f_\infty^i = 2200, g_0^s = 8, g_\infty^i = 3000$. Applying Theorem 3.1, we get an eigenvalue interval $\lambda_1 \in \big(0.00442, 0.10593\big)$ and $\lambda_2 \in \big(0.00611, 0.07022\big)$ in which the three-point fractional order boundary value problem (4.1)-(4.4) has at least one positive solution.

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