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ANTI FUZZY $k-$IDEALS AND ANTI HOMOMORPHISMS OF $\Gamma-$SEMIRINHS

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Abstract. In this paper the notion of anti fuzzy ideal, anti fuzzy $k-$ideal of $\Gamma-$semirings and the concept of anti homomorphism of $\Gamma-$semirings has been introduced. We study the properties of anti fuzzy ideal, anti fuzzy $k-$ideal, anti homomorphic image and pre-image of fuzzy ideal, anti fuzzy ideal and anti fuzzy $k-$ideal of a $\Gamma-$semiring.

1. Introduction

The notion of a semiring is an algebraic structure with two associative binary operations where one distributes over the other, was first introduced by H. S. Vandiver [**16**] in 1934 but semirings had appeared in studies on the theory of ideals of rings.An universal algebra $S = (S, +, \cdot)$ is called semiring if and only if $(S, +), (S, \cdot)$ are semigroups which are connected by distributive laws, *i.e.*, $a(b+c) = ab + ac,\ (a + b)c = ac + bc,$ for all $a, b, c \in S$. Though semiring is a generalization of a ring, ideals of semiring do not coincide with ring ideals. For example an ideal of semiring need not be the kernel of some semiring homomorphism. To solve this problem Herniksen [**4**] defined $k-$ideals and Iizuka [**5**] defined $h-$ideals in semirings to obtain analogous of ring results for semiring. In structure, semirings lie between semigroups and rings. The results which hold in rings but not in semigroups hold in semirings, since semiring is a generalization of ring. The study of rings shows that multiplicative structure of ring is an independent of additive structure whereas in semiring multiplicative structure of semiring is not an independent of additive structure of semiring. The additive and the multiplicative structure of a semiring play an important role in determining the structure of a semiring. The theory of rings and theory of semigroups have considerable impact on the development of

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theory of semirings. Semirings play an important role in studying matrices and determinants. Semirings are useful in the areas of theoretical computer science as well as in the solution of graph theory, optimization theory, in particular for studying automata,coding theory and formal languages. Semiring theory has many applications in other branches.

As a generalization of ring, the notion of a $\Gamma-$ring was introduced by N. Nobusawa [**13**] in 1964. In 1981, M. K. Sen [**15**] introduced the notion of a $\Gamma-$semigroup as a generalization of semigroup. The notion of a ternary algebraic system was introduced by Lehmer [**8**] in 1932, Lister [**10**] introduced the notion of a ternary ring . In 1995, M. Murali Krishna Rao [**11**] introduced the notion of a $\Gamma-$semiring as a generalization of $\Gamma-$ring, ring, ternary semiring and semiring. The fuzzy set theory was developed by L. A. Zadeh [**16**] in 1965. In 1982, W. J. Liu [6] defined and studied fuzzy subrings as well as fuzzy ideals in rings. Biswas [**3**] introduced the concept of anti fuzzy subgroups. K. H. Kim, Y. B. Jun [**6**] introduced the concept of anti fuzzy ideals in near rings. In this paper the notion of anti fuzzy ideal, anti fuzzy $k-$ideal of $\Gamma-$semirings and the concept of anti homomorphism of $\Gamma-$semirings has been introduced. The properties of homomorphic, anti homomorphic image and pre image of fuzzy ideal, anti fuzzy ideal and anti fuzzy $k-$ideal of a $\Gamma-$semiring are studied.

2. Preliminaries

In this section we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1. ([**1**]) A set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) will be called semiring provided

(i) addition is a commutative operation.
(ii) multiplication distributes over addition both from the left and from the right.
(iii) there exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for each $x \in S$.

Definition 2.2. ([**11**]) Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. Then we call M a $\Gamma-$semiring, if there exists a mapping $M \times \Gamma \times M \to M$ (images of (x, α, y) will be denoted by $x\alpha y, x, y \in M, \alpha \in \Gamma$) such that it satisfies the following axioms for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$ (ii) $(x + y)\alpha z = x\alpha z + y\alpha z$
(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$ (iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Every semiring R is a $\Gamma-$semiring with $\Gamma = R$ and ternary operation $x\gamma y$ defined as the usual semiring multiplication.

We illustrate the definition of $\Gamma-$semiring by the following examples.

Example 2.1. ([**11**]) Let S be a semiring and $M_{p,q}(S)$ denote the additive abelian semigroup of all $p \times q$ matrices whose entries are from S. Then $M_{p,q}(S)$ is a $\Gamma-$semiring with $\Gamma = M_{p,q}(S)$ and the ternary operation defined by the usual matrix multiplication as $x\alpha y = x(\alpha^t)y$, where α^t denotes the transpose of the matrix α; for all x, y and $\alpha \in M_{p,q}(S)$.

EXAMPLE 2.2. ([**11**]) Let M be the additive semi group of all $m \times n$ matrices over the set of non negative rational numbers and $\Gamma-$be the additive semigroup of all $n \times m$ matrices over the set of non negative integers, then with respect to usual matrix multiplication M is a $\Gamma-$semiring.

EXAMPLE 2.3. ([**11**]) Let X and Y be abelian semigroups with identity elements. Let $M = Hom(X,Y), \Gamma = Hom(Y,X)$ for all $a, b \in M, \alpha \in \Gamma$. Define $a\alpha b$ be the usual composition map. Then M is a $\Gamma-$semiring.

A $\Gamma-$semiring M is said to have zero element if there exist an element $0 \in M$ such that $0+x = x = x+0$ and $0\alpha x = x\alpha 0 = 0$, for all $x \in M, \alpha \in \Gamma$. A $\Gamma-$semiring M is said to be commutative $\Gamma-$semiring if $x\alpha y = y\alpha x$, for all $x, y \in M$ and $\alpha \in \Gamma$. A subset A of $\Gamma-$semiring M is a left (right) ideal of M if A is an additive semigroup of M and the set $M\Gamma A = \{x\alpha y \mid x \in M, \alpha \in \Gamma, y \in A\}(A\Gamma M)$ is contained in A. If A is both left and right ideals then A is an ideal of M. An ideal I of a $\Gamma-$semiring M is called a $k-$ideal, if $b \in M, a+b$ and $a \in I$ then $b \in I$. A function $f : R \to S$ where R and S are $\Gamma-$semirings is said to be a $\Gamma-$semiring homomorphism if

$$f(a+b) = f(a) + f(b), f(a\alpha b) = f(a)\alpha f(b), \text{ for all } a, b \in R, \alpha \in \Gamma.$$

Let S be a nonempty set, a mapping $f : S \to [0,1]$ is called a fuzzy subset of S. Let f be a fuzzy subset of S. For $t \in [0,1]$, the set $f_t = \{x \in S \mid f(x) \geqslant t\}$ is called level subset of S with respect to f. The complement of a fuzzy subset μ of a $\Gamma-$semiring M is denoted by μ^c and is defined as $\mu^c(x) = 1 - \mu(x)$, for all $x \in M$. A fuzzy subset μ of $\Gamma-$semiring M is called a fuzzy left(right) of M if it satisfies

$$\mu(x+y) \geqslant min\{\mu(x), \mu(y)\}, \mu(x\alpha y) \geqslant \mu(y)\big(\mu(x\alpha y) \geqslant \mu(x)\big),$$

for all $x, y \in M, \alpha \in \Gamma$. If μ is a fuzzy left (right) ideal of $\Gamma-$semiring M then $\mu(0) \geqslant \mu(x)$, for all $x \in M$. A fuzzy subset f of $\Gamma-$semiring M is called a fuzzy ideal of M, if for all $x, y \in M, \alpha \in \Gamma$,

$$f(x+y) \geqslant min\{f(x), f(y)\}, f(x\alpha y) \geqslant max\{f(x), f(y)\}.$$

A fuzzy ideal f of $\Gamma-$semring M with zero 0 is said to be a $k-$fuzzy ideal of M if $f(x+y) = f(0)$ and $f(y) = f(0) \Rightarrow f(x) = f(0)$, for all $x, y \in M$. A fuzzy ideal f of $\Gamma-$semiring M is said to be a fuzzy $k-$ideal of M if $f(x) \geqslant min\{f(x+y), f(y)\}$, for all $x, y \in M$. Let S and T be two sets and $\phi : S \to T$ be any function. A fuzzy subset f of S is called $\phi-$invariant if $\phi(x) = \phi(y) \Rightarrow f(x) = f(y)$.

3. Anti fuzzy ideal and anti fuzzy k-ideal of $\Gamma-$semirings

In this section we introduce the notion of anti fuzzy ideal and anti fuzzy $k-$ideals of $\Gamma-$semirings and study some of their properties.

DEFINITION 3.1. A fuzzy subset μ of a $\Gamma-$semiring M is called an anti fuzzy left(right) of M if it satisfies

$$\mu(x+y) \leqslant max\{\mu(x), \mu(y)\}, \ \mu(x\alpha y) \leqslant \mu(y)\big(\mu(x\alpha y) \leqslant \mu(x)\big),$$

for all $x, y \in M, \alpha \in \Gamma$.

Definition 3.2. A fuzzy subset μ of a $\Gamma-$semiring M is called an anti fuzzy ideal of M if μ is both an anti fuzzy left and an anti fuzzy right ideal of M.

Definition 3.3. Let M be a $\Gamma-$semiring. An anti fuzzy ideal μ of M is said to be an anti fuzzy-$k-$ideal of M if $\mu(x) \leqslant max\{\mu(x+y), \mu(y)\}$, for all $x, y \in M$.

Example 3.1. Let M be the set of all rational numbers and Γ be the set of all rational numbers. Define $+$ on M as usual addition and ternary operation $(a, \alpha, b) \to a\alpha b$ as usual multiplication. Then M is a $\Gamma-$semiring. Let μ be a fuzzy subset of M is defined by

$$\mu(x) = \begin{cases} 1, & \text{if } x \text{ is an integer;} \\ 0, & \text{if } x \text{ is not an integer} \end{cases}$$

Then μ is an anti fuzzy $k-$ideal of $\Gamma-$semiring M.

Definition 3.4. Let M be a $\Gamma-$semiring. A fuzzy subset μ of M is said to be a $k-$anti fuzzy ideal of M if μ is an anti fuzzy ideal and $\mu(x+y) = \mu(0), \mu(y) = \mu(0) \Rightarrow \mu(x) = \mu(0)$.

Definition 3.5. Let M be a $\Gamma-$semiring. If μ is an anti fuzzy-ideal of $\Gamma-$semiring M, for any $t \in [0,1], \mu_t$ is defined by $\mu_t = \{x \in M \mid \mu(x) \leqslant t\}$ then μ_t is called anti level subset.

Theorem 3.1. *If μ is a $k-$anti fuzzy ideal of $\Gamma-$semiring M then $L = \{x \mid x \in M.\mu(x) = 0\}$ is either empty or a $k-$ideal of $\Gamma-$semiring M.*

Proof. Suppose μ is a $k-$anti fuzzy ideal of $\Gamma-$semiring M,
$L = \{x \mid x \in M.\mu(x) = 0\}$, $L \neq \phi, x, y \in L$ and $\alpha \in \Gamma$. Then
$\mu(x) = 0, \mu(y) = 0, \mu(x+y) \leqslant \max\{\mu(x), \mu(y)\} = 0 \Rightarrow \mu(x+y) = 0$.
Therefore $x + y \in L$. Now $x \in L, y \in M, \alpha \in \Gamma$ then $\mu(x) = 0$ and $\mu(x\alpha y) \leqslant \min\{\mu(x), \mu(y)\} = 0 \Rightarrow x\alpha y \in L$. Let $x, x+y \in L$. then $\mu(x) = 0, \mu(x+y) = 0 \Rightarrow \mu(y) = 0$. Therefore $y \in L$. Hence L is a $k-$ideal of $\Gamma-$semiring M. □

Theorem 3.2. *Let μ be an anti fuzzy ideal of $\Gamma-$semiring M. Then μ is an anti fuzzy $k-$ ideal of $\Gamma-$semiring M if and only if non-empty μ_t is a $k-$ideal of $\Gamma-$semiring M, for any $t \in (0,1]$.*

Proof. Suppose μ is an anti fuzzy ideal of $\Gamma-$semiring M. Let $x, y \in \mu_t$. Then $\mu(x) \leqslant t, \mu(y) \leqslant t, \mu(x+y) \leqslant \max\{\mu(x), \mu(y)\} \leqslant t \Rightarrow x+y \in \mu_t$. Hence μ_t is an ideal of $\Gamma-$semiring M. Let $y, x+y \in \mu_t$. Then $\mu(y) \leqslant t,\ \mu(x+y) \leqslant t$. Since μ is an anti fuzzy $k-$ideal of M.
We have $\mu(x) \leqslant \max\{\mu(x+y), \mu(y)\} \leqslant t \Rightarrow \mu(x) \leqslant t \Rightarrow x \in \mu_t$.
Hence μ_t is a $k-$ideal of $\Gamma-$semiring M.

Conversely suppose that μ_t is a $k-$ideal of $\Gamma-$semiring M, for any $t \in (0,1]$. Let $x, a \in M$ and $\mu(a) = t_1, \mu(x+a) = t_2$, put $t = max\{t_1, t_2\}$. Then $a \in \mu_t$ and $x+a \in \mu_t$. Since μ_t is a $k-$ideal, we have $x \in \mu_t$, therefore $\mu(x) \leqslant \max\{\mu(x+a), \mu(a)\}$ for all $x, a \in M$. Hence μ is an anti fuzzy $k-$ideal of $\Gamma-$semiring M. □

THEOREM 3.3. *Let f be an anti fuzzy ideal of $\Gamma-$semiring M. If $t \in [0,1]$. If f_t is a $k-$ideal of $\Gamma-$semiring then f is a $k-$anti fuzzy ideal of $\Gamma-$semiring M.*

PROOF. Let f be an anti fuzzy ideal of $\Gamma-$semiring $M, f(x+y) = f(0), f(y) = f(0)$ and $x, y \in M, \alpha \in \Gamma$. Then $x+y \in f_{f(0)}$ and $y \in f_{f(0)} \Rightarrow x \in f_{f(0)}$. Since $f_{f(0)}$ is a $k-$ideal then $f(x) \leqslant f(0), f(0) = f(x\alpha 0) \leqslant min\{f(x), f(0)\} \leqslant f(x) \Rightarrow f(0) \leqslant f(x)$. Hence $f(x) = f(0)$. Therefore f is a $k-$anti fuzzy ideal of $\Gamma-$semiring M. □

THEOREM 3.4. *Let μ be an anti fuzzy ideal of $\Gamma-$semiring M, μ_{t_1}, μ_{t_2} with $t_1 < t_2$ are anti level subsets of μ. Then μ_{t_1} and μ_{t_2} are equal if and only if there is no $x \in M$ such that $t_1 \leqslant \mu(x) \leqslant t_2$.*

PROOF. Suppose $t_1 < t_2$ in $[0,1]$ and $\mu_{t_1} = \mu_{t_1}$. If there is a $x \in M$ such that $t_1 < \mu(x) \leqslant t_2$ then μ_{t_1} is a proper subset of μ_{t_2}, which is a contradiction.

Conversely suppose there is no $x \in M$ such that $t_1 \leqslant \mu(x) \leqslant t_2 \Rightarrow \mu_{t_1} \subseteq \mu_{t_2}$, if $x \in \mu_{t_2}$ then $\mu(x) \leqslant t_2$ and $\mu(x) \ngtr t_1 \Rightarrow \mu(x) \leqslant t_1 \Rightarrow x \in \mu_{t_1}$. Hence $\mu_{t_1} = \mu_{t_2}$. □

THEOREM 3.5. *Let μ be a fuzzy subset of $\Gamma-$semiring M. μ is an anti fuzzy $k-$ideal of M if and only if μ^c is a fuzzy $k-$ideal of M.*

PROOF. Let M be a $\Gamma-$semiring and μ be an anti fuzzy $k-$ideal of M and $x, y \in M, \alpha \in \Gamma$.

$$\begin{aligned}\mu^c(x+y) &= 1-\mu(x+y) \geqslant 1-\max\{\mu(x),\mu(y)\} = \min\{1-\mu(x), 1-\mu(y)\}\\ &= \min\{\mu^c(x), \mu^c(y)\}\\ \mu^c(x\alpha y) &= 1-\mu(x\alpha y) = \max\{\mu^c(x), \mu^c(y)\}.\end{aligned}$$

Hence μ^c is a fuzzy ideal of M.

$$\begin{aligned}\mu^c(x) &= 1-\mu(x) \geqslant 1-\max\{\mu(x+y),\mu(y)\} = \min\{1-\mu(x+y), 1-\mu(y)\}\\ &= \min\{\mu^c(x+y), \mu^c(y)\}.\end{aligned}$$

Therefore μ^c is a fuzzy $k-$ideal of $\Gamma-$semiring M.

Conversely suppose that μ^c is a fuzzy $k-$ideal of $\Gamma-$semiring M. Let $x, y \in M, \alpha \in \Gamma$.

$$\mu(x+y) = 1-\mu^c(x+y) \leqslant 1-\min\{\mu^c(x), \mu^c(y)\} = \max\{\mu(x), \mu(y)\}$$
$$\mu(x\alpha y) = 1-\mu^c(x\alpha y) \leqslant 1-\max\{\mu^c(x), \mu^c(y)\} = \min\{\mu(x), \mu(y)\}.$$

Hence μ is an anti fuzzy ideal of $\Gamma-$semiring M.

$$\begin{aligned}\mu(x) &= 1-\mu^c(x) \leqslant 1-\min\{\mu^c(x+y), \mu^c(y)\} = \max\{1-\mu^c(x+y), 1-\mu^c(y)\}\\ &= \max\{\mu(x+y), \mu(y)\}.\end{aligned}$$

Hence μ is an anti fuzzy $k-$ideal of $\Gamma-$semiring M. □

THEOREM 3.6. *Let A be a non-empty subset of $\Gamma-$semiring M, $t \in [0,1]$ and a fuzzy subset μ in M such that*

$$\mu(x) = \begin{cases} t, & \text{if } x \in A;\\ 1, & \text{if } x \notin A.\end{cases}$$

Then μ is an anti fuzzy ideal of $\Gamma-$semiring M if and only if $\mu_t = A$ is an ideal of $\Gamma-$semiring M.

Proof. Suppose μ is an anti fuzzy ideal of $\Gamma-$semiring M and $x, y \in A$. Then $\mu(x) = t,\ \mu(y) = t,\ \mu(x+y) \leqslant \max\{\mu(x), \mu(y)\} = t \Rightarrow \mu(x+y) = t \Rightarrow x+y \in A$. Let $x, y \in M, \alpha \in \Gamma$. $\mu(x\alpha y) \leqslant \min\{\mu(x), \mu(y)\} = t \Rightarrow x\alpha y \in A$. Hence $\mu_t = A$ is an ideal of M.

Conversely, let A be an ideal of M, $x, y \in M$ and $\alpha \in \Gamma$.

case(i) If $x, y \in A, x+y, x\alpha y \in A \Rightarrow \mu(x\alpha y) = t$.
Then
$\mu(x+y) \leqslant \max\{\mu(x), \mu(y)\},\ \mu(x\alpha y) \leqslant \min\{\mu(x), \mu(y)\}$.

case(ii) If
$x, y \notin A,\ \ x+y \notin A,\ \ x\alpha y \in A \Rightarrow \mu(x) = t,\ \ \mu(y) = 1,\ \mu(x+y) = 1$, $\mu(x\alpha y) = t$.
Then
$\mu(x+y) \leqslant \max\{\mu(x), \mu(y)\},\ \mu(x\alpha y) \leqslant \min\{\mu(x), \mu(y)\}$.

case(iii) If $x \in A, y \notin A,\ x+y \notin A \Rightarrow \mu(x) = t,\ \mu(y) = 1,\ \mu(x+y) = 1$, $x\alpha y \in A \Rightarrow \mu(x\alpha y) = t$.
Then
$\mu(x+y) \leqslant \max\{\mu(x), \mu(y)\},\ \mu(x\alpha y) \leqslant \min\{\mu(x), \mu(y)\}$.

case(iv) If
$y \in A, x \notin A, x\alpha y \in A,\ x+y \notin A \Rightarrow \mu(x) = 1, \mu(y) = t, \mu(x+y) = t, \mu(x\alpha y) = 1$.
Then
$\mu(x+y) \leqslant \max\{\mu(x), \mu(y)\},\ \mu(x\alpha y) \leqslant \min\{\mu(x), \mu(y)\}$.

Therefore μ is an anti fuzzy ideal of $\Gamma-$semiring M. □

Corollary 3.1. *I is an ideal of $\Gamma-$semiring M if and only if χ_I is a fuzzy ideal of $\Gamma-$semiring M.*

Lemma 3.1. *If μ is an anti fuzzy ideal of $\Gamma-$semiring M and $\mu(x+y) = 1$ then $\mu(x) = 1$ or $\mu(y) = 1$.*

Proof. Suppose μ is an anti fuzzy ideal of $\Gamma-$semiring M and $\mu(x+y) = 1$. $\mu(x+y) \leqslant \max\{\mu(x), \mu(y)\} \Rightarrow 1 \leqslant \max\{\mu(x), \mu(y)\} \Rightarrow \mu(x) = 1$ or $\mu(y) = 1$. □

Theorem 3.7. *If f and g be anti fuzzy $k-$ideals of $\Gamma-$semiring M then $f \cap g$ is also an anti fuzzy $k-$ideal of $\Gamma-$semiring M.*

PROOF. Suppose f and g are anti fuzzy $k-$ideals of $\Gamma-$semiring M and $x, y \in M, \alpha \in \Gamma$.

$$\begin{aligned} f \cap g(x+y) &= \min\{f(x+y), g(x+y)\} \\ &\leqslant \min\{\max\{f(x), f(y)\}, \max\{g(x), g(y)\}\} \\ &= \max\{\min\{f(x), g(x)\}, \min\{f(y), g(y)\}\} \\ &= \max\{f \cap g(x), f \cap g(y\} \end{aligned}$$

$$\begin{aligned} f \cap g(x\alpha y) &= \min\{f(x\alpha y), g(x\alpha y)\} \\ &\leqslant \min\{\min\{f(x), f(y)\}, \min\{g(x), g(y)\}\} \\ &= \min\{\min\{f(x), g(x)\}, \min\{f(y), g(y)\}\} \\ &= \min\{f \cap g(x), f \cap g(y\} \end{aligned}$$

$$\begin{aligned} f \cap g(x) &= \min\{f(x), g(x)\} \\ &\leqslant \min\{\max\{f(x+y), f(y)\}, \max\{g(x+y), g(y)\}\} \\ &= \max\{\min\{f(x+y), g(x+y)\}, \min\{f(y), g(y)\}\} \\ &= \min\{f \cap g(x), f \cap g(y\}. \end{aligned}$$

Hence $f \cap g$ is also an anti fuzzy $k-$ideal of $\Gamma-$semiring M. □

THEOREM 3.8. *If f and g are anti fuzzy $k-$ideals of $\Gamma-$semiring M then $f \cup g$ is also an anti fuzzy $k-$ideal of $\Gamma-$semiring M.*

PROOF. Let f and g be anti fuzzy $k-$ideals of $\Gamma-$semiring M and $x, y \in M, \alpha \in \Gamma$.

$$\begin{aligned} f \cup g(x+y) &= \max\{f(x+y), g(x+y)\} \\ &\leqslant \max\{\max\{f(x), f(y)\}, \max\{g(x), g(y)\}\} \\ &= \max\{\max\{f(x), g(x)\}, \max\{f(y), g(y)\}\} \\ &= \max\{f \cup g(x), f \cup g(y)\} \end{aligned}$$

$$\begin{aligned} f \cup g(x\alpha y) &= \max\{f(x\alpha y), g(x\alpha y)\} \\ &\leqslant \max\{\min\{f(x), f(y)\}, \min\{g(x), g(y)\}\} \\ &= \min\{\max\{f(x), g(x)\}, \max\{f(y), g(y)\}\} \\ &= \min\{f \cup g(x), f \cup g(y)\} \end{aligned}$$

$$\begin{aligned} f \cup g(x) &= \max\{f(x), g(x)\} \\ &\geqslant \max\{\max\{f(x+y), f(y)\}, \max\{g(x+y), g(y)\}\} \\ &= \max\{\max\{f(x+y), g(x+y)\}, \max\{f(y), g(y)\}\} \\ &= \max\{f \cup g(x), f \cup g(y)\}. \end{aligned}$$

Hence $f \cup g$ is also an anti fuzzy $k-$ideal of $\Gamma-$semiring M. □

DEFINITION 3.6. An anti fuzzy $k-$ideal μ of $\Gamma-$semiring M is said to be normal if $\mu(0) = 0$

Theorem 3.9. *Let μ be an anti fuzzy $k-$ideal of $\Gamma-$semiring M. If μ^+ be a fuzzy subset of M is defined by $\mu^+(x) = \mu(x) - \mu(0)$ for all $x \in M$ then μ^+ is a normal anti fuzzy $k-$ideal of M which contains μ.*

Proof. Let μ be an anti fuzzy $k-$ideal of $\Gamma-$semiring M and μ^+ be a fuzzy subset of M, defined by $\mu^+(x) = \mu(x) - \mu(0)$ for all $x \in M$. For any $x, y \in M, \alpha \in \Gamma$,

$$\begin{aligned}
\mu^+(x+y) &= \mu(x+y) - \mu(0) \\
&\leqslant \max\{\mu(x), \mu(y)\} - \mu(0) \\
&= \max\{\mu(x) - \mu(0), \mu(y) - \mu(0)\} \\
&= \max\{\mu^+(x), \mu^+(y)\} \\
\mu^+(x\alpha y) &= \mu(x\alpha y) - \mu(0) \\
&\leqslant \min\{\mu(x), \mu(y)\} - \mu(0) \\
&= \min\{\mu(x) - \mu(0), \mu(y) - \mu(0)\} \\
&= \min\{\mu^+(x), \mu^+(y)\}
\end{aligned}$$

We have

$$\begin{aligned}
&\mu(x) \leqslant \max\{\mu(x+y), \mu(y)\} \\
&\Rightarrow \mu(x) - \mu(0) \leqslant \max\{\mu(x+y) - \mu(0), \mu(y) - \mu(0)\} \\
&\Rightarrow \mu^+(x) \leqslant \max\{\mu^+(x+y), \mu^+(y)\}.
\end{aligned}$$

Hence μ^+ is an anti fuzzy $k-$ideal of M. Clearly μ^+ contains μ and μ^+ is a normal anti fuzzy $k-$ideal of M. □

The proof of the following theorem is a straightforward verification.

Theorem 3.10. *Let M be a $\Gamma-$semiring and I be an ideal of $\Gamma-$semiring M. Then $M/_I = \{x + I \mid x \in M\}$ is a $\Gamma-$semiring with the mapping $* : M/_I \times \Gamma \times M/_I \to M/_I$ is defined by $x + I * \alpha * y + I = x\alpha y + I$ and $x + I + y + I = x + y + I$, for all $x, y \in M, \alpha \in \Gamma$.*

Theorem 3.11. *Let I be an ideal of $\Gamma-$semiring M and μ be an anti fuzzy ideal of $\Gamma-$semiring M. Then the fuzzy subset λ of $M/_I$ is defined by $\lambda(x + I) = \inf_{y \in I} \mu(x + y)$ is an anti fuzzy ideal of $\Gamma-$semiring $M/_I$.*

PROOF. Let I be an ideal of $\Gamma-$semiring M and μ be an anti fuzzy ideal of $\Gamma-$semiring M and $x, y \in M/_I, \alpha \in \Gamma$. Then

$$\begin{aligned}
\lambda(x+I+y+I) &= \lambda(x+y+I)\\
&= \inf_{a\in I} \mu(x+y+a)\\
&= \inf_{a=u+v,u,v\in I} \mu(x+y+u+v)\\
&= \inf_{a=u+v\in I} \mu(x+u+y+v)\\
&\leqslant \inf_{u,v\in I} \max\{\mu(x+u), \mu(y+v)\}\\
&= \max\{\inf_{u\in I} \mu(x+u), \inf_{v\in I} \mu(y+v)\}\\
&= \max\{\lambda(x+I), \lambda(y+I)\}\\
\lambda(x+I\ \alpha\ y+I) &= \lambda(x\alpha y+I)\\
&= \inf_{a\in I} \mu(x\alpha y+a) \leqslant \inf_{a\in I} \max\{\mu(x\alpha y), \mu(a)\})\\
&\leqslant \inf_{a\in I} \max\{\min\{\mu(x), \mu(y)\}, \min\{\mu(a), \mu(a)\}\}\\
&= \inf_{a\in I} \max\{\min\{\mu(x), \mu(a)\}, \min\{\mu(y), \mu(a)\}\}\\
&= \min\{\inf_{a\in I}(x+a), \inf_{a\in I}(y+a)\}\\
&= \min\{\lambda(x+I), \lambda(y+I)\}.
\end{aligned}$$

Hence λ is an anti fuzzy ideal of $\Gamma-$semiring $M/_I$. □

THEOREM 3.12. *Let I be an ideal of $\Gamma-$semiring M and μ be a fuzzy subset of M. If the fuzzy subset λ of $M/_I$ is defined by $\lambda(x+I) = \mu(x)$, for all $x \in I$ is an anti fuzzy left ideal of $\Gamma-$semiring $M/_I$ then μ is an anti fuzzy left ideal of $\Gamma-$semiring M.*

PROOF. Let I be an ideal of $\Gamma-$semiring M and μ be a fuzzy subset of M. Suppose the fuzzy subset λ of $M/_I$ is defined by $\lambda(x+I) = \mu(x)$, for all $x \in I$ is an anti fuzzy left ideal of $\Gamma-$semiring $M/_I$ and $x, y \in M$ and $\alpha \in \Gamma$.

$$\begin{aligned}
\mu(x+y) &= \lambda(x+y+I)\\
&= \lambda(x+I+y+I)\\
&\leqslant \max\{\lambda(x+I), \lambda(y+I)\}\\
&= \max\{\mu(x), \mu(y)\}\\
\mu(x\alpha y) &= \lambda(x+I\alpha y+I)\\
&\leqslant \lambda(y+I)\\
&= \mu(y).
\end{aligned}$$

Hence μ is an anti fuzzy left ideal of $\Gamma-$semiring M. □

Let μ be a fuzzy subset of M and $a \in M$. Then the set $\{b \in M \mid \mu(b) \leqslant \mu(a)\}$ is denoted by I_a.

THEOREM 3.13. *Let μ be an anti fuzzy $k-$left ideal of $\Gamma-$semiring M with zero 0. If $a \in M$ then I_a is an anti fuzzy left $k-$ideal of $\Gamma-$semiring M.*

PROOF. Let μ be an anti fuzzy $k-$ideal of $\Gamma-$semiring M and $a \in M$. We have $\mu(0) \leqslant \mu(x)$ for all $x \in M$. Therefore $0 \in I_a$. Let $b, c \in I_a$. Then $\mu(b) \leqslant \mu(a)$ and $\mu(c) \leqslant \mu(a)$.

$$\begin{aligned} \mu(b+c) &\leqslant \max\{\mu(b), \mu(c)\} \\ &\leqslant \max\{\mu(a), \mu(a)\} \\ &= \mu(a). \end{aligned}$$

Then $b + c \in I_a$. Suppose $b \in I_a, c \in M, \alpha \in \Gamma$. Then $\mu(b) \leqslant \mu(a)$. Now $\mu(c\alpha b) \leqslant \mu(b) \leqslant \mu(a)$. Hence $c\alpha b \in I_a$.

Suppose $x \in I_a$ and $x + y \in I_a$. Then $\mu(x) \leqslant \mu(a), \mu(x+y) \leqslant \mu(a)$. Then $\mu(y) \leqslant \max\{\mu(x+y), \mu(x)\} \leqslant \mu(a) \Rightarrow y \in I_a$. Hence I_a is an anti fuzzy left $k-$ideal of $\Gamma-$semiring M. □

The proof of the following theorem is similar to the proof of the Theorem 3.13

THEOREM 3.14. *Let μ be an anti fuzzy right $k-$ideal of $\Gamma-$semiring M and $a \in M$. Then I_a is a fuzzy right $k-$ideal of $\Gamma-$semiring M.*

COROLLARY 3.2. *Let μ be an anti fuzzy right $k-$ideal of $\Gamma-$semiring M and $a \in M$. Then I_a is a fuzzy $k-$ideal of $\Gamma-$semiring M.*

DEFINITION 3.7. A family of fuzzy subsets $\{\mu_i \mid i \in I\}$ of $\Gamma-$semiring M, then $\bigvee\limits_{i \in I} \mu_i$ is defined by

$$\bigvee_{i \in I} \mu_i(x) = \sup\{\mu_i(x) \mid i \in I\}, \text{ for all } x \in M.$$

THEOREM 3.15. *If $\{\mu_i(x) \mid i \in I\}$ is a family of anti fuzzy ideals of $\Gamma-$semiring M then $\bigvee\limits_{i \in I} \mu_i$ is an anti fuzzy ideal of $\Gamma-$semiring M.*

PROOF. Let $\{\mu_i(x) \mid i \in I\}$ be a family of anti fuzzy ideal of $\Gamma-$semiring M and $x, y \in M, \alpha \in \Gamma$. Then we have

$$\begin{aligned} \left(\bigvee_{i \in I} \mu_i\right)(x+y) &= \sup\{\mu_i(x+y) \mid i \in I\} \\ &\leqslant \sup\{\max\{\mu_i(x),\ \mu_i(y)\} \mid i \in I\} \\ &= \max\{\sup_{i \in I} \mu_i(x), \sup_{i \in I} \mu_i(y)\} \\ &= \max\left\{\bigvee_{i \in I} \mu_i(x), \bigvee_{i \in I} \mu_i(y)\right\} \end{aligned}$$

$$\begin{aligned}\left(\bigvee_{i\in I}\mu_i\right)(x\alpha y) &= \sup\{\mu_i(x\alpha y) \mid i\in I\}\\ &\leqslant \sup\{\min\{\mu_i(x)\mu_i(y)\} \mid i\in I\}\\ &= \min\{\sup_{i\in I}\mu_i(x), \sup_{i\in I}\mu_i(y)\}\\ &= \min\left\{\bigvee_{i\in I}\mu_i(x), \bigvee_{i\in I}\mu_i(y)\right\}.\end{aligned}$$

Hence $\bigvee_{i\in I}\mu_i$ is an anti fuzzy ideal of $\Gamma-$semiring. □

DEFINITION 3.8. Let μ be a fuzzy subset of X and $\alpha\in[0,1-\sup\{\mu(x)\mid x\in X\}]$ The mapping $\mu^T_\alpha : X\to[0,1]$ is called a fuzzy translation of μ if $\mu^T_\alpha(x)=\mu(x)+\alpha$

THEOREM 3.16. *A fuzzy subset μ is an anti fuzzy $k-$ideal of $\Gamma-$semiring M if and only if μ^T_α is an anti fuzzy $k-$ideal of $\Gamma-$semiring M*

PROOF. Suppose μ is an anti fuzzy $k-$ideal of $\Gamma-$semiring M and $x,y\in M,\gamma\in\Gamma$.

$$\begin{aligned}\mu^T_\alpha(x+y) &= \mu(x+y)+\alpha\\ &\leqslant \max\{\mu(x),\mu(y)\}+\alpha\\ &= \max\{\mu(x)+\alpha,\mu(y)+\alpha\}\\ &= \max\{\mu^T_\alpha(x),\mu^T_\alpha(y)\}\end{aligned}$$

$$\begin{aligned}\mu^T_\alpha(x\gamma y) &= \mu(x\gamma y)+\alpha\\ &\leqslant \min\{\mu(x),\mu(y)\}+\alpha\\ &= \min\{\mu(x)+\alpha,\mu(y)+\alpha\}\\ &= \min\{\mu^T_\alpha(x),\mu^T_\alpha(y)\}\end{aligned}$$

$$\begin{aligned}\mu^T_\alpha(x) &= \mu(x)+\alpha\\ &\leqslant \max\{\mu(x+y),\mu(y)\}+\alpha\\ &= \max\{\mu(x+y)+\alpha,\mu(y)+\alpha\}\\ &= \max\{\mu^T_\alpha(x+y),\mu^T_\alpha(y)\}.\end{aligned}$$

Hence μ^T_α is an anti fuzzy $k-$ideal of $\Gamma-$semiring M.

Conversely suppose μ_α^T is an anti fuzzy $k-$ideal of $\Gamma-$semiring $M,\ x, y \in M$ and $\alpha \in \Gamma$.

$$\begin{aligned}
\mu(x+y)+\alpha &= \mu_\alpha^T(x+y)\\
&\leqslant \max\{\mu_\alpha^T(x), \mu_\alpha^T(y)\}\\
&= \max\{\mu(x)+\alpha, \mu(y)+\alpha\}\\
&= \max\{\mu(x), \mu(y)\}+\alpha\\
\text{Therefore}&\\
\mu(x+y) &\leqslant \max\{\mu(x), \mu(y)\}.\\
\mu(x\gamma y)+\alpha &= \mu_\alpha^T(x\gamma y)\\
&\leqslant \min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\}\\
&= \min\{\mu(x)+\alpha, \mu(y)+\alpha\}\\
&= \min\{\mu(x), \mu(y)\}+\alpha\\
\text{Therefore}&\\
\mu(x\gamma y) &\leqslant \min\{\mu(x), \mu(y)\}.\\
\mu(x)+\alpha &= \mu_\alpha^T(x)\\
&\leqslant \max\{\mu_\alpha^T(x+y), \mu_\alpha^T(y)\}\\
&= \max\{\mu(x+y)+\alpha, \mu(y)+\alpha\}\\
&= \max\{\mu(x+y), \mu(y)\}+\alpha\\
\text{Therefore}&\\
\mu(x) &\geqslant \min\{\mu(x+y), \mu(y)\}.
\end{aligned}$$

Hence μ is an anti fuzzy $k-$ideal of $\Gamma-$semiring M. □

DEFINITION 3.9. Let μ be a fuzzy subset X and $\beta \in [0, 1]$. Then mapping $\mu_\beta^M : X \to [0, 1]$ is called a fuzzy multiplication of μ if $\mu_\beta^M(x) = \beta\mu(x)$.

THEOREM 3.17. *μ is an anti fuzzy $k-$ideal of $\Gamma-$semiring M if and only if μ_β^M is an anti fuzzy $k-$ideal of $\Gamma-$semiring M.*

PROOF. Suppose μ is an anti fuzzy $k-$ideal of $\Gamma-$semiring M and $x, y \in M, \gamma \in \Gamma$. Then

$$\begin{aligned}
\mu_\beta^M(x+y) &= \beta\mu(x+y)\\
&\leqslant \beta\max\{\mu(x), \mu(y)\}\\
&= \max\{\beta\mu(x), \beta\mu(y)\}\\
&= \max\{\mu_\beta^M(x), \mu_\beta^M(y)\}.
\end{aligned}$$

$$\begin{aligned}\mu_\beta^M(x\gamma y) &= \beta\mu(x\gamma y)\\ &\leqslant \beta\min\{\mu(x),\mu(y)\}\\ &= \min\{\beta\mu(x),\beta\mu(y)\}\\ &= \min\{\mu_\beta^M(x),\mu_\beta^M(y)\}.\end{aligned}$$

$$\begin{aligned}\mu_\beta^M(x) &= \beta\mu(x)\\ &\leqslant \beta\max\{\mu(x+y),\mu(y)\}\\ &= \max\{\beta\mu(x+y),\beta\mu(y)\}\\ &= \max\{\mu_\beta^M(x+y),\mu_\beta^M(y)\}.\end{aligned}$$

Hence μ_β^M is an anti fuzzy ideal of $\Gamma-$semiring M.

Conversely, suppose that μ_β^M is an anti fuzzy ideal of $\Gamma-$semiring M and $x, y \in M, \gamma \in \Gamma$. Then

$$\begin{aligned}\mu_\beta^M(x+y) &\leqslant \max\{\mu_\beta^M(x),\mu_\beta^M(y)\}\\ \Rightarrow\ \beta\mu(x+y) &\leqslant \max\{\beta\mu(x),\beta\mu(y)\}\\ &= \beta\max\{\mu(x),\mu(y)\}\\ \text{Therefore}&\\ \mu(x+y) &\leqslant \max\{\mu(x),\mu(y)\}.\\ \mu_\beta^M(x\gamma y) &\leqslant \min\{\mu_\beta^M(x),\mu_\beta^M(y)\}\\ &= \beta\min\{\mu(x),\mu(y)\}\\ \beta\mu(x\gamma y) &= \beta\min\{\mu(x),\mu(y)\}\\ \text{Therefore}&\\ \mu(x\gamma y) &= \min\{\mu(x),\mu(y)\}.\\ \mu_\beta^M(x) &\leqslant \max\{\mu_\beta^M(x+y),\mu_\beta^M(y)\}\\ &= \max\{\beta\mu(x+y),\beta\mu(y)\}\\ &= \beta\max\{\mu(x+y),\mu(y)\}\\ \beta\mu(x) &= \beta\max\{\mu(x+y),\mu(y)\}\\ \text{Therefore}&\\ \mu(x) &\leqslant \max\{\mu(x+y),\mu(y)\}.\end{aligned}$$

Hence μ is an anti fuzzy $k-$ideal of $\Gamma-$semiring M. □

Definition 3.10. Let μ be a fuzzy subset X and $\alpha \in [0, 1 - \sup\{\mu(x) \mid x \in X\}]$, $\beta \in [0, 1]$. Then mapping $\mu_{\beta,\alpha}^{MT} : X \to [0, 1]$ is called magnified translation of μ if $\mu_{\beta,\alpha}^{MT}(x) = \beta\mu(x) + \alpha$, for all $x \in X$.

Theorem 3.18. *μ is an anti fuzzy $k-$ideal of $\Gamma-$semiring M if and only if $\mu_{\beta,\alpha}^{MT} : X \to [0, 1]$ is an anti fuzzy $k-$ideal of $\Gamma-$semiring M.*

PROOF. Suppose μ is an anti fuzzy $k-$ideal of $\Gamma-$semiring M.
$\Leftrightarrow$ μ_β^M is an anti fuzzy $k-$ideal of $\Gamma-$semiring M, by Theorem 3.17
$\Leftrightarrow$ $\mu_{\beta,\alpha}^{MT}$ is an anti fuzzy $k-$ideal of $\Gamma-$semiring M, by Theorem 3.16
Hence the theorem. $\square$

4. Homomorphic, anti homomorphic image and pre-image of fuzzy ideals and anti fuzzy ideals of $\Gamma-$semiring

In this section the concept of an anti homomorphism of $\Gamma-$semirings has been introduced. The properties of homomorphic, anti homomorphic image and pre-image of fuzzy ideals and anti fuzzy ideals of $\Gamma-$semiring are studied.

DEFINITION 4.1. A function $f : M \to N$ where M and N are $\Gamma-$semirings is called an anti $\Gamma-$semiring homomorphism if

$$f(a+b) = f(a) + f(b), f(a\alpha b) = f(b)\alpha f(a), \text{ for all } a, b \in M, \alpha \in \Gamma$$

THEOREM 4.1. *Let M and N be $\Gamma-$semirings and $\phi : M \to N$ be an onto homomorphism. If f is a ϕ invariant anti fuzzy $k-$ideal of M then $\phi(f)$ is an anti fuzzy $k-$ideal of N.*

PROOF. Let M and N be $\Gamma-$semirings, $\phi : M \to N$ be an onto homomorphism, f be a ϕ invariant anti fuzzy ideal of M and $a \in M$. Suppose $x \in N, t \in \phi^{-1}(x)$ and $x = \phi(a)$. Then $a \in \phi^{-1}(x) \Rightarrow \phi(t) = x = \phi(a)$, since f is ϕ invariant, $f(t) = f(a) \Rightarrow \phi(f)(x) = \inf\limits_{t\in\phi^{-1}(x)} f(t) = f(a)$. Hence $\phi(f)(x) = f(a)$. Let $x, y \in N$. Then there exists $a, b \in M$ such that $\phi(a) = x, \phi(b) = y \Rightarrow \phi(a+b) = x + y \Rightarrow \phi(f)(x+y) = f(a+b) \leqslant \max\{f(a), f(b)\} = \min\{\phi(f)(x), \phi(f)(y)\}$. Since f is an anti fuzzy $k-$left ideal we have $f(a) \leqslant \max\{f(a+b), f(b)\} \Rightarrow \phi(f)(x) = \max\{\phi(f)(x+y), \phi(f)(y)\}$, for all $x, y \in M$ is an anti fuzzy $k-$left ideal of N. $\square$

THEOREM 4.2. *Let $f : M \to N$ be a homomorphism of $\Gamma-$semirings and μ be an anti fuzzy ideal of M. If $\eta \circ f = \mu$ then η is an anti fuzzy ideal of N.*

PROOF. Let $f : M \to N$ be a homomorphism of $\Gamma-$semirings, μ be an anti fuzzy ideal of $M, \eta \circ f = \mu$ and $a, b \in N, \alpha \in \Gamma$. Then there exist $x, y \in M$ such that $f(x) = a$ and $f(y) = b$.

$$\begin{aligned}\mu(a+b) &= \eta(f(x)+f(y)) = \eta(f(x+y)) = \mu(x+y) \leqslant \max\{\mu(x), \mu(y)\}\\ &= \max\{\eta(f(x)), \eta(f(y))\}\end{aligned}$$

$$\begin{aligned}\mu(a\alpha b) &= \eta(f(x)\alpha f(y)) = \eta(f(x\alpha y)) = \mu(x\alpha y) \leqslant \min\{\mu(x), \mu(y)\}\\ &= \min\{\eta(f(x)), \eta(f(y))\}.\end{aligned}$$

Hence η is an anti fuzzy left ideal of $\Gamma-$semiring N. $\square$

DEFINITION 4.2. Let M and N be two $\Gamma-$semirings and f be a function from M into N. If μ is a fuzzy ideal of N then the pre-image of μ under f is the fuzzy subset of M is defined by $f^{-1}(\mu)(x) = \mu(f(x))$ for all $x \in M$.

THEOREM 4.3. *Let $f : M \to N$ be an onto homomorphism of $\Gamma-$semirings. If μ is an anti fuzzy $k-$left ideal N then $f^{-1}(\mu)$ is an anti fuzzy $k-$left ideal of M.*

PROOF. Suppose $f : M \to N$ is an onto homomorphism of $\Gamma-$semirings and μ is an anti fuzzy $k-$left ideal of N and $x_1, x_2 \in M, \alpha \in \Gamma$.

$$\begin{aligned} f^{-1}(\mu)(x_1 + x_2) &= \mu(f(x_1 + x_2) = \mu\big(f(x_1) + f(x_2)\big) \\ &\leqslant \max\{\mu(f(x_1)), \mu(f(x_2))\} = \max\{\mu(f^{-1}(x_1)), \mu(f^{-1}(x_2))\} \\ f^{-1}(\mu)(x_1\alpha x_2) &= \mu(f(x_1\alpha x_2)) \leqslant \min\{\mu(f(x_1)), \mu(f(x_2))\} \\ &= \min\{\mu(f^{-1}(x_1)), \mu(f^{-1}(x_2))\} \\ f^{-1}(\mu)(x) &= \mu(f(x)) \leqslant \max\{\mu(f(x + y)), \mu(f(y))\} \\ &= \max\{\mu(f^{-1}(x + y)), \mu(f^{-1}(x + y))\}. \end{aligned}$$

Hence $f^{-1}(\mu)$ is an anti fuzzy $k-$left ideal of M. □

THEOREM 4.4. *Let $f : M \to N$ be an onto anti homomorphism of $\Gamma-$semirings if η is a fuzzy left ideal of N and μ is the pre-image of η under f. Then μ is a fuzzy right ideal of M.*

PROOF. Let $f : M \to N$ be an onto anti homomorphism of $\Gamma-$semirings. If η is a fuzzy left ideal of N and μ is the pre-image of η under f and $x, y \in M, \alpha \in \Gamma$.

$$\begin{aligned} \mu(x + y) &= \eta\big(f(x + y)\big) = \eta\big(f(x) + f(y)\big) \geqslant \min\{\eta(f(x)), \eta(f(y))\} \\ &= \min\{\mu(x), \mu(y)\} \\ \mu(x\alpha y) &= \eta\big(f(x\alpha y)\big) = \eta\big(f(y)\alpha f(x)\big) \geqslant \eta(f(x)) = \mu(x). \end{aligned}$$

Hence μ is a fuzzy right ideal of $\Gamma-$semiring M. □

The following proof of the theorem is similar to Theorem 4.4

THEOREM 4.5. *Let $f : M \to N$ be an onto anti homomorphism of $\Gamma-$semirings. if η is a fuzzy right ideal of N and μ is the pre-image of η under f. Then μ is a fuzzy left ideal of M.*

THEOREM 4.6. *Let $f : M \to N$ be an onto anti homomorphism of $\Gamma-$semirings. If μ is an anti fuzzy left $k-$ideal of N then $f^{-1}(\mu)$ is an anti fuzzy right $k-$ideal of M.*

PROOF. Let $f : M \to N$ be an onto anti homomorphism of $\Gamma-$semirings, μ be an anti fuzzy left $k-$ideal of N and $x_1, x_2 \in M, \alpha \in \Gamma$.

$$\begin{aligned} f^{-1}(\mu)(x_1 + x_2) &= \mu(f(x_1 + x_2) = \mu\big(f(x_1) + f(x_2)\big) \\ &\leqslant \max\{\mu(f(x_1)), \mu(f(x_2))\} \\ &= \max\{f^{-1}(\mu)(x_1), f^{-1}(\mu)(x_2)\} \\ f^{-1}(\mu)(x_1\alpha x_2) &= \mu(f(x_1\alpha x_2)) \leqslant \min\{\mu(f(x_1)), \mu(f(x_2))\} \\ &= \min\{f^{-1}(\mu)(x_1), f^{-1}(\mu)(x_2)\} \\ f^{-1}(\mu)(x) &= \mu(f(x)) \leqslant \max\{\mu(f(x + y)), \mu(f(y))\} \\ &= \max\{f^{-1}(\mu)(x + y), f^{-1}(\mu)(x + y)\}. \end{aligned}$$

Hence $f^{-1}(\mu)$ is an anti fuzzy $k-$right ideal of M. □

THEOREM 4.7. *Let $f : M \to N$ be an onto anti homomorphism of $\Gamma-$semirings. The anti homomorphic image of an anti fuzzy left ideal of M is an anti fuzzy right ideal of N.*

PROOF. Let $f : M \to N$ be an onto anti homomorphism of $\Gamma-$semirings, μ be an anti fuzzy left ideal of M and η be a fuzzy subset of N such that $\eta \circ f = \mu$. Let $a, b \in N, \alpha \in \Gamma$ then there exist $x, y \in M$ such that $f(x) = a$ and $f(y) = b$.

$$\begin{aligned} \eta(a + b) &= \eta(f(x) + f(y)) = \eta(f(x + y)) \\ &= \mu(x + y) \leqslant \max\{\mu(x), \mu(y)\} \\ &= \max\{\eta(f(x)), \eta(f(y))\}. \\ \eta(a\alpha b) &= \eta(f(x)\alpha f(y)) = \eta(f(y\alpha x)) \\ &= \mu(y\alpha x) \leqslant \mu(x) \\ &= \eta(f(x)) = \eta(a). \end{aligned}$$

Hence η is an anti fuzzy right ideal of $\Gamma-$semiring N. □

THEOREM 4.8. *Let $f : M \to N$ be an onto anti homomorphism of $\Gamma-$semiring. If η is an anti fuzzy left $k-$ideal of N and μ is the pre-image of η under f then μ is an anti fuzzy right $k-$ideal of M.*

PROOF. Let $f : M \to N$ be an onto anti homomorphism of $\Gamma-$semiring, η be is an anti fuzzy left $k-$ideal of N and μ be the pre-image of η under f and $x, y \in M, \alpha \in \Gamma$.

$$\begin{aligned} \mu(x + y) &= \eta(f(x + y)) = \eta(f(x) + f(y)) \leqslant \max\{\eta(f(x)), \eta(f(y))\} \\ &= \max\{\mu(x), \mu(y)\} \\ \mu(x\alpha y) &= \eta(f(x\alpha y)) = \eta(f(x)\alpha f(y)) \leqslant \eta(f(x)) = \mu(x) \\ \mu(x) &= \eta(f(x)) \leqslant \max\{\eta(f(x + y)), \eta(f(x))\} = \max\{\mu(x + y), \mu(y)\} \end{aligned}$$

Hence μ is an anti fuzzy right $k-$ideal of $\Gamma-$semiring. □

THEOREM 4.9. *Let M and N be $\Gamma-$semirings and $\phi : M \to N$ be an onto anti homomorphism. If f is a ϕ invariant fuzzy ideal of M then $\phi(f)$ is a fuzzy ideal of N.*

PROOF. Let M and N be $\Gamma-$semirings and $\phi : M \to N$ be an anti homomorphism and f be a ϕ invariant fuzzy ideal of M. If $x = \phi(a) \Rightarrow \phi^{-1}(x) = a$. Let $t \in \phi^{-1}(x)$ then $\phi(t) = x = \phi(a)$, since f is ϕ invariant, $f(t) = f(a) \Rightarrow \phi(f)(x) = \inf_{t\in\phi^{-1}(x)} f(t) = f(a)$. Hence $\phi(f)(x) = f(a)$.

Let $x, y \in N$. Then there exist $a, b \in M$ such that

$$\begin{aligned}\phi(a) = x, \phi(b) = y \Rightarrow \phi(a+b) =& x+y \\ \Rightarrow \phi(f)(x+y) = f(a+b) &\geqslant \min\{f(a), f(b)\} \\ &= \min\{\phi(f)(x), \phi(f)(y)\} \\ \phi(f)(x\alpha y) = f(b\alpha a) &\geqslant \max\{f(a), f(b)\} \\ &= \max\{\phi(f)(x), \phi(f)(y)\}.\end{aligned}$$

Hence $\phi(f)$ is a fuzzy ideal of N. □

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