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SPECTRA OF GRAPH OPERATIONS BASED ON CORONA AND NEIGHBORHOOD CORONA OF GRAPH G AND K_1

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Abstract. The corona $G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining each i-th vertex of G to every vertex in the i-th copy of H. The neighborhood corona $G \star H$ of two graphs G and H is the graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining the neighbors of the i-th vertex of G to every vertex in the i-th copy of H. In this paper we compute the adjacency spectrum (respectively, Laplacian spectrum, signless Laplacian) of four types of graph operations on G and H, called as N- vertex corona, C-vertex neighborhood corona, C-edge corona and N-edge corona, based on the corona and neighborhood corona of G and K_1. As an application, our results enable us to construct infinitely many pairs of cospectral graphs and also integral graphs.

1. Introduction

All graphs considered in this paper are simple graphs. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set E(G). The adjacency matrix of G, denoted by A(G), is the $n \times n$ matrix $[a_{ij}]$, where $a_{ij} = 1$, if the vertices v_i and v_j are adjacent in G, otherwise $a_{ij} = 0$. If all the eigenvalues of A(G) are integers then the graph G is said to be an integral graph [**10**]. The Laplacian matrix of the graph G, denoted by L(G), is defined as D(G) - A(G), where D(G) is the diagonal degree matrix of G. The signless Laplacian matrix of the graph G, denoted by Q(G) is defined as D(G)+A(G). The adjacency spectrum $\sigma(G)$, Laplacian spectrum $\mu(G)$,

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and signless Laplacian spectrum $\gamma(G)$ are defined as follows:

$$\sigma(G) = (\lambda_1(G), \lambda_2(G), \cdots, \lambda_n(G)),$$
$$\mu(G) = (\mu_1(G), \mu_2(G), \cdots, \mu_n(G)),$$
$$\gamma(G) = (\gamma_1(G), \gamma_2(G), \cdots, \gamma_n(G)),$$

where $\lambda_i(G)$, $\mu_i(G)$ and $\gamma_i(G)$ are the eigenvalues of A(G), L(G) and Q(G), respectively. Also

$$\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G),$$
$$\mu_1(G) = 0 \leqslant \mu_2(G) \leqslant \cdots \leqslant \mu_n(G),$$

and

$$\gamma_1(G) \geqslant \gamma_2(G) \geqslant \cdots \geqslant \gamma_n(G).$$

Studies on different spectra of graphs can be found in [**1, 5, 6, 7, 8, 15, 19**] and therein references. Two graphs are said to be adjacency cospectral (Laplacian cospectral, signless Laplacian cospectral, respectively) if they have the same adjacency spectrum (Laplacian spectrum, signless Laplacian spectrum, respectively).

The corona $G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining each i-th vertex of G to every vertex in the i-th copy of H. The corona of two graphs was first introduced by Frucht and Harary in [**9**]. The neighborhood corona $G \star H$ of two graphs G and H is the graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining the neighbors of the i-th vertex of G to every vertex in the i-th copy of H. The neighborhood corona of two graphs was introduced by Indulal [**12**]. More information about the corona and neighborhood corona can be found in [**3, 4, 12, 17**]. Several graph operations such as disjoint union, NEPS, corona, edge corona, neighborhood corona, subdivision vertex corona, subdivision edge corona, subdivision neighborhood corona, etc., have been introduced and their spectrum were studied by various Mathematicians. Details about the spectra of some new graph operations can be found in [**3, 11, 12, 16, 17, 18**]. Recently, Lan and Zhou [**14**] have introduced four new graph operations called as R-vertex corona, R-edge corona, R-vertex neighborhood corona and R-edge neighborhood corona and provided a complete information about the spectra of these four graph operations.

Motivated by the above works, we define four new graph operations called as N-vertex corona, C-vertex neighborhood corona, C-edge corona and N-edge corona, based on the corona and neighborhood corona of a graph G and K_1. Further we compute their spectrum in some cases. The paper is organized as follows: In Section 3, we give the adjacency spectra (respectively, Laplacian spectra, signless Laplacian spectra) of N- vertex corona and C-vertex neighborhood corona of two graphs G and H. In Section 4, we give the adjacency spectra (respectively, Laplacian spectra, signless Laplacian spectra) of C-edge corona and N-edge corona of two graphs G and H. In Section 5, using the results obtained in Sections 3 and 5 we give methods to construct infinitely many pairs of cospectral graphs and also integral graphs.

2. Preliminaries

In this section, we give some definitions and lemmas which are useful to prove our main results.

Let G_1 and G_2 be two graphs on n and m vertices respectively. Let $N(G_1) = G_1 \star K_1$, $C(G_1) = G_1 \circ K_1$. We define four new graph operations on G_1 and G_2 as follows:

Definition 2.1. The N-vertex corona $G_1 \circledast G_2$ of two graphs G_1 and G_2 is the graph obtained by taking one copy of $N(G_1)$, $|V(G_1)|$ copies of G_2 and joining each i-th vertex of G_1 to every vertex of the i-th copy of G_2.

Definition 2.2. The C-vertex neighborhood corona $G_1 \odot G_2$ of two graphs G_1 and G_2 is the graph obtained by taking one copy of $C(G_1)$, $|V(G_1)|$ copies of G_2 and joining each neighbors of the i-th vertex of G_1 to every vertex of i-th copy of G_2.

Definition 2.3. The N-edge corona $G_1 \boxtimes G_2$ of two graphs G_1 and G_2 is the graph obtained by taking one copy of $N(G_1)$, $|E(G_1)|$ copies of G_2 and joining each terminal vertex of i-th edge of G_1 to every vertex of the i-th copy of G_2.

Definition 2.4. The C-edge corona $G_1 \boxdot G_2$ of two graphs G_1 and G_2 is the graph obtained by taking one copy of $C(G_1)$, $|E(G_1)|$ copies of G_2 and joining each terminal vertex of i-th edge of G_1 to every vertex of i-th copy of G_2.

Let $A = (a_{ij})$ be a $n \times m$ matrix, $B = (b_{ij})$ be a $p \times q$ matrix then the Kronecker product [**7**] $A \otimes B$ of A and B is the np by mq matrix obtained by replacing each entry a_{ij} of A by $a_{ij}B$. It is well-known that $(A \otimes B)(C \otimes D) = AC \otimes BD$, whenever the products AC and BD exist.

The M-coronal $\Gamma_M(x)$ of a square matrix M of order n [**4, 18**] is defined as follows:

$$\Gamma_M(x) = e^T(xI_n - M)^{-1}e,$$

where e is the column vector of size n with all its entries are 1. If M is a square matrix of order n such that sum of entries in each row a constant 'r' then it is easy to see that $\Gamma_M(x) = n/(x-r)$. Further for a complete bipartite graph $K_{p,q}$ we have [**4**]

$$\Gamma_{A(K_{p,q})}(x) = \frac{(p+q)x + 2pq}{x^2 - pq}.$$

Lemma 2.1. [**7**] *If M, N, P, Q are matrices with M being a non-singular matrix, then*

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |M||Q - PM^{-1}N|.$$

3. Spectra of N-vertex corona and C-vertex neighborhood corona

In this section, we determine the characteristic polynomial of N- vertex corona and C-vertex neighborhood corona of two graphs G_1 and G_2 in terms of coronal of a matrix. Also we compute the adjacency spectrum (respectively, Laplacian spectrum, signless Laplacian spectrum) of C- edge corona and N-edge corona of two graphs G_1 and G_2 in some cases.

THEOREM 3.1. *Let G_1 and G_2 be two graphs on n and m vertices respectively. Then*

$$f(A(G_1 \circledast G_2),\ x) = \prod_{i=1}^{m}(x-\lambda_i(G_2))^n \prod_{i=1}^{n}(x-\Gamma_{A(G_2)}(x)-\lambda_i(G_1))x-\lambda_i^2(G_1).$$

PROOF. With suitable labelling of the vertices of G, the adjacency matrix $A(G_1 \circledast G_2)$ can be formulated as follows:

$$A(G_1 \circledast G_2) = \begin{pmatrix} I_n \otimes A(G_2) & 0 & I_n \otimes e \\ 0 & 0 & A(G_1) \\ I_n \otimes e^T & A(G_1) & A(G_1) \end{pmatrix},$$

where e is the column vector of size m, with all its entries are 1, I_n is the identity matrix of order n. Now,

$$f(A(G_1 \circledast G_2),\ x) = det \begin{pmatrix} I_n \otimes (xI_m - A(G_2)) & 0 & -I_n \otimes e \\ 0 & xI_n & -A(G_1) \\ -I_n \otimes e^T & -A(G_1) & xI_n - A(G_1) \end{pmatrix}.$$

By Lemma 2.1, we have

$$f(A(G_1 \circledast G_2),\ x) = \prod_{i=1}^{m}(x-\lambda_i(G_2))^n \ detS, \tag{3.1}$$

where

$$S = \begin{pmatrix} xI_n & -A(G_1) \\ -A(G_1) & (x-\Gamma_{A(G_2)}(x))I_n - A(G_1) \end{pmatrix}.$$

Again using Lemma 2.1, we see that

$$\begin{aligned} detS &= x^n det((x-\Gamma_{A(G_2)}(x))I_n - A(G_1) - A^2(G_1)/x) \\ &= \prod_{i=1}^{n}(x-\Gamma_{A(G_2)}(x)-\lambda_i(G_1))x-\lambda_i^2(G_1). \end{aligned} \tag{3.2}$$

So by (3.1) and (3.2) the result follows. $\square$

As $\Gamma_M(x) = \dfrac{n}{(x-r)}$, where M is the square matrix of order n with each of its row sum a constant 'r' and $\Gamma_{K_{p,q}}(x) = \dfrac{(p+q)x+2pq}{x^2-pq}$, proofs of the following two corollaries follows immediately by the above theorem.

Corollary 3.1. *Let G_1 be an arbitrary graph and G_2 be a r-regular graph on n and m vertices respectively. Then the adjacency spectrum of $G = (G_1 \circledast G_2)$ is given by:*

a. $\lambda_i(G_2)$, *with multiplicity n for* $i = 2, \cdots, m$.
b. three roots of the polynomial
$x^3 - (r + \lambda_i(G_1))x^2 + (r\lambda_i(G_1) - m - \lambda_i^2(G_1))x + r\lambda_i^2(G_1)$, *for* $i = 1, \cdots, n$.

Corollary 3.2. *Let G_1 be an arbitrary graph on n vertices. Then the adjacency spectrum of $G_1 \circledast K_{p,q}$ is given by:*

(a) 0 *with multiplicity* $n(p+q-2)$.
(b) four roots of the polynomial
$x^4 - \lambda_i(G_1)x^3 - \left(\lambda_i^2(G_1) + pq + p + q\right)x^2 + \left(\lambda_i(G_1) - 2\right)pqx + \lambda_i^2(G_1)pq$,
for $i = 1, \cdots, n$.

Theorem 3.2. *Let G_1 be r_1-regular on n vertices and G_2 be an arbitrary graph on m vertices. Then the Laplacian spectrum of $G_1 \circledast G_2$ is given by:*

a. $\mu_i(G_2) + 1$ *with multiplicity n, for* $i = 2, \cdots, m$.
b. three roots of the polynomial
$x^3 - (\mu_i(G_1) + m + 2\,r_1 + 1)\,x^2 - \left(\mu_i(G_1)^2 - 3\,\mu_i(G_1)r_1 - mr_1 - \mu_i(G_1) - 2\,r_1\right)x$
$+ \mu_i(G_1)^2 - 3\,\mu_i(G_1)r_1$, *for* $i = 1, \cdots, n$.

Proof. With suitable labelling of the vertices of G, the adjacency matrix $L(G_1 \circledast G_2)$ can be formulated as follows:

$$L(G_1 \circledast G_2) = \begin{pmatrix} I_n \otimes (I_m + L(G_2)) & 0 & -I_n \otimes e \\ 0 & r_1 I_n & -A(G_1) \\ -I_n \otimes e^T & -A(G_1) & (r_1+m)I_n + L(G_1) \end{pmatrix},$$

where e is the column vector of size m with all its entries are 1, I_n is the identity matrix of order n. Now,

$$f(L(G_1 \circledast G_2)) = det \begin{pmatrix} I_n \otimes ((x-1)I_m - L(G_2)) & 0 & I_n \otimes e \\ 0 & (x-r_1)I_n & A(G_1) \\ I_n \otimes e^T & A(G_1) & (x-r_1-m)I_n - L(G_1) \end{pmatrix}.$$

By Lemma 2.1, it follows that

$$f(L(G_1 \circledast G_2)) = \prod_{i=1}^{m}(x - \mu_i(G_2) - 1)^n \; detS, \tag{3.3}$$

where

$$S = \begin{pmatrix} (x-r_1)I_n & A(G_1) \\ A(G_1) & (x-\Gamma_{L(G_2)}(x-1)-r_1-m)I_n - L(G_1) \end{pmatrix}.$$

Again using Lemma 2.1, we see that

$$\begin{aligned} detS &= (x-r_1)^n det((x-\Gamma_{L(G_2)}(x-1)-r_1-m)I_n - L(G_1) - A^2(G_1)/(x-r_1)) \\ &= \prod_{i=1}^{n}(x-m/(x-1)-r_1-m-\mu_i(G_1))(x-r_1)-(\mu_i(G_1)-r_1)^2. \end{aligned} \tag{3.4}$$

So, by (3.3) and (3.4) the desired result follows. □

Let t(G) denote the number of spanning trees of G. It is well known [**7**] that for a connected graph G on n vertices, t(G) is given by

$$t(G) = \frac{\mu_2(G)\cdots\mu_n(G)}{n}. \tag{3.5}$$

Corollary 3.3. *Let G_1 be r_1-regular graph on n vertices and G_2 be an arbitrary graph on m vertices. Then the number of spanning trees of $G_1 \circledast G_2$ is given by*

$$t(G_1 \circledast G_2) = r_1 t(G_1)\prod_{i=2}^{n}(3r_1-\mu_i(G_1))\prod_{i=2}^{m}(\mu_i(G_2)+1)^n.$$

Proof. Proof follows directly from the above theorem and (3.5). □

Theorem 3.3. *Let G_1 be r_1-regular on n vertices and G_2 be r_2-regular graph on m vertices. Then the signless Laplacian spectrum of $G_1 \circledast G_2$ is given by:*
a. $\gamma_i(G_2)+1$ with multiplicity n, for $i = 2, \cdots, m$.
b. three roots of the polynomial

$$(x-2r_2-1)(x-r_1)(x-r_1-m-\gamma_i(G_1))-m(x-r_1)-(\gamma_i(G_1)-r_1)^2(x-2r_2-1),$$

for $i = 1, \cdots, n$.

Proof. With suitable labelling of the vertices of G, the adjacency matrix $Q(G_1 \circledast G_2)$ can be formulated as follows:

$$Q(G_1 \circledast G_2) = \begin{pmatrix} I_n \otimes (I_m + Q(G_2)) & 0 & I_n \otimes e \\ 0 & r_1 I_n & A(G_1) \\ I_n \otimes e^T & A(G_1) & (r_1+m)I_n + Q(G_1) \end{pmatrix},$$

where e is the column vector of size m with all its entries are 1, I_n is the identity matrix of order n. Rest of the proof is similar to the proof of Theorem 3.2. □

As the proofs of the following theorems are similar to that of Theorem 3.1, we omit the details.

Theorem 3.4. *Let G_1 and G_2 be two graphs on n and m vertices. Then*

$$f(A(G_1 \odot G_2), x) = \prod_{i=1}^{m}(x - \lambda_i(G_2))^n \prod_{i=1}^{n}(x^2 - (\lambda_i(G_1) + \lambda_i^2(G_1)\Gamma_{A(G_2)}(x))x - 1).$$

Using the fact that $\Gamma_M(x) = \dfrac{n}{(x-r)}$, where M is the square matrix of order n with each of its row sum a constant 'r' and $\Gamma_{K_{p,q}}(x) = \dfrac{(p+q)x + 2pq}{x^2 - pq}$, in the above theorem we have the following two corollaries.

Corollary 3.4. *Let G_1 be an arbitrary graph and G_2 be a r-regular graph, on n and m vertices, respectively. Then the adjacency spectrum of $G_1 \odot G_2$ is given by:*

(a) $\lambda_i(G_2)$ with multiplicity n, for $i = 2, \cdots, m$.

(b) three roots of the polynomial
$x^3 - (\lambda_i(G_1) + r)x^2 - (\lambda_i^2(G_1)m - \lambda_i(G_1)r + 1))x + r$, *for $i = 1, \cdots, n$.*

Corollary 3.5. *Let G_1 be an arbitrary graph on n vertices. Then the adjacency spectrum of $G_1 \odot K_{p,q}$ is given by:*

(a) 0 with multiplicity $n(p+q-2)$.

(b) four roots of the polynomial
$x^4 - \lambda_i(G_1)x^3 - \left(\lambda_i(G_1)^2 p + \lambda_i(G_1)^2 q + pq + 1\right)x^2 + \left(-2\,\lambda_i(G_1)^2 + \lambda_i(G_1)\right)pqx + pq$, *for $i = 1, \cdots, n$.*

Theorem 3.5. *Let G_1 be r_1-regular and G_2 be an arbitrary graph on n and m vertices respectively. Then the Laplacian spectrum of $G_1 \odot G_2$ is given by:*

a. $\mu_i(G_2) + r_1$ with multiplicity n, for $i = 2, \cdots, m$.

b. three roots of the polynomial
$x^3 - (mr_1 + \lambda_i(G_1) + r_1 + 2)\,x^2 + ((-\lambda_i(G_1)m + 2\,mr_1 + r_1 + 1)\lambda_i(G_1) + mr_1 + 2\,r_1)\,x + (\lambda_i(G_1)m - 2\,mr_1 - r_1)\lambda_i(G_1)$, *for $i = 1, \cdots, n$.*

By the above theorem and (3.5), we have the following corollary:

Corollary 3.6. *Let G_1 be a r_1-regular graph and G_2 be an arbitrary graph on n and m vertices, respectively. Then the number of spanning trees of $G_1 \odot G_2$ is given by:*

$$t(G_1 \odot G_2) = r_1 t(G_1)\prod_{i=2}^{n}(2mr_1 - \mu_i(G_1) + r_1)\prod_{i=2}^{m}(\mu_i(G_2) + r_1)^n.$$

Theorem 3.6. *Let G_1 be r_1-regular and G_2 be r_2-regular graph on n and m vertices, respectively. Then the signless Laplacian spectrum of $G_1 \odot G_2$ is given by:*

a. $\gamma_i(G_2) + r_1$ with multiplicity n, for $i = 2, \cdots, m$.

b. three roots of the polynomial
$x^3 - (mr_1 + \gamma_i(G_1) + 2\,r_2 + r_1 + 2)\,x^2 + ((-\gamma_i(G_1)m + 2\,mr_1 + 2\,r_2 + r_1 + 1)\gamma_i(G_1) + mr_1 + 4\,r_2 + 2\,r_1 + 2\,r_1 m r_2)\,x + \gamma_i(G_1)^2 m - 2\,\gamma_i(G_1)mr_1 - 2\,r_1 m r_2 - 2\,\gamma_i(G_1)r_2 - \gamma_i(G_1)r_1$, *for $i = 1, \cdots, n$.*

4. Spectra of C-edge corona and N-edge corona

In this section, we determine the characteristic polynomial of C- edge corona and N-edge corona of two graphs G_1 and G_2 in terms of coronal of a matrix. Also we compute the adjacency spectrum (respectively, Laplacian spectrum, signless Laplacian spectrum) of C- edge corona and N-edge corona of two graphs G_1 and G_2 in some cases.

THEOREM 4.1. *Let G_i $(i = 1, 2)$ be r_i- regular graphs with n_i vertices and m_i edges, respectively. Then*

$$f(A(G_1 \boxdot G_2), x) = \prod_{i=1}^{n_2}(x - \lambda_i(G_2))^{m_1} \prod_{i=1}^{n_1}(x^2 - (\lambda_i(G_1) + (\lambda_i(G_1) + r_1)\Gamma_{A(G_2)}(x))x - 1).$$

PROOF. With suitable labelling of the vertices of $G_1 \boxdot G_2$, the adjacency matrix $A(G_1 \boxdot G_2)$ can be formulated as follows:

$$A(G_1 \boxdot G_2) = \begin{pmatrix} I_{m_1} \otimes A(G_2) & 0 & B \otimes e \\ 0 & 0 & I_{n_1} \\ B^T \otimes e^T & I_{n_1} & A(G_1) \end{pmatrix},$$

where e is the column vector of size n_2 with all its entries are 1, I_{n_1} is the identity matrix of order n_1 and B is the incidence matrix of G_1. Now,

$$f(A(G_1 \boxdot G_2)) = det \begin{pmatrix} I_{m_1} \otimes (xI_{n_2} - A(G_2)) & 0 & -B \otimes e \\ 0 & xI_{n_1} & -I_{n_1} \\ -B^T \otimes e^T & -I_{n_1} & xI_{n_1} - A(G_1) \end{pmatrix}.$$

By Lemma 2.1, we have

$$f(A(G_1 \boxdot G_2)) = \prod_{i=1}^{n_2}(x - \lambda_i(G_2))^{m_1}\ detS, \tag{4.1}$$

where

$$S = \begin{pmatrix} xI_{n_1} & -I_{n_1} \\ -I_{n_1} & xI_{n_1} - A(G_1) - (A(G_1) + r_1 I_{n_1})\Gamma_{A(G_2)}(x) \end{pmatrix}.$$

Again employing Lemma 2.1 to detS, we see that

$$\begin{aligned} detS &= x^{n_1} det(xI_{n_1} - A(G_1) - (A(G_1) + r_1 I_{n_1})\Gamma_{A(G_2)}(x) - I_{n_1}/x) \\ &= \prod_{i=1}^{n_1}(x^2 - (\lambda_i(G_1) + (\lambda_i(G_1) + r_1)\Gamma_{A(G_2)}(x))x - 1). \qquad (4.2) \end{aligned}$$

So by (4.1) and (4.2) the result follows. □

Using the fact that $\Gamma_M(x) = \dfrac{n}{(x-r)}$, where M is the square matrix of order n with each of its row sum a constant 'r' and $\Gamma_{K_{p,q}}(x) = \dfrac{(p+q)x+2pq}{x^2-pq}$, in the above theorem we have the following two corollaries.

Corollary 4.1. *Let G_i $(i=1,2)$ be r_i-regular graphs with n_i vertices and m_i edges, respectively. Then the adjacency spectrum of $G_1 \boxdot G_2$ is given by:*

(a) $\lambda_i(G_2)$ with multiplicity m_1, for $i = 2, \cdots, n_2$.
(b) r_2 with multiplicity $m_1 - n_1$.
(c) three roots of the polynomial
$x^3 - (\lambda_i(G_1) + r_2)\, x^2 + (\lambda_i(G_1)r_2 - n_2\lambda_i(G_1) - r_1 n_2 - 1)\, x + r_2,$
for $i = 1, \cdots, n_1$.

Corollary 4.2. *Let G_1 be r_1-regular graph $(r_1 \geqslant 2)$ with n_1 vertices and m_1 edges. Then the adjacency spectrum of $G_1 \boxdot K_{p,q}$ is given by:*

(a) 0 with multiplicity $m_1(p+q-2)$.
(b) $\pm\sqrt{pq}$ with multiplicity $m_1 - n_1$.
(c) four roots of the polynomial
$x^4 - \lambda_i(G_1)x^3 - (\lambda_i(G_1)p + \lambda_i(G_1)q + pq + pr_1 + qr_1 + 1)\, x^2 - pq\,(\lambda_i(G_1) + 2\,r_1)\, x + pq,$
for $i = 1, \cdots, n_1$.

Theorem 4.2. *Let G_1 be r_1- regular graph $(r_1 \geqslant 2)$ and G_2 be an arbitrary graph with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then the Laplacian spectrum of $G_1 \boxdot G_2$ is given by:*

a. $\mu_i(G_2) + 2$ with multiplicity m_1, for $i = 2, \cdots, n_2$.
b. 2 with multiplicity $m_1 - n_1$.
c. three roots of the polynomial
$(x-1)((x-2)(x - n_2 r_1 - \mu_i(G_1) - 1) + (\mu_i(G_1) - 2r_1)n_2) - (x-2),$
for $i = 1, \cdots, n_1$.

Proof. With suitable labelling of the vertices of $G_1 \boxdot G_2$, the adjacency matrix $L(G_1 \boxdot G_2)$ can be formulated as follows:

$$L(G_1 \boxdot G_2) = \begin{pmatrix} I_{m_1} \otimes (2I_{n_2} + L(G_2)) & 0 & -B \otimes e \\ 0 & I_{n_1} & -I_{n_1} \\ -B^T \otimes e^T & -I_{n_1} & (mr_1+1)I_{n_1} + L(G_1) \end{pmatrix},$$

where e is the column vector of size n_2 with all its entries are 1, I_{n_1} is the identity matrix of order n_1 and B is the incidence matrix of G_1. Now,

$$f(L(G_1 \boxdot G_2)) = det \begin{pmatrix} I_{m_1} \otimes ((x-2)I_{n_2} - L(G_2)) & 0 & B \otimes e \\ 0 & (x-1)I_{n_1} & I_{n_1} \\ B^T \otimes e^T & I_{n_1} & (x - mr_1 - 1)I_{n_1} - L(G_1) \end{pmatrix}.$$

By Lemma 2.1, we have

$$f(L(G_1 \boxdot G_2)) = \prod_{i=1}^{n_2}(x-\mu_i(G_2)-2)^{m_1}\ detS, \tag{4.3}$$

where

$$S = \begin{pmatrix} (x-1)I_{n_1} & I_{n_1} \\ I_{n_1} & (x-mr_1-1)I_{n_1} - L(G_1) - (A(G_1)+r_1I_{n_1})\Gamma_{L(G_2)}(x-2) \end{pmatrix}.$$

Again employing Lemma 2.1 to detS, we see that

$$\begin{aligned} detS &= (x-1)^{n_1} det((x-n_2r_1-1)I_{n_1} - L(G_1) - (A(G_1)+r_1I_{n_1})\Gamma_{L(G_2)}(x-2) - I_{n_1}/(x-1)) \\ &= \prod_{i=1}^{n_1}(x-1)(x-n_2r_1-\mu_i(G_1)-1+(\mu_i(G_1)-2r_1)n_2/(x-2))-1. \end{aligned} \tag{4.4}$$

So by (4.3) and (4.4) the result follows. □

COROLLARY 4.3. *Let G_1 be r_1- regular graph ($r_1 \geqslant 2$) and G_2 be an arbitrary graph with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then the number of spanning trees of $G_1 \boxdot G_2$ is given by:*

$$t(G_1 \boxdot G_2) = t(G_1)2^{m_1-n_1+1}(n_2+2)^{n_1}\prod_{i=2}^{n_2}(\mu_i(G_2)+2)^{m_1}.$$

PROOF. By (3.5) and above theorem, we obtain the desired result. □

THEOREM 4.3. *Let G_1 be r_1- regular graph ($r_1 \geqslant 2$) and G_2 be an r_1 graph with n_1, n_2 vertices and m_1, m_2 edges. Then the signless Laplacian spectrum of $G_1 \boxdot G_2$ is given by:*

a. $\gamma_i(G_2)+2$ *with multiplicity* m_1, *for* $i = 2,\cdots,n_2$.
b. $2(r_2+1)$ *with multiplicity* m_1-n_1.
c. three roots of the polynomial
$x^3-(r_1n_2+\gamma_i(G_1)+2r_2+4)x^2+(2r_1n_2r_2+2\gamma_i(G_1)r_2-\gamma_i(G_1)n_2+3r_1n_2$
$+3\gamma_i(G_1)+4r_2+4)x-2r_1n_2r_2-2\gamma_i(G_1)r_2+\gamma_i(G_1)n_2-2r_1n_2-2\gamma_i(G_1)$,
for $i = 1,\cdots,n_1$.

PROOF. With suitable labelling of the vertices of $G_1 \boxdot G_2$, the adjacency matrix $Q(G_1 \boxdot G_2)$ can be formulated as follows:

$$Q(G_1 \boxdot G_2) = \begin{pmatrix} I_{m_1}\otimes(2I_{n_2}+Q(G_2)) & 0 & B\otimes e \\ 0 & I_{n_1} & I_{n_1} \\ B^T\otimes e^T & I_{n_1} & (mr_1+1)I_{n_1}+Q(G_1) \end{pmatrix},$$

where e is the column vector of size n_2 with all its entries are 1, I_{n_1} is the identity matrix of order n_1 and B is the incidence matrix of G_1. Rest of the proof is similar to the proof of Theorem 4.2. □

As the proofs of the following theorems are similar to that of Theorem 4.1 we omit the details.

THEOREM 4.4. *Let G_i (i=1,2) be r_i- regular graphs with n_i vertices and m_i edges, respectively. Then*

$$f(A(G_1 \boxtimes G_2), x) = \prod_{i=1}^{n_2}(x-\lambda_i(G_2))^{m_1} \prod_{i=1}^{n}(x-(\lambda_i(G_1)+r_1)\Gamma_{A(G_2)}(x)-\lambda_i(G_1))x-\lambda_i^2(G_1).$$

As $\Gamma_M(x) = \dfrac{n}{(x-r)}$, where M is the square matrix of order n with each of its row sum a constant 'r' and $\Gamma_{K_{p,q}}(x) = \dfrac{(p+q)x+2pq}{x^2-pq}$, proofs of the following two corollaries follows immediately by the above theorem.

COROLLARY 4.4. *Let G_i $(i = 1, 2)$ be r_i-regular graphs with $r_1 \geqslant 2$, n_i vertices and m_i edges. Then the adjacency spectrum of $G = (G_1 \boxtimes G_2)$ is given by:*
a. $\lambda_i(G_2)$, *with multiplicity* m_1 *for* $i = 2, \cdots, n_2$.
b. r_2 *with multiplicity* $m_1 - n_1$.
c. three roots of the polynomial
$x^3 - (r_2 + \lambda_i(G_1))x^2 - (\lambda_i^2(G_1) + \lambda_i(G_1)(n_2 - r_2) + n_2r_1)x + r_2\lambda_i^2(G_1)$, *for* $i = 1, \cdots, n_1$.

COROLLARY 4.5. *Let G_1 be r_1-regular graphs with n_1 vertices and m_1 edges. Then the adjacency spectrum of $G = (G_1 \boxtimes K_{p,q})$ is given by:*
a. 0, *with multiplicity* $m_1(p + q - 2)$.
b. $\pm\sqrt{pq}$ *with multiplicity* $m_1 - n_1$.
c. four roots of the polynomial
$x^4 - \lambda_i(G_1)x^3 + \left(-\lambda_i(G_1)^2 + (-p-q)\,\lambda_i(G_1) + (-q-r_1)\,p - qr_1\right)x^2$
$-\,pq\,(\lambda_i(G_1) + 2\,r_1)\,x + \lambda_i(G_1)^2 pq$, *for* $i = 1, \cdots, n_1$.

THEOREM 4.5. *Let G_1 be r_1-regular graph $(r_1 \geqslant 2)$ with n_1 vertices and m_1 edges. Then for an arbitrary graph G_2 on n_2 vertices, the Laplacian spectrum of $G_1 \boxtimes G_2$ is given by:*
a. $\mu_i(G_2) + 2$ *with multiplicity* m_1, *for* $i = 2, \cdots, n_2$.
b. 2 *with multiplicity* $m_1 - n_1$.
c. three roots of the polynomial
$(x-2)(x-r_1)(x-\mu_i(G_1)-r_1-n_2r_1) + n_2(\mu_i(G_1)-2r_1)(x-r_1)$
$-\,(\mu_i(G_1)-r_1)^2(x-2)$, *for* $i = 1, \cdots, n_1$.

Applying (3.5) to the above theorem we have the following result:

COROLLARY 4.6. *Let G_1 be r_1-regular graph $(r_1 \geqslant 2)$ with n_1 vertices and m_1 edges. Then for an arbitrary graph G_2 on n_2 vertices, the number of spanning trees of $G_1 \boxtimes G_2$ is given by:*

$$t(G_1 \boxtimes G_2) = r_1 t(G_1) 2^{m_1-n_1+1} \prod_{i=2}^{n_1}(6r_1 + n_2r_1 - 2\mu_i(G_1)) \prod_{i=2}^{n_2}(\mu_i(G_2)+2)^{m_1}.$$

THEOREM 4.6. *Let G_i $(i = 1, 2)$ be r_i-regular graphs with n_i vertices, m_i edges and $r_1 \geqslant 2$. Then the signless Laplacian spectrum of $G_1 \boxtimes G_2$ is given by:*

a. $\gamma_i(G_2) + 2$ *with multiplicity* m_1, *for* $i = 2, \cdots, n_2$.

b. $2(r_2 + 1)$ *with multiplicity* $m_1 - n_1$.

c. *three roots of the polynomial*
$(x-2r_2-2)(x-r_1)(x-\gamma_i(G_1)-r_1-n_2r_1)-n_2\gamma_i(G_1)(x-r_1)-(\gamma_i(G_1)-r_1)^2(x-2r_2-2)$, *for* $i = 1, \cdots, n_1$.

5. Applications

The notion of integral graph was first introduced by Harary and Schwenk in 1974 [**10**]. In general, the problem of characterizing integral graphs is a difficult task. In [**2**] constructions and properties of integral graphs are discussed in detail. The graphs K_n, $K_{m,n}$ (mn a perfect square), C_6, the cocktail parity graph $CP(n) = n\bar{K}_2$, are all examples of integral graphs. Moreover, some graph operations such as cartesian product, direct product and strong product when applied on integral graphs produce again integral graphs. For other related works see [**13, 20, 21**] and therein references. In this section, we apply our graph operations based on corona and neighborhood corona on known integral graphs to produce class of new integral graphs. At the end of the section, as an application we give methods to construct infinitely many pairs of cospectral graphs.

From Corollaries 3.1 and 3.4 it follows that

a. If G is an integral graph of order n, then $G \circledast mK_1$ is integral if and only if $5\lambda_i^2(G) + 4m$ is a perfect square, for $i = 1, 2, \cdots, n$.

b. If G is an integral graph of order n, then $G \odot mK_1$ is integral if and only if $\lambda_i^2(G)(4m + 1) + 4$ is a perfect square, for all $i = 1, 2, \cdots, n$.

In particular, we have the following:

i. $K_n \circledast mK_1$ is integral if and only if 4m+5 and $5n^2 - 10n + 4m + 5$ are perfect squares.
ii. $K_{p,q} \circledast mK_1$ is integral if and only if pq, m and $5pq + 4m$ are perfect squares.
iii. If $G \circledast mK_1$ is an integral graph then $(K_2 \otimes G) \circledast mK_1$ is integral, where $\otimes$ denotes the direct product of two graphs.

iv. $K_n \odot mK_1$ is integral if and only if 4m+5 and $(n - 1)^2(4m + 1) + 4$ are perfect squares.

v. $K_{p,q} \odot mK_1$ is integral if and only if pq and $pq(4m + 1) + 4$ are perfect squares.

vi. If $G \odot mK_1$ is an integral graph then $(K_2 \otimes G) \odot mK_1$ is integral, where $\otimes$ denotes the direct product of two graphs.

The above observations enable us to construct some class of new integral graphs.

EXAMPLE 5.1. In the following table we give infinite ordered pairs (m,n) for which the graph $K_n \circledast mK_1$ is integral.

n	m
$6k+3$	$(5k^2+5k+1)(45k^2+15k-1),\ k=1,2,\cdots$
$6k-1$	$(5k^2-5k+1)(45k^2-15k-1),\ k=1,2,\cdots$
$2k+1$	$k^4-3k^2+1,\ k=2,3,\cdots$
2	$k^2+3k+1,\ k=1,2,\cdots$

EXAMPLE 5.2. a. $K_{n,n} \circledast n^2K_1$ is integral graph for all n.
b. $K_{1,n^2} \circledast n^2K_1$ is integral graph for all n.

EXAMPLE 5.3. a. If $n=2k$ and $m=k^2-k-1$, then $K_n \odot mK_1$ is integral for $k=1,2,\cdots$.
b. If $n=2k-1$ and $m=k^2-k-1$, then $K_{n,n} \odot mK_1$ and $K_{1,n^2} \odot mK_1$ is integral for $k=1,2,\cdots$.

From Corollaries 4.1 and 4.4 it follows that

a. If G is an integral r-regular graph ($r \geqslant 2$) on n vertices, then $G \boxdot mK_1$ is integral if and only if $\lambda_i(G)_i^2+4m(\lambda_i(G)+r)+4$ is a perfect square, for $i=1,2,\cdots n$.

b. If G is an integral r-regular graph ($r \geqslant 2$) on n vertices, then $G \boxtimes mK_1$ is integral if and only if $5\lambda_i^2(G)+4m(\lambda_i(G)+r)$ is a perfect square, for $i=1,2,\cdots,n$.

In particular, we have the following

i. $K_n \boxdot mK_1$ ($n \geqslant 2$) is integral if and only if $4m(n-2)+5$ and $(n-1)^2+8m(n-1)+4$ are perfect squares.
ii. $K_{n,n} \boxdot mK_1$ ($n \geqslant 2$)is integral if and only if $mn+1$, $n^2+8mn+4$, n^2+4 are perfect squares.
iii. $K_n \boxtimes mK_1$ ($n \geqslant 2$) is integral if and only if $4m(n-2)+5$ and $5(n-1)^2+8m(n-1)$ are perfect squares.
iv. $K_{n,n} \boxtimes mK_1$ ($n \geqslant 2$) is never an integral graph for all n and m.

The above observations enables us to construct some new class of integral graphs.

EXAMPLE 5.4. If $m=k^2-k-1$, then $K_3 \boxtimes mK_1$ is integral for all $k=2,3,\cdots$.

Now, we give methods to construct infinite family of cospectral graphs.
From Theorem 3.1 and 3.4 one can easily notice that

a. If G_1 and G_2 are adjacency cospectral graphs and H is an arbitrary graph, then
   i. $G_1 \circledast H$ and $G_2 \circledast H$ are adjacency cospectral.

   ii. $G_1 \odot H$ and $G_2 \odot H$ are adjacency cospectral.
b. If G is an arbitrary graph and H_1, H_2 are adjacency cospectral graphs with $\Gamma_{A(H_1)}(x)=\Gamma_{A(H_2)}(x)$, then
   i. $G \circledast H_1$ and $G \circledast H_2$ are adjacency cospectral.
   ii. $G \odot H_1$ and $G \odot H_2$ are adjacency cospectral.

Similary using Theorem 3.2, 3.5 and 3.3, 3.6 one can construct Laplacian cospectral and signless Laplacian cospectral graphs.
Also from Theorem 4.1 and 4.4 we have the following results:

a. If G_1 and G_2 are adjacency regular cospectral graphs and H is an arbitrary graph, then
   i. $G_1 \boxdot H$ and $G_2 \boxdot H$ are adjacency cospectral.

   ii. $G_1 \boxtimes H$ and $G_2 \boxtimes H$ are adjacency cospectral.

b. If G is an arbitrary regular graph and H_1, H_2 are adjacency cospectral graphs with $\Gamma_{A(H_1)}(x) = \Gamma_{A(H_2)}(x)$, then
   i. $G \boxdot H_1$ and $G \boxdot H_2$ are adjacency cospectral.
   ii. $G \boxtimes H_1$ and $G \boxtimes H_2$ are adjacency cospectral.

Similary using Theorem 4.2, 4.5 and 4.3, 4.6 one can construct Laplacian cospectral and signless Laplacian cospectral graphs.

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