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SOME DEGREE AND DISTANCE BASED TOPOLOGICAL INDICES OF VERTEX-EDGE CORONA OF TWO GRAPHS

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ABSTRACT. In this paper, we establish vertex-edge corona of two graphs and compute Wiener index, Zagreb indices, Degree distance index and Gutman index of vertex-edge corona of two graphs.

1. Introduction

All graphs considered here are finite, undirected and simple connected graphs. Let $G = (V,\ E)$ be a graph with vertex set $V := V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E := E(G) = \{e_1, e_2, \cdots, e_m\}$. Throughout the paper we denote the degree of a vertex v in G by $d_G(v)$ and the shortest distance between two vertices v_i and v_j in G by $d_G(v_i,\ v_j)$. A topological index of a graph G is a numerical quantity derived from the structure of the graph G, so that it remains invariant under graph isomorphism. The Wiener index also known as the first topological index [**29**] was introduced by a chemist H. Wiener in 1947, and is defined as

$$W(G) = \sum_{\{v_i,\ v_j\} \subseteq V(G)} d_G(v_i,\ v_j).$$

The Wiener index was used to model certain physico-chemical properties of hydrocarbon molecules and is a useful tool for designing quantitative structure-property relations in organic chemistry. This index was also used to establish correlations with various physico-chemical and thermodynamic parameters such as boiling point, density, critical pressure of chemical compounds. Studies on Wiener index can be found in [**11, 12, 20, 23**]. The first and second Zagreb indices are the degree

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based topological indices introduced in 1972, by I. Gutman and N. Trinajstić [**19**] and these are defined as

$$M_1(G) = \sum_{v_i \in V(G)} d^2_{_G}(v_i) = \sum_{e_i = v_l v_m \in E(G)} (d_{_G}(v_l) + d_{_G}(v_m))$$

and

$$M_2(G) = \sum_{e_i = v_l v_m \in E(G)} d_{_G}(v_l) d_{_G}(v_m),$$

respectively. They observed that total π-electron energy depends on molecular structures and approximate expressions for total π-electron energy involves M_1 and M_2. Zagreb indices are extremely useful for deriving multi linear regression models [**8**]. Informations regarding Zagreb indices can be found in [**17, 18, 22**]. The Degree distance and Gutman indices of a graph G are defined as

$$DD(G) = \sum_{\{v_i,\ v_j\} \subseteq V(G)} (d_{_G}(v_i) + d_{_G}(v_j)) d_{_G}(v_i,\ v_j)$$

and

$$Gut(G) = \sum_{\{v_i,\ v_j\} \subseteq V(G)} d_{_G}(v_i) d_{_G}(v_j) d_{_G}(v_i,\ v_j).$$

The degree distance index was introduced independently by A. A. Dobrynin, A. A. Kochetova [**13**] and Gutman [**16**]. The Degree distance closely related to Wiener index. The degree distance is structure descriptors based on molecular topology of quantitative relations between structure and activity [**26**] and its physico-chemical applications includes the prediction of boiling points and
calculation of velocity of ultrasound in organic materials. The Gutman index was introduced by Gutman [**16**]. The Gutman index which is a variant of the well-known Wiener index, reflects precisely the same structural features of a molecular as the Wiener index [**28**]. For details about Degree distance and Gutman indices we refer to [**3, 14, 25**]. The vertex Padmakar-Ivan(PI) index [**4**] of a graph G is defined as

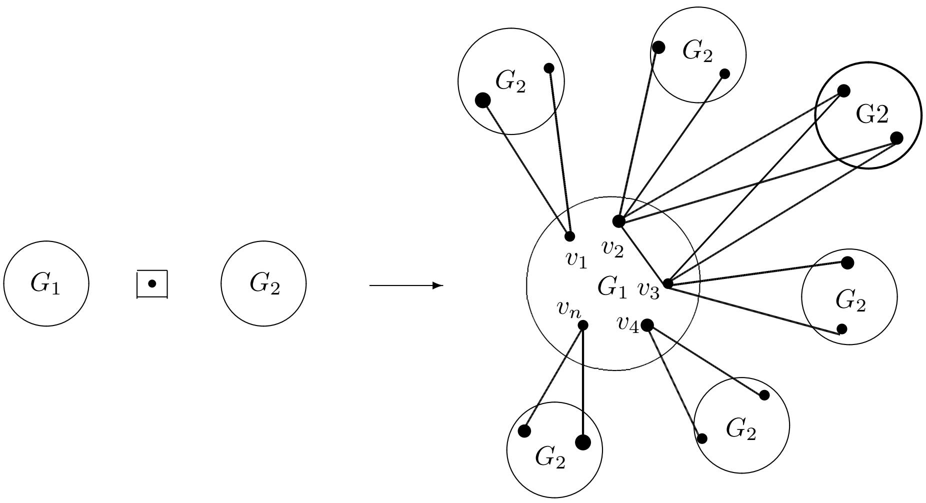
$$PI(G) = \sum_{e_i = v_l v_m \in E(G)} [n_{e_i}(v_l|G) + n_{e_i}(v_m|G)],$$

where $n_{e=uv}(u|G)$ is the number of vertices in G that are closer to u than v in G. Ashrafi and Loghman [**6, 7**] have studied mathematical properties of the Padmakar-Ivan index and its applications in chemistry and nanoscience. For, more details see [**5, 10, 24**].
Let $e = uv$ and $f = xy$ be two distinct edges in G. The distance $d_G(e, f)$ between two edges e and f is given by $min\{d_G(x,u), d_G(x,v), d_G(y,u), d_G(y,v)\}$. The edge-Wiener index $W_e(G)$ [**9**] of a graph G, is the sum of all distances between unordered pairs of vertices in the line graph $L(G)$ of G. Equivalently, edge-Wiener index of G is the Wiener index of the line graph $L(G)$ of G. i.e,

$$W_e(G) = W(L(G)) = \sum_{\{xy,uv\} \in E(G)} (min\{d_G(x,u), d_G(x,v), d_G(y,u), d_G(y,v)\} + 1).$$

The corona $G_1 \circ G_2$ [**15**] of two graphs G_1 and G_2, is the graph obtained by taking one copy of G_1, $|V(G_1)|$ copies of G_2 and joining i-th vertex of G_1 to every vertex in the i-th copy of G_2. The edge corona $G_1 \diamond G_2$ [**21**] of two graphs G_1 and G_2, is the graph obtained by taking one copy of G_1 and $|E(G_1)|$ copies of G_2 and joining end vertices of i-th edge of G_1 to every vertex in the i-th copy of G_2. Recently, various graph invariants of corona product of two graphs have been studied [**1, 2, 27, 30**]. Motivated by this, we establish vertex-edge corona of two graphs. Let G_1 be a graph with vertex set $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$ and edge set $E(G_1) = \{e_1, e_2, \ldots, e_{m_1}\}$. Let G_2 be a graph with vertex set $V(G_2) = \{u_1, u_2, \ldots, u_{n_2}\}$ and edge set $E(G_2) = \{e'_1, e'_2, \ldots, e'_{m_2}\}$. The vertex-edge corona of two graphs G_1 and G_2 denoted by $G_1 \boxdot G_2$, is the graph obtained by taking one copy of G_1 , $|V(G_1)|$ copies of G_2 and $|E(G_1)|$ copies of G_2, then joining i-th vertex of G_1 to every vertex in the i-th vertex copy of G_2 and also joining end vertices of j-th edge of G_1 to every vertex in the j-th edge copy of G_2, where $1 \leqslant i \leqslant n_1$ and $1 \leqslant j \leqslant m_1$.
Note that the vertex-edge corona $G_1 \boxdot G_2$ of G_1 and G_2 has $n_1 + n_2(m_1 + n_1)$ vertices and $m_1 + m_1(m_2 + 2n_2) + n_1(m_2 + n_2)$ edges.
We denote, the vertex set of the j-th edge copy of G_2 by $V_{j_e}(G_2) = \{u_{j2}, u_{j2}, \ldots, u_{jn_2}\}$ and the vertex set of the i-th vertex copy of G_2 by $V_{i_v}(G_2) = \{w_{i1}, w_{i2}, \ldots, w_{in_2}\}$. Also, we denote by $E_{j_e}(G_2)$ and $E_{i_v}(G_2)$, the edge set of the j-th edge and i-th vertex copy of G_2, respectively.

FIGURE 1



Generalised vertex-edge corona of G_1 and G_2 graphs.

In this paper, we compute some topological indices of vertex-edge corona of two graphs. In the Section 2, we establish formulas for the Wiener index and Zagreb indices of vertex-edge corona of two graphs. Also, we compute the same for some

standard graphs. In the Section 3, we compute degree distance and Gutman index of vertex-edge corona of two graphs.
The following Lemmas follow immediately from the definition of vertex-edge corona of two graphs.

Lemma 1.1. *We have,*

a. $d_G(v_i) = (n_2+1)d_{G_1}(v_i) + n_2, \ \forall\ v_i \in V(G_1)$.
b. $d_G(u_{ij}) = d_{G_2}(u_j) + 2, \ \forall\ u_{ij} \in V_{i_e}(G_2)$.
c. $d_G(w_{ij}) = d_{G_2}(w_j) + 1, \ \forall\ w_{ij} \in V_{i_v}(G_2)$.

Lemma 1.2. *We have,*

a. $d_G(v_i,\ v_j) = d_{G_1}(v_i,\ v_j),\ \forall\ v_i, v_j \in V(G_1)$.
b. $d_G(u_{ij},\ u_{ik}) = \begin{cases} 1, & \forall\ u_ju_k \in E_e(G_2), \\ 2, & \forall\ u_ju_k \notin E_e(G_2). \end{cases}$
c. $d_G(w_{ij},\ w_{ik}) = \begin{cases} 1, & \forall\ w_jw_k \in E_v(G_2), \\ 2, & \forall\ w_jw_k \notin E_v(G_2). \end{cases}$
d. $d_G(u_{ij},\ u_{km}) = d_{G_1}(e_i,\ e_k) + 2\ ,\ \ \forall\ i \neq k$.
e. $d_G(w_{ij},\ w_{km}) = d_{G_1}(v_i,\ v_k) + 2,\ \ \forall\ i \neq k$.
f. For $e_i = v_lv_m$,

$$d_{G_1}(u_{ij},\ v_k) = \begin{cases} \dfrac{d_{G_1}(v_l,\ v_k) + d_{G_1}(v_k,\ v_m) + 1}{2}, & if\ d(v_l,\ v_k) \neq d(v_k,\ v_m), \\ \dfrac{d_{G_1}(v_l,\ v_k) + d_{G_1}(v_k,\ v_m)}{2} + 1, & if\ d(v_l,\ v_k) = d(v_k,\ v_m) \end{cases}$$

$\forall\ u_{ij} \in V_{i_e}(G_2),\ v_k \in V(G_1)$.
g. $d_G(w_{ij},\ v_k) = d_G(v_i,\ v_k) + 1,\ \forall\ w_{ij} \in V_{i_v}(G_2), v_k \in V(G_1)$.
h. For $e_i = v_lv_m$.

$$d_G(u_{ij},\ w_{km}) = \begin{cases} \dfrac{d_{G_1}(v_l,\ v_k) + d_{G_1}(v_k,\ v_m)\ + 3}{2}, & if \ \ d(v_l,\ v_k) \neq d(v_k,\ v_m), \\ \dfrac{d_{G_1}(v_l,\ v_k) + d_{G_1}(v_k,\ v_m)}{2} + 2, & if \ \ d(v_l,\ v_k) = d(v_k,\ v_m) \end{cases}$$

$\forall\ u_{ij} \in V_{i_e}(G_2),\ w_{km} \in V_{k_v}(G_2)$.

2. Wiener index, Zagreb indices of vertex-edge corona of two graphs

In this section, we compute Wiener index, first and second Zagreb indices of vertex-edge corona of two graphs.

Theorem 2.1. *The Wiener index of $G := G_1 \boxdot G_2$ is given by*

$$\begin{aligned} W(G_1 \boxdot G_2) = {} & (1+n_2)^2W(G_1) + n_2^2W_e(G_1) + \frac{n_2}{2}(n_2+1)(DD(G_1) - PI(G_1)) \\ & + n_1^2n_2(n_2+1) + n_2^2m_1((1/2) + 3n_1 + (1/2)m_1) - n_2(m_1+n_1) \\ & - m_2(n_1+m_1). \end{aligned}$$

PROOF. The Wiener index of G is given by,

$$W(G) = \sum_{\{u,\ v\}\subseteq V(G)} d_G(u\ ,v)$$

$$= A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8, \tag{2.1}$$

where

$$A_1 = \sum_{\{v_i,\ v_j\}\subseteq V(G_1)} d_G(v_i,\ v_j),$$

$$A_2 = \sum_{e_i\in E(G_1)} \sum_{\{u_{ij},\ u_{ik}\}\subseteq V_{i_e}(G_2)} d_G(u_{ij},\ u_{ik}),$$

$$A_3 = \sum_{v_i\in V(G_1)} \sum_{\{w_{ij},\ w_{ik}\}\subseteq V_{i_v}(G_2)} d_G(w_{ij},\ w_{ik}),$$

$$A_4 = \sum_{\{e_i,\ e_k\}\in E(G_1)} \sum_{\substack{u_{ij}\in V_{i_e}(G_2)\\ u_{km}\in V_{k_e}(G_2)}} d_G(u_{ij},\ u_{km}),$$

$$A_5 = \sum_{\{v_i,\ v_k\}\in V(G_1)} \sum_{\substack{w_{ij}\in V_{i_v}(G_2)\\ w_{km}\in V_{k_v}(G_2)}} d_G(w_{ij},\ w_{km}),$$

$$A_6 = \sum_{e_i\in E(G_1)} \sum_{\substack{u_{ij}\in V_{i_e}(G_2)\\ v_k\in V(G_1)}} d_G(u_{ij},\ v_k),$$

$$A_7 = \sum_{v_i\in V(G_1)} \sum_{\substack{w_{ij}\in V_{i_v}(G_2)\\ v_k\in V(G_1)}} d_G(w_{ij},\ v_k),$$

$$A_8 = \sum_{e_i\in E(G_1)} \sum_{\substack{u_{ij}\in V_{i_e}(G_2)\\ w_{km}\in V_k(G_2)}} d_G(u_{ij},\ w_{km}).$$

From Lemma 1.2, we have the following.

$$A_1 = \sum_{\{v_i,\ v_j\}\subseteq V(G_1)} d_G(v_i,\ v_j)$$

$$= \sum_{\{v_i,\ v_j\}\subseteq V(G_1)} d_{G_1}(v_i,\ v_j)$$

$$= W(G_1), \tag{2.2}$$

$$A_2 = \sum_{e_i\in E(G_1)} \sum_{\{u_{ij},\ u_{ik}\}\subseteq V_{i_e}(G_2)} d_G(u_{ij},\ u_{ik})$$

$$= \sum_{e_i\in E(G_1)} \left\{ 2 \sum_{\{u_j,\ u_k\}\subseteq V(G_2)} - \sum_{u_j u_k\in E(G_2)} \right\}$$

$$= m_1(n_2(n_2-1) - m_2), \tag{2.3}$$

$$\begin{aligned}
A_3 &= \sum_{v_i \in V(G_1)} \sum_{\{w_{ij},\ w_{ik}\} \subseteq V_{i_v}(G_2)} d_G(w_{ij},\ w_{ik}) \\
&= \sum_{v_i \in V(G_1)} \left\{ 2 \sum_{\{w_j,\ w_k\} \subseteq V(G_2)} - \sum_{w_j w_k \in E(G_2)} \right\} \\
&= n_1(n_2(n_2-1) - m_2), \qquad (2.4)
\end{aligned}$$

$$\begin{aligned}
A_4 &= \sum_{\{e_i,\ e_k\} \in E(G_1)} \sum_{\substack{u_{ij} \in V_{i_e}(G_2) \\ u_{km} \in V_{k_e}(G_2)}} d_G(u_{ij},\ u_{km}) \\
&= \sum_{\{e_i,\ e_k\} \in E(G_1)} \sum_{\substack{u_{ij} \in V_{i_e}(G_2) \\ u_{km} \in V_{k_e}(G_2)}} (d_{G_1}(e_i,\ e_k) + 2) \\
&= n_2^2 \sum_{\{e_i,\ e_k\} \in E(G_1)} (d_{G_1}(e_i,\ e_k) + 2) \\
&= n_2^2 (W_e(G_1) + m_1(m_1 - 1)/2), \qquad (2.5)
\end{aligned}$$

$$\begin{aligned}
A_5 &= \sum_{\{v_i,\ v_k\} \in V(G_1)} \sum_{\substack{w_{ij} \in V_{i_v}(G_2) \\ w_{km} \in V_{k_v}(G_2)}} d_G(w_{ij},\ w_{km}) \\
&= \sum_{\{v_i,\ v_k\} \in V(G_1)} \sum_{\substack{w_{ij} \in V_{i_v}(G_2) \\ w_{km} \in V_{k_v}(G_2)}} (d_{G_1}(v_i,\ v_k) + 2) \\
&= n_2^2 \sum_{\{v_i,\ v_k\} \in V(G_1)} (d_{G_1}(v_i,\ v_k) + 2) \\
&= n_2^2 (W(G_1) + n_1(n_1 - 1)), \qquad (2.6)
\end{aligned}$$

$$\begin{aligned}
A_6 = &\sum_{e_i \in E(G_1)} \sum_{\substack{u_{ij} \in V_{i_e}(G_2) \\ v_k \in V(G_1)}} d_G(u_{ij},\ v_k) \\
= &\sum_{e_i \in E(G_1)} n_2 \left\{ \sum_{\substack{v_k \in V(G_1) \\ d(v_l,\ v_k) \neq d(v_k,\ v_m)}} \frac{d_{G_1}(v_l,\ v_k) + d_{G_1}(v_k,\ v_m) + 1}{2} \right. \\
&\left. + \sum_{\substack{v_k \in V(G_1) \\ d(v_l,\ v_k) = d(v_k,\ v_m)}} \left(\frac{d_{G_1}(v_l,\ v_k) + d_{G_1}(v_k,\ v_m)}{2} + 1 \right) \right\}
\end{aligned}$$

$$
\begin{aligned}
&= n_2 \sum_{e_i\in E(G_1)} \left\{ \sum_{v_k\in V(G_1)} \frac{d_{G_1}(v_l,\ v_k)+d_{G_1}(v_k,\ v_m)}{2} \right\}\\
&+ n_2 \sum_{e_i\in E(G_1)} \left\{ \frac{2n_1-(n_{e_i}(v_l/G_1)+n_{e_i}(v_m/G_1))}{2} \right\}\\
&= \frac{n_2}{2} \left\{ \sum_{e_i\in E(G_1)} \sum_{v_k\in V(G_1)} \frac{d_{G_1}(v_l,\ v_k)+d_{G_1}(v_k,\ v_m)}{2} + 2n_1m_1 - PI(G_1) \right\}\\
&= \frac{n_2}{2} \left\{ \sum_{v_k\in V(G_1)} \sum_{v_i\in V(G_1)} \sum_{v_iv_j\in E(G_1)} d_{G_1}(v_i,\ v_k) + 2n_1m_1 - PI(G_1) \right\}\\
&= \frac{n_2}{2} \left\{ \sum_{v_i\in V(G_1)} \sum_{v_k\in V(G_1)} d_{G_1}(v_i) d_{G_1}(v_i,\ v_k) + 2n_1m_1 - PI(G_1) \right\}\\
&= \frac{n_2}{2} \left\{ \sum_{\{v_i,\ v_k\}\subseteq V(G_1)} (d_{G_1}(v_i)+d_{G_1}(v_k)) d_{G_1}(v_i,\ v_k) + 2n_1m_1 - PI(G_1) \right\}\\
&= \frac{n_2}{2} \left\{ DD(G_1) + 2n_1m_1 - PI(G_1) \right\}, \qquad (2.7)
\end{aligned}
$$

$$
\begin{aligned}
A_7 &= \sum_{v_i\in V(G_1)} \sum_{\substack{w_{ij}\in V_{i_v}(G_2)\\ v_k\in V(G_1)}} d_G(w_{ij},\ v_k)\\
&= \sum_{v_i\in V(G_1)} \sum_{\substack{w_{ij}\in V_{i_v}(G_2)\\ v_k\in V(G_1)}} (d_{G_1}(v_i,\ v_k)+1)\\
&= n_2 \sum_{v_i\in V(G_1)} \sum_{v_k\in V(G_1)} (d_{G_1}(v_i,\ v_k)+1)\\
&= n_2(2W(G_1)+n_1^2), \qquad (2.8)
\end{aligned}
$$

$$
\begin{aligned}
A_8 &= \sum_{e_i\in E(G_1)} \sum_{\substack{u_{ij}\in V_{i_e}(G_2)\\ w_{km}\in V_k(G_2)}} d_G(u_{ij},\ w_{km})\\
&= \sum_{e_i\in E(G_1)} n_2^2 \left\{ \sum_{\substack{v_k\in V(G_1)\\ d(v_l,\ v_k)\neq d(v_k,\ v_m)}} \frac{d_{G_1}(v_l,\ v_k)+d_{G_1}(v_k,\ v_m)+3}{2} \right.
\end{aligned}
$$

$$+\sum_{\substack{v_k\in V(G_1)\\ d(v_l,\ v_k)=d(v_k,\ v_m)}}\frac{d_{G_1}(v_l,\ v_k)+d_{G_1}(v_k,\ v_m)}{2}+2\Bigg\}$$

$$=\frac{n_2^2}{2}\sum_{e_i\in E(G_1)}\Bigg\{\sum_{v_k\in V(G_1)}d_{G_1}(v_l,\ v_k)+d_{G_1}(v_k,\ v_m)+\sum_{\substack{v_k\in V(G_1)\\ d(v_l,\ v_k)\neq d(v_k,\ v_m)}}3$$

$$+\sum_{\substack{v_k\in V(G_1)\\ d(v_l,\ v_k)=d(v_k,\ v_m)}}4\Bigg\}$$

$$=\frac{n_2^2}{2}\Bigg\{\sum_{e_i\in E(G_1)}\Bigg\{\sum_{v_k\in V(G_1)}d_{G_1}(v_l,\ v_k)+d_{G_1}(v_k,\ v_m)\Bigg\}$$

$$+\sum_{e_i\in E(G_1)}\left\{\frac{4n_1-(n_{e_i}(v_l/G_1)+n_{e_i}(v_m/G_1))}{2}\right\}\Bigg\}$$

$$=\frac{n_2^2}{2}\Bigg\{\sum_{e_i\in E(G_1)}\sum_{v_k\in V(G_1)}\frac{d_{G_1}(v_l,\ v_k)+d_{G_1}(v_k,\ v_m)}{2}+4n_1m_1-PI(G_1)\Bigg\}$$

$$=\frac{n_2^2}{2}\Bigg\{\sum_{v_k\in V(G_1)}\sum_{v_i\in V(G_1)}\sum_{v_iv_j\in E(G_1)}d_{G_1}(v_i,\ v_k)+4n_1m_1-PI(G_1)\Bigg\}$$

$$=\frac{n_2^2}{2}\Bigg\{\sum_{v_i\in V(G_1)}\sum_{v_k\in V(G_1)}d_{G_1}(v_i)d_{G_1}(v_i,\ v_k)+4n_1m_1-PI(G_1)\Bigg\}$$

$$=\frac{n_2^2}{2}\Bigg\{\sum_{\{v_i,\ v_k\}\subseteq V(G_1)}(d_{G_1}(v_i)+d_{G_1}(v_k))d_{G_1}(v_i,\ v_k)+4n_1m_1-PI(G_1)\Bigg\}$$

$$=\frac{n_2^2}{2}(DD(G_1)+4n_1m_1-PI(G_1)). \tag{2.9}$$

Substituting (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), (2.8) and (2.9) in (2.1), we obtain the required result. □

COROLLARY 2.1. *Let G_1 be a bipartite graph. Then, the Wiener index of $G_1 \boxdot G_2$ is given by*

$$\begin{aligned}W(G_1 \boxdot G_2) = (1+n_2)^2 W(G_1) + n_2^2 W_e(G_1) + \frac{n_2}{2}(n_2+1)(DD(G_1) - n_1 m_1) \\ + n_1^2 n_2(n_2+1) + n_2^2 m_1((1/2) + 3n_1 + (1/2)m_1) - n_2(m_1+n_1) \\ - m_2(n_1+m_1).\end{aligned}$$

In the next theorem, we compute first Zagreb index of vertex-edge corona of G_1 and G_2 graphs.

THEOREM 2.2. *The First zagreb index of $G := G_1 \boxdot G_2$ is given by*

$$\begin{aligned}M_1(G) = (n_2+1)^2 M_1(G_1) + M_1(G_2)(m_1+n_1) + 8m_1(n_2+m_2) + n_2^2(4m_1+n_1) \\ + n_1(n_2+4m_2).\end{aligned}$$

PROOF. We have,

$$\begin{aligned}M_1(G) &= \sum_{v \in V(G)} d_G^2(v) \\ &= \sum_{v_i \in V(G_1)} d_G^2(v_i) + \sum_{e_i \in E(G_1)} \sum_{u_{ij} \in V(G_2)} d_G^2(u_{ij}) + \sum_{v_i \in V(G_1)} \sum_{w_{ij} \in V(G_2)} d_G^2(w_{ij}) \\ &= A_1 + A_2 + A_3. \qquad (2.10)\end{aligned}$$

We compute A_1, A_2 and A_3 using Lemma 1.1 as follows.

$$\begin{aligned}A_1 &= \sum_{v_i \in V(G_1)} d_G^2(v_i) \\ &= \sum_{v_i \in V(G_1)} ((n_2+1)d_{G_1}(v_i) + n_2)^2 \\ &= \sum_{v_i \in V(G_1)} (n_2+1)^2 d_{G_1}^2(v_i) + n_2^2 + 2(n_2+1)n_2 d_{G_1}(v_i) \\ &= (n_2+1)^2 M_1(G_1) + n_2^2 n_1 + 2n_2(n_2+1)2m_1 \\ &= (n_2+1)^2 M_1(G_1) + 4n_2 m_1(n_2+1) + n_1 n_2^2, \qquad (2.11)\end{aligned}$$

$$\begin{aligned}
A_2 &= \sum_{e_i \in E(G_1)} \sum_{u_{ij} \in V(G_2)} d^2_{_G}(u_{ij}) \\
&= \sum_{e_i \in E(G_1)} \sum_{u_{ij} \in V_{i_e}(G_2)} (d_{_{G_2}}(u_{ij}) + 2)^2 \\
&= \sum_{e_i \in E(G_1)} \sum_{u_j \in V_e(G_2)} (d_{_{G_2}}(u_j)^2 + 4 + 4d_{_{G_2}}(u_j)) \\
&= \sum_{e_i \in E(G_1)} (M_1(G_2) + 4n_2 + 8m_2) \\
&= m_1(M_1(G_2) + 4n_2 + 8m_2), \qquad (2.12)
\end{aligned}$$

$$\begin{aligned}
A_3 &= \sum_{v_i \in V(G_1)} \sum_{w_{ij} \in V_{i_v}(G_2)} d^2_{_G}(w_{ij}) \\
&= \sum_{v_i \in V(G_1)} \sum_{w_{ij} \in V_{i_v}(G_2)} (d_{_{G_2}}(w_{ij}) + 1)^2 \\
&= \sum_{v_i \in V(G_1)} \sum_{w_{ij} \in V_{i_v}(G_2)} (d^2_{_{G_2}}(w_{ij}) + 1 + 2d_{_{G_2}}(w_{ij})) \\
&= \sum_{v_i \in V(G_1)} (M_1(G_2) + n_2 + 4m_2) \\
&= n_1(M_1(G_2) + n_2 + 4m_2). \qquad (2.13)
\end{aligned}$$

Using (2.11), (2.12) and (2.13) in (2.10), we complete the proof of the theorem. □

Next, we establish formula for second Zagreb index of vertex-edge corona of G_1 and G_2 graphs.

THEOREM 2.3. *The second Zagreb index of vertex-edge corona $G := G_1 \boxdot G_2$ is given by*

$$\begin{aligned}
M_2(G) = (n_2+1)^2 M_2(G_1) &+ M_1(G_1)(n_2+1)(3n_2+2m_2) + M_2(G_2)(m_1+n_1) \\
&+ m_1 n_2(7n_2+2) + M_1(G_2)(2m_1+n_1) + n_1(n_2^2 + m_2(2n_2+1)) \\
&+ 8m_1 m_2(n_2+1).
\end{aligned}$$

PROOF. From the definition of $M_2(G)$, we have

$$\begin{aligned}
M_2(G) &= \sum_{e_i = uv \in E(G)} d_{_G}(u) d_{_G}(v) \\
&= A_1 + A_2 + A_3 + A_4 + A_5, \qquad (2.14)
\end{aligned}$$

where

$$
\begin{aligned}
A_1 &= \sum_{v_iv_j \in E(G_1)} d_G(v_i)d_G(v_j),\\
A_2 &= \sum_{e_i \in E(G_1)} \sum_{u_{ij}u_{ik} \in E_{i_e}(G_2)} d_G(u_{ij})d_G(u_{ik}),\\
A_3 &= \sum_{e_i = v_lv_m \in E(G_1)} \sum_{u_{ij} \in V_{i_e}(G_2)} (d_G(v_l) + d_G(v_m))d_G(u_{ij}),\\
A_4 &= \sum_{v_i \in V(G_1)} \sum_{w_{ij}w_{ik} \in E_{i_v}(G_2)} d_G(w_{ij})d_G(w_{ik}),\\
A_5 &= \sum_{v_i \in V(G_1)} \sum_{w_{ij} \in V_{i_v}(G_2)} d_G(v_i)d_G(w_{ij}).
\end{aligned}
$$

Employing Lemma 1.1, we compute $A_1, A_2, A_3, A_4,$ and A_5.

$$
\begin{aligned}
A_1 &= \sum_{v_iv_j \in E(G_1)} d_G(v_i)d_G(v_j)\\
&= \sum_{v_iv_j \in E(G_1)} (n_2+1)d_{G_1}(v_i) + n_2)((n_2+1)d_{G_1}(v_j) + n_2)\\
&= \sum_{v_iv_j \in E(G_1)} [(n_2+1)^2 d_{G_1}(v_i)d_{G_1}(v_j) + n_2(n_2+1)(d_{G_1}(v_i) + d_{G_1}(v_j)) + n_2^2]
\end{aligned}
$$

(2.15)
$$
= (n_2+1)^2 M_2(G_1) + n_2(n_2+1)M_1(G_1) + m_1n_2^2,
$$

$$
\begin{aligned}
A_2 &= \sum_{e_i \in E(G_1)} \sum_{u_{ij}u_{ik} \in E_{i_e}(G_2)} d_G(u_{ij})d_G(u_{ik})\\
&= \sum_{e_i \in E(G_1)} \sum_{u_ju_k \in E(G_2)} (d_{G_2}(u_j) + 2)(d_{G_2}(u_k) + 2)\\
&= \sum_{e_i \in E(G_1)} \sum_{u_ju_k \in E(G_2)} (d_{G_2}(u_j)d_{G_2}(u_k) + 2d_{G_2}(u_j) + 2d_{G_2}(u_k) + 4)\\
&= \sum_{e_i \in E(G_1)} \{M_2(G_2) + 2M_1(G_1) + 4m_2\}
\end{aligned}
$$

(2.16)
$$
= m_1(M_2(G_2) + 2M_1(G_2) + 4m_2),
$$

$$
A_3 = \sum_{e_i = v_lv_m \in E(G_1)} \sum_{u_{ij} \in V_{i_e}(G_2)} (d_G(v_l) + d_G(v_m))d_G(u_{ij})
$$

$$= \sum_{e_i=v_lv_m\in E(G_1)} \sum_{u_j\in V(G_2)} [(n_2+1)d_{G_1}(v_i)+n_2+(n_2+1)d_{G_1}(v_m)+n_2]$$
$$(d_{G_2}(u_j)+2)$$
$$= \sum_{e_i=v_lv_m\in E(G_1)} \sum_{u_j\in V(G_2)} [(n_2+1)(d_{G_1}(v_i)+d_{G_1}(v_m))+2n_2](d_{G_2}(u_j)+2)$$
$$= \sum_{e_i=v_lv_m\in E(G_1)} [(n_2+1)(d_{G_1}(v_i)+d_{G_1}(v_m))+2n_2](2m_2+2n_2)$$
(2.17)
$$= 2(n_2+1)M_1(G_1)(m_2+n_2)+4m_1(n_2m_2+n_2^2),$$

$$A_4 = \sum_{v_i\in V(G_1)} \sum_{w_{ij}w_{ik}\in E_{i_v}(G_2)} d_G(w_{ij})d_G(w_{ik})$$
$$= \sum_{v_i\in V(G_1)} \sum_{w_jw_k\in E(G_2)} (d_{G_2}(w_j+1)(d_{G_2}(w_k)+1))$$
$$= \sum_{v_i\in V(G_1)} \sum_{w_jw_k\in E(G_2)} (d_{G_2}(w_j)d_{G_2}(w_k)+d_{G_2}(w_j)+d_{G_2}(w_k)+1)$$
$$= \sum_{v_i\in V(G_1)} (M_2(G_2)+M_1(G_2)+m_2)$$
$$= n_1(M_2(G_2)+M_1(G_2)+m_2), \tag{2.18}$$

$$A_5 = \sum_{v_i\in V(G_1)} \sum_{w_{ij}\in V_{i_v}(G_2)} d_G(v_i)d_G(w_{ij})$$
$$= \sum_{v_i\in V(G_1)} \sum_{w_j\in V(G_2)} ((n_2+1)d_{G_1}(v_i)+n_2)(d_{G_2}(w_j)+1)$$
$$= \sum_{v_i\in V(G_1)} \sum_{w_{ij}\in V(G_2)} ((n_2+1)(d_{G_1}(v_i)d_{G_2}(w_j)+d_{G_1}(v_i))+n_2d_{G_2}(w_j)+n_2)$$
(2.19)
$$= 2m_1(n_2+1)(2m_2+n_2)+n_1n_2(2m_2+n_2).$$

Using (2.15), (2.16), (2.17), (2.18), (2.19) in (2.14), we determine the required result. □

It is well-known that $M_1(P_n) = 4n-6$ $(n \geqslant 2)$, $M_1(C_n) = 4n$, $M_1(K_n) = n(n-1)^2$, $M_2(P_n) = 4n-8$ $(n \geqslant 3)$, $M_2(C_n) = 4n$, $M_2(K_n) = n(n-1)^3/2$. Using these facts, in Theorems (2.10) and (2.14), we obtain the following corollaries.

Corollary 2.2. *We have*

a. $M_1(P_n \boxdot P_m) = 9m^2n - 10m^2 + 37mn - 32m - 20n + 8.$
b. $M_1(P_n \boxdot C_m) = 9m^2n - 10m^2 + 37mn - 32m + 4n - 6.$
c. $M_1(P_n \boxdot K_m) = 2m^3n - m^3 + 11m^2n - 12m^2 + 13mn - 17m + 4n - 6.$
d. $M_1(C_n \boxdot C_m) = 9m^2n + 37mn + 4n.$

e. $M_1(C_n \boxdot P_m) = 9m^2n + 37mn - 20n.$
f. $M_1(C_n \boxdot K_m) = 2m^3n + 11m^2n + 13mn + 4n.$
g. $M_1(K_n \boxdot K_m) = m^2n^3 + m^2n^2 + 2mn^3 - m^2n - (3/2)mn^2 + n^3 - (1/2)mn - 2n^2 + n + (1/2)m^3(n + n^2).$
h. $M_1(K_n \boxdot P_m) = m^2n^3 + 2mn^3 + 6mn^2 + n^3 + mn - 9n^2 - 2n.$
i. $M_1(K_n \boxdot C_m) = m^2n^3 + 2mn^3 + 6mn^2 + n^3 + mn - 2n^2 + n.$

COROLLARY 2.3. *We have*

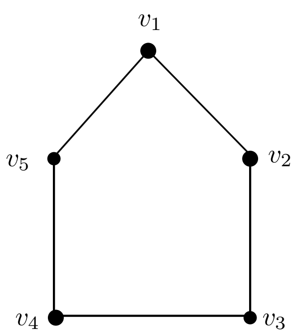
a. $M_2(P_n \boxdot P_m) = 42m^2n - 53m^2 + 29mn + 12n^2 - 40m - 55n + 32$
b. $M_2(P_n \boxdot C_m) = 42m^2n - 53m^2 + 59mn - 68m + 4n - 8.$
c. $M_2(P_n \boxdot K_m) = 9nm^3 + m^4n + (41/2)m^2n + (31/2)nm - (61/2)m^2 - (55/2)m - 8 + 4n - (21/2)m^3 - (1/2)m^4.$
d. $M_2(C_n \boxdot P_m) = 42m^2n + 41mn - 47n.$
e. $M_2(C_n \boxdot C_m) = 42m^2n + 59mn + 4n.$
f. $M_2(C_n \boxdot K_m) = (41/2)nm^2 + (31/2)nm + 4n + 9nm^3 + m^4n.$
g. $M_2(K_n \boxdot P_m) = (1/2)m^2n^4 + (7/2)m^2n^3 + (1/2)n^4 - (7/2)n^3 - (17/2)n^2 + 2nm - (7/2)n + mn^4 - m^2n^2 + 4mn^2.$
h. $M_2(K_n \boxdot C_m) = mn^4 - m^2n^2 + 4mn^2 + (1/2)m^2n^4 + (7/2)m^2n^3 + (1/2)n^4 - (3/2)n^3 + (3/2)n^2 + 2nm - (1/2)n + 2mn^3.$
i. $M_2(K_n \boxdot K_m) = mn^4 - (1/4)m^2n^2 - (9/4)mn^2 + (1/2)m^2n^4 + (3/2)m^2n^3 + (1/4)nm^2 - (3/4)nm^3 + (1/4)m^4n + (1/2)n^4 - (3/2)n^3 - (3/4)m^3n^2 + (1/4)m^4n^2 + n^2m(m-1)^2 + m^3n^3 + (3/2)n^2 + (5/4)nm - (1/2)n - mn^3.$

3. Degree Distance and Gutman index of vertex-edge corona of two graphs

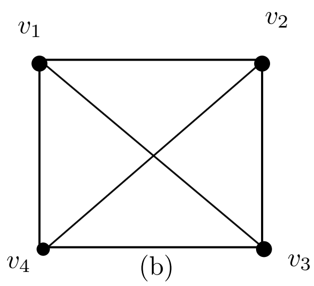
For a graph G, we define

$$C(G) := \sum_{\substack{w \in V(G) \\ e=uv \in E(G) \\ d_G(w,v)=d_G(u,w)}} d_G(w).$$

EXAMPLE 3.1.



(a)



(b)

(a): We have,

$$\begin{aligned}C(G) &= d_G(v_1) + d_G(v_2) + d_G(v_3) + d_G(v_4) + d_G(v_5)\\ &= 2+2+2+2+2\\ &= 10.\end{aligned}$$

(b): We have,

$$\begin{aligned}C(G) &= d_G(v_1) + d_G(v_2) + d_G(v_2) + d_G(v_3) + d_G(v_4) + d_G(v_4)\\ &= 2+3+3+2+3+3\\ &= 16.\end{aligned}$$

In this section, we compute the degree distance and Gutman index of vertex-edge corona of two graphs.

THEOREM 3.1. *The degree distance of vertex-edge corona $G := G_1 \boxdot G_2$ of G_1 and G_2 is given by*

$$\begin{aligned}DD(G) = &(3n_2^2 + 2n_2m_2 + 3n_2 + m_2 + 1)DD(G_1) - M_1(G_2)(n_1 + m_1) - 8m_1m_2\\ &+ 4n_2W_e(G_1)(n_2 + m_2) + 4W(G_1)(n_2^2 + n_2m_2 + n_2 + m_2) + m_1^2n_2^2\\ &- 2n_1(n_2 - 3m_2) + n_1^2(3n_2^2 + 4n_2m_2 + n_2 + 2m_2) + 2m_1n_2(m_1m_2 - 2)\\ &+ 2m_1^2n_2 + n_2^2(9n_1m_1 + 3m_1^2 + 2m_1) - PI(G_1)(2n_2^2 + n_2 + 2n_2m_2 + m_2)\\ &+ 4n_1n_2m_1(2m_2 + 1) + n_2(n_2 + 1)(\frac{1}{2}C(G_1) + Gut(G_1))\\ &+ 2m_1m_2(n_1 + n_2).\end{aligned}$$

PROOF.

$$\begin{aligned}DD(G) &= \sum_{\{x,\ y\}\subseteq V(G)} (d_G(x) + d_G(y))\ d_G(x,\ y)\\ &= A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8, \qquad (3.1)\end{aligned}$$

where

$$A_1 = \sum_{\{v_i,\ v_j\} \subseteq V(G_1)} (d_G(v_i) + d_G(v_j)) d_G(v_i,\ v_j),$$

$$A_2 = \sum_{e_i \in E(G_1)} \sum_{\{u_{ij},\ u_{ik}\} \subseteq V_{i_e}(G_2)} (d_G(u_{ij}) + d_G(u_{ik})) d_G(u_{ij},\ u_{ik}),$$

$$A_3 = \sum_{v_i \in V(G_1)} \sum_{\{w_{ij},\ w_{ik}\} \subseteq V_{i_v}(G_2)} (d_G(w_{ij}) + d_G(w_{ik})) d_G(w_{ij},\ w_{ik}),$$

$$A_4 = \sum_{\{e_i,\ e_k\} \subseteq E(G_1)} \sum_{\substack{u_{ij} \in V_{i_e}(G_2) \\ u_{km} \in V_{k_e}(G_2)}} (d_G(u_{ij}) + d_G(u_{km})) d_G(u_{ij},\ u_{km}),$$

$$A_5 = \sum_{\{v_i,\ v_k\} \subseteq V(G_1)} \sum_{\substack{w_{ij} \in V_{i_v}(G_2) \\ w_{km} \in V_{k_v}(G_2)}} (d_G(w_{ij}) + d_G(w_{km})) d_G(w_{ij},\ w_{km}),$$

$$A_6 = \sum_{e_i = v_l v_m \in E(G_1)} \sum_{\substack{u_{ij} \in V_{i_e}(G_2) \\ v_k \in V(G_1)}} (d_G(u_{ij}) + d_G(v_k)) d_G(u_{ij},\ v_k),$$

$$A_7 = \sum_{v_i \in V(G_1)} \sum_{w_{ij} \in V_{i_v}(G_2)} (d_G(w_{ij}) + d_G(v_k)) d_G(w_{ij},\ v_k),$$

$$A_8 = \sum_{e_i = v_l v_m \in E(G_1)} \sum_{\substack{u_{ij} \in V_{i_e}(G_2) \\ w_{km} \in V_{k_v}(G_2)}} (d_G(u_{ij}) + d_G(w_{km})) d_G(u_{ij},\ w_{km}).$$

Using Lemmas 1.1 and 1.2, we have,

$$\begin{aligned} A_1 &= \sum_{\{v_i,\ v_j\} \subseteq V(G_1)} (d_G(v_i) + d_G(v_j)) d_G(v_i,\ v_j) \\ &= \sum_{\{v_i,\ v_j\} \subseteq V(G_1)} ((n_2+1) d_{G_1}(v_i) + n_2 + (n_2+1) d_{G_1}(v_j) + n_2) d_{G_1}(v_i,\ v_j) \\ &= \sum_{\{v_i,\ v_j\} \subseteq V(G_1)} ((n_2+1)(d_{G_1}(v_i) + d_{G_1}(v_j)) + 2n_2) d_{G_1}(v_i,\ v_j) \\ &= (n_2+1) DD(G_1) + 2n_2 W(G_1), \end{aligned} \tag{3.2}$$

$$\begin{aligned} A_2 &= \sum_{e_i \in E(G_1)} \sum_{\{u_{ij},\ u_{ik}\} \subseteq V_{i_e}(G_2)} (d_G(u_{ij}) + d_G(u_{ik})) d_G(u_{ij},\ u_{ik}) \\ &= \sum_{e_i \in E(G_1)} \sum_{\{u_{ij},\ u_{ik}\} \subseteq V_{i_e}(G_2)} (d_{G_2}(u_{ij}) + 2 + d_{G_2}(u_{ik}) + 2) d_G(u_{ij},\ u_{ik}) \\ &= \sum_{e_i \in E(G_1)} \sum_{\{u_{ij},\ u_{ik}\} \subseteq V_{i_e}(G_2)} (d_{G_2}(u_{ij}) + d_{G_2}(u_{ik}) + 4) d_G(u_{ij},\ u_{ik}) \end{aligned}$$

$$= \sum_{e_i \in E(G_1)} \left\{ 2 \sum_{\{u_j,\ u_k\} \subseteq V(G_2)} (d_{G_2}(u_j) + d_{G_2}(u_k) + 4) \right.$$

$$\left. - \sum_{u_j u_k \in E(G_2)} (d_{G_2}(u_j) + d_{G_2}(u_k) + 4) \right\}$$

$$= \sum_{e_i \in E(G_1)} \left\{ 2(n_2 - 1) \sum_{u_j \in V(G_2)} d_{G_2}(u_j) + 4(n_2 - 1)n_2 - M_1(G_2) - 4m_2 \right\}$$

$$= \sum_{e_i \in E(G_1)} \{2(n_2 - 1)2m_2 + 4(n_2 - 1)n_2 - M_1(G_2) - 4m_2\}$$

$$= m_1(4(n_2 - 1)(m_2 + n_2) - 4m_2 - M_1(G_2)), \tag{3.3}$$

$$A_3 = \sum_{v_i \in V(G_1)} \sum_{\{w_{ij},\ w_{ik}\} \subseteq V_{i_v}(G_2)} (d_G(w_{ij}) + d_G(w_{ik})) d_G(w_{ij},\ w_{ik})$$

$$= \sum_{v_i \in V(G_1)} \sum_{\{w_{ij},\ w_{ik}\} \subseteq V_{i_v}(G_2)} (d_{G_2}(w_{ij}) + 1 + d_{G_2}(w_{ik}) + 1) d_G(w_{ij},\ w_{ik})$$

$$= \sum_{v_i \in V(G_1)} \sum_{\{w_{ij},\ w_{ik}\} \subseteq V_{i_v}(G_2)} (d_{G_2}(w_{ij}) + d_{G_2}(w_{ik}) + 2) d_G(w_{ij},\ w_{ik})$$

$$= \sum_{v_i \in V(G_1)} \left\{ 2 \sum_{\{w_j,\ w_k\} \subseteq V(G_2)} (d_{G_2}(w_j) + d_{G_2}(w_k) + 2) \right.$$

$$\left. - \sum_{w_j w_k \in E(G_2)} (d_{G_2}(w_j) + d_{G_2}(w_k) + 2) \right\}$$

$$= \sum_{v_i \in V(G_1)} \left\{ 2(n_2 - 1) \sum_{w_j \in V(G_2)} d_{G_2}(w_j) + 2(n_2 - 1)n_2 - M_1(G_2) - 2m_2 \right\}$$

$$= \sum_{v_i \in V(G_1)} (2(n_2 - 1)2m_2 + 2(n_2 - 1)n_2 - M_1(G_2) - 2m_2)$$

$$= n_1(2(n_2 - 1)(2m_2 + n_2) - M_1(G_2) - 2m_2), \tag{3.4}$$

$$A_4 = \sum_{\{e_i,\ e_k\} \subseteq E(G_1)} \sum_{\substack{u_{ij} \in V_{i_e}(G_2) \\ u_{km} \in V_{k_e}(G_2)}} (d_G(u_{ij}) + d_G(u_{km})) d_G(u_{ij},\ u_{km})$$

$$= \sum_{\{e_i,\ e_k\} \subseteq E(G_1)} \left\{ \sum_{\substack{u_{ij} \in V_{i_e}(G_2) \\ u_{km} \in V_{k_e}(G_2)}} (d_{G_2}(u_{ij}) + 2 + d_{G_2}(u_{km}) + 2)(d_{G_1}(e_i,\ e_k) + 2) \right\}$$

$$= \sum_{\{e_i,\ e_k\}\subseteq E(G_1)} \left\{ \sum_{\substack{u_{ij}\in V_{i_e}(G_2)\\ u_{km}\in V_{k_e}(G_2)}} \left[(d_{G_2}(u_{ij}) + d_{G_2}(u_{km}) + 4)\left(d_{G_1}(e_i,\ e_k) + 2\right)\right] \right\}$$

$$= \sum_{\{e_i,\ e_k\}\subseteq E(G_1)} (4m_2n_2 + 4n_2^2)(d_{G_1}(e_i,\ e_k) + 2)$$

(3.5)

$$= 4n_2(m_2 + n_2)(W_e(G_1) + m_1(m_1 - 1)/2),$$

$$A_5 = \sum_{\{v_i,\ v_k\}\subseteq V(G_1)} \sum_{\substack{w_{ij}\in V_{i_v}(G_2)\\ w_{km}\in V_{k_v}(G_2)}} (d_G(w_{ij}) + d_G(w_{km}))d_G(w_{ij},\ w_{km})$$

$$= \sum_{\{v_i,\ v_k\}\subseteq V(G_1)} \left\{ \sum_{\substack{w_{ij}\in V_{i_v}(G_2)\\ w_{km}\in V_{k_v}(G_2)}} (d_{G_2}(w_{ij}) + 1 + d_{G_2}(w_{km}) + 1)(d_{G_1}(v_i,\ v_k) + 2) \right\}$$

$$= \sum_{\{v_i,\ v_k\}\subseteq V(G_1)} (4m_2n_2 + 2n_2^2)(d_{G_1}(v_i,\ v_k) + 2)$$

(3.6)

$$= 2n_2(2m_2 + n_2)(W(G_1) + n_1(n_1 - 1)).$$

$$A_6 = \sum_{e_i=v_lv_m\in E(G_1)} \sum_{\substack{u_{ij}\in V_{i_e}(G_2)\\ v_k\in V(G_1)}} (d_G(u_{ij}) + d_G(v_k))d_G(u_{ij},\ v_k)$$

$$= \sum_{e_i=v_lv_m\in E(G_1)} \sum_{\substack{u_{ij}\in V_{i_e}(G_2)\\ v_k\in V(G_1)}} (d_{G_2}(u_{ij}) + 2 + (n_2 + 1)d_{G_1}(v_k) + n_2)d_G(u_{ij},\ v_k)$$

$$= \sum_{e_i=v_lv_m\in E(G_1)} \sum_{v_k\in V(G_1)} (2m_2 + 2n_2 + n_2^2 + n_2(n_2 + 1)d_{G_1}(v_k))d_G(u_{ij},\ v_k)$$

$$= \sum_{e_i=v_lv_m\in E(G_1)} \left\{ \left\{ \sum_{\substack{v_k\in V(G_1)\\ d(v_l,\ v_k)\neq d(v_k,\ v_m)}} \frac{d_{G_1}(v_l,\ v_k) + d_{G_1}(v_k,\ v_m) + 1}{2} \right\} \right.$$

$$\left. + \left\{ \sum_{\substack{v_k\in V(G_1)\\ d(v_l,\ v_k)= d(v_k,\ v_m)}} \left\{ \frac{d_{G_1}(v_l,\ v_k) + d_{G_1}(v_k,\ v_m)}{2} + 1 \right\} \right\} (2m_2 + 2n_2 + n_2^2) \right.$$

$$+\left\{\sum_{\substack{v_k\in V(G_1)\\ d(v_l,\ v_k)\neq d(v_k,\ v_m)}} \frac{d_{G_1}(v_l,\ v_k)+d_{G_1}(v_k,\ v_m)+1}{2}\right\}$$

$$+\left\{\sum_{\substack{v_k\in V(G_1)\\ d(v_l,\ v_k)= d(v_k,\ v_m)}} \left\{\frac{d_{G_1}(v_l,\ v_k)+d_{G_1}(v_k,\ v_m)}{2}+1\right\}\right\}(n_2(n_2+1)d_{G_1}(v_k))\Bigg\}$$

$$=\frac{(2m_2+2n_2+n_2^2)}{2}\left\{\sum_{e_i\in E(G_1)}\sum_{v_k\in V(G_1)}(d_{G_1}(v_l,\ v_k)+d_{G_1}(v_k,\ v_m))\right.$$

$$\left.+\sum_{e_i\in E(G_1)}(2n_1-(n_{e_i}(v_l/G_1)+n_{e_i}(v_m/G_1)))\right\}$$

$$+\frac{n_2(n_2+1)}{2}\sum_{e_i\in E(G_1)}\left\{\sum_{\substack{v_k\in V(G_1)\\ d(v_l,\ v_k)\neq d(v_k,\ v_m)}} d_{G_1}(v_k)+\sum_{\substack{v_k\in V(G_1)\\ d(v_l,\ v_k)= d(v_k,\ v_m)}} 2\ d_{G_1}(v_k)\right.$$

$$\left.+\sum_{v_k\in V(G_1)}(d_{G_1}(v_l,\ v_k)+d_{G_1}(v_k,\ v_m))d_{G_1}(v_k)\right\}$$

$$=\frac{(2m_2+2n_2+n_2^2)}{2}\left\{\sum_{v_k\in V(G_1)}\sum_{v_i\in V(G_1)}\sum_{v_iv_j\in E(G_1)} d_{G_1}(v_i,\ v_k)+2n_1m_1\right.$$

$$\left.-PI(G_1)\right\}$$

$$+\frac{n_2(n_2+1)}{2}\sum_{e_i\in E(G_1)}\left\{\sum_{v_k\in V(G_1)}(d_{G_1}(v_l,\ v_k)+d_{G_1}(v_k,\ v_m))d_{G_1}(v_k)\right.$$

$$\left.+\sum_{\substack{v_k\in V(G_1)\\ d(v_l,\ v_k)\neq d(v_k,\ v_m)}} d_{G_1}(v_k)+\sum_{\substack{v_k\in V(G_1)\\ d(v_l,\ v_k)= d(v_k,\ v_m)}} 2\ d_{G_1}(v_k)\right\}$$

$$=\frac{(2m_2+2n_2+n_2^2)}{2}\left\{\sum_{v_i\in V(G_1)}\sum_{v_k\in V(G_1)} d_{G_1}(v_i)d_{G_1}(v_i,\ v_k)+2n_1m_1-PI(G_1))\right\}$$

$$+\frac{n_2(n_2+1)}{2}\left\{\sum_{e_i=v_lv_m\in E(G_1)}\sum_{v_k\in V(G_1)}(d_{G_1}(v_l,\ v_k)+d_{G_1}(v_k,\ v_m))\ d_{G_1}(v_k)\right.$$

$$\left.+\sum_{e_i\in E(G_1)}2m_1+\frac{1}{2}\sum_{e_i\in E(G_1)}\sum_{\substack{v_k\in V(G_1)\\ d(v_l,\ v_k)= d(v_k,\ v_m)}} d_{G_1}(v_k)\right\}$$

$$= \frac{(2m_2 + 2n_2 + n_2^2)}{2} \left\{ \sum_{\{v_i,\ v_k\} \subseteq V(G_1)} ((d_{G_1}(v_i) + d_{G_1}(v_k))d_{G_1}(v_i,\ v_k) \right.$$

$$\left. +2n_1m_1 - PI(G_1))\right\}$$

$$+ \frac{n_2(n_2+1)}{2} \left\{ \sum_{e_i = v_l v_m \in E(G_1)} \sum_{v_k \in V(G_1)} (d_{G_1}(v_l,\ v_k) + d_{G_1}(v_k,\ v_m))\ d_{G_1}(v_k) \right.$$

$$\left. +2m_1^2 + \frac{1}{2}C(G_1) \right\}$$

Using (2.9) in above equation, we get

$$= \frac{(2m_2 + 2n_2 + n_2^2)}{2}(DD(G_1) + 2n_1m_1 - PI(G_1)) + n_2(n_2+1)(Gut(G_1) + m_1^2)$$

(3.7)

$$+ n_2(n_2+1)\frac{1}{2}C(G_1).$$

$$A_7 = \sum_{v_i \in V(G_1)} \sum_{w_{ij} \in V_{i_v}(G_2)} (d_G(w_{ij}) + d_G(v_k))d_G(w_{ij},\ v_k)$$

$$= \sum_{v_i \in V(G_1)} \sum_{\substack{w_j \in V(G_2) \\ v_k \in V(G_1)}} (d_{G_2}(w_j) + 1 + (n_2+1)d_{G_1}(v_k) + n_2)(d_{G_1}(v_i,\ v_k) + 1)$$

$$= \sum_{v_i \in V(G_1)} d_{G_1}(v_i,\ v_k) \sum_{\substack{w_j \in V(G_2) \\ v_k \in V(G_1)}} (d_{G_2}(w_j) + (n_2+1) + (n_2+1)d_{G_1}(v_k))$$

$$+ \sum_{v_i \in V(G_1)} \sum_{\substack{w_j \in V(G_2) \\ v_k \in V(G_1)}} (d_{G_2}(w_j) + (n_2+1) + (n_2+1)d_{G_1}(v_k))$$

$$= \sum_{v_i \in V(G_1)} \sum_{v_k \in V(G_1)} d_{G_1}(v_i,\ v_k)\{2m_2 + n_2(n_2+1) + n_2(n_2+1)d_{G_1}(v_k)\}$$

$$+ \sum_{v_i \in V(G_1)} \sum_{v_k \in V(G_1)} \{2m_2 + n_2(n_2+1) + n_2(n_2+1)d_{G_1}(v_k)\}$$

$$= 4m_2W(G_1) + W(G_1)n_2(n_2+1) + DD(G_1)n_2(n_2+1) + 2m_2n_1^2$$

$$+ n_1^2n_2(n_2+1) + 2m_1n_1n_2(n_2+1)$$

(3.8) $$= (2m_2 + n_2(1+n_2))(2W(G_1) + n_1^2) + n_2(n_2+1)(DD(G_1) + 2m_1n_1),$$

$$A_8 = \sum_{e_i = v_l v_m \in E(G_1)} \sum_{\substack{u_{ij} \in V_{i_e}(G_2) \\ w_{km} \in V_{k_v}(G_2)}} (d_G(u_{ij}) + d_G(w_{km}))d_G(u_{ij},\ w_{km})$$

$$= \sum_{e_i=v_lv_m\in E(G_1)} \sum_{\substack{u_{ij}\in V_{i_e}(G_2)\\ w_{km}\in V_{k_v}(G_2)}} (d_{G_2}(u_{ij}) + 2 + d_{G_2}(w_{km}) + 1)d_G(u_{ij},\ w_{km})$$

$$= \sum_{e_i=v_lv_m\in E(G_1)} \sum_{\substack{u_{ij}\in V_{i_e}(G_2)\\ w_{km}\in V_{k_v}(G_2)}} (d_{G_2}(u_{ij}) + d_{G_2}(w_{km}) + 3)d_G(u_{ij},\ w_{km})$$

$$= \sum_{e_i\in E(G_1)} (4m_2n_2 + 3n_2^2)d_G(u_{ij},\ w_{km})$$

$$= (4m_2n_2 + 3n_2^2) \sum_{e_i\in E(G_1)} \left\{ \sum_{\substack{v_k\in V(G_1)\\ d(v_l,\ v_k)\neq d(v_k,\ v_m)}} \frac{d_{G_1}(v_l,\ v_k) + d_{G_1}(v_k,\ v_m) + 3}{2} \right.$$

$$\left. + \sum_{\substack{v_k\in V(G_1)\\ d(v_l,\ v_k)=d(v_k,\ v_m)}} \left\{ \frac{d_{G_1}(v_l,\ v_k) + d_{G_1}(v_k,\ v_m)}{2} + 2 \right\} \right\}$$

$$= (4m_2n_2 + 3n_2^2) \sum_{e_i\in E(G_1)} \frac{1}{2} \left\{ \sum_{v_k\in V(G_1)} d_{G_1}(v_l,\ v_k) + d_{G_1}(v_k,\ v_m) \right.$$

$$\left. + \sum_{\substack{v_k\in V(G_1)\\ d(v_l,\ v_k)\neq d(v_k,\ v_m)}} 3 + \sum_{\substack{v_k\in V(G_1)\\ d(v_l,\ v_k)=d(v_k,\ v_m)}} 4 \right\}$$

Using (2.9) in above equation, we get

(3.9)
$$= \frac{(4m_2n_2 + 3n_2^2)}{2}(DD(G_1) + 4n_1m_1 - PI(G_1)).$$

Substituting (3.2), (3.3), (3.4), (3.5), (3.6), (3.7), (3.8) and (3.9) in (3.1). Hence the proof. □

Corollary 3.1. *Let G_1 be a bipartite graph. Then,*

$$\begin{aligned} DD(G) &= (3n_2^2 + 2n_2m_2 + 3n_2 + m_2 + 1)DD(G_1) - M_1(G_2)(n_1 + m_1) - 8m_1m_2\\ &+ 4n_2W_e(G_1)(n_2 + m_2) + 4W(G_1)(n_2^2 + n_2m_2 + n_2 + m_2) + m_1^2n_2^2\\ &- 2n_1(n_2 - 3m_2) + n_1^2(3n_2^2 + 4n_2m_2 + n_2 + 2m_2) + 2m_1n_2(m_1m_2 - 2)\\ &+ 2m_1^2n_2 + n_2^2(9n_1m_1 + 3m_1^2 + 2m_1) - n_1m_1(2n_2^2 + n_2 + 2n_2m_2 + m_2)\\ &+ 4n_1n_2m_1(2m_2 + 1) + n_2(n_2 + 1)Gut(G_1) + 2m_1m_2(n_1 + n_2). \end{aligned}$$

THEOREM 3.2. *The Gutman index of vertex-edge corona $G := G_1 \boxdot G_2$ of G_1 and G_2 is given by*

$$\begin{aligned}Gut(G) &= Gut(G_1)(n_2(3n_2+2m_2+4)+2m_2+1) - M_2(G_2)(m_1+n_1)\\&- 2PI(G_1)(n_2(n_2+2m_2)+m_2^2) + C(G_1)(n_2(n_2+m_2+1)+m_2)\\&+ 2DD(G_1)(n_2(2n_2+3m_2+1)+m_2(m_2+1)) - M_1(G_2)(2n_1+3m_1)\\&+ (W(G_1)+W_e(G_1))(4n_2(n_2+2m_2)+4m_2^2) + (n_2^2+m_2^2)(8n_1m_1+2m_1)\\&+ m_1^2(4n_2^2+6n_2m_2+2m_2^2+2n_2+2m_2) + (2n_2^2+6n_2m_2+4m_2^2)n_1^2\\&+ 4m_1m_2(n_1+n_2-3) + n_1(n_2-5m_2) + 2n_2m_1(9n_1m_2+n_1-2).\end{aligned}$$

PROOF.

$$\begin{aligned}Gut(G) &= \sum_{\{x,\ y\}\subseteq V(G)} d_G(x)\ d_G(y)\ d_G(x,\ y)\\&= A_1+A_2+A_3+A_4+A_5+A_6+A_7+A_8, \qquad (3.10)\end{aligned}$$

where

$$\begin{aligned}A_1 &= \sum_{\{v_i,\ v_j\}\subseteq V(G_1)} d_G(v_i)\ d_G(v_j)\ d_G(v_i,\ v_j),\\A_2 &= \sum_{e_i\in E(G_1)}\ \sum_{\{u_{ij},\ u_{ik}\}\subseteq V_{i_e}(G_2)} d_G(u_{ij})\ d_G(u_{ik})\ d_G(u_{ij},\ u_{ik}),\\A_3 &= \sum_{v_i\in V(G_1)}\ \sum_{\{w_{ij},\ w_{ik}\}\subseteq V_{i_v}(G_2)} d_G(w_{ij})\ d_G(w_{ik})\ d_G(w_{ij},\ w_{ik}),\\A_4 &= \sum_{\{e_i,\ e_k\}\in E(G_1)}\ \sum_{\substack{u_{ij}\in V_{i_e}(G_2)\\u_{km}\in V_{k_e}(G_2)}} d_G(u_{ij})\ d_G(u_{km})\ d_G(u_{ij},\ u_{km}),\\A_5 &= \sum_{\{v_i,\ v_k\}\subseteq V(G_1)}\ \sum_{\substack{w_{ij}\in V_{i_v}(G_2)\\w_{km}\in V_{k_v}(G_2)}} d_G(w_{ij})\ d_G(w_{km})\ d_G(w_{ij},\ w_{km}),\\A_6 &= \sum_{e_i=v_lv_m\in E(G_1)}\ \sum_{\substack{u_{ij}\in V_{i_e}(G_2)\\v_k\in V(G_1)}} d_G(u_{ij})\ d_G(v_k)\ d_G(u_{ij},\ v_k),\\A_7 &= \sum_{v_i\in V(G_1)}\ \sum_{\substack{w_{ij}\in V_{i_v}(G_2)\\v_k\in V(G_1)}} d_G(w_{ij})\ d_G(v_k) d_G(w_{ij},\ v_k),\\A_8 &= \sum_{e_i=v_lv_m\in E(G_1)}\ \sum_{\substack{u_{ij}\in V_{i_e}(G_2)\\w_{km}\in V_{k_v}(G_2)}} d_G(u_{ij}) d_G(w_{km}) d_G(u_{ij},\ w_{km}).\end{aligned}$$

We compute A_1, A_2, A_3, A_4, and A_5 using Lemmas 1.1 and 1.2 as follows:

$$A_1 = \sum_{\{v_i,\ v_j\}\subseteq V(G_1)} d_G(v_i)\ d_G(v_j)\ d_G(v_i,\ v_j)$$
$$= \sum_{\{v_i,\ v_j\}\subseteq V(G_1)} ((n_2+1)^2 d_{G_1}(v_i) d_{G_1}(v_j) + (n_2+1)n_2(d_{G_1}(v_i)+d_{G_1}(v_j)) + n_2^2)$$
$$d_{G_1}(v_i,\ v_j)$$
$$= (n_2+1)^2 Gut(G_1) + (n_2+1)n_2 DD(G_1) + n_2^2 W(G_1), \tag{3.11}$$

$$A_2 = \sum_{e_i\in E(G_1)} \sum_{\{u_{ij},\ u_{ik}\}\subseteq V_{i_e}(G_2)} d_G(u_{ij})\ d_G(u_{ik})\ d_G(u_{ij},\ u_{ik})$$
$$= \sum_{e_i\in E(G_1)} \sum_{\{u_{ij},\ u_{ik}\}\subseteq V_{i_e}(G_2)} ((d_G(u_{ij}+2)(d_G(u_{ik})+2)))d_G(u_{ij},\ u_{ik})$$
$$= \sum_{e_i\in E(G_1)} \sum_{\{u_{ij},\ u_{ik}\}\subseteq V_{i_e}(G_2)} (d_{G_2}(u_{ij})d_{G_2}(u_{ik}) + 2(d_{G_2}(u_{ij}) + d_{G_2}(u_{ik})) + 4)$$
$$d_G(u_{ij},\ u_{ik})$$
$$= \sum_{e_i\in E(G_1)} \left\{ 2 \sum_{\{u_j,\ u_k\}\subseteq V(G_2)} (d_{G_2}(u_j)d_{G_2}(u_k) + 2(d_{G_2}(u_j) + d_{G_2}(u_k)) + 4) \right.$$
$$\left. - \sum_{u_ju_k\in E(G_2)} (d_{G_2}(u_j)d_{G_2}(u_k) + 2(d_{G_2}(u_j) + d_{G_2}(u_k)) + 4) \right\}$$
$$= \sum_{e_i\in E(G_1)} (4m_2^2 - M_1(G_2) + 4(n_2-1)2m_2 + 4(n_2-1)n_2 - M_2(G_2)$$
$$- 2M_1(G_2) - 4m_2)$$
$$= m_1(4m_2^2 + 4(n_2-1)(2m_2+n_2) - 3M_1(G_2) - M_2(G_2) - 4m_2), \tag{3.12}$$

$$A_3 = \sum_{v_i\in V(G_1)} \sum_{\{w_{ij},\ w_{ik}\}\subseteq V_{i_v}(G_2)} d_G(w_{ij})\ d_G(w_{ik})\ d_G(w_{ij},\ w_{ik})$$
$$= \sum_{v_i\in V(G_1)} \sum_{\{w_{ij},\ w_{ik}\}\subseteq V_{i_v}(G_2)} (d_G(w_{ij})+1)(d_G(w_{ik})+1)d_G(w_{ij},\ w_{ik})$$
$$= \sum_{v_i\in V(G_1)} \sum_{\{w_{ij},\ w_{ik}\}\subseteq V_{i_v}(G_2)} (d_{G_2}(w_{ij})d_{G_2}(w_{ik}) + (d_{G_2}(w_{ij}) + d_{G_2}(w_{ik})) + 1)$$
$$d_G(w_{ij},\ w_{ik})$$

$$= \sum_{v_i \in V(G_1)} \Bigg\{ 2 \sum_{\{w_j,\ w_k\} \subseteq V(G_2)} (d_{G_2}(w_j) d_{G_2}(w_k) + (d_{G_2}(w_j) + d_{G_2}(w_k)) + 1)$$

$$- \sum_{w_j w_k \in E(G_2)} (d_{G_2}(w_j) d_{G_2}(w_k) + (d_{G_2}(w_j) + d_{G_2}(w_k)) + 1) \Bigg\}$$

$$= \sum_{v_i \in V(G_1)} (4m_2^2 - M_1(G_2) + 2(n_2 - 1)2m_2 + (n_2 - 1)n_2 - M_2(G_2)$$

$$- M_1(G_2) - m_2)$$

$$= n_1(4m_2^2 + (n_2 - 1)(4m_2 + n_2) - 2M_1(G_2) - M_2(G_2) - m_2), \tag{3.13}$$

$$A_4 = \sum_{\{e_i,\ e_k\} \in E(G_1)} \sum_{\substack{u_{ij} \in V_{i_e}(G_2) \\ u_{km} \in V_{k_e}(G_2)}} d_G(u_{ij})\ d_G(u_{km})\ d_G(u_{ij},\ u_{km})$$

$$= \sum_{\{e_i,\ e_k\} \subseteq E(G_1)} \left\{ \sum_{\substack{u_{ij} \in V_{i_e}(G_2) \\ u_{km} \in V_{k_e}(G_2)}} ((d_{G_2}(u_{ij}) + 2)(d_{G_2}(u_{km}) + 2))(d_{G_1}(e_i,\ e_k) + 2) \right\}$$

$$= \sum_{\{e_i,\ e_k\} \subseteq E(G_1)} \sum_{\substack{u_{ij} \in V_{i_e}(G_2) \\ u_{km} \in V_{k_e}(G_2)}} (d_{G_2}(u_{ij}) d_{G_2}(u_{km}) + 2d_{G_2}(u_{ij}) + 2d_{G_2}(u_{km}) + 4)$$

$$(d_{G_1}(e_i,\ e_k) + 2)$$

$$= \sum_{\{e_i,\ e_k\} \subseteq E(G_1)} (4m_2^2 + 4n_2^2 + 4m_2 n_2 + 4m_2 n_2)(d_{G_1}(e_i,\ e_k) + 2)$$

$$= 4(m_2 + n_2)^2 (W_e(G_1) + m_1(m_1 - 1)/2), \tag{3.14}$$

$$A_5 = \sum_{\{v_i,\ v_k\} \subseteq V(G_1)} \sum_{\substack{w_{ij} \in V_{i_v}(G_2) \\ w_{km} \in V_{k_v}(G_2)}} d_G(w_{ij})\ d_G(w_{km})\ d_G(w_{ij},\ w_{km})$$

$$= \sum_{\{v_i,\ v_k\} \subseteq V(G_1)} \left\{ \sum_{\substack{w_{ij} \in V_{i_v}(G_2) \\ w_{km} \in V_{k_v}(G_2)}} (d_{G_2}(w_{ij}) + 1)(d_{G_2}(w_{km}) + 1)(d_{G_1}(v_i,\ v_k) + 2) \right\}$$

$$= \sum_{\{v_i,\ v_k\} \subseteq V(G_1)} \sum_{\substack{w_{ij} \in V_{i_v}(G_2) \\ w_{km} \in V_{k_v}(G_2)}} (d_{G_2}(w_{ij}) d_{G_2}(w_{km}) + d_{G_2}(w_{ij}) + d_{G_2}(w_{km}) + 1)$$
$$(d_{G_1}(v_i,\ v_k) + 2)$$
$$= \sum_{\{v_i,\ v_k\} \subseteq V(G_1)} (4m_2^2 + n_2^2 + 4m_2 n_2)\ (d_{G_1}(v_i,\ v_k) + 2)$$
(3.15)
$$= (2m_2 + n_2)^2 (W(G_1) + n_1(n_1 - 1)).$$

$$A_6 = \sum_{e_i = v_l v_m \in E(G_1)} \sum_{\substack{u_{ij} \in V_{i_e}(G_2) \\ v_k \in V(G_1)}} d_G(u_{ij})\ d_G(v_k)\ d_G(u_{ij},\ v_k)$$
$$= \sum_{e_i = v_l v_m \in E(G_1)} \sum_{\substack{u_{ij} \in V_{i_e}(G_2) \\ v_k \in V(G_1)}} (d_{G_2}(u_{ij}) + 2)((n_2 + 1) d_{G_1}(v_k) + n_2) d_G(u_{ij},\ v_k)$$
$$= \sum_{e_i = v_l v_m \in E(G_1)} \sum_{\substack{u_{ij} \in V_{i_e}(G_2) \\ v_k \in V(G_1)}} ((d_{G_2}(u_{ij}) + 2)(n_2 + 1) d_{G_1}(v_k) + n_2(d_{G_2}(u_{ij}) + 2))$$
$$d_G(u_{ij},\ v_k)$$
$$= \sum_{e_i = v_l v_m \in E(G_1)} \left\{ \sum_{\substack{v_k \in V(G_1) \\ d(v_l,\ v_k) \neq d(v_k,\ v_m)}} \frac{d_{G_1}(v_l,\ v_k) + d_{G_1}(v_k,\ v_m) + 1}{2} \right.$$
$$\left. + \sum_{\substack{v_k \in V(G_1) \\ d(v_l,\ v_k) = d(v_k,\ v_m)}} 1 + \sum_{\substack{v_k \in V(G_1) \\ d(v_l,\ v_k) = d(v_k,\ v_m)}} \frac{d_{G_1}(v_l,\ v_k) + d_{G_1}(v_k,\ v_m)}{2} \right\}$$
$$((n_2 + 1) 2m_2 d_{G_1}(v_k) + 2n_2 m_2 + 2(n_2 + 1) d_{G_1}(v_k) + 2n_2^2)$$
$$= \sum_{e_i = v_l v_m \in E(G_1)} \left\{ \sum_{v_k \in V(G_1)} \frac{d_{G_1}(v_l,\ v_k) + d_{G_1}(v_k,\ v_m)}{2} (2(n_2^2 + m_2 n_2)) \right.$$
$$+ \left\{ \sum_{\substack{v_k \in V(G_1) \\ d(v_l,\ v_k) \neq d(v_k,\ v_m)}} \frac{1}{2} + \sum_{\substack{v_k \in V(G_1) \\ d(v_l,\ v_k) = d(v_k,\ v_m)}} 1 \right\} (2(n_2^2 + m_2 n_2))$$
$$+ \sum_{v_k \in V(G_1)} \frac{d_{G_1}(v_l,\ v_k) + d_{G_1}(v_k,\ v_m)}{2} (2(n_2 + 1) d_{G_1}(v_k)(m_2 + n_2))$$
$$\left. + \left\{ \sum_{\substack{v_k \in V(G_1) \\ d(v_l,\ v_k) \neq d(v_k,\ v_m)}} \frac{1}{2} + \sum_{\substack{v_k \in V(G_1) \\ d(v_l,\ v_k) = d(v_k,\ v_m)}} 1 \right\} (2(n_2 + 1) d_{G_1}(v_k)(m_2 + n_2)) \right\}$$

$$= \sum_{e_i=v_lv_m\in E(G_1)} \left\{ \sum_{v_k\in V(G_1)} \frac{d_{G_1}(v_l,\ v_k)+d_{G_1}(v_k,\ v_m)}{2}(2(n_2^2+m_2n_2)) \right.$$
$$+\frac{1}{2}\sum_{e_i=v_lv_m\in E(G_1)} (2n_1-(n_{e_i}(v_l/G_1)+n_{e_i}(v_m/G_1))(2(n_2^2+m_2n_2)))$$
$$+\frac{1}{2}\sum_{e_i=v_lv_m\in E(G_1)}\sum_{v_k\in V(G_1)} (d_{G_1}(v_l,\ v_k)+d_{G_1}(v_k,\ v_m))(2(n_2+1)d_{G_1}(v_k)(m_2+n_2))$$
$$+\frac{1}{2}\sum_{e_i\in E(G_1)}\sum_{\substack{v_k\in V(G_1)\\ d(v_l,\ v_k)\neq d(v_k,\ v_m)}} (2(n_2+1)d_{G_1}(v_k)(m_2+n_2))$$
$$\left. +\sum_{e_i\in E(G_1)}\sum_{\substack{v_k\in V(G_1)\\ d(v_l,\ v_k)= d(v_k,\ v_m)}} (2(n_2+1)d_{G_1}(v_k)(m_2+n_2)) \right\}$$

Using (2.9) in above equation, we get

$$= (m_2n_2+n_2^2)(DD(G_1)+2n_1m_1-PI(G_1))+(n_2+1)(m_2+n_2)(2Gut(G_1)+2m_1^2)$$
$$+(n_2+1)(m_2+n_2)C(G_1), \tag{3.16}$$

$$A_7 = \sum_{v_i\in V(G_1)}\sum_{\substack{w_{ij}\in V_{i_v}(G_2)\\ v_k\in V(G_1)}} d_G(w_{ij})\ d_G(v_k)d_G(w_{ij},\ v_k)$$
$$= \sum_{v_i\in V(G_1)}\sum_{\substack{w_{ij}\in V_{i_v}(G_2)\\ v_k\in V(G_1)}} (d_{G_2}(w_{ij})+1)((n_2+1)d_{G_1}(v_k)+n_2)d_G(w_{ij},\ v_k)$$
$$= \sum_{v_i\in V(G_1)}\sum_{\substack{w_j\in V(G_2)\\ v_k\in V(G_1)}} ((d_{G_2}(w_j)d_{G_1}(v_k)(n_2+1)+n_2d_{G_2}(w_j)+(n_2+1)d_{G_1}(v_k)+n_2)$$
$$(d_{G_1}(v_i,\ v_k)+1)$$
$$= \sum_{v_i\in V(G_1)}\sum_{v_k\in V(G_1)} (2m_2d_{G_1}(v_k)(n_2+1)+n_22m_2+(n_2+1)n_2d_{G_1}(v_k)+n_2^2)$$
$$(d_{G_1}(v_i,\ v_k)+1)$$
$$= (2m_2(n_2+1)+(n_2+1)n_2)\sum_{v_i\in V(G_1)}\sum_{v_k\in V(G_1)} d_{G_1}(v_k)d_{G_1}(v_i,\ v_k)$$
$$+\sum_{v_i\in V(G_1)}\sum_{v_k\in V(G_1)} n_2(n_2+2m_2)+n_2(n_2+2m_2)\sum_{v_i\in V(G_1)}\sum_{v_k\in V(G_1)} d_{G_1}(v_i,\ v_k)$$
$$+(2m_2(n_2+1)+(n_2+1)n_2)\sum_{v_i\in V(G_1)}\sum_{v_k\in V(G_1)} d_{G_1}(v_k)$$
$$= (DD(G_1)+2m_1n_1)(n_2+1)(2m_2+n_2)+n_2(n_2+2m_2)(2W(G_1)+n_1^2), \tag{3.17}$$

$$
\begin{aligned}
A_8 &= \sum_{e_i=v_lv_m\in E(G_1)} \sum_{\substack{u_{ij}\in V_{i_e}(G_2)\\ w_{km}\in V_{k_v}(G_2)}} d_G(u_{ij})d_G(w_{km})d_G(u_{ij},\ w_{km})\\
&= \sum_{e_i=v_lv_m\in E(G_1)} \sum_{\substack{u_{ij}\in V_{i_e}(G_2)\\ w_{km}\in V_{k_v}(G_2)}} (d_{G_2}(u_{ij})+2)(d_{G_2}(w_{km})+1)d_G(u_{ij},\ w_{km})\\
&= \sum_{e_i=v_lv_m\in E(G_1)} \sum_{\substack{u_{ij}\in V_{i_e}(G_2)\\ w_{km}\in V_{k_v}(G_2)}} (d_{G_2}(u_{ij})d_{G_2}(w_{km})+d_{G_2}(u_{ij})+2d_{G_2}(w_{km})+2)\\
&\quad d_G(u_{ij},\ w_{km})\\
&= \sum_{e_i=v_lv_m\in E(G_1)} \left\{ \sum_{\substack{v_k\in V(G_1)\\ d(v_l,\ v_k)\neq d(v_k,\ v_m)}} \frac{d_{G_1}(v_l,\ v_k)+d_{G_1}(v_k,\ v_m)+3}{2} \right.\\
&\quad \left. + \sum_{\substack{v_k\in V(G_1)\\ d(v_l,\ v_k)= d(v_k,\ v_m)}} \left\{\frac{d_{G_1}(v_l,\ v_k)+d_{G_1}(v_k,\ v_m)}{2}+2\right\}\right\}(4m_2^2+6m_2n_2+2n_2^2)\\
&= \sum_{e_i=v_lv_m\in E(G_1)} \frac{1}{2}\sum_{v_k\in V(G_1)} d_{G_1}(v_l,\ v_k)+d_{G_1}(v_k,\ v_m)(4m_2^2+6m_2n_2+2n_2^2)\\
&\quad + \sum_{e_i\in E(G_2)} (4m_2^2+6m_2n_2+2n_2^2)\left\{ \sum_{\substack{v_k\in V(G_1)\\ d(v_l,\ v_k)\neq d(v_k,\ v_m)}} \frac{3}{2} + \sum_{\substack{v_k\in V(G_1)\\ d(v_l,\ v_k)= d(v_k,\ v_m)}} 2\right\}
\end{aligned}
$$

Using (2.9) in above equation, we get

$$
\begin{aligned}
&= \frac{4m_2^2+6m_2n_2+2n_2^2}{2}(DD(G_1)+4n_1m_1-PI(G_1))\\
&= (2m_2^2+3m_2n_2+n_2^2)(DD(G_1)+4n_1m_1-PI(G_1)). \qquad (3.18)
\end{aligned}
$$

Substituting (3.11), (3.12), (3.13), (3.14), (3.15), (3.17), (**??**) and (3.18) in (3.10), we obtain the required result. □

COROLLARY 3.2. *Let G_1 be a bipartite graph. Then,*

$$
\begin{aligned}
Gut(G) &= Gut(G_1)(n_2(3n_2+2m_2+4)+2m_2+1)-M_2(G_2)(m_1+n_1)\\
&\quad -2n_1m_1(n_2(n_2+2m_2)+m_2^2)+(n_2^2+m_2^2)(8n_1m_1+2m_1)\\
&\quad +2DD(G_1)(n_2(2n_2+3m_2+1)+m_2(m_2+1))-M_1(G_2)(2n_1+3m_1)\\
&\quad +(W(G_1)+W_e(G_1))(4n_2(n_2+2m_2)+4m_2^2)+n_1(n_2-5m_2)\\
&\quad +m_1^2(4n_2^2+6n_2m_2+2m_2^2+2n_2+2m_2)+(2n_2^2+6n_2m_2+4m_2^2)n_1^2\\
&\quad +4m_1m_2(n_1+n_2-3)+2n_2m_1(9n_1m_2+n_1-2).
\end{aligned}
$$

It is well-known that

$W(P_n) = \dfrac{n(n^2-1)}{6}$, $W(C_n) = \dfrac{n^3}{8}$, $W(C_{2n+1}) = \dfrac{n(n^2-1)}{8}$, $W(K_n) = \dfrac{n(n-1)}{2}$,
$W_e(K_n) = \dfrac{n(n-1)(n^2-3n+2)}{4}$,
$DD(P_n) = \dfrac{n(n-1)(2n-1)}{3}$, $DD(C_n) = \dfrac{n^3}{2}$, $DD(C_{2n+1}) = \dfrac{n(n^2-1)}{2}$,
$DD(K_n) = n(n-1)^2$,
$PI(P_n) = n(n-1)$, $PI(C_{2n+1}) = 2n(2n+1)$, $PI(K_n) = n(n-1)$.
$C(C_{2n+1}) = 4n+2$, $C(K_n) = n(n-1)^2(n-2)/2$.
$Gut(P_n) = (n-1)(2n^2-4n+3)/3$, $Gut(C_n) = n^3/2$, $Gut(C_{n+1}) = n(n^2-1)/2$.
Using these facts, in Theorems 2.1, 3.1 and 3.3, we obtain the following corollaries.

Corollary 3.3. *We have*

a. $W(K_n \boxdot P_m) = (3/4)m^2n^3 - nm + (1/2)n^3 + (1/8)n^2m^2 + (3/8)m^2n^4 + n^2 - (1/2)n^2m - (1/4)m^2n.$

b. $W(K_n \boxdot C_m) = (1/8)n^2m^2 - (1/4)m^2n - (1/2)n^2m - nm + (1/2)n^2 - (1/2)n + (3/8)m^2n^4 + (3/4)m^2n^3 + (1/2)mn^3.$

c. $W(K_n \boxdot K_m) = (3/4)m^2n^3 - (1/4)nm + (1/2)mn^3 - (1/8)n^2m^2 + (3/8)m^2n^4 - (1/2)n + (1/2)n^2 + (1/4)n^2m - (1/2)m^2n.$

d. $DD(K_n \boxdot P_m) = 5m^2n^2 - (5/2)m^2n - 13mn^2 + (19/2)nm - 6n - n^3 + (1/2)m^2n^3 + 4m^2n^4 + 3mn^3 - (1/2)mn^4 + 5n^2.$

e. $DD(K_n \boxdot C_m) = 5m^2n^2 - (5/2)m^2n - (21/2)mn^2 + (17/2)nm + n + n^3 + (1/2)m^2n^3 + 4m^2n^4 + 4mn^3 + mn^4 - 2n^2.$

f. $DD(K_n \boxdot K_m) = (1/4)m^2n^2 + 5m^2n - 3mn^2 - 5nm + n + (3/4)n^4m^3 + (3/4)m^3n^2 - m^3n + (1/2)m^3n^3 + n^3 + (7/4)m^2n^4 + mn^3 + mn^4 - 2n^2.$

g. $Gut(K_n \boxdot P_m) = 4nm + (9/2)n - (17/2)m^2n^3 - 3mn^4 + (19/2)m^2n^2 + 3mn^3 - (7/2)m^2n - 22mn^2 + (21/2)m^2n^4 - (13/2)n^3 + 23n^2.$

h. $Gut(K_n \boxdot C_m) = 5mn^4 + (21/2)m^2n^2 - 8mn^3 - (9/2)m^2n - 12mn^2 + (21/2)m^2n^4 - (3/2)n^3 - nm - (1/2)n - (9/2)m^2n^3 + (3/2)n^2 + (1/2)n^4.$

i. $Gut(K_n \boxdot K_m) = (5/4)nm - (1/2)n + (3/8)m^4n^4 + (7/4)m^3n^4 + 2m^3n^2 - (1/4)m^3n - (5/8)m^4n^2 - (1/2)m^4n - m^3n^3 + (1/4)m^4n^3 - (19/4)m^2n^3 + 2mn^4 + (7/8)m^2n^2 - 5mn^3 + (31/4)mn^2 + (23/8)m^2n^4 - (3/2)n^3 - (5/2)n^2 + (1/2)n^4.$

By varying G_1 and G_2 we can obtain many corollaries for standard graphs.

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