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Some properties of Hermite matrix polynomials

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Abstract. The main aim in this paper, we use the differential operators and matrix polynomial sets, generating matrix functions, integrals representation to derive the properties of Hermite matrix polynomials. Finally, we obtain an expansion of Hermite matrix polynomials in a series of Laguerre matrix polynomials and the Christoffel's formula of summation is established.

1. Introduction

Theory of special functions plays an important role in the formalism of mathematical physics. Hermite and Chebyshev polynomials in [**19**] are among the most important special functions, with very diverse applications to physics, engineering and mathematical physics ranging from abstract number theory to problems of physics and engineering. Recently, the Hermite matrix polynomials have been introduced and studied in a number of papers [**1, 12, 13, 20, 15, 16, 17, 36**]. This approach has indeed allowed the derivation of the Hermite matrix polynomials of variables and its extension to the Hermite, Gegenbauer, Bessel and pseudo Chebyshev matrix polynomials in [**14, 18, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 35**].

Our primary purpose in this paper deals with the introduction and study of Hermite matrix polynomials taking advantage of those recently treated in [**1, 2**]. The organization of the paper is as follows: In Section 2, differential operators and matrix polynomial sets are proved for Hermite matrix polynomials. Section 3, generating matrix functions for Hermite matrix polynomials are established. Integrals of representation for Hermite matrix polynomials are shown in Section 4. Finally, we obtain an expansion of Hermite matrix polynomials in a series of Laguerre matrix polynomials and the Christoffel's formula of summation is established in Section 5.

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If D_0 is the complex plane cut along the negative real axis and $\log(z)$ denotes the principle logarithm of z [**5**], then $z^{\frac{1}{2}}$ represents $\exp(\frac{1}{2}\log(z))$. Its spectrum $\sigma(A)$ denotes the set of all eigenvalues of A. If A is a matrix in $\mathbb{C}^{N\times N}$ with $\sigma(A) \subset D_0$, then $A^{\frac{1}{2}} = \sqrt{A} = \exp(\frac{1}{2}\log(A))$ denotes the image by $z^{\frac{1}{2}}$ of the matrix functional calculus acting on the matrix A. The two-norm of A is denoted by $||A||_2$ and it is defined by

$$||A||_2 = \sup_{x\neq 0} \frac{||Ax||_2}{||x||_2}$$

where for a vector y in $\mathbb{C}^N$, $||y||_2 = (y^T y)^{\frac{1}{2}}$ is the Euclidean norm of y.

If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane and if A is a matrix in $\mathbb{C}^{N\times N}$ with $\sigma(A) \subset \Omega$, then from the properties of the matrix functional calculus [**5**], it follows that

$$f(A)g(A) = g(A)f(A). \tag{1.1}$$

Hence, if B in $\mathbb{C}^{N\times N}$ is a matrix for which $\sigma(B) \subset \Omega$ and also if $AB = BA$, then

$$f(A)g(B) = g(B)f(A). \tag{1.2}$$

Let A be a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfying the condition [**12, 13**]

$$Re(z) > 0, \quad \text{for all } z \in \sigma(A). \tag{1.3}$$

It has been seen by Defez and Jódar [**2**] that if $A(k,n)$ and $B(k,n)$ are matrices in $\mathbb{C}^{N\times N}$ for $n \geqslant 0$, $k \geqslant 0$, it follows (in an analogous way to the proof of Lemma **11** of [**19**]) that

$$\begin{aligned} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k), \\ \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]} A(k,n-2k). \end{aligned} \tag{1.4}$$

Similarly to (1.4), we can write

$$\begin{aligned} \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+k), \\ \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]} A(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+2k). \end{aligned} \tag{1.5}$$

In the following, we will apply the above results to Hermite matrix polynomials and we will see that the results, summarized in this section, can be exploited to state quite general results.

1.1. Hermite matrix polynomials. One of the most direct ways of exploring generalized classes of Hermite matrix polynomials is to start from modified forms of the ordinary Hermite matrix polynomials and generating matrix function.

The Hermite matrix polynomials $H_n(x,A)$ of single variable was defined by using the generating matrix function [**1, 6, 12, 13, 20**] in the following form

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x,A)=\exp\left(xt\sqrt{2A}-t^2I\right) \tag{1.6}$$

where A is a positive stable matrix in $\mathbb{C}^{N\times N}$ and I is the identity matrix in $\mathbb{C}^{N\times N}$.

The Hermite matrix polynomials are explicitly expressed as follows

$$H_n(x,A)=n!\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k}{k!(n-2k)!}(x\sqrt{2A})^{n-2k},\quad n\geqslant 0. \tag{1.7}$$

It is clear that

$$\begin{aligned}&H_{-1}(x,A)=\mathbf{0},\;\; H_0(x,A)=I,\;\; H_1(x,A)=x\sqrt{2A}\\ &\text{and }\; H_n(-x,A)=(-1)^nH_n(x,A).\end{aligned}$$

where $\mathbf{0}$ is the null matrix in $\mathbb{C}^{N\times N}$.

From (1.6) and (1.7). It is easy to prove that

$$\begin{aligned}&\frac{d}{dz}H_n(x,A)=n\sqrt{2A}H_{n-1}(x,A),\\ &H_{n+1}(x,A)=\left[z\sqrt{2A}-\frac{2}{\sqrt{2A}}\frac{d}{dz}\right]H_n(x,A).\end{aligned} \tag{1.8}$$

The matrix differential equations satisfied by $H_n(x,A)$ can be straightforwardly deduced by introducing the shift operators

$$\begin{aligned}&\widehat{P}=\frac{1}{\sqrt{2A}}\frac{d}{dx},\\ &\widehat{M}=x\sqrt{2A}-\frac{2}{\sqrt{2A}}\frac{d}{dx}\end{aligned} \tag{1.9}$$

which act on $H_n(x,A)$ according to the rules

$$\begin{aligned}&\widehat{P}H_n(x,A)=nH_{n-1}(x,A),\\ &\widehat{M}H_n(x,A)=H_{n+1}(x,A).\end{aligned} \tag{1.10}$$

Using the identity

$$\widehat{M}\widehat{P}H_n(x,A)=nH_n(x,A) \tag{1.11}$$

from (1.11), we find that $H_n(x,A)$ satisfy the following matrix differential equations of second order [**6, 20, 21**]

$$\left[\frac{d^2}{dx^2}I-\frac{z}{2}\frac{d}{dx}(\sqrt{2A})^2+\frac{n}{2}(\sqrt{2A})^2\right]H_n(x,A)=\mathbf{0}. \tag{1.12}$$

The Hermite matrix polynomials are defined through the operational rule [**1**] in the form

$$H_n(x,A)=\exp\big(-\frac{1}{(\sqrt{2A})^2}\frac{d^2}{dx^2}\big)\big(x\sqrt{2A}\big)^n. \tag{1.13}$$

In addition, the inverse of (1.13) allows concluding that

$$(x\sqrt{2A})^n=\exp\left(\frac{1}{(\sqrt{2A})^2}\frac{d^2}{dx^2}\right)H_n(x,A). \tag{1.14}$$

For the sake of clarity, we recall that an expansion of $z^n\ I$ in a series of Hermite matrix polynomials was in [**2, 6**]

$$(x\sqrt{2A})^n=\sum_{k=0}^{[\frac{1}{2}n]}\frac{n!}{k!(n-2k)!}H_{n-2k}(x,A). \tag{1.15}$$

Furthermore, the n^{th} Laguerre matrix polynomials $L_n^{(A,\lambda)}(x)$ is defined by [**7, 12**]

$$L_n^{(A,\lambda)}(x)=\sum_{k=0}^{n}\frac{(-1)^k(A+I)_n[(A+I)_k]^{-1}\lambda^k x^k}{k!(n-k)!};n\geqslant 0 \tag{1.16}$$

where A is a matrix in $\mathbb{C}^{N\times N}$ such that $-k$ is not an eigenvalue of A, for every integer $k>0$ and λ is a complex number such that $Re(\lambda)>0$. In (1.16), putting $\lambda=1$ gives

$$L_n^{(A)}(x)=\sum_{k=0}^{n}\frac{(-1)^k(A+I)_n[(A+I)_k]^{-1}x^k}{k!(n-k)!}. \tag{1.17}$$

So, we derive an expansion of x^nI in a series of Laguerre matrix polynomials in the form

$$x^nI=n!\sum_{k=0}^{n}\frac{(-1)^k(A+I)_n[(A+I)_k]^{-1}}{(n-k)!}L_k^{(A)}(x). \tag{1.18}$$

The next section is devoted to the differential operators and matrix polynomial sets of Hermite matrix polynomials, treated within the context of the point of view so far developed.

2. Differential operators and matrix polynomial sets

Let $\varphi_n(x,A)$; $n=0,1,2,...$, be any simple set of matrix polynomials. Let us define the set of matrix polynomials $T_n(x,A)$, $n\geqslant 0$, by

$$T_0(x,A)D\varphi_1(x,A)=\varphi_0(x,A);D=\frac{d}{dx} \tag{2.1}$$

and

$$T_n(x,A)D^{n+1}\varphi_{n+1}(x,A)=\varphi_n(x,A)-\sum_{k=0}^{n-1}T_k(x,A)D^{k+1}\varphi_{n+1}(x);n\geqslant 1. \tag{2.2}$$

Because $\varphi_n(x,A)$ is of degree precisely n for each n, it follows that $T_n(x,A)$ is uniquely defined and is of degree $\leqq n$. Note that $D\varphi_1(x,A)$ is constant, as is

$\varphi_0(x,A)$, so $T_0(x,A)$ is constant. For $n \geqq 1$ each $T_n(x)$ is defined by (2.2) in terms of pervious elements of the set, $T_k(x,A)$ for $0 \leqq k \leqq (n-1)$. Because $D^{n+1}\varphi_{n+1}(x,A)$ is constant and the degree of $T_k(x,A)D^{k+1}\varphi_{k+1}(x,A)$ exceeds the degree of $T_k(x,A)$ by exactly $(n-k)$, each member of (2.2) has degree at most n.

Theorem 2.1. *For the simple set of matrix polynomials $\varphi_n(x,A)$ there exists a unique differential operator of the form*

$$J(x,D,A) = \sum_{k=0}^{\infty} T_k(x,A)D^{k+1} \tag{2.3}$$

in which $T_k(x,A)$ is a matrix polynomials of degree $\leqq k$, for which

$$J(x,D,A)\varphi_n(x,A) = \varphi_{n-1}(x,A); \quad n \geqq 1. \tag{2.4}$$

It is important that J be independent of n.

Proof. The requirement (2.4) demands that, for $n \geqq 1$

$$\sum_{k=0}^{n-1} T_k(x,A)D^{k+1}\varphi_n(x,A) = \varphi_{n-1}(x,A)$$

this is merely a restatement of (2.1) and (2.2). Equations (2.1) and (2.2) as we saw, determine $T_k(x,A)$ uniquely. □

We say that the matrix polynomials set $\varphi_n(x,A)$ belongs to the operator J and that J is operator associated with the set $\varphi_n(x,A)$. There is only one such operator associated with a given $\varphi_n(x,A)$, but there are infinitely many sets of matrix polynomials belonging to the same operator.

Theorem 2.2. *A necessary and sufficient condition that two simple sets of matrix polynomials $\varphi_n(x,A)$ and $\psi_n(x,A)$ belonging to the same operator J is that there exists a sequence of numbers b_k independent of n, such that*

$$\psi_n(x,A) = \sum_{k=0}^{n} b_k\varphi_{n-k}(x,A). \tag{2.5}$$

Assume (2.5) to hold. There exists an operator J to which $\varphi_n(x,A)$ belongs. That $\psi_n(x,A)$ belongs to the same operator follows from

$$J\psi_n(x,A) = \sum_{k=0}^{n} b_k J\varphi_{n-k}(x,A) = \sum_{k=0}^{n} b_k\varphi_{n-k-1}(x,A) = \psi_{n-1}(x,A).$$

Next assume that $\varphi_n(x,A)$ and $\psi_n(x,A)$ belong to the same operator J. We need to show that the b_k of (2.5) exists. We know, because $\varphi_n(x,A)$ and $\psi_n(x,A)$ are simple sets of matrix polynomials, that there exist the relations

$$\psi_n(x,A) = \sum_{k=0}^{n-1} A(k,n)\varphi_{n-k}(x,A), \tag{2.6}$$

in general, the coefficients $A(k,n)$ depend upon n as well as on k. Since $\varphi_n(x,A)$ and $\psi_n(x,A)$ belong to J, we may apply J to each member of (2.6) and obtain

$$\psi_{n-1}(x,A)=\sum_{k=0}^{n-1}A(k,n)\varphi_{n-k-1}(x,A);\quad n\geqq 1. \tag{2.7}$$

Recall that $J\varphi_0(x,A)=0$, which is the reason that the term $A(k,n)$ dropped out when the operator J was applied to (2.6). We may shift index from n to $(n+1)$ in (2.7) to get

$$\psi_n(x,A)=\sum_{k=0}^{n}A(k,n+1)\varphi_{n-k}(x,A);\quad n\geqq 0. \tag{2.8}$$

Comparing (2.6) and (2.8), we see that

$$A(k,n)=A(k,n+1)$$

for all k, n. Then $A(k,n)=b_k$, independent of n.

Not every operator of the form (2.3) is associated with some matrix polynomial set in the sense we have defined. For the operator J of the form (2.3) to be associated with some simple set, it is necessary and sufficient that J transform every matrix polynomial of degree precisely n into a matrix polynomial of degree precisely $(n-1)$.

Example 2.1. Determine the operator associated with the set $\varphi_n(x,A)=\frac{H_n(x,A)}{(n!)^2}$, in which $H_n(x,A)$ is the Hermite matrix polynomials.

Here, we have

$$\begin{aligned}
\varphi_0(x,A)&=H_0(x,A)=I,\\
\varphi_1(x,A)&=H_1(x,A)=x\sqrt{2A},\\
\varphi_2(x,A)&=\frac{1}{4}H_2(x,A)=\frac{1}{4}(x\sqrt{2A})^2-\frac{1}{2}I,\\
\varphi_3(x,A)&=\frac{1}{36}H_3(x,A)=\frac{1}{36}(x\sqrt{2A})^3-\frac{1}{6}x\sqrt{2A},\\
\varphi_4(x,A)&=\frac{1}{(4!)^2}H_4(x,A)=\frac{1}{(4!)^2}(x\sqrt{2A})^4-\frac{12}{(4!)^2}(x\sqrt{2A})^2+\frac{12}{(4!)^2}I
\end{aligned}$$

and

$$\varphi_5(x,A)=\frac{1}{(5!)^2}H_5(x,A)=\frac{1}{(5!)^2}(x\sqrt{2A})^5-\frac{20}{(5!)^2}(x\sqrt{2A})^3+\frac{60}{(5!)^2}(x\sqrt{2A}),\quad \text{etc.}$$

We seek an operator J of the form

$$J=\sum_{k=0}^{\infty}T_k(x,A)D^{k+1}$$

such that $J\varphi_n=\varphi_{n-1}$ for $n\geqq 1$. Then

$$T_0(x,A)D\varphi_1=\varphi_0 \text{ or } T_0(x,A)\sqrt{2A}=I$$

so that $T_0(x,A) = (\sqrt{2A})^{-1}$. Next we have

$$\sum_{k=0}^{1} T_k(x,A) D^{k+1} \varphi_2(x,A) = \varphi_1(x,A),$$

$$[T_0(x,A)D + T_1(x,A)D^2]\varphi_2(x,A) = \varphi_1(x,A)$$

or

$$[(\sqrt{2A})^{-1}D + T_1(x,A)D^2][\frac{1}{4}(x\sqrt{2A})^2 - \frac{1}{2}I] = x\sqrt{2A}.$$

Then, we get

$$\frac{x\sqrt{2A}}{2} + T_1(x,A)\frac{(\sqrt{2A})^2}{2} = x\sqrt{2A}.$$

So that $T_1(x,A) = x(\sqrt{2A})^{-1}$. In turn

$$[T_0(x,A)D + T_1(x,A)D^2 + T_2(x,A)D^3]\varphi_3(x,A) = \varphi_2(x,A)$$

or

$$[(\sqrt{2A})^{-1}D + x(\sqrt{2A})^{-1}D^2 + T_2(x,A)D^3][\frac{1}{36}(x\sqrt{2A})^3 - \frac{1}{6}x\sqrt{2A}] = \frac{1}{4}(x\sqrt{2A})^2 - \frac{1}{2}I,$$

we obtain

$$(\sqrt{2A})^{-1}[\frac{x^2}{12}(\sqrt{2A})^3 - \frac{1}{6}\sqrt{2A}] + x(\sqrt{2A})^{-1}[\frac{x}{6}(\sqrt{2A})^3] + T_2(x,A)[\frac{1}{6}(\sqrt{2A})^3] = \frac{1}{4}(x\sqrt{2A})^2 - \frac{1}{2}I \tag{2.9}$$

from which $T_2(x,A) = -2(\sqrt{2A})^{-3}$.

If we continue the above procedure, we find that $T_3(x,A) = 0$, $T_4(x,A) = 0$, and we begin to suspect that J may terminate. Let us therefore define

$$J_1 = (\sqrt{2A})^{-1}D + x(\sqrt{2A})^{-1}D^2 - 2(\sqrt{2A})^{-2}D^3$$

operate on $\varphi_n(x,A)$ with J_1 and see whether the result is $\varphi_{n-1}(x,A)$.

Now, we have

$$\begin{aligned} J_1\varphi_n(x,A) = J_1\frac{H_n(x,A)}{(n!)^2} &= \frac{1}{(n!)^2}\left[(\sqrt{2A})^{-1}D + x(\sqrt{2A})^{-1}D^2 - 2(\sqrt{2A})^{-2}D^3\right]H_n(x,A) \\ &= \frac{1}{(n!)^2}\left[(\sqrt{2A})^{-1}H_n'(x,A) + x(\sqrt{2A})^{-1}H_n''(x,A) - 2(\sqrt{2A})^{-3}H_n'''(x,A)\right] \\ &= -\frac{1}{(n!)^2}(\sqrt{2A})^{-3}\left[2H_n'''(x,A) - x(\sqrt{2A})^2H_n''(x,A) - (\sqrt{2A})^2H_n'(x,A)\right]. \end{aligned}$$

From Hermite's matrix differential equation

$$\left[\frac{d^2}{dx^2}I - \frac{x}{2}(\sqrt{2A})^2\frac{d}{dx} + \frac{n}{2}(\sqrt{2A})^2\right]H_n(x,A) = \mathbf{0},$$

$$2H_n''(x,A) - x(\sqrt{2A})^2H_n'(x,A) + n(\sqrt{2A})^2H_n(x,A) = \mathbf{0}$$

we obtain

$$2H_n'''(x,A) - x(\sqrt{2A})^2H_n''(x,A) - (\sqrt{2A})^2H_n'(x,A) + n(\sqrt{2A})^2H_n'(x,A) = \mathbf{0}.$$

So that we have

$$J_1\varphi_n(x,A) = -\frac{(\sqrt{2A})^{-3}}{(n!)^2}\Big[-n(\sqrt{2A})^2 H'_n(x,A)\Big] = \frac{(\sqrt{2A})^{-1}H'_n(x,A)}{(n!)(n-1)!}.$$

But we also know that $H'_n(x,A) = n\sqrt{2A}H_{n-1}(x,A)$. Hence, we have

$$J_1\varphi_n(x,A) = \frac{nH_{n-1}(x,A)}{(n!)(n-1)!} = \frac{H_{n-1}(x,A)}{((n-1)!)^2} = \varphi_{n-1}(x,A)$$

as desired. Therefore $\varphi_n(x,A) = \frac{H_n(x,A)}{(n!)^2}$ belongs to operator J_1 of (2.9).

3. Generating matrix functions for Hermite matrix polynomials

Now, we can see state that the generating matrix functions for Hermite matrix polynomials with on their properties and prove the following.

THEOREM 3.1. *Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1.3), then*

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}H_{2n}(x,A)t^n = e^t\cos(x\sqrt{2tA}) \tag{3.1}$$

and

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}H_{2n+1}(x,A)t^n = \frac{e^t\sin(x\sqrt{2tA})}{\sqrt{t}}. \tag{3.2}$$

PROOF. By using (1.7), consider the series in the form

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}H_{2n}(x,A)t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^{n+k}(x\sqrt{2A})^{2n-2k}}{k!(2n-2k)!}t^n\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{n+2k}(x\sqrt{2A})^{2n}}{k!(2n)!}t^{n+k}\\ &= e^t\cos(x\sqrt{2tA}).\end{aligned}$$

Therefore, (3.1) follows. The series can be given

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}H_{2n+1}(x,A)t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{2n+1}{2}]}\frac{(-1)^{n+k}(x\sqrt{2A})^{2n+1-2k}}{k!(2n+1-2k)!}t^n\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{n+2k}(x\sqrt{2A})^{2n+1}}{k!(2n+1)!}t^{n+k}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{n+2k}(x\sqrt{2A})^{2n+1}}{k!(2n+1)!}t^{n+k}\\ &= t^{-\frac{1}{2}}e^t\sin(x\sqrt{2tA}).\end{aligned}$$

The proof of Theorem **3.1** is completed. □

In the following theorem, we obtain another generating matrix function for Hermite matrix polynomials as follows.

THEOREM 3.2. *Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1.3), then*

$$\sum_{n=0}^{\infty}\frac{H_{2n}(x,A)}{(2n)!}t^{2n}=e^{-t^2}\cosh(xt\sqrt{2A}) \tag{3.3}$$

and

$$\sum_{n=0}^{\infty}\frac{H_{2n+1}(x,A)}{(2n+1)!}t^{2n+1}=e^{-t^2}\sinh(xt\sqrt{2A}). \tag{3.4}$$

PROOF. Using (1.7), we consider the series

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{H_{2n}(x,A)}{(2n)!}t^{2n}&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k(x\sqrt{2A})^{2n-2k}}{k!(2n-2k)!}t^{2n}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(x\sqrt{2A})^{2n}}{k!(2n)!}t^{2n+2k}\\&=e^{-t^2}\cosh(xt\sqrt{2A}).\end{aligned}$$

Therefore, (3.3) follows. The series can be given in the form

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{H_{2n+1}(x,A)}{(2n+1)!}t^{2n+1}&=\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{2n+1}{2}]}\frac{(-1)^k(x\sqrt{2A})^{2n+1-2k}}{k!(2n+1-2k)!}t^{2n+1}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(x\sqrt{2A})^{2n+1}}{k!(2n+1)!}t^{2n+2k+1}\\&=e^{-t^2}\sinh(xt\sqrt{2A}).\end{aligned}$$

Thus the proof of Theorem **3.2** is completed. □

The above relations will be used, along with the Hermite matrix polynomials to derive the following theorem.

THEOREM 3.3. *Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1.3). Then the Hermite matrix polynomials have the following generating matrix function*

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n H_{2n}(x,A)t^{2n}}{(2n)!}=(1+t^2)^{-\alpha}\,{}_1F_1\left(\alpha;\frac{1}{2};\frac{x^2t^2(\sqrt{2A})^2}{4(1+t^2)}\right) \tag{3.5}$$

where α is a positive integer and $|t^2|<1$.

PROOF. Let us consider the sum and using (1.7), yields

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(\alpha)_n H_{2n}(x,A)t^{2n}}{(2n)!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k(\alpha)_n(x\sqrt{2A})^{2n-2k}t^{2n}}{k!(2n-2k)!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(\alpha)_{n+k}(x\sqrt{2A})^{2n}t^{2n+2k}}{k!(2n)!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(\alpha+n)_k(\alpha)_n(xt\sqrt{2A})^{2n}t^{2k}}{k!(2n)!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\alpha)_n(xt\sqrt{2A})^{2n}}{(2n)!}\frac{(-1)^k(\alpha+n)_k t^{2k}}{k!}\\ &= \sum_{n=0}^{\infty}\frac{(\alpha)_n(xt\sqrt{2A})^{2n}}{(2n)!}\frac{1}{(1+t^2)^{\alpha+n}}\\ &= (1+t^2)^{-\alpha}\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!2^{2n}(\frac{1}{2})_n}\frac{(xt\sqrt{2A})^{2n}}{(1+t^2)^n}\\ &= (1+t^2)^{-\alpha}\ {}_1F_1\left(\alpha;\frac{1}{2};\frac{x^2t^2(\sqrt{2A})^2}{4(1+t^2)}\right).\end{aligned}$$

The proof of Theorem **3.3** is completed. □

4. Integrals of Hermite matrix polynomials

Now, we can see state that the integrals of Hermite matrix polynomials with on their properties and prove the following.

THEOREM 4.1. *Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1.3), then we have*

$$\exp\left(-\frac{Ax^2}{2}\right)=\frac{2}{\sqrt{\pi}}\int_0^{\infty}\exp\left(-t^2\right)\cos(xt\sqrt{2A})dt. \tag{4.1}$$

PROOF. From Taylor's expansion, we have

$$\cos(x\sqrt{2A})=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(x\sqrt{2A})^{2n}. \tag{4.2}$$

Since the summation in the right-hand side of the above equality is finite, then the series and the integral can be permuted. From the definition of Gamma function, we have

$$\int_0^{\infty}e^{-t^2}t^{2n}dt=\frac{1}{2}\Gamma(n+\frac{1}{2}) \tag{4.3}$$

and from Legendre duplication formula, we have

$$\Gamma(n+\frac{1}{2})=\frac{(2n)!\sqrt{\pi}}{2^{2n}n!} \tag{4.4}$$

then, we have used the Gamma function as well as Legendre duplication formula

$$\int_0^\infty \exp\Big(-t^2\Big)\cos(xt\sqrt{2A})dt = \int_0^\infty \exp\Big(-t^2\Big)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(xt\sqrt{2A})^{2n}dt$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(x\sqrt{2A})^{2n}\int_0^\infty \exp\Big(-t^2\Big)t^{2n}dt = \sum_{n=0}^{\infty}\frac{(-1)^n}{2(2n)!}(x\sqrt{2A})^{2n}\Gamma(n+\frac{1}{2})$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2\,(2n)!}(x\sqrt{2A})^{2n}\frac{(2n)!\sqrt{\pi}}{n!2^{2n}} = \sum_{n=0}^{\infty}\frac{(-1)^n\sqrt{\pi}}{2\,n!2^{2n}}(x\sqrt{2A})^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n\sqrt{\pi}}{2\,n!}(\frac{x\sqrt{2A}}{2})^{2n}$$

The proof of Theorem **4.1** is completed. □

THEOREM 4.2. *Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1.3), then we have*

$$H_{2n}(x,A) = \frac{(-1)^n 2^{2n+1}}{\sqrt{\pi}}\exp\left(\frac{Ax^2}{2}\right)\int_0^\infty \exp\Big(-t^2\Big)t^{2n}\cos(xt\sqrt{2A})dt \tag{4.5}$$

and

$$H_{2n+1}(x,A) = \frac{(-1)^n 2^{2n+2}}{\sqrt{\pi}}\exp\left(\frac{Ax^2}{2}\right)\int_0^\infty \exp\Big(-t^2\Big)t^{2n+1}\sin(xt\sqrt{2A})dt. \tag{4.6}$$

PROOF. Differentiating (4.3) $2n$ times with respect to x, we get

$$\frac{d^{2n}}{dx^{2n}}\exp\left(-\frac{Ax^2}{2}\right) = \frac{2}{\sqrt{\pi}}\int_0^\infty \exp\Big(-t^2\Big)(-1)^n(t\sqrt{2A})^{2n}\cos(xt\sqrt{2A})dt. \tag{4.7}$$

But, from Rodrigues's formula $H_n(x,A) = (-1)^n(\frac{A}{2})^{-\frac{n}{2}}\exp\left(\frac{Ax^2}{2}\right)\frac{d^n}{dx^n}\exp\left(-\frac{Ax^2}{2}\right)$, we have $H_{2n}(x,A) = (\frac{A}{2})^{-n}\exp\left(\frac{Ax^2}{2}\right)\frac{d^{2n}}{dx^{2n}}\exp\left(-\frac{Ax^2}{2}\right)$. Then, we have

$$H_{2n}(x,A) = \frac{(-1)^n 2^{2n+1}}{\sqrt{\pi}}\exp\left(\frac{Ax^2}{2}\right)\int_0^\infty \exp\Big(-t^2\Big)t^{2n}\cos(xt\sqrt{2A})dt.$$

Differentiating (4.3) $2n+1$ times with respect to x, we get

$$\frac{d^{2n+1}}{dx^{2n+1}}\exp\left(-\frac{Ax^2}{2}\right) = \frac{2}{\sqrt{\pi}}\int_0^\infty \exp\Big(-t^2\Big)(-1)^n(t\sqrt{2A})^{2n+1}\sin(xt\sqrt{2A})dt \tag{4.8}$$

and $H_{2n+1}(x,A) = (-1)^{2n+1}(\frac{A}{2})^{-\frac{2n+1}{2}}\exp\left(\frac{Ax^2}{2}\right)\frac{d^{2n+1}}{dx^{2n+1}}\exp\left(-\frac{Ax^2}{2}\right)$. Then we get

$$H_{2n+1}(x,A) = \frac{(-1)^n 2^{2n+2}}{\sqrt{\pi}}\exp\left(\frac{Ax^2}{2}\right)\int_0^\infty \exp\Big(-t^2\Big)t^{2n+1}\sin(xt\sqrt{2A})dt.$$

The proof of Theorem **4.2** is completed. □

5. Expansion of Hermite matrix polynomials in a series of Laguerre matrix polynomials

In this section, the Hermite matrix polynomials can be expanded in a series of Laguerre matrix polynomials. For the sake of clarity, we recall that if A is a matrix in $\mathbb{C}^{N\times N}$ satisfies the condition (1.3), let us employ (1.5), (1.7) and (1.18) in expanding the Hermite matrix polynomial in a series of Laguerre matrix polynomials. We consider the series

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{H_n(x,A)t^n}{n!}&=\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^k(x\sqrt{2A})^{n-2k}}{k!(n-2k)!}t^n\\ &=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(x\sqrt{2A})^{n}}{k!n!}t^{n+2k} \qquad (5.1)\\ &=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{n}\frac{(-1)^{k+s}(\sqrt{2A})^{n}}{k!(n-s)!}(A+I)_n[(A+I)_s]^{-1}L_s^{(A)}(x)t^{n+2k}\end{aligned}$$

by using (1.5), becomes

$$\sum_{n=0}^{\infty}\frac{H_n(x,A)t^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^{k+s}(\sqrt{2A})^{n+s}}{k!n!}(A+I)_{n+s}[(A+I)_s]^{-1}L_s^{(A)}(x)t^{n+s+2k}.$$

From (1.4), we have

$$\sum_{n=0}^{\infty}\frac{H_n(x,A)t^n}{n!}=\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^{k+s}(\sqrt{2A})^{n+s-2k}}{k!(n-2k)!}(A+I)_{n+s-2k}[(A+I)_s]^{-1}L_s^{(A)}(x)t^{n+s}.$$

The reciprocal gamma function denoted by $\Gamma^{-1}(z)=\frac{1}{\Gamma(z)}$ is an entire function of the complex variable z. Then for any matrix A in $\mathbb{C}^{N\times N}$, the image of $\Gamma^{-1}(z)$ acting on A denoted by $\Gamma^{-1}(A)$ is a well-defined matrix. Then $\Gamma(A)$ is an invertible matrix, its inverse coincides with $\Gamma^{-1}(A)$ and one gets the formula [**19**]

$$\begin{aligned}(A)_n&=A(A+I)...(A+(n-1)I)\\ &=\Gamma(A+nI)\Gamma^{-1}(A);\quad n\geqslant 1;\quad (A)_0=I.\end{aligned}\qquad (5.2)$$

From (5.2), it is easy to find that

$$(A)_{n-k}=(-1)^k(A)_n[(I-A-nI)_k]^{-1};\quad 0\leqslant k\leqslant n. \qquad (5.3)$$

In accordance with (5.2) and (5.3), one gets

$$(A+I)_{n+s-2k}=2^{-2k}(A+I)_{n+s}[(-\frac{1}{2}(A+(n+s-1)I))_k]^{-1}[(-\frac{1}{2}(A+(n+s)I))_k]^{-1}.$$

Thus, we know that

$$\frac{1}{(n-2k)!}=\frac{(-n)_{2k}}{n!}.$$

Therefore, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{H_n(x,A)t^n}{n!} =& \sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^{k+s}(\sqrt{2A})^{n+s-2k}(-n)_{2k}}{k!n!}\\ & 2^{-2k}(A+I)_{n+s}[(-\frac{1}{2}(A+(n+s-1)I))_k]^{-1}[(-\frac{1}{2}(A+(n+s)I))_k]^{-1}\\ & [(A+I)_s]^{-1}L_s^{(A)}(x)t^{n+s}.\end{aligned}$$

Also, we recall the following relation

$$(-n)_{2k} = 2^{2k}(-\frac{n}{2})_k(-\frac{n-1}{2})_k$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{H_n(x,A)t^n}{n!} =& \sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^{k+s}(\sqrt{2A})^{n+s-2k}}{k!n!}(-\frac{n}{2})_k(-\frac{n-1}{2})_k\\ & (A+I)_{n+s}[(-\frac{1}{2}(A+(n+s-1)I))_k]^{-1}[(-\frac{1}{2}(A+(n+s)I))_k]^{-1}\\ & [(A+I)_s]^{-1}L_s^{(A)}(x)t^{n+s}\\ =& \sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s(\sqrt{2A})^{n+s}}{n!}(A+I)_{n+s}[(A+I)_s]^{-1}\\ & {}_2F_2\Big(-\frac{1}{2}nI,-\frac{1}{2}(n-1)I;-\frac{1}{2}(A+(n+s)I),-\frac{1}{2}(A+(n+s-1)I);-\frac{1}{(\sqrt{2A})^2}\Big)\\ & L_s^{(A)}(x)t^{n+s}\\ =& \sum_{n=0}^{\infty}\sum_{s=0}^{n}\frac{(-1)^s(\sqrt{2A})^{n}}{(n-s)!}(A+I)_{n}[(A+I)_s]^{-1}\\ & {}_2F_2\Big(-\frac{1}{2}(n-s)I,-\frac{1}{2}(n-s-1)I;-\frac{1}{2}(A+nI),-\frac{1}{2}(A+(n-1)I);-\frac{1}{(\sqrt{2A})^2}\Big)\\ & L_s^{(A)}(x)t^{n}.\end{aligned}$$

Again we collect power of t

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{H_n(x,A)t^n}{n!} =& \sum_{n=0}^{\infty}\sum_{s=0}^{n}\frac{(-1)^s(\sqrt{2A})^{n}}{(n-s)!}(A+I)_{n}[(A+I)_s]^{-1}\\ & {}_2F_2\Big(-\frac{1}{2}(n-s)I,-\frac{1}{2}(n-s-1)I;-\frac{1}{2}(A+nI),-\frac{1}{2}(A+(n-1)I);-\frac{1}{(\sqrt{2A})^2}\Big)\\ & L_s^{(A)}(x)t^{n}\end{aligned}$$

Therefore, we obtain an expansion of Hermite matrix polynomials as a series of Laguerre matrix polynomials in the form

$$\begin{aligned}H_n(x,A) =& n!\sum_{s=0}^{n}\frac{(-1)^s(\sqrt{2A})^n}{(n-s)!}(A+I)_n[(A+I)_s]^{-1}\\ &{}_2F_2\bigg(-\frac{1}{2}(n-s)I,-\frac{1}{2}(n-s-1)I;-\frac{1}{2}(A+nI),-\frac{1}{2}(A+(n-1)I);-\frac{1}{(\sqrt{2A})^2}\bigg)\\ &L_s^{(A)}(x)\end{aligned}$$

From the above we may conclude that

$$\begin{aligned}H_n(x,A) =& n!(A+I)_n(\sqrt{2A})^n\sum_{s=0}^{n}(-n)_s[(A+I)_s]^{-1}\\ &{}_2F_2\bigg(-\frac{1}{2}(n-s)I,-\frac{1}{2}(n-s-1)I;-\frac{1}{2}(A+nI),-\frac{1}{2}(A+(n-1)I);-\frac{1}{(\sqrt{2A})^2}\bigg)\\ &L_s^{(A)}(x)\end{aligned} \tag{5.4}$$

5.1. The Christoffel's formula of summation. Here, we establish Christoffel's formula of summation which will be required in the consideration of an expansion in a series of Hermite matrix polynomials $H_n(x,A)$. Let A be a matrix in $C^{N\times N}$ satisfying the condition (1.3) and (1.8), then the Hermite matrix polynomials becomes

$$H_{n+1}(x,A) = x\sqrt{2A}H_n(x,A) - 2nH_{n-1}(x,A), \quad n \geqslant 1. \tag{5.5}$$

We wish to prove the identity

$$\sum_{i=0}^{n}\frac{H_i(x,A)H_i(y,A)}{i!2^i} = \frac{H_n(x,A)H_{n+1}(y,A)-H_{n+1}(x,A)H_n(y,A)}{n!2^n\sqrt{2A}(y-x)}. \tag{5.6}$$

Form (5.5), substituting i for n and multiplying (5.5) by $\frac{H_i(y,A)}{i!2^{i+1}}$, we get

$$\frac{1}{i!2^{i+1}}H_{i+1}(x,A)H_i(y,A) - \frac{x\sqrt{2A}}{i!2^{i+1}}H_i(x,A)H_i(y,A) + \frac{1}{(i-1)!2^i}H_{i-1}(x,A)H_i(y,A) = \mathbf{0}. \tag{5.7}$$

Interchanging x and y, we have

$$\frac{1}{i!2^{i+1}}H_{i+1}(y,A)H_i(x,A) - \frac{y\sqrt{2A}}{i!2^{i+1}}H_i(y,A)H_i(x,A) + \frac{1}{(i-1)!2^i}H_{i-1}(y,A)H_i(x,A) = \mathbf{0}. \tag{5.8}$$

Subtracting the result from (5.5) and (5.6), we have

$$\begin{aligned}&\frac{(y-x)\sqrt{2A}}{i!2^{i+1}}H_i(y,A)H_i(x,A) = \frac{1}{i!2^{i+1}}\Big[H_{i+1}(y,A)H_i(x,A)-H_{i+1}(x,A)H_i(y,A)\Big]\\ &+\frac{1}{(i-1)!2^i}\Big[H_{i-1}(y,A)H_i(x,A)-H_{i-1}(x,A)H_i(y,A)\Big].\end{aligned} \tag{5.9}$$

Setting $i = 0, 1, 2, ..., n$, we obtain

$$i=0;\quad \frac{\sqrt{2A}(y-x)}{2}H_0(y,A)H_0(x,A)=\frac{1}{2}\Big[H_1(y,A)H_0(x,A)-H_1(x,A)H_0(y,A)\Big], \tag{5.10}$$

$$\begin{aligned} i=1;\quad \frac{(y-x)\sqrt{2A}}{2^2}H_1(y,A)H_1(x,A)&=\frac{1}{2^2}\Big[H_2(y,A)H_1(x,A)-H_2(x,A)H_1(y,A)\Big]\\ &+\frac{1}{2}\Big[H_0(y,A)H_1(x,A)-H_0(x,A)H_1(y,A)\Big]. \end{aligned} \tag{5.11}$$

$$\begin{aligned} i=2;\quad \frac{(y-x)\sqrt{2A}}{2!2^3}H_2(y,A)H_2(x,A)&=\frac{1}{2!2^3}\Big[H_3(y,A)H_2(x,A)-H_3(x,A)H_2(y,A)\Big]\\ &+\frac{1}{2^2}\Big[H_1(y,A)H_2(x,A)-H_1(x,A)H_2(y,A)\Big], \end{aligned} \tag{5.12}$$

$$\begin{aligned} i=3;\quad \frac{(y-x)\sqrt{2A}}{3!2^4}H_3(y,A)H_3(x,A)&=\frac{1}{3!2^4}\Big[H_4(y,A)H_3(x,A)-H_4(x,A)H_3(y,A)\Big]\\ &+\frac{1}{2!2^3}\Big[H_2(y,A)H_3(x,A)-H_2(x,A)H_3(y,A)\Big] \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} i=n;\quad \frac{(y-x)\sqrt{2A}}{n!2^{n+1}}H_n(y,A)H_n(x,A)&=\frac{1}{n!2^{n+1}}\Big[H_{n+1}(y,A)H_n(x,A)-H_{n+1}(x,A)H_n(y,A)\Big]\\ &+\frac{1}{(n-1)!2^n}\Big[H_{n-1}(y,A)H_n(x,A)-H_{n-1}(x,A)H_n(y,A)\Big] \end{aligned} \tag{5.14}$$

whence (5.6) follows by addition. Hence the Christoffel formula of summation (5.6) is established.

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References

[1] R.S. Batahan; A new extension of Hermite matrix polynomials and its applications. *Linear Algebra Appl.*, **419** (2006), 82-92.

[2] E. Defez and L. Jódar; Some applications of the Hermite matrix polynomials series expansions. *J. Comp. Appl. Math.*, **99** (1998), 105-117.

[3] E. Defez and L.Jódar; Chebyshev matrix polynomials and second order matrix differential equations. *Utilitas Math.*, **61** (2002), 107-123.

[4] E. Defez, L. Jódar and A. Law; Jacobi matrix differential equation, polynomial solutions, and their properties. *Comput. Math. Applicat.*, **48** (2004), 789-803.
[5] N. Dunford and J. Schwartz; *Linear Operators, part I, General Theory.* Interscience Publishers, INC. New York, 1957.
[6] L. Jódar and R. Company; Hermite matrix polynomials and second order matrix differential equations. *J. Approx. Theory Appl.*, **12** (1996), 20-30.
[7] L. Jódar, R. Company and E. Navarro; Laguerre matrix polynomials and system of second-order differential equations. *Appl. Num. Math.*, **15** (1994), 53-63.
[8] L. Jódar, R. Company and E. Ponsoda. Orthogonal matrix polynomials and systems of second order differential equations. *Diff. Equations Dynam. Syst.*, **3** (1995), 269-288.
[9] L. Jódar and J.C. Cortés. On the hypergeometric matrix function, *J. Comp. Appl. Math.*, **99** (1998), 205-217.
[10] L. Jódar and J.C. Cortés; Closed form general solution of the hypergeometric matrix differential equation. *Math. Computer Modell.*, **32** (2000), 1017-1028.
[11] L. Jódar and E. Defez, Some new matrix formulas related to Hermite matrix polynomials theory. *Proceedings Int. Workshop Orth. Poly. Math. Phys.*, **Leganés.** (1996), 93-101.
[12] L. Jódar and E. Defez; A connection between Laguerre's and Hermite's matrix polynomials. *Appl. Math. Lett.*, **11** (1998), 13-17.
[13] L. Jódar and E. Defez; On Hermite matrix polynomials and Hermite matrix function. *J. Approx. Theory Appl.*, **14** (1998), 36-48.
[14] G.S. Khammash, M.T. Mohamed and A. Shehata; On pseudo Chebyshev matrix polynomials of two variables. *Inter. J. Math. Archive*, **2** (2011), 1022-1026.
[15] M.S. Metwally, M.T. Mohamed and A. Shehata; On Hermite-Hermite matrix polynomials. *Math. Bohemica*, **133** (2008), 421-434.
[16] M.S. Metwally, M.T. Mohamed and A. Shehata; Generalizations of two-index two-variable Hermite matrix polynomials. *Demonstratio Mathematica*, **42** (2009), 687-701.
[17] M.S. Metwally, M.T. Mohamed and A. Shehata, On pseudo Hermite matrix polynomials of two variables. *Banach J. Math. Anal.*, **4** (2010),169-178.
[18] M.T. Mohamed and A. Shehata; A study of Appell's matrix functions of two complex variables and some properties. *J. Advan. Appl. Math. Sci.*, **9** (2011), 23-33.
[19] E.D. Rainville; *Special Functions.* The Macmillan Company, New York, 1960.
[20] K.A.M. Sayyed, M.S. Metwally and R.S. Batahan; On generalized Hermite matrix polynomials. *Electron. J. Linear Algebra*, **10** (2003), 272-279.
[21] K.A.M. Sayyed, M.S. Metwally and R.S. Batahan; Gegenbauer matrix polynomials and second order matrix differential equations. *Divulgaciones Matemáticas*, **12** (2004), 101-115.
[22] A. Shehata; A new extension of Gegenbauer matrix polynomials and their properties. *Bulletin Inter. Math. Virtual Institute*, **2** (2012), 29-42.
[23] A. Shehata; On p and q-Horn's matrix function of two complex variables. *Appl. Math.*, **2 (12)** (2011),1437-1442.
[24] Shehata, A. On Tricomi and Hermite-Tricomi matrix functions of complex variable. *Communications Math. Applications*, **2 (2-3)** (2011), 97-109.
[25] Shehata, A. A new extension of Hermite-Hermite matrix polynomials and their properties. *Thai Journal of Mathematics*, **10 (2)** (2012), 433-444.
[26] Shehata, A. On Rices matrix polynomials. *Afrika Matematika*, **25(3)** (2014), 757-777.
[27] Shehata, A. On Rainville's matrix polynomials. *Sylwan Journal*, **158(9)**, (2014), 158-178.
[28] Shehata, A. Some relations on Humbert matrix polynomials. *Mathematica Bohemica*, (in press).
[29] Shehata, A. Some relations on Konhauser matrix polynomials. *Miskolc Mathematical Notes*, (in press).
[30] Shehata, A. New kinds of Hypergeometric matrix functions. *British Journal of Mathematics and Computer Science*, **5 (1)** (2015), 92-102.
[31] A. Shehata; Some relations on Gegenbauer matrix polynomials. *Review of Computer Engineering Research.*

[32] A. Shehata; Connections between Legendre with Hermite and Laguerre matrix polynomials. *Gazi University Journal of Science(G. U. J. Sci.)*, (in press).
[33] A. Shehata; On a new family of the extended generalized Bessel-type matrix polynomials. *Mitteilungen Klosterneuburg Journal*, **65(2)** (2015), 100-121.
[34] Shehata, A. and Çekim, B. Some relations on Hermite-Hermite matrix polynomials. *University Politechnica of Bucharest Scientific Bulletin- Series A- Applied Mathematics and Physics*, (in press).
[35] Upadhyaya, L.M., and Shehata, A. On Legendre matrix polynomials and its applications. *Inter. Trans. Mathematical Sci. Computer (ITMSC)*, **4 (2)** (2011), 291-310.
[36] Upadhyaya, L.M., and Shehata, A. A new extension of generalized Hermite matrix polynomials. *Bulletin Malaysian Mathematical Sci. Soc.*, (in press).
[37] A. Sinap and W.V. Assche; Orthogonal matrix polynomials and applications. *J. Comp. Appl. Math.*, **66** (1996), 27-52.

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