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A Comparison of Various Types of Soft Compatible Maps and Common Fixed Point Theorem - I

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Abstract. In this article, first we generalize a few notions like compatible maps, compatible maps of type (α), compatible maps of type (β), weakly compatible maps in soft metric spaces and then we give an accounts for comparison of these soft compatible maps. Finally, we demonstrate the utility of these new concepts by proving common fixed point theorem for six soft continuous self maps on a complete soft metric space.

1. Introduction

The soft set theory, initiated by Molodtsov [**3**] in 1999, is one of the branch of mathematics, which targets to explain phenomena and concepts like ambiguous, undefined, vague and imprecise meaning. Soft set theory provides a very general frame work with the involvement of parameters. Since soft set theory has a rich potential, applications of soft set theory in other disciplines and real life problems are progressing rapidly.

In recent time, researchers have contributed a lot towards soft theory . Maji. et al. [**5, 6**] introduced several operations on soft sets and applied it to decision making problems. A new definition of soft set parameterization reduction and a comparison of it with attribute reduction in rough set theory was given by Chen [**2**].

Then the idea of soft topological space was first given by M.Shabir and M.Naz [**4**] and mappings between soft sets were described by P.Majumdar and S.K.Samanta [**7**]. Later, many researches about soft topological spaces were studied in [**12, 13, 14**]. In these studies, the concept of soft point is expressed by different approaches. Then S. Das and S. K. Samanta [**8**] introduce the notion of soft metric space and

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investigated some basic properties of this space. Later on different kinds of fixed point theorem have been established in soft metric spaces.

In this article, a generalization of compatible maps, compatible maps of type (α), compatible maps of type (β), weakly compatible maps are introduced in soft metric spaces and then we give an accounts for comparison of these soft compatible maps. Finally, we have proved a common fixed point theorem for six soft continuous self maps on a complete soft metric space. In order to add validity and weight to the argument that our concept is viable and meaningful in the context of fixed point theory in soft metric spaces, we have presented a series of Propositions and Examples in support of pertinent results.

2. Preliminaries

Definition 2.1. [**3**] Let X be an initial universe set and E be a set of parameters. A pair (F, E) is called a soft set over X if and only if F is a mapping from E into the set of all subsets of the set X, i.e., $F : E \to P(X)$, where $P(X)$ is the power set of X.

Definition 2.2. [**8**] A soft set (P, E) over X is said to be a soft point if there is exactly one $e \in E$, such that $P(e) = \{x\}$ for some $x \in X$ and $P(e') = \phi$, $\forall\, e' \in E - \{e\}$. It will be denoted by $\widetilde{x}_e$.

Definition 2.3. [**8, 10**] Two soft point $\widetilde{x}_e, \widetilde{y}_{e'}$ are said to be equal if $e = e'$ and $P(e) = P(e')$ i.e., $x = y$. Thus $\widetilde{x}_e \neq \widetilde{y}_{e'} \Leftrightarrow x \neq y$ or $e \neq e'$.

Definition 2.4. [**9**] Let R be the set of real numbers, $\wp(R)$ be the collection of all nonempty bounded subsets of R and A be a set of parameters. Then a mapping $F : A \to \wp(R)$ is called a soft real set. It is denoted by (F, A).
If in particular (F, A) is a singleton soft set, then identifying (F, A) with the corresponding soft element, it will be called a soft real number.

Definition 2.5. [**9**] Let (F, A) be a soft real set and $\overline{0}$ be the soft real set such that $\overline{0}(\lambda) = \{0\}$, $\forall\, \lambda \in A$. Then (F, A) is said to be a
(i) positive soft real set if $\overline{0} <_s (F, A)$,
(ii) negative soft real set if $(F, A) <_s \overline{0}$,
(iii) non-negative soft real set if $F(\lambda)$ is a subset of the set of non-negative real numbers for each $\lambda \in A$,
(iv) non-positive soft real set if $F(\lambda)$ is a subset of the set of non-positive real numbers for each $\lambda \in A$.

Definition 2.6. [**11**] Let τ be the collection of soft sets over X, then τ is said to be a soft topology on X if
1. $\widetilde{\Phi}, \widetilde{X} \in \tau$.
2. the union of any number of soft sets in τ belongs to τ
3. the intersection of any finite number of soft sets in τ belongs to τ. The triplet (X, τ, E) is called a soft topological space over X.

Definition 2.7. [**1**] Let (X, τ, E) and (Y, τ', E) be two soft topological spaces, $f : (X, \tau, E) \to (Y, \tau', E)$ be a mapping. For each soft neighborhood

(H, E) of $(\widetilde{f(x)}_e, E)$, if there exists a soft neighborhood (F, E) of $(\widetilde{x}_e, E)$ such that $f((F, E)) \subset (H, E)$, then f is said to be soft continuous mapping at $(\widetilde{x}_e, E)$.
If f is soft continuous mapping for all $(\widetilde{x}_e, E)$, then f is called soft continuous mapping.

Let $\widetilde{X}$ be the absolute soft set i.e., $F(e) = X, \forall e \in E$, where $(F, E) = \widetilde{X}$ and $SP(\widetilde{X})$ be the collection of all soft points of $\widetilde{X}$ and $\mathbb{R}(E)^*$ denote the set of all non-negative soft real numbers.

Definition 2.8. [**8**] A mapping $\widetilde{d} : SP(\widetilde{X}) \times SP(\widetilde{X}) \to \mathbb{R}(E)^*$, is said to be a soft metric on the soft set $\widetilde{X}$ if $\widetilde{d}$ satisfies the following conditions:

(M_1) $\widetilde{d}(\widetilde{x}_{e_1}, \widetilde{y}_{e_2}) \widetilde{\geqslant} \overline{0} \quad \forall\ \widetilde{x}_{e_1}, \widetilde{y}_{e_2} \widetilde{\in} \widetilde{X}$;

(M_2) $\widetilde{d}(\widetilde{x}_{e_1}, \widetilde{y}_{e_2}) = \overline{0}$ *if and only if* $\widetilde{x}_{e_1} = \widetilde{y}_{e_2}$;

(M_3) $\widetilde{d}(\widetilde{x}_{e_1}, \widetilde{y}_{e_2}) = \widetilde{d}(\widetilde{y}_{e_2}, \widetilde{x}_{e_1}) \quad \forall\ \widetilde{x}_{e_1}, \widetilde{y}_{e_2} \widetilde{\in} \widetilde{X}$;

(M_4) For all $\widetilde{x}_{e_1}, \widetilde{y}_{e_2}, \widetilde{z}_{e_3} \widetilde{\in} \widetilde{X}$, $\widetilde{d}(\widetilde{x}_{e_1}, \widetilde{z}_{e_3}) \widetilde{\leqslant} \widetilde{d}(\widetilde{x}_{e_1}, \widetilde{y}_{e_2}) + \widetilde{d}(\widetilde{y}_{e_2}, \widetilde{z}_{e_3})$.

The soft set $\widetilde{X}$ with a soft metric $\widetilde{d}$ on $\widetilde{X}$ is called a soft metric space and denoted by $(\widetilde{X}, \widetilde{d}, E)$.

Definition 2.9. [**8**] Let $(\widetilde{X}, \widetilde{d}, E)$ be a soft metric space and $(\widetilde{\varepsilon})$ be a non-negative soft real number. $B(\widetilde{x}_e, \widetilde{\varepsilon}) = \{\widetilde{y}_{e'} \widetilde{\in} \widetilde{X} : \widetilde{d}(\widetilde{x}_e, \widetilde{y}_{e'}) \widetilde{<} \widetilde{\varepsilon}\} \subset SP(\widetilde{X})$ is called the soft open ball with center at $\widetilde{x_e}$ and radius $\widetilde{\varepsilon}$ and $B(\widetilde{x}_e, \widetilde{\varepsilon}) = \{\widetilde{y}_{e'} \widetilde{\in} \widetilde{X} : \widetilde{d}(\widetilde{x}_e, \widetilde{y}_{e'}) \widetilde{\leqslant} \widetilde{\varepsilon}\} \subseteq SP(\widetilde{X})$ is called the soft closed ball with center at $\widetilde{x}_e$ and radius $\widetilde{\varepsilon}$.

Definition 2.10. [**8**] Let $(\widetilde{X}, \widetilde{d}, E)$ be a soft metric space and (F, E) be a non-null soft subset of $\widetilde{X}$ in $(\widetilde{X}, \widetilde{d}, E)$. Then (F, E) is said to be soft open in $\widetilde{X}$ with respect to $\widetilde{d}$ if and only if all soft points of (F, E) is interior points of (F, E).

Definition 2.11. [**7**] Let $(\widetilde{X}, \widetilde{d}, E)$ and $(\widetilde{Y}, \widetilde{\rho}, E')$ be two soft metric spaces. The mapping $(f, \varphi) : (\widetilde{X}, \widetilde{d}, E) \to (\widetilde{Y}, \widetilde{\rho}, E')$ is a soft mapping, where $f : X \to Y$, $\varphi : E \to E'$ are two mappings.

Definition 2.12. [**8**] Let $\{\widetilde{x}_{\lambda, n}\}_n$ be a sequence of soft points in a soft metric space $(\widetilde{X}, \widetilde{d}, E)$. The sequence $\{\widetilde{x}_{\lambda, n}\}_n$ is said to be convergent in $(\widetilde{X}, \widetilde{d}, E)$ if there is a soft point $\widetilde{y}_\mu \widetilde{\in} \widetilde{X}$ such that $\widetilde{d}(\widetilde{x}_{\lambda, n}, \widetilde{y}_\mu) \to \overline{0}$ as $n \to \infty$.
This means for every $\widetilde{\epsilon} \widetilde{>} \overline{0}$, chosen arbitrarily, $\exists$ a natural number $N = N(\widetilde{\epsilon})$, such that $\overline{0} \widetilde{\leqslant} d(\widetilde{x}_{\lambda, n}, \widetilde{y}_\mu) \widetilde{<} \widetilde{\epsilon}$, whenever $n > N$.

Definition 2.13. [**8**] (Cauchy Sequence). A sequence $\{\widetilde{x}_{\lambda, n}\}_n$ of soft points in $(\widetilde{X}, \widetilde{d}, E)$ is considered as a Cauchy sequence in $\widetilde{X}$ if corresponding to every $\widetilde{\epsilon} \widetilde{>} \overline{0}$, $\exists\ m \in \mathbb{N}$ such that $d(\widetilde{x}_{\lambda, i}, \widetilde{x}_{\lambda, j}) \widetilde{\leqslant} \widetilde{\epsilon}, \forall i, j \geqslant m$, i.e.,
$d(\widetilde{x}_{\lambda, i}, \widetilde{x}_{\lambda, j}) \to \overline{0}$ as $i, j \to \infty$.

DEFINITION 2.14. [8] (Complete Metric Space). A soft metric space $(\widetilde{X}, \widetilde{d}, E)$ is called complete if every Cauchy Sequence in $\widetilde{X}$ converges to some point of $\widetilde{X}$. The soft metric space $(\widetilde{X}, \widetilde{d}, E)$ is called incomplete if it is not complete.

3. Soft Compatible maps:

DEFINITION 3.1. Let $(A, \psi), (B, \varphi) : (\widetilde{X}, \widetilde{d}, E) \rightarrow (\widetilde{X}, \widetilde{d}, E)$ be soft mappings. (A, ψ) and (B, φ) are said to be soft compatible if

$$\lim_{n \to \infty} \widetilde{d}((A, \psi)(B, \varphi)\widetilde{x}^{n}_{\lambda_n}, (B, \varphi)(A, \psi)\widetilde{x}^{n}_{\lambda_n}) = \overline{0}$$

whenever $\{\widetilde{x}^{n}_{\lambda_n}\}$ is a sequence in $\widetilde{X}$ such that

$$\lim_{n \to \infty}(A, \psi)\widetilde{x}^{n}_{\lambda_n} = \lim_{n \to \infty}(B, \varphi)\widetilde{x}^{n}_{\lambda_n} = \widetilde{z}_{\nu} \text{ for some } \widetilde{z}_{\nu} \text{ in } \widetilde{X}.$$

DEFINITION 3.2. Let $(A, \psi), (B, \varphi) : (\widetilde{X}, \widetilde{d}, E) \rightarrow (\widetilde{X}, \widetilde{d}, E)$ be soft mappings. (A, ψ) and (B, φ) are said to be soft compatible of type (α) if

$$\lim_{n \to \infty} \widetilde{d}((A, \psi)(B, \varphi)\widetilde{x}^{n}_{\lambda_n}, (B, \varphi)(B, \varphi)\widetilde{x}^{n}_{\lambda_n}) = \overline{0}$$

and

$$\lim_{n \to \infty} \widetilde{d}((B, \varphi)(A, \psi)\widetilde{x}^{n}_{\lambda_n}, (A, \psi)(A, \psi)\widetilde{x}^{n}_{\lambda_n}) = \overline{0}$$

whenever $\{\widetilde{x}^{n}_{\lambda_n}\}$ is a sequence in $\widetilde{X}$ such that

$$\lim_{n \to \infty}(A, \psi)\widetilde{x}^{n}_{\lambda_n} = \lim_{n \to \infty}(B, \varphi)\widetilde{x}^{n}_{\lambda_n} = \widetilde{z}_{\nu}$$

for some $\widetilde{z}_{\nu}$ in $\widetilde{X}$.

DEFINITION 3.3. Let $(A, \psi), (B, \varphi) : (\widetilde{X}, \widetilde{d}, E) \rightarrow (\widetilde{X}, \widetilde{d}, E)$ be soft mappings. (A, ψ) and (B, φ) are said to be soft compatible of type (β) if

$$\lim_{n \to \infty} \widetilde{d}((A, \psi)(A, \psi)\widetilde{x}^{n}_{\lambda_n}, (B, \varphi)(B, \varphi)\widetilde{x}^{n}_{\lambda_n}) = \overline{0}$$

whenever $\{\widetilde{x}^{n}_{\lambda_n}\}$ is a sequence in $\widetilde{X}$ such that

$$\lim_{n \to \infty}(A, \psi)\widetilde{x}^{n}_{\lambda_n} = \lim_{n \to \infty}(B, \varphi)\widetilde{x}^{n}_{\lambda_n} = \widetilde{z}_{\nu}$$

for some $\widetilde{z}_{\nu}$ in $\widetilde{X}$

DEFINITION 3.4. Two soft mappings (A, ψ) and (B, φ) from a soft metric space $(\widetilde{X}, \widetilde{d}, E)$ in to it self are said to be weakly soft compatible if they commute at their coincidence soft points. i.e., for $\widetilde{x}_{\lambda} \in \widetilde{X}$,
$(A, \psi)\widetilde{x}_{\lambda} = (B, \varphi)\widetilde{x}_{\lambda} \Longrightarrow (A, \psi)(B, \varphi)\widetilde{x}_{\lambda} = (B, \varphi)(A, \psi)\widetilde{x}_{\lambda}$.

PROPOSITION 3.1. *Let $(\widetilde{X}, \widetilde{d}, E)$ be a soft metric space and (A, ψ) and (B, φ) be soft continuous maps from $\widetilde{X}$ in to itself. Then (A, ψ) and (B, φ) are soft compatible if and only if they are soft compatible of type (β).*

Proof. Suppose that (A, ψ) and (B, φ) are soft compatible and let $\{\widetilde{x}^{\,n}_{\lambda_n}\}$ be a soft sequence in $\widetilde{X}$ such that

$$\lim_{n\to\infty}(A,\psi)\widetilde{x}^{\,n}_{\lambda_n} = \lim_{n\to\infty}(B,\varphi)\widetilde{x}^{\,n}_{\lambda_n} = \widetilde{z}_{\nu}$$

for some $\widetilde{z}_{\nu_n}$ in $\widetilde{X}$.
Since (A, ψ) and (B, φ) be soft continuous, we have

$$\lim_{n\to\infty}(A,\psi)(A,\psi)\widetilde{x}^{\,n}_{\lambda_n} = \lim_{n\to\infty}(A,\psi)(B,\varphi)\widetilde{x}^{\,n}_{\lambda_n} = (A,\psi)\,\widetilde{z}_{\nu}\,.$$

$$\lim_{n\to\infty}(B,\varphi)(A,\psi)\widetilde{x}^{\,n}_{\lambda_n} = \lim_{n\to\infty}(B,\varphi)(B,\varphi)\widetilde{x}^{\,n}_{\lambda_n} = (B,\varphi)\,\widetilde{z}_{\nu}\,.$$

Further, since (A, ψ) and (B, φ) are soft compatible,

$$\lim_{n\to\infty}\widetilde{d}((A,\psi)(B,\varphi)\widetilde{x}^{\,n}_{\lambda_n},(B,\varphi)(A,\psi)\widetilde{x}^{\,n}_{\lambda_n}) = \overline{0}$$

Now, since we have

$$\begin{aligned}
&\widetilde{d}((A,\psi)(A,\psi)\widetilde{x}^{\,n}_{\lambda_n},(B,\varphi)(B,\varphi)\widetilde{x}^{\,n}_{\lambda_n})\\
&\qquad\widetilde{\leqslant}\ \widetilde{d}((A,\psi)(A,\psi)\widetilde{x}^{\,n}_{\lambda_n},(A,\psi)(B,\varphi)\widetilde{x}^{\,n}_{\lambda_n}) +\\
&\qquad\quad \widetilde{d}((A,\psi)(B,\varphi)\widetilde{x}^{\,n}_{\lambda_n},(B,\varphi)(B,\varphi)\widetilde{x}^{\,n}_{\lambda_n})\\
&\qquad\widetilde{\leqslant}\ \widetilde{d}((A,\psi)(A,\psi)\widetilde{x}^{\,n}_{\lambda_n},(A,\psi)(B,\varphi)\widetilde{x}^{\,n}_{\lambda_n}) +\\
&\qquad\quad \widetilde{d}((A,\psi)(B,\varphi)\widetilde{x}^{\,n}_{\lambda_n},(B,\varphi)(A,\psi)\widetilde{x}^{\,n}_{\lambda_n}) +\\
&\qquad\quad \widetilde{d}((B,\varphi)(A,\psi)\widetilde{x}^{\,n}_{\lambda_n},(B,\varphi)(B,\varphi)\widetilde{x}^{\,n}_{\lambda_n}),
\end{aligned}$$

it follows that

$$\lim_{n\to\infty}\widetilde{d}((A,\psi)(A,\psi)\widetilde{x}^{\,n}_{\lambda_n},(B,\varphi)(B,\varphi)\widetilde{x}^{\,n}_{\lambda_n}) = \overline{0}.$$

Therefore, (A, ψ) and (B, φ) are soft compatible of type (β).
Conversely, suppose that (A, ψ) and (B, φ) are soft compatible of type (β) and let $\{\widetilde{x}^{\,n}_{\lambda_n}\}$ be a sequence in $\widetilde{X}$ such that

$$\lim_{n\to\infty}(A,\psi)x^{n}_{\lambda_n} = \lim_{n\to\infty}(B,\varphi)x^{n}_{\lambda_n} = \widetilde{z}_{\nu} \text{ for some } \widetilde{z}_{\nu_n} \text{ in } \widetilde{X}.$$

Since (A, ψ) and (B, φ) are soft continuous, we have

$$\lim_{n\to\infty}(A,\psi)(A,\psi)\widetilde{x}^{\,n}_{\lambda_n} = \lim_{n\to\infty}(A,\psi)(B,\varphi)\widetilde{x}^{\,n}_{\lambda_n} = (A,\psi)\,\widetilde{z}_{\nu}\,,$$

$$\lim_{n\to\infty}(B,\varphi)(A,\psi)\widetilde{x}^{\,n}_{\lambda_n} = \lim_{n\to\infty}(B,\varphi)(B,\varphi)\widetilde{x}^{\,n}_{\lambda_n} = (B,\varphi)\,\widetilde{z}_{\nu}\,.$$

Further, since (A, ψ) and (B, φ) are soft compatible of type (β), we have

$$\lim_{n\to\infty}\widetilde{d}((A,\psi)(A,\psi)\widetilde{x}^{\,n}_{\lambda_n},(B,\varphi)(B,\varphi)\widetilde{x}^{\,n}_{\lambda_n}) = \overline{0}.$$

Thus, from the inequality

$$\begin{aligned}
&\widetilde{d}((A,\psi)(B,\varphi)\widetilde{x}^{\,n}_{\lambda_n},(B,\varphi)(A,\psi)\widetilde{x}^{\,n}_{\lambda_n})\\
&\qquad\widetilde{\leqslant}\ \widetilde{d}((A,\psi)(B,\varphi)\widetilde{x}^{\,n}_{\lambda_n},(A,\psi)(A,\psi)\widetilde{x}^{\,n}_{\lambda_n}) +\\
&\qquad\quad \widetilde{d}((A,\psi)(A,\psi)\widetilde{x}^{\,n}_{\lambda_n},(B,\varphi)(A,\psi)\widetilde{x}^{\,n}_{\lambda_n})\\
&\qquad\widetilde{\leqslant}\ \widetilde{d}((A,\psi)(B,\varphi)\widetilde{x}^{\,n}_{\lambda_n},(A,\psi)(A,\psi)\widetilde{x}^{\,n}_{\lambda_n}) +
\end{aligned}$$

$$\widetilde{d}((A,\psi)(A,\psi)\widetilde{x}^{\,n}_{\lambda_n},(B,\varphi)(B,\varphi)\widetilde{x}^{\,n}_{\lambda_n})+$$
$$\widetilde{d}((B,\varphi)(B,\varphi)\widetilde{x}^{\,n}_{\lambda_n},(B,\varphi)(A,\psi)\widetilde{x}^{\,n}_{\lambda_n}),$$

it follows that

$$\lim_{n\to\infty}\widetilde{d}((A,\psi)(B,\varphi)\widetilde{x}^{\,n}_{\lambda_n},(B,\varphi)(A,\psi)\widetilde{x}^{\,n}_{\lambda_n})=\overline{0}.$$

Therefore, (A,ψ) and (B,φ) are soft compatible. This completes the proof. □

PROPOSITION 3.2. *Let $(\widetilde{X},\widetilde{d},E)$ be a soft metric space and (A,ψ) and (B,φ) be soft continuous maps from $\widetilde{X}$ in to itself. Then (A,ψ) and (B,φ) are soft compatible if and only if they are soft compatible of type (α).*

PROOF. The proof is same as that of the Proposition (3.1). □

PROPOSITION 3.3. *Let $(\widetilde{X},\widetilde{d},E)$ be a soft metric space and (A,ψ) and (B,φ) be soft compatible of type (α). If one of (A,ψ) and (B,φ) is soft continuous, then (A,ψ) and (B,φ) are soft compatible maps of type (β).*

PROOF. Suppose that (A,ψ) and (B,φ) are soft compatible of type (α) and let $\{\widetilde{x}^n_{\lambda_n}\}$ be a sequence in $\widetilde{X}$ such that

$$\lim_{n\to\infty}(A,\psi)\widetilde{x}^n_{\lambda_n}=\lim_{n\to\infty}(B,\varphi)\widetilde{x}^n_{\lambda_n}=z_\nu \text{ for some } z_{\nu_n}\in\widetilde{X}.$$

Suppose (A,ψ) is soft continuous. Then we have

$$\lim_{n\to\infty}(A,\psi)(A,\psi)\widetilde{x}^n_{\lambda_n}=\lim_{n\to\infty}(A,\psi)(B,\varphi)\widetilde{x}^n_{\lambda_n}=(A,\psi)\widetilde{z}_\nu.$$

Further, since (A,ψ) and (B,φ) are soft compatible of type (α), we therefore have

$$\lim_{n\to\infty}\widetilde{d}((A,\psi)(B,\varphi)\widetilde{x}^n_{\lambda_n},(B,\varphi)(B,\varphi)x^n_{\lambda_n})=\overline{0},$$
$$\lim_{n\to\infty}\widetilde{d}((B,\varphi)(A,\psi)\widetilde{x}^n_{\lambda_n},(A,\psi)(A,\psi)\widetilde{x}^n_{\lambda_n})=\overline{0}.$$

Thus, from the inequality

$$\begin{aligned}\widetilde{d}((A,\psi)(A,\psi)\widetilde{x}^n_{\lambda_n},(B,\varphi)(B,\varphi)\widetilde{x}^n_{\lambda_n})\\ \leqslant \widetilde{d}((A,\psi)(A,\psi)\widetilde{x}^n_{\lambda_n},(A,\psi)(B,\varphi)\widetilde{x}^n_{\lambda_n})+\\ \widetilde{d}((A,\psi)(B,\varphi)\widetilde{x}^n_{\lambda_n},(B,\varphi)(B,\varphi)\widetilde{x}^n_{\lambda_n}),\end{aligned}$$

it follows that

$$\lim_{n\to\infty}\widetilde{d}((A,\psi)(A,\psi)\widetilde{x}^n_{\lambda_n},(B,\varphi)(B,\varphi)\widetilde{x}^n_{\lambda_n})=\overline{0}$$

Therefore, (A,ψ) and (B,φ) are soft compatible of type (β). □

PROPOSITION 3.4. *Let $(\widetilde{X},\widetilde{d},E)$ be a soft metric space and (A,ψ) and (B,φ) be soft continuous maps from $\widetilde{X}$ in to itself. If (A,ψ) and (B,φ) are soft compatible of type (β), then they are soft compatible of type (α).*

PROOF. Suppose that (A, ψ) and (B, φ) are soft compatible of type (β) and let $\{\widetilde{x}^n_{\lambda_n}\}$ be a sequence in $\widetilde{X}$ such that

$$\lim_{n \to \infty}(A, \psi)\widetilde{x}^n_{\lambda_n} = \lim_{n \to \infty}(B, \varphi)\widetilde{x}^n_{\lambda_n} = \widetilde{z}_\nu \text{ for some } \widetilde{z}_\nu \in \widetilde{X}.$$

Since (A, ψ) and (B, φ) are soft continuous, we have

$$\lim_{n \to \infty}(A, \psi)(A, \psi)\widetilde{x}^n_{\lambda_n} = \lim_{n \to \infty}(A, \psi)(B, \varphi)\widetilde{x}^n_{\lambda_n} = (A, \psi)\widetilde{z}_\nu.$$

$$\lim_{n \to \infty}((B, \varphi)(B, \varphi)\widetilde{x}^n_{\lambda_n} = \lim_{n \to \infty}(B, \varphi)(A, \psi)\widetilde{x}^n_{\lambda_n} = (B, \varphi)\widetilde{z}_\nu.$$

Further, since (A, ψ) and (B, φ) are soft compatible of type (β),

$$\lim_{n \to \infty} \widetilde{d}((A, \psi)(A, \psi)\widetilde{x}^n_{\lambda_n}, (B, \varphi)(B, \varphi)\widetilde{x}^n_{\lambda_n}) = \overline{0}.$$

Thus, from the inequality

$$\begin{aligned}\widetilde{d}((A, \psi)(B, \varphi)\widetilde{x}^n_{\lambda_n}, (B, \varphi)(B, \varphi)\widetilde{x}^n_{\lambda_n}) \\ \leqslant \widetilde{d}((A, \psi)(B, \varphi)\widetilde{x}^n_{\lambda_n}, (A, \psi)(A, \psi)\widetilde{x}^n_{\lambda_n}) + \\ \widetilde{d}((A, \psi)(A, \psi)\widetilde{x}^n_{\lambda_n}, (B, \varphi)(B, \varphi)\widetilde{x}^n_{\lambda_n}),\end{aligned}$$

it follows that

$$\lim_{n \to \infty} \widetilde{d}((A, \psi)(B, \varphi)\widetilde{x}^n_{\lambda_n}, (B, \varphi)(B, \varphi)\widetilde{x}^n_{\lambda_n}) = \overline{0}.$$

Similarly, we also have

$$\lim_{n \to \infty} \widetilde{d}((B, \varphi)(A, \psi)\widetilde{x}^n_{\lambda_n}, (A, \psi)(A, \psi)\widetilde{x}^n_{\lambda_n}) = \overline{0}.$$

Therefore, (A, ψ) and (B, φ) are soft compatible of type (α). □

PROPOSITION 3.5. *Let $(\widetilde{X}, \widetilde{d}, E)$ be a soft metric space and (A, ψ) and (B, φ) be soft maps from $\widetilde{X}$ in to itself. If (A, ψ) and (B, φ) be soft compatible of type (β) and $(A, \psi)\widetilde{z}_\nu = (B, \varphi)\widetilde{z}_\nu$ for some $\widetilde{z}_\nu \widetilde{\in} \widetilde{X}$, then*

$$\begin{aligned}(A, \psi)(B, \varphi)\widetilde{z}_\nu = (B, \varphi)(B, \varphi)\widetilde{z}_\nu = \\ (B, \varphi)(A, \psi)\widetilde{z}_\nu = (A, \psi)(A, \psi)\widetilde{z}_\nu.\end{aligned}$$

PROOF. Suppose that $\widetilde{x}^n_{\lambda_n}$ be a soft sequence in $\widetilde{X}$ defined by $\{\widetilde{x}^n_{\lambda_n}\} = \widetilde{z}_\nu$ for some $\widetilde{z}_\nu \in \widetilde{X}$ and $n = 1, 2, \cdots$ and $(A, \psi)\widetilde{z}_\nu = (B, \varphi)\widetilde{z}_\nu$.
Then we have

$$\lim_{n \to \infty}(A, \psi)\widetilde{x}^n_{\lambda_n} = \lim_{n \to \infty}(B, \varphi)\widetilde{x}^n_{\lambda_n} = \widetilde{z}_\nu \text{ for some } \widetilde{z}_\nu \in \widetilde{X}.$$

Since (A, ψ) and (B, φ) are soft compatible of type (β), we have

$$\begin{aligned}\widetilde{d}((A, \psi)(A, \psi)\widetilde{z}_\nu, (B, \varphi)(B, \varphi)\widetilde{z}_\nu) = \\ \lim_{n \to \infty} \widetilde{d}((A, \psi)(A, \psi)\widetilde{x}^n_{\lambda_n}, (B, \varphi)(B, \varphi)\widetilde{x}^n_{\lambda_n}) = \overline{0}\end{aligned}$$

and so $(A, \psi)(A, \psi)\widetilde{z}_\nu = (B, \varphi)(B, \varphi)\widetilde{z}_\nu$. On the other hand, from $(A, \psi)\widetilde{z}_\nu = (B, \varphi)\widetilde{z}_\nu$, it follows that $(A, \psi)(A, \psi)\widetilde{z}_\nu = (A, \psi)(B, \varphi)\widetilde{z}_\nu$ and $(B, \varphi)(A, \psi)\widetilde{z}_\nu = (B, \varphi)(B, \varphi)\widetilde{z}_\nu$. Therefore, we have

$$(A, \psi)(B, \varphi)\widetilde{z}_\nu = (A, \psi)(A, \psi)\widetilde{z}_\nu = (B, \varphi)(B, \varphi)\widetilde{z}_\nu = (B, \varphi)(A, \psi)\widetilde{z}_\nu .$$

This completes the proof. □

PROPOSITION 3.6. *Let* $(\widetilde{X}, \widetilde{d}, E)$ *be a soft metric space. Let* (A, ψ) *and* (B, φ) *be soft compatible of type* (β) *from* $\widetilde{X}$ *in to itself and let* $\{\widetilde{x}_\lambda^n\}$ *be a sequence in* $\widetilde{X}$ *such that*

$$\lim_{n \to \infty}(A, \psi)\widetilde{x}_\lambda^n = \lim_{n \to \infty}(B, \varphi)\widetilde{x}_\lambda^n = \widetilde{z}_\nu \text{ for some } \widetilde{z}_\nu \in \widetilde{X} .$$

Then we have the following:
(1) $\lim_{n \to \infty}(B, \varphi)(B, \varphi)\widetilde{x}_{\lambda_n}^n = (A, \psi)\widetilde{z}_\nu$ *if* (A, ψ) *is soft continuous at* $\widetilde{z}_\nu$;
(2) $\lim_{n \to \infty}(A, \psi)(A, \psi)\widetilde{x}_{\lambda_n}^n = (B, \varphi)\widetilde{z}_\nu$ *if* (B, φ) *is soft continuous at* $\widetilde{z}_\nu$;
(3) $(A, \psi)(B, \varphi)\widetilde{z}_\nu = (B, \varphi)(A, \psi)\widetilde{z}_\nu$ *and* $(A, \psi)\widetilde{z}_\nu = (B, \varphi)\widetilde{z}_\nu$ *if* (A, ψ) *and* (B, φ) *are soft continuous at* $\widetilde{z}_\nu$.

PROOF. Suppose that (A, ψ) is soft continuous at $\widetilde{z}_\nu$.
From $\lim_{n \to \infty}(A, \psi)\widetilde{x}_{\lambda_n}^n = \widetilde{z}_\nu$ for some $\widetilde{z}_\nu \in \widetilde{X}$, it follows that

$$\lim_{n \to \infty}(A, \psi)(A, \psi)\widetilde{x}_{\lambda_n}^n = (A, \psi)\widetilde{z}_\nu .$$

Further, since (A, ψ) and (B, φ) are soft compatible of type (β),

$$\lim_{n \to \infty} \widetilde{d}((A, \psi)(A, \psi)\widetilde{x}_{\lambda_n}^n, (B, \varphi)(B, \varphi)x_{\lambda_n}^n) = \overline{0} .$$

Thus, from the inequality

$$\begin{aligned}\widetilde{d}((B, \varphi)(B, \varphi)\widetilde{x}_{\lambda_n}^n, (A, \psi)\widetilde{z}_\nu) &\leqslant \widetilde{d}((B, \varphi)(B, \varphi)\widetilde{x}_{\lambda_n}^n, (A, \psi)(A, \psi)\widetilde{x}_{\lambda_n}^n) + \\ &\quad \widetilde{d}((A, \psi)(A, \psi)\widetilde{x}_{\lambda_n}^n, (A, \psi)\widetilde{z}_\nu)),\end{aligned}$$

it follows that

$$\lim_{n \to \infty} \widetilde{d}((B, \varphi)(B, \varphi)\widetilde{x}_{\lambda_n}^n, (A, \psi)\widetilde{z}_\nu)) = \overline{0} .$$

Therefore we have $\lim_{n \to \infty}(B, \varphi)(B, \varphi)\widetilde{x}_{\lambda_n}^n = (A, \psi)\widetilde{z}_\nu$.
(2) The proof of $\lim_{n \to \infty}(A, \psi)(A, \psi)\widetilde{x}_{\lambda_n}^n = (B, \varphi)\widetilde{z}_\nu$ directly follows the first part.
(3) Suppose that (A, ψ) and (B, φ) are soft continuous at $\widetilde{z}_\nu$. Since (A, ψ) is soft continuous at $\widetilde{z}_\nu$, by (1), we have $\lim_{n \to \infty}(B, \varphi)(B, \varphi)\widetilde{x}_{\lambda_n}^n = (A, \psi)\widetilde{z}_\nu$. On the other hand, since $(B, \varphi)\widetilde{x}_{\lambda_n}^n = \widetilde{z}_\nu$ for some $\widetilde{z}_\nu \in \widetilde{X}$ and (B, φ) is

soft continuous at $\widetilde{z}_\nu$, we have $\lim_{n \to \infty} (B, \varphi)(B, \varphi)\widetilde{x}^n_{\lambda_n} = (B, \varphi)\widetilde{z}_\nu$. Therefore, by the uniqueness of the limit, we have $(A, \psi)\widetilde{z}_\nu = (B, \varphi)\widetilde{z}_\nu$ and so, from the Proposition (3.5), it follows that $(A, \psi)(B, \varphi)\widetilde{z}_\nu = (B, \varphi)(A, \psi)\widetilde{z}_\nu$. This completes the proof. □

PROPOSITION 3.7. *Let* $(\widetilde{X}, \widetilde{d}, E)$ *be a soft metric space and* (A, ψ) *and* (B, φ) *be soft continuous maps from* $\widetilde{X}$ *in to itself. Then if* (A, ψ) *and* (B, φ) *are soft compatible of type* (α), *they are weakly soft compatible.*

PROOF. Let $(A, \psi)\widetilde{z}_\nu = (B, \varphi)\widetilde{z}_\nu$ for some $\widetilde{z}_\nu \in \widetilde{X}$ and also let $x^n_{\lambda_n} = \widetilde{z}_\nu$, $n = 1, 2, \cdots$, a constant soft sequence. Then we have

$$(A, \psi)\widetilde{x}^n_{\lambda_n} \to (A, \psi)\widetilde{z}_\nu \text{ and } (B, \varphi)x^n_{\lambda_n} \to (B, \varphi)\widetilde{z}_\nu \text{ as } n \to \infty .$$

Now, by soft compatibility of type (α), we have

$$\widetilde{d}((A, \psi)(B, \varphi)\widetilde{z}_\nu , (B, \varphi)(B, \varphi)\widetilde{z}_\nu) = \widetilde{d}((A, \psi)(B, \varphi)\widetilde{x}^n_{\lambda_n} , (B, \varphi)(B, \varphi)\widetilde{x}^n_{\lambda_n}) \to \widetilde{0},$$

which implies that $(A, \psi)(B, \varphi)\widetilde{z}_\nu = (B, \varphi)(B, \varphi)\widetilde{z}_\nu$.
Similarly,

$$\widetilde{d}((B, \varphi)(A, \psi)\widetilde{z}_\nu , (A, \psi)(A, \psi)\widetilde{z}_\nu) = \widetilde{d}((B, \varphi)(A, \psi)x^n_{\lambda_n}) , (A, \psi)(A, \psi)\widetilde{x}^n_{\lambda_n})) \to \widetilde{0},$$

implies, $(B, \varphi)(A, \psi)\widetilde{z}_\nu = (A, \psi)(A, \psi)\widetilde{z}_\nu$.
Now, $(A, \psi)\widetilde{z}_\nu = (B, \varphi)\widetilde{z}_\nu \Rightarrow (A, \psi)(A, \psi)\widetilde{z}_\nu = (A, \psi)(B, \varphi)\widetilde{z}_\nu$
$\Rightarrow (B, \varphi)(A, \psi)\widetilde{z}_\nu = (A, \psi)(B, \varphi)\widetilde{z}_\nu$
Then we get $(A, \psi)(B, \varphi)\widetilde{z}_\nu = (B, \varphi)(A, \psi)\widetilde{z}_\nu$
Hence, (A, ψ) and (B, φ) are weakly soft compatible. □

PROPOSITION 3.8. *Let* $(\widetilde{X}, \widetilde{d}, E)$ *be a soft metric space and* (A, ψ) *and* (B, φ) *be soft continuous maps from* $\widetilde{X}$ *in to itself. If* (A, ψ) *and* (B, φ) *are soft compatible of type* (β), *then they are weakly soft compatible.*

PROOF. The proof directly follows from the previous proposition (3.7). □

EXAMPLE 3.1. Let $(\widetilde{X}, \widetilde{d}, E)$ be a soft metric space with the following metrics

$$d(x, y) = | x - y |, d_1(x, y) = \min\{| x - y |, 1\} \text{ and}$$

$$\widetilde{d}(\widetilde{x}_\lambda , \widetilde{y}_\mu) = \frac{1}{2}\{d_1(\lambda, \mu) + d(x, y)\},$$

where $\widetilde{X} = [0, 20]$ and $E = \mathbb{R}^+$. Let function $A, B : X \to X$ and $\psi, \varphi : R \to R$ are defined as follows

$$A\widetilde{x} = \begin{cases} 0 & \text{if x=0} \quad , \\ \text{x + 16} & \text{if } 0 < x \leqslant 4 \\ \text{x - 4} & \text{if } 4 < x \leqslant 20 \end{cases} \qquad B\widetilde{x} = \begin{cases} 0 & \text{if } x \in \{0\} \cup (4, 20] \\ 3 & \text{if } 0 < x \leqslant 4 \end{cases},$$

$$\psi(y) = \begin{cases} \frac{1}{y^2} & \text{if } y \neq 0 \\ 1 & \text{if } y = 0 \end{cases} \qquad \varphi(y) = \begin{cases} \frac{1}{y} & \text{if } y \neq 0 \\ 1 & \text{if } y = 0 \end{cases}$$

Take $\widetilde{x}^n_{y_n} = (4 + \frac{1}{n})_n, \forall n \in \mathbb{N}$. Then

$(A, \psi)\widetilde{x}^n_{y_n} = (x_n - 4)_{\frac{1}{y_n^2}} \to (0)_0$ and $(B, \varphi)\widetilde{x}^n_{y_n}(0)_{\frac{1}{n}} \to (0)_0$ as $n \to \infty$

$(A, \psi)(B, \varphi)(0)_0 = (A, \psi)(0)_0 = (0)_1 = (B, \varphi)(0)_0 = (B, \varphi)(A, \psi)(0)_0$

$(A, \psi)(B, \varphi)(0)_1 = (A, \psi)(0)_1 = (0)_1 = (B, \varphi)(0)_1 = (B, \varphi)(A, \psi)(0)_1$.

Clearly, (A, ψ) and (B, φ) are weakly soft compatible maps, since they commute at their coincidence soft point $(0)_0$ and $(0)_1$. On the other hand, we have $(A, \psi)(B, \varphi)\widetilde{x}^n_{y_n} = (A, \psi)(0)_1 = (0)_1$ or $(A, \psi)(B, \varphi)\widetilde{x}^n_{y_n} = (A, \psi)(0)_{\frac{1}{y}} = (0)_{y^2}$, $(A, \psi)(A, \psi)\widetilde{x}^n_{y_n} = (A, \psi)(x_{n-4})_{\frac{1}{y^2}} = (x_{n+12})_{y^4}$, $(B, \varphi)(A, \psi)\widetilde{x}^n_{y_n} = (B, \varphi)(x_{n-4})_{\frac{1}{y^2}} = (3)_{y^2}$, $(B, \varphi)(B, \varphi)\widetilde{x}^n_{y_n} = (B, \varphi)(0)_1 = (0)_1$ or $(B, \varphi)(B, \varphi)\widetilde{x}^n_{y_n} = (B, \varphi)(0)_{\frac{1}{y}} = (0)_y$

Consequently,

$$\lim_{n \to \infty} \widetilde{d}((A, \psi)(B, \varphi)\widetilde{x}^n_{\lambda_n}, (B, \varphi)(A, \psi)\widetilde{x}^n_{\lambda_n})$$
$$= \lim_{n \to \infty} \widetilde{d}((0)_1, (3)_{n^2}) \neq \overline{0}$$

that is (A, ψ) and (B, φ) are not soft compatible. Moreover, we have

$$\lim_{n \to \infty} \widetilde{d}((B, \varphi)(A, \psi)\widetilde{x}^n_{\lambda_n}, (A, \psi)(A, \psi)\widetilde{x}^n_{\lambda_n}) =$$
$$\lim_{n \to \infty} \widetilde{d}((3)_{y^2}, x_{n+12})_{y^4}) \neq \overline{0}$$

Similarly,

$$\lim_{n \to \infty} \widetilde{d}((A, \psi)(B, \varphi)\widetilde{x}^n_{\lambda_n}, (B, \varphi)(B, \varphi)\widetilde{x}^n_{\lambda_n})$$
$$= \lim_{n \to \infty} \widetilde{d}((0)_1, (0)_1) = \overline{0}$$

or

$$\lim_{n \to \infty} \widetilde{d}((A, \psi)(B, \varphi)\widetilde{x}^n_{\lambda_n}, (B, \varphi)(B, \varphi)\widetilde{x}^n_{\lambda_n}) =$$
$$\lim_{n \to \infty} \widetilde{d}((0)_{y^2}, (0)_y) \neq \overline{0}$$

thus, (A, ψ) and (B, φ) are not soft compatible of type (α). We have

$$\lim_{n \to \infty} \widetilde{d}((A, \psi)(A, \psi)\widetilde{x}^n_{\lambda_n}, (B, \varphi)(B, \varphi)\widetilde{x}^n_{\lambda_n}) =$$
$$\lim_{n \to \infty} \widetilde{d}\big(x_{n+12})_{y^4}, (0)_1\big) \neq \overline{0}$$

or

$$\lim_{n \to \infty} \widetilde{d}\big((A, \psi)(A, \psi)\widetilde{x}^n_{\lambda_n}, (B, \varphi)(B, \varphi)\widetilde{x}^n_{\lambda_n}\big) =$$
$$\lim_{n \to \infty} \widetilde{d}\big(x_{n+12})_{y^4}, (0)_y\big) \neq 0$$

thus, (A,ψ) and (B,φ) are not soft compatible of type (β).

4. Fixed point theorem:

THEOREM 4.1. *Let* $(\widetilde{X},\widetilde{d},E)$ *be a complete soft metric space and let* $(A,\psi),(B,\varphi),(S,\phi),(T,\xi),(P,\tau),$ *and* (Q,ζ) *be soft mappings from* $\widetilde{X}$ *in to itself such that the following conditions are satisfied :*

(a) $\quad (P,\tau)(\widetilde{X},E)\subset(S,\phi)(T,\xi)(\widetilde{X},E),$

$$(Q,\zeta)(\widetilde{X},E)\subset(A,\psi)(B,\varphi)(\widetilde{X},E)$$

(b) $\quad (A,\psi)(B,\varphi)=(B,\varphi)(A,\psi),\ (S,\phi)(T,\xi)=(T,\xi)(S,\phi),$

$$(P,\tau)(B,\varphi)=(B,\varphi)(P,\tau),\ (Q,\zeta)(T,\xi)=(T,\xi)(Q,\zeta)$$

(c) *either* (P,τ) *or* $(A,\psi)(B,\varphi)$ *is soft continuous;*

(d) $((P,\tau),(A,\psi)(B,\varphi))$ *is soft compatible of type* (B,φ) *and* $((Q,\zeta),(S,\phi)(T,\xi))$ *is weakly soft compatible;*

(e) *there exists* $q\in(\overline{0},\overline{1})$ *such that for every* $\widetilde{x}_\lambda,\widetilde{y}_\mu\in\widetilde{X}$

$$\widetilde{d}((P,\tau)\widetilde{x}_\lambda,(Q,\zeta)\widetilde{y}_\mu)\widetilde{\leqslant}\widetilde{q}\max\Big\{\widetilde{d}((A,\psi)(B,\varphi)\widetilde{x}_\lambda,(S,\phi)(T,\xi)\widetilde{y}_\mu),$$
$$\widetilde{d}((P,\tau)\widetilde{x}_\lambda,(A,\psi)(B,\varphi)\widetilde{x}_\lambda),\widetilde{d}((Q,\zeta)\widetilde{y}_\mu,(S,\phi)(T,\xi)\widetilde{y}_\mu),$$
$$\widetilde{d}((P,\tau)\widetilde{x}_\lambda,(S,\phi)(T,\xi)\widetilde{y}_\mu)\}$$

Then $(A,\psi),(B,\varphi),(S,\phi),(T,\xi),(P,\tau)$ *and* (Q,ζ) *have a unique common fixed soft point in* $\widetilde{X}$.

PROOF. Let $\widetilde{x}^0_{\lambda_0}\in SP(\widetilde{X})$. From (a) there exist $\widetilde{x}^1_{\lambda_1},\widetilde{x}^2_{\lambda_2}\in SP(\widetilde{X})$ such that $(P,\tau)\widetilde{x}^0_{\lambda_0}=(S,\phi)(T,\xi)\widetilde{x}^1_{\lambda_1}$ and $(Q,\zeta)\widetilde{x}^1_{\lambda_1}=(A,\psi)(B,\varphi)\widetilde{x}^2_{\lambda_2}$. Inductively, we can construct sequence $\{\widetilde{x}^n_{\lambda_n}\}$ and $\{\widetilde{y}^n_{\mu_n}\}$ in $SP(\widetilde{X})$ such that

$$(P,\tau)\widetilde{x}^{2n-2}_{\lambda_{2n-2}}=(S,\phi)(T,\xi)\widetilde{x}^{2n-1}_{\lambda_{2n-1}}=\widetilde{y}^{\,2n-1}_{\mu_{2n-1}}$$

and

$$(Q,\zeta)\widetilde{x}^{2n-1}_{\lambda_{2n-1}}=(A,\psi)(B,\varphi)\widetilde{x}^{2n}_{\lambda_{2n}}=\widetilde{y}^{2n}_{\mu_{2n}}$$

Step1. Put $\widetilde{x}_\lambda=\widetilde{x}^{2n}_{\lambda_{2n}}$ and $\widetilde{y}_\mu=\widetilde{x}^{2n+1}_{\lambda_{2n+1}}$ in (e), we get

$$\widetilde{d}\Big((P,\tau)\widetilde{x}^{2n}_{\lambda_{2n}},(Q,\zeta)\widetilde{x}^{2n+1}_{\lambda_{2n+1}}\Big)$$
$$\widetilde{\leqslant}\widetilde{q}\max\Big\{\widetilde{d}\Big((A,\psi)(B,\varphi)\widetilde{x}^{2n}_{\lambda_{2n}},(S,\phi)(T,\xi)\widetilde{x}^{2n+1}_{\lambda_{2n+1}}\Big),$$
$$\widetilde{d}((P,\tau)\widetilde{x}^{2n}_{\mu_{2n}},(A,\psi)(B,\varphi)\widetilde{x}^{2n}_{\mu_{2n}}),\widetilde{d}((Q,\zeta)\widetilde{x}^{2n+1}_{\lambda_{2n+1}},(S,\phi)(T,\xi)\widetilde{x}^{2n+1}_{\lambda_{2n+1}}),$$
$$\widetilde{d}((P,\tau)\widetilde{x}^{2n}_{\lambda_{2n}},(S,\phi)(T,\xi)\widetilde{x}^{2n+1}_{\lambda_{2n+1}})\Big\}$$
$$=\widetilde{q}\max\Big\{\widetilde{d}\Big(\widetilde{y}^{2n}_{\mu_{2n}},\widetilde{y}^{2n+1}_{\mu_{2n+1}}\Big),\widetilde{d}\Big(\widetilde{y}^{2n+1}_{\mu_{2n+1}},y^{2n}_{\mu_{2n}}\Big),\widetilde{d}\Big(\widetilde{y}^{2n+2}_{\mu_{2n+2}},\widetilde{y}^{2n+1}_{\mu_{2n+1}}\Big),$$
$$\widetilde{d}\Big(y^{2n+1}_{\mu_{2n+1}},\widetilde{y}^{2n+1}_{\mu_{2n+1}}\Big)\Big\}$$
$$=\widetilde{q}\max\Big\{\widetilde{d}\Big(\widetilde{y}^{2n}_{\mu_{2n}},\widetilde{y}^{2n+1}_{\mu_{2n+1}}\Big),\widetilde{d}\Big(\widetilde{y}^{2n+2}_{\mu_{2n+2}},\widetilde{y}^{2n+1}_{\mu_{2n+1}}\Big)\Big\}$$

$\Rightarrow \widetilde{d}\left(\widetilde{y}^{2n+1}_{\mu_{2n+1}}, \widetilde{y}^{2n+2}_{\mu_{2n+2}}\right) \widetilde{\leqslant} \widetilde{q}\, \widetilde{d}\left(\widetilde{y}^{2n}_{\mu_{2n}}, \widetilde{y}^{2n+1}_{\mu_{2n+1}}\right)$

Similarly, we have

$$\widetilde{d}\left(\widetilde{y}^{2n+2}_{\mu_{2n+2}}, \widetilde{y}^{2n+3}_{\mu_{2n+3}}\right) \widetilde{\leqslant} \widetilde{q}\, \widetilde{d}\left(y^{2n+1}_{\mu_{2n+1}}, \widetilde{y}^{2n+2}_{\mu_{2n+2}}\right)$$

Thus we have

$$\widetilde{d}\left(\widetilde{y}^{n}_{\mu_{n}}, \widetilde{y}^{n+1}_{\mu_{n+1}}\right) \widetilde{\leqslant} \widetilde{q}\, \widetilde{d}\left(\widetilde{y}^{n-1}_{\mu_{n-1}}, \widetilde{y}^{n}_{\mu_{n}}\right) \widetilde{\leqslant} \cdots \leqslant \widetilde{q}^{\,n} \widetilde{d}\left(\widetilde{y}^{0}_{\mu_0}, \widetilde{y}^{1}_{\mu_1}\right)$$

For $m, n \in N$, we have suppose $n > m$. Then we have

$$\begin{aligned} \widetilde{d}\left(\widetilde{y}^{n}_{\mu_n}, \widetilde{y}^{m}_{\mu_m}\right) &\widetilde{\leqslant} \widetilde{d}\left(\widetilde{y}^{m}_{\mu_m}, \widetilde{y}^{m+1}_{\mu_{m+1}}\right) + \widetilde{d}\left(\widetilde{y}^{m+1}_{\mu_{m+1}}, \widetilde{y}^{m+2}_{\mu_{m+2}}\right) + \cdots + \\ &\qquad \widetilde{d}\left(\widetilde{y}^{n-1}_{\mu_{n-1}}, \widetilde{y}^{n}_{\mu_n}\right) \\ &\widetilde{\leqslant} \left(\widetilde{q}^{m} + \widetilde{q}^{\,m+1} + \cdots + \widetilde{q}^{\,n-1}\right) \widetilde{d}\left(\widetilde{y}^{0}_{\mu_0}, \widetilde{y}^{1}_{\mu_1}\right) \\ &= \widetilde{q}^{\,m} \frac{1 - \widetilde{q}^{\,(n-m)}}{1 - \widetilde{q}} \widetilde{d}\left(\widetilde{y}^{0}_{\mu_0}, \widetilde{y}^{1}_{\mu_1}\right) \longrightarrow \overline{0}\,(m \to \infty) \end{aligned}$$

and hence $\{\widetilde{y}^{n}_{\mu_n}\}$ is a soft Cauchy sequence in $\widetilde{X}$.
Since $(\widetilde{X}, \widetilde{d}, E)$ is complete, $\{\widetilde{y}^{n}_{\mu_n}\}$ converges to some soft point $\widetilde{z}_\nu \in \widetilde{X}$.
Also its subsequences converges to the same soft point, i.e.,

$$\left\{(Q, \zeta)\,\widetilde{x}^{2n+1}_{\lambda_{2n+1}}\right\} \longrightarrow \widetilde{z}_\nu \quad \text{and} \quad \left\{(S, \phi)(T, \xi)\,\widetilde{x}^{2n+1}_{\lambda_{2n+1}}\right\} \longrightarrow \widetilde{z}_\nu \tag{4.1}$$

$$\left\{(P, \tau)\,\widetilde{x}^{2n}_{\lambda_{2n}}\right\} \longrightarrow \widetilde{z}_\nu \quad \text{and} \quad \left\{(A, \psi)(B, \varphi)\widetilde{x}^{2n}_{\lambda_{2n}}\right\} \longrightarrow \widetilde{z}_\nu \tag{4.2}$$

Case 1. Suppose $(A, \psi)(B, \varphi)$ is soft continuous, we have

$$\left(((A, \psi)(B, \varphi))^2\, \widetilde{x}^{2n}_{\lambda_{2n}}\right) \longrightarrow ((A, \psi)(B, \varphi)\, \widetilde{z}_\nu)$$

and

$$\left(((A, \psi)(B, \varphi)(P, \tau))\,\widetilde{x}^{2n}_{\lambda_{2n}}\right) \longrightarrow ((A, \psi)(B, \varphi)\, \widetilde{z}_\nu)\ .$$

As $((P, \tau), (A, \psi)(B, \varphi))$ is soft compatible pair of type (β), we have

$$\widetilde{d}((P, \tau)^2\, \widetilde{x}^{2n}_{\lambda_{2n}}, ((A, \psi)(B, \varphi))^2\, \widetilde{x}^{2n}_{\lambda_{2n}}) \to \overline{0} \text{ as } n \to \infty$$

or

$$\widetilde{d}((P, \tau)^2\, \widetilde{x}^{2n}_{\lambda_{2n}}, (A, \psi)(B, \varphi)\, \widetilde{z}_\nu) = \overline{0}$$

Therefore, $\left((P, \tau)(P, \tau)\,\widetilde{x}^{2n}_{\lambda_{2n}}\right) \longrightarrow ((A, \psi)(B, \varphi)\, \widetilde{z}_\nu)$

Step 2. Put $\widetilde{x}_\lambda = (P, \tau)\,\widetilde{x}^{2n}_{\lambda_{2n}}$ and $\widetilde{y}_\mu = \widetilde{x}^{2n+1}_{\lambda_{2n+1}}$ in (e), we get

$$\begin{aligned} &\widetilde{d}\left((P, \tau)(P, \tau)\,\widetilde{x}^{2n}_{\lambda_{2n}}, (Q, \zeta)\,\widetilde{x}^{2n+1}_{\lambda_{2n+1}}\right) \\ &\quad \widetilde{\leqslant} \widetilde{q} \max \Big\{ \widetilde{d}\left((A, \psi)(B, \varphi)(P, \tau)\,\widetilde{x}^{2n}_{\lambda_{2n}}, (S, \phi)(T, \xi)\,\widetilde{x}^{2n+1}_{\lambda_{2n+1}}\right), \\ &\qquad\qquad \widetilde{d}\left((P, \tau)(P, \tau)\,\widetilde{x}^{2n}_{\lambda_{2n}}, (A, \psi)(B, \varphi)(P, \tau)\,\widetilde{x}^{2n}_{\lambda_{2n}}\right), \\ &\qquad\qquad\qquad \widetilde{d}\left((Q, \zeta)\,\widetilde{x}^{2n+1}_{\lambda_{2n+1}}, (S, \phi)(T, \xi)\, x^{2n+1}_{\lambda_{2n+1}}\right), \end{aligned}$$

$$\widetilde{d}\Big((P,\tau)(P,\tau)\widetilde{x}^{2n}_{\lambda_{2n}},(S,\phi)(T,\xi)\widetilde{x}^{2n+1}_{\lambda_{2n+1}}\Big)\Big\}$$

Taking $n\to\infty$,we get

$$\widetilde{d}((A,\psi)(B,\varphi)\widetilde{z}_\nu,\widetilde{z}_\nu)\,\widetilde{\leqslant}\,\widetilde{q}\max\Big\{\widetilde{d}((A,\psi)(B,\varphi))\widetilde{z}_\nu,\widetilde{z}_\nu),$$
$$\widetilde{d}((A,\psi)(B,\varphi)\widetilde{z}_\nu,(A,\psi)(B,\varphi))\widetilde{z}_\nu),\widetilde{d}(\widetilde{z}_\nu,\widetilde{z}_\nu),$$
$$\widetilde{d}((A,\psi)(B,\varphi)\widetilde{z}_\nu,\widetilde{z}_\nu)\Big\}$$
$$\widetilde{\leqslant}\,\widetilde{q}\,\widetilde{d}((A,\psi)(B,\varphi)\widetilde{z}_\nu,\widetilde{z}_\nu)\,\widetilde{<}\,\widetilde{d}((A,\psi)(B,\varphi)\widetilde{z}_\nu,\widetilde{z}_\nu)$$

which is a contradiction. Hence,

$$(A,\psi)(B,\varphi)\widetilde{z}_\nu=\widetilde{z}_\nu \tag{4.3}$$

Step 3. Put $\widetilde{x}_\lambda=\widetilde{z}_\nu$ and $\widetilde{y}_\mu=\widetilde{x}^{2n+1}_{\lambda_{2n+1}}$ in (e), we get

$$\widetilde{d}\Big((P,\tau)\widetilde{z}_\nu,(Q,\zeta)\widetilde{x}^{2n+1}_{\lambda_{2n+1}}\Big)$$
$$\widetilde{\leqslant}\,\widetilde{q}\max\Big\{\widetilde{d}\Big((A,\psi)(B,\varphi))\widetilde{z}_\nu,(S,\phi)(T,\xi)\widetilde{x}^{2n+1}_{\lambda_{2n+1}}\Big),$$
$$\widetilde{d}((P,\tau)\widetilde{z}_\nu,(A,\psi)(B,\varphi))\widetilde{z}_\nu),\widetilde{d}\Big((Q,\zeta)\widetilde{x}^{2n+1}_{\lambda_{2n+1}},(S,\phi)(T,\xi)\widetilde{x}^{2n+1}_{\lambda_{2n+1}}\Big),$$
$$\widetilde{d}\Big((P,\tau)\widetilde{z}_\nu,(S,\phi)(T,\xi)\widetilde{x}^{2n+1}_{\lambda_{2n+1}}\Big)\Big\}$$

Taking $n\to\infty$ and using equation (4.1), we get

$$\widetilde{d}((P,\tau)\widetilde{z}_\nu,\widetilde{z}_\nu)\,\widetilde{\leqslant}\,\widetilde{q}\max\Big\{\widetilde{d}(\widetilde{z}_\nu,\widetilde{z}_\nu),\ \widetilde{d}((P,\tau)\widetilde{z}_\nu,\ \widetilde{z}_\nu),\widetilde{d}(\widetilde{z}_\nu,\widetilde{z}_\nu),$$
$$\widetilde{d}((P,\tau)\widetilde{z}_\nu,\widetilde{z}_\nu)\Big\}$$
$$\widetilde{\leqslant}\,\widetilde{q}\,\widetilde{d}((P,\tau)\widetilde{z}_\nu,\widetilde{z}_\nu)\,\widetilde{<}\,\widetilde{d}((P,\tau)\widetilde{z}_\nu,\widetilde{z}_\nu)$$

which is a contradiction. Hence $(P,\tau)\widetilde{z}_\nu=\widetilde{z}_\nu$

Therefore, $(A,\psi)(B,\varphi)\widetilde{z}_\nu=(P,\tau)\widetilde{z}_\nu=\widetilde{z}_\nu$

Step 4. Put $\widetilde{x}_\lambda=(B,\varphi)\widetilde{z}_\nu$ and $\widetilde{y}_\mu=\widetilde{x}^{2n+1}_{\lambda_{2n+1}}$ in (e), we get

$$\widetilde{d}\Big((P,\tau)(B,\varphi)\widetilde{z}_\nu,(Q,\zeta)\widetilde{x}^{2n+1}_{\lambda_{2n+1}}\Big)$$
$$\widetilde{\leqslant}\,\widetilde{q}\max\Big\{\widetilde{d}\Big((A,\psi)(B,\varphi)(B,\varphi)\widetilde{z}_\nu,(S,\phi)(T,\xi)\widetilde{x}^{2n+1}_{\lambda_{2n+1}}\Big),$$
$$\widetilde{d}((P,\tau)(B,\varphi)\widetilde{z}_\nu,(A,\psi)(B,\varphi)(B,\varphi)\widetilde{z}_\nu),$$
$$\widetilde{d}\Big((Q,\zeta)\widetilde{x}^{2n+1}_{\lambda_{2n+1}},(S,\phi)(T,\xi)\widetilde{x}^{2n+1}_{\lambda_{2n+1}}\Big),$$
$$\widetilde{d}\Big((P,\tau)(B,\varphi)\widetilde{z}_\nu,(S,\phi)(T,\xi)\widetilde{x}^{2n+1}_{\lambda_{2n+1}}\Big)\Big\}$$

As $(B,\varphi)(P,\tau)=(P,\tau)(B,\varphi)$, $(A,\psi)(B,\varphi)=(B,\varphi)(A,\psi)$

So we have

$(P,\tau)((B,\varphi)\widetilde{z}_\nu)=(B,\varphi)((P,\tau)\widetilde{z}_\nu)=(B,\varphi)\widetilde{z}_\nu$ and

$$((A,\psi)(B,\varphi))((B,\varphi)\widetilde{z}_\nu)=((B,\varphi)(A,\psi))((B,\varphi)\widetilde{z}_\nu)=$$

$$(B, \varphi)((A, \psi))(B, \varphi)\widetilde{z}_\nu) = (B, \varphi)\widetilde{z}_\nu$$

Taking $n \to \infty$ and using (4.1), we get

$$\begin{aligned}
&\widetilde{d}((B, \varphi)\widetilde{z}_\nu, \widetilde{z}_\nu) \\
&\widetilde{\leqslant} \widetilde{q} \max\Big\{ \widetilde{d}((B, \varphi)\widetilde{z}_\nu, \widetilde{z}_\nu), \ \widetilde{d}((B, \varphi)\widetilde{z}_\nu, (B, \varphi)\widetilde{z}_\nu), \widetilde{d}(\widetilde{z}_\nu, \widetilde{z}_\nu), \\
&\qquad \widetilde{d}((B, \varphi)\widetilde{z}_\nu, \widetilde{z}_\nu)\} \\
&\widetilde{\leqslant} \widetilde{q}\widetilde{d}((B, \varphi)\widetilde{z}_\nu, \widetilde{z}_\nu) \widetilde{<} \widetilde{d}((B, \varphi)\widetilde{z}_\nu, \widetilde{z}_\nu)
\end{aligned}$$

which is a contradiction. Hence $(B, \varphi)\widetilde{z}_\nu = \widetilde{z}_\nu$ and from (4.3), also we have

$$(A, \psi)(B, \varphi)\widetilde{z}_\nu = \widetilde{z}_\nu \Rightarrow (A, \psi)\widetilde{z}_\nu = \widetilde{z}_\nu .$$

Therefore,

$$\text{Therefore, } (A, \psi)\widetilde{z}_\nu = (B, \varphi)\widetilde{z}_\nu = (P, \tau)\widetilde{z}_\nu = \widetilde{z}_\nu \tag{4.4}$$

Step 5. As $(P, \tau)(\widetilde{X}) \subset (S, \phi)(T, \xi)(\widetilde{X})$ there exists $\widetilde{u}_\varrho \in (\widetilde{X})$ such that $\widetilde{z}_\nu = (P, \tau)\widetilde{z}_\nu = (S, \phi)(T, \xi)\widetilde{u}_\varrho$

Putting $\widetilde{x}_\mu = \widetilde{x}^{2n}_{\lambda_{2n}}$ and $\widetilde{y}_\mu = \widetilde{u}_\varrho$ in (e), we get

$$\begin{aligned}
&\widetilde{d}\big((P, \tau)\widetilde{x}^{2n}_{\lambda_{2n}}, (Q, \zeta)\widetilde{u}_\varrho\big) \\
&\widetilde{\leqslant} \widetilde{q} \max\Big\{ \widetilde{d}\big((A, \psi)(B, \varphi)\widetilde{x}^{2n}_{\lambda_{2n}}, (S, \phi)(T, \xi)\widetilde{u}_\varrho\big), \\
&\widetilde{d}\big((P, \tau)\widetilde{x}^{2n}_{\lambda_{2n}}, (A, \psi)(B, \varphi)\widetilde{x}^{2n}_{\lambda_{2n}}, \ \widetilde{d}((Q, \zeta)\widetilde{u}_\varrho, (S, \phi)(T, \xi)\widetilde{u}_\varrho), \\
&\qquad \widetilde{d}\big((P, \tau)\widetilde{x}^{2n}_{\lambda_{2n}}, (S, \phi)(T, \xi)\widetilde{u}_\varrho\big)\Big\}
\end{aligned}$$

Taking $n \to \infty$ and using (4.1) and (4.2), we get

$$\begin{aligned}
&\widetilde{d}(\widetilde{z}_\nu, (Q, \zeta)\widetilde{u}_\varrho) \widetilde{\leqslant} \widetilde{q} \max\Big\{ \widetilde{d}(\widetilde{z}_\nu, \widetilde{z}_\nu), \ \widetilde{d}(\widetilde{z}_\nu, \widetilde{z}_\nu), \widetilde{d}((Q, \zeta)\widetilde{u}_\varrho, \widetilde{z}_\nu), \\
&\qquad \widetilde{d}(\widetilde{z}_\nu, \widetilde{z}_\nu)\Big\} \\
&\widetilde{\leqslant} \widetilde{q}\widetilde{d}(\widetilde{z}_\nu, (Q, \zeta)\widetilde{u}_\varrho) \widetilde{<} \widetilde{d}(\widetilde{z}_\nu, (Q, \zeta)\widetilde{u}_\varrho)
\end{aligned}$$

which is a contradiction. Hence $(Q, \zeta)\widetilde{u}_\varrho = \widetilde{z}_\nu$

$$\textit{i.e.}, \ (S, \phi)(T, \xi)\widetilde{u}_\varrho = \widetilde{z}_\nu = (Q, \zeta)\widetilde{u}_\varrho$$

Since $((Q, \zeta), (S, \phi)(T, \xi))$ is weak compatible, therefore, we have

$$\begin{aligned}
&(Q, \zeta)(S, \phi)(T, \xi)\widetilde{u}_\varrho = (S, \phi)(T, \xi)(Q, \zeta)\widetilde{u}_\varrho \\
&\Longrightarrow (Q, \zeta)\widetilde{z}_\nu = (S, \phi)(T, \xi)\widetilde{z}_\nu
\end{aligned}$$

Step 6. Putting $\widetilde{x}_\lambda = \widetilde{x}^{2n}_{\lambda_{2n}}$ and $\widetilde{y}_\mu = \widetilde{z}_\nu$ in (e), we get

$$\begin{aligned}
&\widetilde{d}\big((P, \tau)\widetilde{x}^{2n}_{\lambda_{2n}}, (Q, \zeta)\widetilde{z}_\nu\big) \\
&\widetilde{\leqslant} \widetilde{q} \max\Big\{ \widetilde{d}\big((A, \psi)(B, \varphi)\widetilde{x}^{2n}_{\lambda_{2n}}, (S, \phi)(T, \xi)\widetilde{z}_\nu\big), \\
&\widetilde{d}\big((P, \tau)\widetilde{x}^{2n}_{\lambda_{2n}}, (A, \psi)(B, \varphi)\widetilde{x}^{2n}_{\lambda_{2n}}, \ \widetilde{d}((Q, \zeta)\widetilde{z}_\nu, (S, \phi)(T, \xi)\widetilde{z}_\nu), \\
&\qquad \widetilde{d}\big((P, \tau)\widetilde{x}^{2n}_{\lambda_{2n}}, (S, \phi)(T, \xi)\widetilde{z}_\nu\big)\Big\}
\end{aligned}$$

Taking $n \to \infty$ and using (4.2) and **Step 5.**, we get

$$\widetilde{d}(\widetilde{z}_\nu, (Q,\zeta)\widetilde{z}_\nu) \;\widetilde{\leqslant}\; \widetilde{q}\max\Big\{\widetilde{d}(\widetilde{z}_\nu, (Q,\zeta)\widetilde{z}_\nu),\ \widetilde{d}(\widetilde{z}_\nu, \widetilde{z}_\nu),$$

$$\widetilde{d}((Q,\zeta)\widetilde{z}_\nu, (Q,\zeta)\widetilde{z}_\nu),\ \widetilde{d}(\widetilde{z}_\nu, (Q,\zeta)\widetilde{z}_\nu)\Big\}$$

$$\widetilde{\leqslant}\; \widetilde{q}\widetilde{d}(\widetilde{z}_\nu, (Q,\zeta)\widetilde{z}_\nu) \;\widetilde{<}\; \widetilde{d}(\widetilde{z}_\nu, (Q,\zeta)\widetilde{u}_\varrho),$$

which is a contradiction. Hence $(Q,\zeta)\widetilde{z}_\nu = \widetilde{z}_\nu$

Step 7. Putting $\widetilde{x}_\lambda = \widetilde{x}^{2n}_{\lambda_{2n}}$ and $\widetilde{y}_\mu = (T,\xi)\widetilde{z}_\nu$ in (e), we get

$$\widetilde{d}\big((P,\tau)\widetilde{x}^{2n}_{\lambda_{2n}}, (Q,\zeta)(T,\xi)\widetilde{z}_\nu\big)$$

$$\widetilde{\leqslant}\widetilde{q}\max\Big\{\widetilde{d}\big((A,\psi)(B,\varphi)\widetilde{x}^{2n}_{\lambda_{2n}}, (S,\phi)(T,\xi)^2\widetilde{z}_\nu\big),$$

$$\widetilde{d}\big((P,\tau)\widetilde{x}^{2n}_{\lambda_{2n}}, (A,\psi)(B,\varphi)\widetilde{x}^{2n}_{\lambda_{2n}}\big), \widetilde{d}\big((Q,\zeta)(T,\xi)\widetilde{z}_\nu, (S,\phi)(T,\xi)^2\widetilde{z}_\nu\big),$$

$$\widetilde{d}\big((P,\tau)\widetilde{x}^{2n}_{\lambda_{2n}}, (S,\phi)(T,\xi)^2\widetilde{z}_\nu\big)\Big\}$$

As $(Q,\zeta)(T,\xi) = (T,\xi)(Q,\zeta)$ and $(S,\phi)(T,\xi) = (T,\xi)(S,\phi)$
we have $(Q,\zeta)(T,\xi)\widetilde{z}_\nu = (T,\xi)(Q,\zeta)\widetilde{z}_\nu = (T,\xi)\widetilde{z}_\nu$ and

$$(S,\phi)(T,\xi)((T,\xi)\widetilde{z}_\nu) = (T,\xi)((S,\phi)(T,\xi)\widetilde{z}_\nu) =$$

$$(T,\xi)(Q,\zeta)\widetilde{z}_\nu = (T,\xi)\widetilde{z}_\nu$$

Taking $n \to \infty$ we get

$$\widetilde{d}(\widetilde{z}_\nu, (T,\xi)\widetilde{z}_\nu) \widetilde{\leqslant}\widetilde{q}\max\Big\{\widetilde{d}(\widetilde{z}_\nu, (T,\xi)\widetilde{z}_\nu),\ \widetilde{d}(\widetilde{z}_\nu, \widetilde{z}_\nu),$$

$$\widetilde{d}((T,\xi)\widetilde{z}_\nu, (T,\xi)\widetilde{z}_\nu),\ \widetilde{d}(\widetilde{z}_\nu, (T,\xi)\widetilde{z}_\nu)\Big\}$$

$$\widetilde{\leqslant}\widetilde{q}\widetilde{d}(\widetilde{z}_\nu, (T,\xi)\widetilde{z}_\nu)\widetilde{<}\widetilde{d}(\widetilde{z}_\nu, (T,\xi)\widetilde{z}_\nu)$$

which is a contradiction. Hence $,(T,\xi)\widetilde{z}_\nu = \widetilde{z}_\nu$
Now $(S,\phi)(T,\xi)\widetilde{z}_\nu = (T,\xi)\widetilde{z}_\nu = \widetilde{z}_\nu \Rightarrow (S,\phi)\widetilde{z}_\nu = \widetilde{z}_\nu$
Hence,

$$(S,\phi)\widetilde{z}_\nu = (T,\xi)\widetilde{z}_\nu = (Q,\zeta)\widetilde{z}_\nu = \widetilde{z}_\nu \tag{4.5}$$

Combining (4.4) and (4.5), we get

$$(A,\psi)\widetilde{z}_\nu = (B,\varphi)\widetilde{z}_\nu = (P,\tau)\widetilde{z}_\nu = (Q,\zeta)\widetilde{z}_\nu = (T,\xi)\widetilde{z}_\nu$$

$$= (S,\phi)\widetilde{z}_\nu = \widetilde{z}_\nu$$

Hence $\widetilde{z}_\nu$ is the common fixed soft point of (A,ψ), (B,φ), (S,ϕ), (T,ξ), (P,τ) and (Q,ζ)

Case II. Suppose (P,τ) is soft continuous.
$(P,\tau)^2\widetilde{x}^{2n}_{\lambda_{2n}} \to (P,\tau)\widetilde{z}_\nu$ and $(P,\tau)((A,\psi)(B,\varphi))\widetilde{x}^{2n}_{\lambda_{2n}} \to (P,\tau)\widetilde{z}_\nu$
As $((P,\tau), (A,\psi)(B,\varphi))$ is compatible pair of type (β)
$\widetilde{d}\big((P,\tau)(P,\tau)\widetilde{x}^{2n}_{\lambda_{2n}}, ((A,\psi)(B,\varphi))^2\widetilde{x}^{2n}_{\lambda_{2n}}\big) = \overline{0}$ or
$\widetilde{d}\big((P,\tau)\widetilde{z}_\nu, (A,\psi)(B,\varphi)^2\widetilde{x}^{2n}_{\lambda_{2n}}\big) = \overline{0}$

Therefore $((A,\psi)(B,\varphi))^2\,\widetilde{x}^{2n}_{\lambda_{2n}} \to (P,\tau)\widetilde{z}_{\nu}$

Step 8. Putting $\widetilde{x}_{\lambda} = (P,\tau)\widetilde{x}^{2n}_{\lambda_{2n}}$ and $\widetilde{y}_{\mu} = \widetilde{x}^{2n+1}_{\lambda_{2n+1}}$ in (e), we have

$$\begin{aligned}
&\widetilde{d}\Big((P,\tau)(P,\tau)\widetilde{x}^{2n}_{\lambda_{2n}},(Q,\zeta)\widetilde{x}^{2n+1}_{\lambda_{2n+1}}\Big)\\
&\quad\widetilde{\leqslant}\,\widetilde{q}\max\Big\{\widetilde{d}\Big((A,\psi)(B,\varphi)(P,\tau)\widetilde{x}^{2n}_{\lambda_{2n}},(S,\phi)(T,\xi)\widetilde{x}^{2n+1}_{\lambda_{2n+1}}\Big),\\
&\qquad\widetilde{d}\big((P,\tau)(P,\tau)\widetilde{x}^{2n}_{\lambda_{2n}},(A,\psi)(B,\varphi)(P,\tau)\widetilde{x}^{2n}_{\lambda_{2n}}\big),\\
&\qquad\widetilde{d}\Big((Q,\zeta)\widetilde{x}^{2n+1}_{\lambda_{2n+1}},(S,\phi)(T,\xi)\widetilde{x}^{2n+1}_{\lambda_{2n+1}}\Big),\\
&\qquad\widetilde{d}\Big((P,\tau)(P,\tau)\widetilde{x}^{2n}_{\lambda_{2n}},(S,\phi)(T,\xi)\widetilde{x}^{2n+1}_{\lambda_{2n+1}}\Big)\Big\}
\end{aligned}$$

Taking $n\to\infty$, we get

$$\begin{aligned}
\widetilde{d}((P,\tau)\widetilde{z}_{\nu},\widetilde{z}_{\nu})\,&\widetilde{\leqslant}\,\widetilde{q}\max\Big\{\widetilde{d}((P,\tau)\widetilde{z}_{\nu},\widetilde{z}_{\nu})\ \widetilde{d}((P,\tau)\widetilde{z}_{\nu},(P,\tau)\widetilde{z}_{\nu}),\\
&\qquad\widetilde{d}(\widetilde{z}_{\nu},\widetilde{z}_{\nu}),\ \widetilde{d}((P,\tau)z_{\nu},z_{\nu})\Big\}\\
&\widetilde{\leqslant}\,\widetilde{q}\,\widetilde{d}((P,\tau)\widetilde{z}_{\nu},\widetilde{z}_{\nu})\,\widetilde{<}\,\widetilde{d}((P,\tau)\widetilde{z}_{\nu},\widetilde{z}_{\nu})
\end{aligned}$$

which is a contradiction. Hence $(P,\tau)\,\widetilde{z}_{\nu}=\widetilde{z}_{\nu}$

Step 9. Putting $\widetilde{x}_{\lambda} = (A,\psi)(B,\varphi)\widetilde{x}^{2n}_{\lambda_{2n}}$ and $\widetilde{y}_{\nu} = \widetilde{x}^{2n+1}_{\lambda_{2n+1}}$ in (e), we get

$$\begin{aligned}
&\widetilde{d}\Big((P,\tau)(A,\psi)(B,\varphi)\widetilde{x}^{2n}_{\lambda_{2n}},(Q,\zeta)\widetilde{x}^{2n+1}_{\lambda_{2n+1}}\Big)\\
&\quad\widetilde{\leqslant}\max\widetilde{q}\,\Big\{\widetilde{d}\Big((A,\psi)(B,\varphi)^2\,\widetilde{x}^{2n}_{\lambda_{2n}},(S,\phi)(T,\xi)\widetilde{x}^{2n+1}_{\lambda_{2n+1}}\Big),\\
&\qquad\widetilde{d}\big((P,\tau)(A,\psi)(B,\varphi)\widetilde{x}^{2n}_{\lambda_{2n}},(A,\psi)(B,\varphi)^2\,\widetilde{x}^{2n}_{\lambda_{2n}}\big),\\
&\qquad\widetilde{d}\Big((Q,\zeta)\widetilde{x}^{2n+1}_{\lambda_{2n+1}},(S,\phi)(T,\xi)\widetilde{x}^{2n+1}_{\lambda_{2n+1}}\Big),\\
&\qquad\widetilde{d}\Big((P,\tau)(A,\psi)(B,\varphi)\widetilde{x}^{2n}_{\lambda_{2n}},(S,\phi)(T,\xi)\widetilde{x}^{2n+1}_{\lambda_{2n+1}}\Big)\Big\}
\end{aligned}$$

Taking $n\to\infty$, we get

$$\begin{aligned}
&\widetilde{d}((A,\psi)(B,\varphi)\widetilde{z}_{\nu},\widetilde{z}_{\nu})\,\widetilde{\leqslant}\,\widetilde{q}\max\Big\{\widetilde{d}((A,\psi)(B,\varphi)\widetilde{z}_{\nu},z_{\nu}),\\
&\widetilde{d}((A,\psi)(B,\varphi)\widetilde{z}_{\nu},(A,\psi)(B,\varphi)\widetilde{z}_{\nu}),\widetilde{d}(\widetilde{z}_{\nu},\widetilde{z}_{\nu}),\ \widetilde{d}((A,\psi)(B,\varphi)\widetilde{z}_{\nu},\widetilde{z}_{\nu})\Big\}\\
&\qquad\widetilde{\leqslant}\,\widetilde{q}\,\widetilde{d}((A,\psi)(B,\varphi)\widetilde{z}_{\nu},\widetilde{z}_{\nu})\,\widetilde{<}\,\widetilde{d}((A,\psi)(B,\varphi)\widetilde{z}_{\nu},\widetilde{z}_{\nu})
\end{aligned}$$

which is a contradiction. Hence $(A,\psi)(B,\varphi)\,\widetilde{z}_{\nu}=\widetilde{z}_{\nu}$

Therefore, $(A,\psi)(B,\varphi)\widetilde{z}_{\nu}=\widetilde{z}_{\nu}=(P,\tau)\,\widetilde{z}_{\nu}$

Now, apply **Step 4.**, to get $(B,\varphi)\widetilde{z}_{\nu}=\widetilde{z}_{\nu}$ and so

$$(A,\psi)\widetilde{z}_{\nu}=(B,\varphi)\widetilde{z}_{\nu}=(P,\tau)\,\widetilde{z}_{\nu}=\widetilde{z}_{\nu}$$

Further using **Steps 5, 6, 7**, we get

$$(Q,\zeta)\widetilde{z}_{\nu}=(S,\phi)(T,\xi)\widetilde{z}_{\nu}=(S,\phi)\,\widetilde{z}_{\nu}=(T,\xi)\widetilde{z}_{\nu}=\widetilde{z}_{\nu}$$

i.e. $\widetilde{z}_{\nu}$ is the common fixed soft point of the six maps (A,ψ), (B,φ), (S,ϕ), (T,ξ), (P,τ) and (Q,ζ) in this case also.

Uniqeness : Let u_{ϱ} be another common fixed soft point of $(A, \psi), (B, \varphi), (S, \phi), (T, \xi), (P, \tau)$ and (Q, ζ).
Then, $(A, \psi)\widetilde{u}_{\varrho} = (B, \varphi)\widetilde{u}_{\varrho} = (S, \phi)\widetilde{u}_{\varrho} = (T, \xi)\widetilde{u}_{\varrho} = (P, \tau)\widetilde{u}_{\varrho} = (Q, \zeta)\widetilde{u}_{\varrho} = \widetilde{u}_{\varrho}$
Put $\widetilde{x}_{\lambda} = \widetilde{z}_{\nu}$ and $\widetilde{y}_{\nu} = \widetilde{u}_{\varrho}$ in (e), we get

$$\widetilde{d}((P, \tau)\widetilde{z}_{\nu}, (Q, \zeta)\widetilde{u}_{\varrho})$$
$$\widetilde{\leqslant}\widetilde{q}\max\Big\{\widetilde{d}((A, \psi)(B, \varphi)\widetilde{z}_{\nu}, (S, \phi)(T, \xi)\widetilde{u}_{\varrho}),$$
$$\widetilde{d}((P, \tau)\widetilde{z}_{\nu}, (A, \psi)(B, \varphi)\widetilde{z}_{\nu}), \widetilde{d}((Q, \zeta)\widetilde{u}_{\varrho}, (S, \phi)(T, \xi)\widetilde{u}_{\varrho}),$$
$$\widetilde{d}((P, \tau)\widetilde{z}_{\nu}, (S, \phi)(T, \xi)\widetilde{u}_{\varrho})\Big\}$$

Taking $n \to \infty$, we get

$$\widetilde{d}(\widetilde{z}_{\nu}, \widetilde{u}_{\varrho}) \widetilde{\leqslant}\widetilde{q}\max\Big\{\widetilde{d}(\widetilde{z}_{\nu}, \widetilde{u}_{\varrho}), \widetilde{d}(\widetilde{z}_{\nu}, \widetilde{z}_{\nu}), \widetilde{d}(\widetilde{u}_{\varrho}, \widetilde{u}_{\varrho}), \widetilde{d}(\widetilde{z}_{\nu}, \widetilde{u}_{\varrho})\Big\}$$
$$\widetilde{\leqslant}\widetilde{q}\widetilde{d}(\widetilde{z}_{\nu}, \widetilde{u}_{\varrho}) \widetilde{<} \widetilde{d}(\widetilde{z}_{\nu}, \widetilde{u}_{\varrho})$$

which is a contradiction. Hence $\widetilde{z}_{\nu} = \widetilde{u}_{\varrho}$
Therefore $\widetilde{u}_{\varrho}$ is the unique common fixed soft point of self maps $(A, \psi), (B, \varphi), (S, \phi), (T, \xi), (P, \tau)$ and (Q, ζ) □

5. Conclusion

In this paper, under different sufficient conditions several types of compatible maps are being compared in soft metric spaces and also fixed point theorem for six soft continuous self maps is established here. One can further try to generalize these concepts in fuzzy soft metric spaces, because, fuzzy soft metric is a generalization of both fuzzy metric space and soft metric space. We hope that the finding in this paper will help researchers to enhance and promote the further study on soft metric space to carry out general framework for their applications in real life.

References

[1] C. Gunduz Aras, A. Sonmez, H. Cakalli, *On Soft Mappings* , arXiv:1305.4545v1 [math.GM] 16. May 2013.

[2] D.Chen, *The parameterization reduction of soft sets and its applications*,Comput. Math. Appl., 49(5-6)(2005), 757-763.

[3] D. Molodtsov, *Soft set-theory-first results*, Comput. Math. Appl. 37(4-5)(1999), 19-31.

[4] M. Shabir and M. Naz, *On soft topological spaces*, Comput. Math. Appl., 61(7)(2011), 1786-1799.

[5] P. K. Maji, A.R.Roy, R. Biswas, *An application of soft sets in a decision making problem*, Comput. Math. Appl., 44(8-9)(2002), 1077-1083.

[6] P.K.Maji, R.Biswas, A.R.Roy, *Soft set theory*, Comput. Math. Appl., 45(4-5)(2003), 555-562.

[7] P. Majumdar, S.K. Samanta, *On soft mappings*, Comput. Math. Appl., 60(9)(2010), 2666-2672.

[8] S. Das and S.K.Samanta, *Soft metric*, Annals of Fuzzy Mathematics and Informatics, 6(1)(2013), 77-94.

[9] S. Das and S.K.Samanta, *Soft Real Sets, Soft Real Numbers and Their Properties* , The Journal of Fuzzy Mathematics, 20(3)(2012), 551-576.

[10] S. Bayramov, C. Gunduz Aras, *Soft locally compact and soft paracompact spaces* , Journal of Mathematics and System Science, 3(2013), 122130.
[11] S. Hussain and B. Ahmad, *Some properties of soft topological spaces* , Computers and Math. with Applications, 62(11)(2011), 4058-4067.
[12] Sanjay Roy and T.K.Samanta, *A Note on Soft Topological Space*, Punjab University Journal of Mathematics, 46(1)(2014), 19-24.
[13] Sanjay Roy and T.K.Samanta, *An introduction to open and closed sets on fuzzy soft topological spaces*, Annals of Fuzzy mathematics and Informatics, 6(2)(2013), 425-431.
[14] Sanjay Roy and T.K.Samanta, *A note on fuzzy soft topological spaces*, Annals of Fuzzy mathematics and Informatics, 3(2)(2012), 305-311.

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