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Hyper-order and Fixed Points of Meromorphic Solutions of Higher Order Non-homogeneous Linear Differential Equations

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Abstract. In this paper, we investigate the order of growth of solutions of the higher order non-homogeneous linear differential equation

$$f^{(k)}+\sum_{j=0}^{k-1}h_j e^{P_j(z)} f^{(j)}= F,$$

where $P_j(z)$ $(j = 0, 1, \cdots , k-1)$ are polynomials with $\deg P_j = n_j \geqslant 1$ and $h_j(z)$ $(j = 0, 1, \cdots , k-1)$ not all vanishing identically, F are meromorphic functions of finite order having only finitely many poles. Under some conditions, we prove that every meromorphic solution $f \not\equiv 0$ of the above equation is of infinite order. We give also some estimates of their hyper-order, exponent of convergence of the zeros and the hyper-exponent of convergence of zeros. Furthermore, we give an estimation for the exponent of convergence of fixed points of solutions and their 1st, 2nd derivatives.

1. Introduction and statement of results

In this paper, we use the standard notations of Nevanlinna's value distribution theory (see, [**11**], [**13**], [**18**]). In addition, we use the notations $\lambda(f)$ and $\lambda\left(\frac{1}{f}\right)$ to denote respectively the exponent of convergence of the zeros and the poles of a meromorphic function f, $\rho(f)$ to denote the order of growth of f. To express the rate of growth of meromorphic solutions of infinite order, we recall the following definition.

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Definition 1.1. ([**12**], [**18**]) Let f be a meromorphic function. Then the hyper-order $\rho_2(f)$ of $f(z)$ is defined by

$$\rho_2(f) = \limsup_{r \longrightarrow +\infty} \frac{\log \log T(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f.

To give estimates of fixed points, we define:

Definition 1.2. ([**7**], [**16**]) Let f be a meromorphic function and let $z_1, z_2, \cdots$ ($|z_j| = r_j,\ 0 < r_1 \leqslant r_2 \leqslant \cdots$) be the sequence of the distinct fixed points of f. The exponent of convergence of the sequence of distinct fixed points of $f(z)$ is defined by

$$\overline{\tau}(f) = \inf \left\{ \tau > 0 : \sum_{j=1}^{+\infty} |z_j|^{-\tau} < +\infty \right\}.$$

Clearly,

$$\overline{\tau}(f) = \overline{\lambda}(f - z) = \limsup_{r \longrightarrow +\infty} \frac{\log \overline{N}\left(r, \frac{1}{f-z}\right)}{\log r},$$

where $\overline{N}\left(r, \frac{1}{f-z}\right)$ is the integrated counting function of distinct fixed points of $f(z)$ in $\{z : |z| \leqslant r\}$.

Definition 1.3. ([**6**]) Let f be a meromorphic function. The hyper-exponent $\lambda_2(f)$ of convergence of zeros and the hyper-exponent $\overline{\lambda}_2(f)$ of convergence of distinct zeros of f are defined respectively by

$$\lambda_2(f) = \limsup_{r \longrightarrow +\infty} \frac{\log \log N\left(r, \frac{1}{f}\right)}{\log r}, \quad \overline{\lambda}_2(f) = \limsup_{r \longrightarrow +\infty} \frac{\log \log \overline{N}\left(r, \frac{1}{f}\right)}{\log r}.$$

Several authors, such as Kwon [**12**], Chen [**8**], Gundersen [**10**] have investigated the second order linear differential equation

$$f'' + A_1(z) e^{P(z)} f' + A_0(z) e^{Q(z)} f = 0, \tag{1.1}$$

where $P(z),\ Q(z)$ are nonconstant polynomials, $A_1(z),\ A_0(z) \not\equiv 0$ are entire functions such that $\rho(A_1) < \deg P(z),\ \rho(A_0) < \deg Q(z)$. Gundersen showed in ([**10**], p. 419) that if $\deg P(z) \neq \deg Q(z)$, then every nonconstant solution of (1.1) is of infinite order. If $\deg P(z) = \deg Q(z)$, then (1.1) may have nonconstant solutions of finite order. For instanse $f(z) = e^z + \frac{1}{2}$ satisfies $f'' + 2e^z f' - 2e^z f = 0$.

In [**17**], Wang and Laine have investigated the growth of higher order non-homogeneous linear differential equations and obtained the following result.

Theorem 1.1. *([**17**]) Suppose that $A_j(z) = h_j(z) e^{P_j(z)} (j = 0, \cdots, k-1)$, where $P_j(z) = a_{j,n} z^n + \cdots + a_{j,0}$ $(j = 0, 1, \cdots, k-1)$ are polynomials with degree $n \geqslant 1$, $h_j(z)$ $(\not\equiv 0)$ $(j = 0, 1, \cdots, k-1)$ are entire functions with order less than n, and that $H(z) \not\equiv 0$ is an entire function of order less than n. If $a_{j,n}$ $(j = 0, 1, \cdots, k-1)$ are distinct complex numbers, then every solution f of the differential equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = H(z)$$

is of infinite order.

In this paper, we consider the higher order nonhomogeneous linear differential equation

$$f^{(k)}+h_{k-1}(z)\,e^{P_{k-1}(z)}f^{(k-1)}+\cdots+h_1(z)\,e^{P_1(z)}f'+h_0(z)\,e^{P_0(z)}f=F(z), \tag{1.2}$$

where $P_j(z)$ are polynomials with degree $n_j \geqslant 1$ $(j=0,1,\cdots,k-1)$ and h_j $(j=0,1,\cdots,k-1)$ not all vanishing identically, F are meromorphic functions having only finitely many poles. We obtain the following results.

THEOREM 1.2. *Let $n_j \geqslant 1$ $(j=0,1,\cdots,k-1)$ be integers and $P_j(z)$ $(j=0,1,\cdots,k-1)$ be polynomials with degree n_j, and let $h_j(z)$ $(j=0,1,\cdots,k-1)$ not all vanishing identically, F be meromorphic functions of finite order having only finitely many poles such that $\rho(F)<\max\{n_j : j=0,1,\cdots,k-1\}=n$ and $\rho(h_j)<n_j$ $(j=0,1,\cdots,k-1)$. Suppose that n_j are distinct integer numbers. Then every meromorphic solution $f\not\equiv 0$ of equation (1.2) is of infinite order and the hyper-order of f satisfies $\rho_2(f)\leqslant n$. Furthermore if $F\not\equiv 0$, then every meromorphic solution f of equation (1.2) satisfies*

$$\overline{\lambda}(f)=\lambda(f)=\rho(f)=\infty,\quad \overline{\lambda}_2(f)=\lambda_2(f)=\rho_2(f)\leqslant n.$$

THEOREM 1.3. *Let $n_j \geqslant 1$ $(j=0,1,\cdots,k-1)$ be integers and $P_j(z)$ $(j=0,1,\cdots,k-1)$ be polynomials with degree n_j, and let $h_j(z)$ $(j=0,1,\cdots,k-1)$ $(h_0\not\equiv 0)$, F be meromorphic functions of finite order having only finitely many poles such that $\max\{\rho(h_j)\ (j=0,1,\cdots,k-1),\ \rho(F)\}<\deg P_j(z)$ $(j=0,1,\cdots,k-1)$, with $P_j(z)\equiv 0$ if $h_j\equiv 0$. If n_j $(j=0,1,\cdots,k-1)$ are distinct integer numbers, then for any meromorphic solution $f\not\equiv 0$ of equation (1.2), we have f, f', f'' all have infinitely many fixed points and satisfy*

$$\overline{\tau}(f)=\overline{\tau}(f')=\overline{\tau}(f'')=\infty.$$

REMARK 1.1. For some papers related to second order nonhomogeneous linear differential equations see ([**1**], [**2**]).

2. Lemmas for the proofs of theorems

First, we recall the following definitions. The linear measure of a set $E\subset(0,+\infty)$ is defined as $m(E)=\int_0^{+\infty}\chi_E(t)\,dt$, and the logarithmic measure of a set $F\subset(1,+\infty)$ is defined by $lm(F)=\int_1^{+\infty}\frac{\chi_F(t)}{t}dt$, where $\chi_H(t)$ is the characteristic function of a set H.

LEMMA 2.1. ([**3**], [**15**]) *Let $P(z)=a_nz^n+\cdots+a_0$ $(a_n=\alpha+i\beta\neq 0)$ be a polynomial with degree $n\geqslant 1$ and $A(z)\not\equiv 0$ be a meromorphic function with $\rho(A)<n$. Set $f(z)=A(z)\,e^{P(z)}$, $z=re^{i\theta}$, $\delta(P,\theta)=\alpha\cos n\theta-\beta\sin n\theta$. Then for any given $\varepsilon>0$, there exists a set $E_1\subset[0,2\pi)$ that has linear measure zero, such that if $\theta\in[0,2\pi)\setminus(E_1\cup E_2)$, where $E_2=\{\theta\in[0,2\pi):\delta(P,\theta)=0\}$ is a finite set, then for sufficiently large $|z|=r$, we have*
(i) if $\delta(P,\theta)>0$, then

$$\exp\{(1-\varepsilon)\,\delta(P,\theta)\,r^n\}\leqslant|f(z)|\leqslant\exp\{(1+\varepsilon)\,\delta(P,\theta)\,r^n\}, \tag{2.1}$$

(ii) if $\delta(P,\theta)<0$, *then*

$$\exp\{(1+\varepsilon)\delta(P,\theta)r^n\}\leqslant|f(z)|\leqslant\exp\{(1-\varepsilon)\delta(P,\theta)r^n\}. \tag{2.2}$$

LEMMA 2.2. ([**9**], *p. 89*) *Let* $f(z)$ *be a transcendental meromorphic function of finite order* ρ. *Let* $\Gamma=\{(k_1,j_1),(k_2,j_2),\cdots,(k_m,j_m)\}$ *denote a set of distinct pairs of integers satisfying* $k_i>j_i\geqslant 0$ $(i=1,2,\cdots,m)$ *and let* $\varepsilon>0$ *be a given constant. Then, there exists a set* $E_3\subset[0,2\pi)$ *that has linear measure zero such that if* $\psi_0\in[0,2\pi)\setminus E_3$, *then there is a constant* $R_0=R_0(\psi_0)>1$ *such that for all* z *satisfying* $\arg z=\psi_0$ *and* $|z|\geqslant R_0$ *and for all* $(k,j)\in\Gamma$, *we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right|\leqslant|z|^{(k-j)(\rho-1+\varepsilon)}.$$

LEMMA 2.3. ([**1**]) *Let* $f(z)$ *be a meromorphic function having only finitely many poles, and suppose that*

$$G(z):=\frac{\log^+\left|f^{(s)}(z)\right|}{|z|^\rho}$$

is unbounded on some ray $\arg z=\theta$ *with constant* $\rho>0$. *Then, there exists an infinite sequence of points* $z_n=r_ne^{i\theta}$ $(n=1,2,\cdots)$ *tending to infinity such that* $G(z_n)\to\infty$ *and*

$$\left|\frac{f^{(j)}(z_n)}{f^{(s)}(z_n)}\right|\leqslant\frac{1}{(s-j)!}(1+o(1))|z_n|^{s-j}\quad(j=0,\cdots,s-1)\quad as\quad n\to\infty.$$

REMARK 2.1. Lemma 2.3 was obtained by Wang and Laine in [**17**] when $f(z)$ is entire function.

LEMMA 2.4. ([**17**]) *Let* $f(z)$ *be an entire function with* $\rho(f)<\infty$. *Suppose that there exists a set* $E_4\subset[0,2\pi)$ *which has linear measure zero, such that* $\log^+\left|f\left(re^{i\theta}\right)\right|\leqslant Mr^\sigma$ *for any ray* $\arg(z)=\theta\in[0,2\pi)\setminus E_4$, *where* M *is a positive constant depending on* θ, *while* σ *is a positive constant independent of* θ. *Then* $\rho(f)\leqslant\sigma$.

LEMMA 2.5. ([**5**]) *Let* $f(z)$ *be a meromorphic function of order* $\rho(f)=\rho<+\infty$. *Then for any given* $\varepsilon>0$, *there exists a set* $E_5\subset(1,+\infty)$ *that has finite linear measure and finite logarithmic measure such that when* $|z|=r\notin[0,1]\cup E_5, r\longrightarrow+\infty$, *we have*

$$|f(z)|\leqslant\exp\left\{r^{\rho+\varepsilon}\right\}.$$

Let $g(z)=\sum\limits_{n=0}^{\infty}a_nz^n$ be an entire function. We define by

$$\mu(r)=\max\{|a_n|r^n;n=0,1,2,\cdots\},$$

the maximum term of g, and define by $\nu_g(r)=\max\{m;\mu(r)=|a_m|r^m\}$ the central index of g.

LEMMA 2.6. ([**19**]) *Let* $f(z)=g(z)/d(z)$ *be a transcendental meromorphic function, where* $g(z)$ *is a transcendental entire function and* $d(z)$ *is a polynomial.*

Then there exists a set $E_6 \subset (1,+\infty)$ *that has finite logarithmic measure such that for all* z *satisfying* $|z| = r \notin [0,1] \cup E_6, r \longrightarrow +\infty$ *and* $|g(z)| = M(r,g)$, *we have*

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^n (1+o(1)),$$

where $n \geqslant 1$ *is positive integer.*

Lemma 2.7. ([**6**]) *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be an entire function of infinite order with the hyper-order* $\sigma_2(f) = \sigma$. *Then*

$$\limsup_{r \longrightarrow +\infty} \frac{\log\log \nu_f(r)}{\log r} = \sigma.$$

Lemma 2.8. ([**14**]) *Let* $g(z)$ *be an entire function of infinite order. Denote* $M(r,g) = \max\{|g(z)| : |z| = r\}$, *then for any sufficiently large number* $\lambda > 0$, *and any* $r \in E_7 \subset (1,\infty)$

$$M(r,g) > c_1 \exp\{c_2 r^{\lambda}\},$$

where $lm(E_7) = \infty$ *and* c_1, c_2 *are positive constants.*

Lemma 2.9. *Suppose that* $k \geqslant 2$ *and* $A_0, A_1, \cdots, A_{k-1}$, F *are meromorphic functions not all vanishing identically having only finitely many poles. Let* $\rho = \max\{\rho(A_j)\,(j = 0,1,\cdots,k-1),\ \rho(F)\} < \infty$ *and let* $f(z)$ *be a meromorphic solution of infinite order of the differential equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1 f' + A_0 f = F. \tag{2.3}$$

Then $\rho_2(f) \leqslant \rho$. *Furthermore if* $F \not\equiv 0$, *then we have*

$$\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) \leqslant \rho.$$

Proof. We assume that f is a meromorphic solution of equation (2.3) of infinite order $\rho(f) = \infty$. By (2.3), we have

$$\left|\frac{f^{(k)}}{f}\right| \leqslant |A_{k-1}| \left|\frac{f^{(k-1)}}{f}\right| + |A_{k-2}| \left|\frac{f^{(k-2)}}{f}\right| + \cdots + |A_1| \left|\frac{f'}{f}\right| + \left|\frac{F}{f}\right| + |A_0|. \tag{2.4}$$

From equation (2.3), we know that the poles of f can only occur at the poles of A_j $(j = 0,1,\cdots,k-1)$ and F. Since A_j $(j = 0,1,\cdots,k-1)$ and F are meromorphic functions having only finitely many poles, then $f(z)$ must have only finitely many poles. Therefore, by Hadamard factorization theorem, we can write f as $f(z) = \frac{g(z)}{d(z)}$, where $d(z)$ is a polynomial and $g(z)$ is a transcendental entire function with $\rho(g) = \rho(f) = \infty$ and $\rho_2(f) = \rho_2(g)$. By Lemma 2.5, Lemma 2.6 and Lemma 2.8, for any small $\varepsilon > 0$ and any sufficiently large number $\lambda > \rho + \varepsilon$, there exist a set $E = E_5 \cup E_6 \subset (1,+\infty)$ that has finite logarithmic measure and a set $E_7 \subset (1,+\infty)$ with $lm(E_7) = \infty$ and positive constants c_1, c_2, such that for all z satisfying $|z| = r \in E_7 \setminus [0,1] \cup E$, $r \longrightarrow +\infty$ with $|g(z)| = M(r,g)$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^j (1+o(1)) \quad (j = 1,\cdots,k), \tag{2.5}$$

$$|A_j(z)| \leqslant \exp\left\{r^{\rho+\varepsilon}\right\},\ j=0,1,\cdots,k-1 \text{ and } |F(z)| \leqslant \exp\left\{r^{\rho+\varepsilon}\right\}, \tag{2.6}$$

$$\left|\frac{F(z)}{f(z)}\right| = \left|\frac{F(z)}{g(z)}\right| |d(z)| < Ar^m \frac{1}{c_1} \exp\left\{r^{\rho+\varepsilon} - c_2 r^{\lambda}\right\} \longrightarrow 0, \tag{2.7}$$

where $A > 0$ is a constant and $m = \deg d \geqslant 1$ is an integer. Substituting (2.5), (2.6), (2.7) into (2.4), we obtain

$$\left|\frac{\nu_g(r)}{z}\right|^k |1+o(1)| \leqslant \sum_{j=1}^{k-1} e^{r^{\rho+\varepsilon}} \left|\frac{\nu_g(r)}{z}\right|^j |1+o(1)| + o(1) + e^{r^{\rho+\varepsilon}},$$

it follow that

$$(\nu_g(r))^k |1+o(1)| \leqslant (k+1) e^{r^{\rho+\varepsilon}} r^k (\nu_g(r))^{k-1} |1+o(1)|,$$

so,

$$\nu_g(r) |1+o(1)| \leqslant (k+1) e^{r^{\rho+\varepsilon}} r^k |1+o(1)| \tag{2.8}$$

holds for all z satisfying $|z| = r \in E_7 \diagdown [0,1] \cup E,\ r \longrightarrow +\infty$ with $|g(z)| = M(r,g)$. Hence, by (2.8) and Lemma 2.7, we obtain that

$$\rho_2(f) = \rho_2(g) = \limsup_{r\longrightarrow+\infty} \frac{\log\log \nu_g(r)}{\log r} \leqslant \rho + \varepsilon.$$

Since $\varepsilon > 0$ being arbitrary, then we get

$$\rho_2(f) \leqslant \rho. \tag{2.9}$$

We know that if f has a zero at z_0 of order $m,\ m > k$ and A_j $(j = 0, 1, \cdots, k-1)$ are analytic at z_0, then $F(z)$ must have a zero at z_0 of order $m - k$. Therefore, we get by $F \not\equiv 0$ that

$$N\left(r, \frac{1}{f}\right) \leqslant k\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k-1} N(r, A_j). \tag{2.10}$$

On the other hand, (2.3) can be rewritten as follows

$$\frac{1}{f} = \frac{1}{F}\left[\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_1\frac{f'}{f} + A_0\right].$$

So

$$m\left(r, \frac{1}{f}\right) \leqslant m\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k-1} m(r, A_j) + \sum_{j=1}^{k} m\left(r, \frac{f^{(j)}}{f}\right) + O(1). \tag{2.11}$$

Hence, by the lemma of logarithmic derivative [**11**], there exists a set $E \subset [0, +\infty)$ having finite linear measure such that for all $r \notin E$, we have

$$m\left(r, \frac{f^{(j)}}{f}\right) = O(\log T(r,f) + \log r)\ \ (j = 1, 2, \cdots, k). \tag{2.12}$$

By (2.10), (2.11) and (2.12), we have

$$T(r,f)=T\left(r,\frac{1}{f}\right)+O(1)\leqslant k\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{F}\right)+\sum_{j=0}^{k-1}N(r,A_j)+m\left(r,\frac{1}{F}\right)$$

$$+\sum_{j=0}^{k-1}m(r,A_j)+\sum_{j=1}^{k}m\left(r,\frac{f^{(j)}}{f}\right)+O(1)$$

$$=k\overline{N}\left(r,\frac{1}{f}\right)+T(r,F)+\sum_{j=0}^{k-1}T(r,A_j)+C\log(rT(r,f)),\ r\notin E, \tag{2.13}$$

where C is a positive constant. For sufficiently large r, we have

$$C\log(rT(r,f))\leqslant\frac{1}{2}T(r,f), \tag{2.14}$$

$$T(r,F)\leqslant r^{\rho+\varepsilon},\ T(r,A_j)\leqslant r^{\rho+\varepsilon}\quad(j=0,1,\cdots,k-1). \tag{2.15}$$

Then for $r\notin E$ sufficiently large, by using (2.14), (2.15) we conclude from (2.13) that

$$T(r,f)\leqslant k\overline{N}\left(r,\frac{1}{f}\right)+(k+1)r^{\rho+\varepsilon}+\frac{1}{2}T(r,f),$$

it follows that

$$T(r,f)\leqslant 2k\overline{N}\left(r,\frac{1}{f}\right)+2(k+1)r^{\rho+\varepsilon},\quad r\notin E. \tag{2.16}$$

Hence, by (2.16) we get

$$\rho_2(f)\leqslant\overline{\lambda}_2(f),$$

then

$$\lambda_2(f)\geqslant\overline{\lambda}_2(f)\geqslant\rho_2(f).$$

Since by definition, we have $\overline{\lambda}_2(f)\leqslant\lambda_2(f)\leqslant\rho_2(f)$, then

$$\overline{\lambda}_2(f)=\lambda_2(f)=\rho_2(f).$$

By (2.9) we obtain

$$\overline{\lambda}_2(f)=\lambda_2(f)=\rho_2(f)\leqslant\rho.$$

□

LEMMA 2.10. *([**1**]) Let $P_j(z)$ $(j=0,1,\cdots,k)$ be polynomials with* $\deg P_0(z)=n$ $(n\geqslant 1)$ *and* $\deg P_j(z)\leqslant n$ $(j=1,2,\cdots,k)$. *Let $A_j(z)$ $(j=0,1,\cdots,k)$ be meromorphic functions with finite order and* $\max\{\rho(A_j):j=0,1,\cdots,k\}<n$ *such that $A_0(z)\not\equiv 0$. We denote*

$$F(z)=A_ke^{P_k(z)}+A_{k-1}e^{P_{k-1}(z)}+\cdots+A_1e^{P_1(z)}+A_0e^{P_0(z)}.$$

If $\deg(P_0(z)-P_j(z))=n$ *for all $j=1,\cdots,k$, then F is a nontrivial meromorphic function with finite order and satisfies $\rho(F)=n$.*

LEMMA 2.11. *([4]) Let A_j $(j = 0, 1, \cdots, k-1)$, $F \not\equiv 0$ be finite order meromorphic functions. If $f(z)$ is an infinite order meromorphic solution of the equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1 f' + A_0 f = F,$$

then f satisfies

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = \infty.$$

3. Proof of the Theorems

PROOF OF THEOREM 1.2. First, we prove that every meromorphic solution $f(z) \not\equiv 0$ of (1.2) is transcendental of order $\rho(f) \geqslant n$. We assume that $f(z) \not\equiv 0$ is a meromorphic solution of equation (1.2) with $\rho(f) < n$. Since $\deg P_j \neq \deg P_i$ $(0 \leqslant i < j \leqslant k-1)$, then there exists exactly one $s \in \{0, 1, \cdots, k-1\}$ such that $h_s \not\equiv 0$ and $\deg P_s(z) = n = \max\{\deg P_j(z) \ \ (j = 0, 1, \cdots, k-1)\}$. We can rewrite equation (1.2) in the form

$$h_{k-1}(z) f^{(k-1)} e^{P_{k-1}(z)} + \cdots + h_s(z) f^{(s)} e^{P_s(z)}$$

$$+ \cdots + h_1(z) f^{'} e^{P_1(z)} + h_0(z) f e^{P_0(z)} = B(z), \tag{3.1}$$

where

$$B(z) = -f^{(k)} + F(z).$$

Since $\sigma = \max\{\rho(h_j) \ \ (j = 0, 1, \cdots, k-1),\ \rho(F)\} < n$ and $\rho(f) < n$, then $h_j(z) f^{(j)}(z)$ $(j = 0, 1, \cdots, k-1)$ and $B(z)$ are meromorphic functions of finite order with $\rho(h_j f^{(j)}) < n$ $(j = 0, 1, \cdots, k-1)$ and $\rho(B) < n$. We have $\deg P_j(z) < n$ $(j = 0, 1, \cdots, k-1;\ j \neq s)$ and $\deg(P_s(z) - P_j(z)) = n$ $(j = 0, 1, \cdots, k-1;\ j \neq s)$. By Lemma 2.10, we find that the order of growth of the left side of the equation (3.1) is n, this contradicts the fact $\rho(B) < n$. Consequently, any meromorphic solution $f \not\equiv 0$ of equation (1.2) is transcendental with order $\rho(f) \geqslant n$.

Now, we prove that $\rho(f) = +\infty$. Suppose, contrary to the assertion, that $f \not\equiv 0$ is a meromorphic solution of (1.2) with $\rho(f) = \rho < \infty$. Then, by the assertion above we have $n \leqslant \rho(f)$. Rewrite equation (1.2) in the form

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F, \tag{3.2}$$

where $A_j(z) = h_j(z) e^{P_j(z)}$ $(j = 0, 1, \cdots, k-1)$. By Lemma 2.2, there exists a set $E_3 \subset [0, 2\pi)$ of linear measure zero, such that if $\theta \in [0, 2\pi) \backslash E_3$, then there is a constant $R_0 = R_0(\theta) > 1$, such that for all z satisfying $\arg(z) = \theta$ and $|z| = r \geqslant R_0$, we have

$$\left| \frac{f^{(j)}(z)}{f^{(i)}(z)} \right| \leqslant |z|^{2\rho}, \quad 0 \leqslant i < j \leqslant k. \tag{3.3}$$

By Lemma 2.1, there is a set $E_1 \subset [0, 2\pi)$ that has linear measure zero, such that if $\theta \in [0, 2\pi) \setminus (E_1 \cup E_2)$, where $E_2 = \{\theta \in [0, 2\pi) : \delta(P_j, \theta) = 0 \ (j = 0, 1, \cdots, k-1)\}$ is a finite set. Then for sufficiently large $|z| = r$, we have $\delta(P_j, \theta) \neq 0$ $(j = 0, 1, \cdots, k-1)$ and $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ satisfy either inequality (2.1) or (2.2). For any fixed $\theta \in [0, 2\pi) \setminus (E_1 \cup E_2 \cup E_3)$, we have two cases: At least one of

$\delta(P_j,\theta)$ $(j=0,1,\cdots,k-1)$ is strictly positive or all $\delta(P_j,\theta)$ $(j=0,1,\cdots,k-1)$ satisfy $\delta(P_j,\theta)<0$. We now discuss these two cases separately.

Case 1. Set $\delta(P_{j_i},\theta)=\delta_{j_i}>0$ for $j_i\in\{j_1,j_2,\cdots,j_m\}\subset\{0,1,\cdots,k-1\}$ and $\delta(P_l,\theta)=\delta_l<0$ $(h_l\not\equiv 0)$ for $l\in\{0,1,\cdots,k-1\}\setminus\{j_1,j_2,\cdots,j_m\}$. Then there exists one $j_s\in\{j_1,j_2,\cdots,j_m\}$ such that

$$\deg P_{j_s}=d_{j_s}=\max\{\deg P_{j_i}:j_i=j_1,j_2,\cdots,j_m\}.$$

By Lemma 2.1, for any given ε $(0<\varepsilon<1)$, we have for sufficiently large r

$$\exp\left\{(1-\varepsilon)\delta_{j_s}r^{d_{j_s}}\right\}\leqslant|A_{j_s}(z)|, \tag{3.4}$$

$$|A_{j_i}(z)|\leqslant\exp\left\{(1+\varepsilon)\delta_{j_i}r^{d_{j_i}}\right\}\ \text{ for } j_i=j_1,j_2,\cdots,j_m \text{ and } j_i\neq j_s,$$

$$|A_l(z)|\leqslant\exp\left\{(1-\varepsilon)\delta_l r^{d_l}\right\}\ \text{ for } l\in\{0,1,\cdots,k-1\}\setminus\{j_1,j_2,\cdots,j_m\}.$$

Denoting $d_{j_t}=\max\{\deg P_{j_i}:j_i=j_1,j_2,\cdots,j_m;\ j_i\neq j_s\}$. Then for sufficiently large r, we have

$$|A_j(z)|\leqslant\exp\left\{(1+\varepsilon)\delta_{j_t}r^{d_{j_t}}\right\}\ \text{ for } j\in\{0,1,\cdots,k-1\} \text{ and } j\neq j_s. \tag{3.5}$$

We now proceed to show that

$$G(z)=\frac{\log^+\left|f^{(j_s)}(z)\right|}{|z|^{\sigma+\varepsilon}} \tag{3.6}$$

is bounded on the ray $\arg z=\theta$. Supposing that this is not the case. Then by Lemma 2.3, there exists an infinite sequence of points $z_m=r_me^{i\theta}$ $(m=1,2,\cdots)$ tending to infinity such that

$$\frac{\log^+\left|f^{(j_s)}(z_m)\right|}{|z_m|^{\sigma+\varepsilon}}\to\infty \tag{3.7}$$

and

$$\left|\frac{f^{(j)}(z_m)}{f^{(j_s)}(z_m)}\right|\leqslant\frac{1}{(j_s-j)!}(1+o(1))|z_m|^{j_s-j}\quad(j=0,\cdots,j_s)\quad\text{as}\quad m\to\infty. \tag{3.8}$$

From (3.7) for any sufficiently large number $M_1>0$ we have

$$\frac{\log^+\left|f^{(j_s)}(z_m)\right|}{|z_m|^{\sigma+\varepsilon}}>M_1,\text{ then }\left|f^{(j_s)}(z_m)\right|>e^{M_1|z_m|^{\sigma+\varepsilon}}\quad\text{as}\quad m\to+\infty. \tag{3.9}$$

Since $F(z)$ is a meromorphic function with only finitely many poles, then by Hadamard factorization theorem, we can write $F(z)=\frac{H(z)}{\pi(z)}$, where $\pi(z)$ is a polynomial and $H(z)$ is an entire function with $\rho(H)=\rho(F)$. From (3.9) for m sufficiently large $(r_m\to+\infty)$, we have

$$\left|\frac{F(z_m)}{f^{(j_s)}(z_m)}\right|=\left|\frac{H(z_m)}{\pi(z_m)f^{(j_s)}(z_m)}\right|\leqslant\frac{|H(z_m)|}{cr_m^de^{M_1|z_m|^{\sigma+\varepsilon}}}\leqslant\frac{|H(z_m)|}{e^{M_1|z_m|^{\sigma+\varepsilon}}},$$

where $c > 0$ is a constant and $d = \deg \pi \geqslant 1$ is an integer. Since $\rho(H) = \rho(F) \leqslant \sigma$, then we have

$$\left|\frac{F(z_m)}{f^{(j_s)}(z_m)}\right| \leqslant \frac{|H(z_m)|}{e^{M_1|z_m|^{\sigma+\varepsilon}}} \to 0 \quad \text{as } m \to +\infty. \tag{3.10}$$

From equation (3.2), we obtain

$$|A_{j_s}(z_m)| \leqslant \left|\frac{f^{(k)}(z_m)}{f^{(j_s)}(z_m)}\right| + |A_{k-1}(z_m)| \left|\frac{f^{(k-1)}(z_m)}{f^{(j_s)}(z_m)}\right|$$

$$+ \cdots + |A_{j_s+1}(z_m)| \left|\frac{f^{(j_s+1)}(z_m)}{f^{(j_s)}(z_m)}\right| + |A_{j_s-1}(z_m)| \left|\frac{f^{(j_s-1)}(z_m)}{f^{(j_s)}(z_m)}\right|$$

$$+ \cdots + |A_1(z_m)| \left|\frac{f'(z_m)}{f^{(j_s)}(z_m)}\right| + |A_0(z_m)| \left|\frac{f(z_m)}{f^{(j_s)}(z_m)}\right| + \left|\frac{F(z_m)}{f^{(j_s)}(z_m)}\right|. \tag{3.11}$$

Using inequalities (3.3), (3.4), (3.5), (3.8) and the limit (3.10), we conclude from the inequality (3.11) that

$$\exp\left\{(1-\varepsilon)\,\delta_{j_s} r_m^{d_{j_s}}\right\} \leqslant r_m^{\alpha} + (k-1)\, r_m^{\alpha} \exp\left\{(1+\varepsilon)\,\delta_{j_t} r_m^{d_{j_t}}\right\} + o(1),$$

where α is a bounded constant satisfying $\alpha > \max\{2\rho,\ (j_s - j)\ (j = 0, \cdots, j_s)\}$. Hence

$$\exp\left\{(1-\varepsilon)\,\delta_{j_s} r_m^{d_{j_s}}\right\} \leqslant (k+1)\, r_m^{\alpha} \exp\left\{(1+\varepsilon)\,\delta_{j_t} r_m^{d_{j_t}}\right\},$$

it follows that

$$\exp\left\{(1-\varepsilon)\,\delta_{j_s} r_m^{d_{j_s}} - (1+\varepsilon)\,\delta_{j_t} r_m^{d_{j_t}}\right\} \leqslant (k+1)\, r_m^{\alpha}.$$

Since $0 < \varepsilon < 1$ and $d_{j_s} > d_{j_t}$, this is a contradiction, provided that r_m is sufficiently large enough. Therefore, $\frac{\log^+\left|f^{(j_s)}(z)\right|}{|z|^{\sigma+\varepsilon}}$ is bounded on the ray $\arg(z) = \theta$. Then there exists a bounded constant $M_2 > 0$ such that

$$\left|f^{(j_s)}(z)\right| \leqslant e^{M_2|z|^{\sigma+\varepsilon}}$$

on the ray $\arg(z) = \theta$. Hence, by (j_s)-fold iterated integration (see, [**1**]), we conclude that

$$|f(z)| \leqslant \frac{1}{j_s!}(1+o(1))\, r^{j_s} \left|f^{(j_s)}(z)\right| \leqslant \frac{1}{j_s!}(1+o(1))\, r^{j_s} e^{M_2|z|^{\sigma+\varepsilon}} \leqslant e^{M_2|z|^{\sigma+2\varepsilon}}$$

on the ray $\arg(z) = \theta$.

Case 2. $\delta(P_j, \theta) = \delta_j < 0$ $(j = 0, 1, \cdots, k-1)$. From (3.2), we get

$$1 \leqslant |A_{k-1}(z)| \left|\frac{f^{(k-1)}(z)}{f^{(k)}(z)}\right| + |A_{k-2}(z)| \left|\frac{f^{(k-2)}(z)}{f^{(k)}(z)}\right|$$

$$+ \cdots + |A_0(z)| \left|\frac{f(z)}{f^{(k)}(z)}\right| + \left|\frac{F(z)}{f^{(k)}(z)}\right|. \tag{3.12}$$

By Lemma 2.1, for any given ε $(0<\varepsilon<1)$ we have

$$|A_j(z)| \leqslant \exp\left\{(1-\varepsilon)\,\delta_j r^{d_j}\right\},\ (j=0,1,\cdots,k-1).$$

Then

$$|A_j(z)| \leqslant \exp\left\{(1-\varepsilon)\,\delta r^{d_{j_t}}\right\},\ (j=0,1,\cdots,k-1), \tag{3.13}$$

where $\delta=\max\{\delta_j : j=0,1,\cdots,k-1\}$ and $d_{j_t}=\min\{\deg P_j : j=0,1,\cdots,k-1\}$. We prove that

$$G(z)=\frac{\log^+\left|f^{(k)}(z)\right|}{|z|^{\sigma+\varepsilon}} \tag{3.14}$$

is bounded on the ray $\arg z=\theta$. Supposing that this is not the case. Then by Lemma 2.3, there exists an infinite sequence of points $z_m=r_m e^{i\theta}$ $(m=1,2,\cdots)$ tending to infinity such that

$$\frac{\log^+\left|f^{(k)}(z_m)\right|}{|z_m|^{\sigma+\varepsilon}} \to \infty \text{ as } m\to\infty \tag{3.15}$$

and

$$\left|\frac{f^{(j)}(z_m)}{f^{(k)}(z_m)}\right| \leqslant \frac{1}{(k-j)!}\,(1+o(1))\,|z_m|^{k-j},\ (j=0,1,\cdots,k-1). \tag{3.16}$$

From (3.15) for any sufficiently large number $M_3>0$ we have

$$\left|f^{(k)}(z_m)\right| > e^{M_3|z_m|^{\sigma+\varepsilon}} \quad\text{as}\quad m\to+\infty. \tag{3.17}$$

By using the same reasoning as above we get from (3.17) that for m sufficiently large $(r_m\to+\infty)$

$$\left|\frac{F(z_m)}{f^{(k)}(z_m)}\right| = \left|\frac{H(z_m)}{\pi(z_m)\ f^{(k)}(z_m)}\right| \leqslant \frac{|H(z_m)|}{c r_m^d e^{M_3|z_m|^{\sigma+\varepsilon}}} \leqslant \frac{|H(z_m)|}{e^{M_3|z_m|^{\sigma+\varepsilon}}},$$

where $c>0$ and $d=\deg\pi\geqslant 1$. Since $\rho(H)=\rho(F)\leqslant\sigma$, then we have

$$\left|\frac{F(z_m)}{f^{(k)}(z_m)}\right| \leqslant \frac{|H(z_m)|}{e^{M_3|z_m|^{\sigma+\varepsilon}}} \to 0 \ \text{ as } m\to\infty. \tag{3.18}$$

Using inequalities (3.13), (3.16) and the limit (3.18), we conclude from the inequality (3.12) that

$$1 \leqslant k\exp\left\{(1-\varepsilon)\,\delta r_m^{d_{j_t}}\right\} r_m^k(1+o(1))+o(1).$$

By $0<\varepsilon<1$, this is a contradiction, provided that r_m is sufficiently large enough. Therefore, $\frac{\log^+\left|f^{(k)}(z)\right|}{|z|^{\sigma+\varepsilon}}$ is bounded on the ray $\arg(z)=\theta$, then there exists a bounded constant $M_4>0$ such that

$$\left|f^{(k)}(z)\right| \leqslant e^{M_4|z|^{\rho+\varepsilon}} \tag{3.19}$$

on the ray $\arg(z)=\theta$. Hence, by (k)-fold iterated integration (see, [**1**]), we conclude that

$$|f(z)| \leqslant \frac{1}{k!}\,(1+o(1))\,r^k\left|f^{(k)}(z)\right|$$

on the ray $\arg(z)=\theta$. Then by using (3.19), we obtain

$$|f(z)| \leqslant \frac{1}{k!}(1+o(1))\, r^k e^{M_4|z|^{\sigma+\varepsilon}} \leqslant e^{M_4 r^{\sigma+2\varepsilon}}$$

on the ray $\arg(z)=\theta$. In both cases, there exists a bounded positive constant $M>0$ such that

$$|f(z)| \leqslant e^{Mr^{\sigma+2\varepsilon}} \tag{3.20}$$

on the ray $\arg(z)=\theta$. From equation (3.2), we know that the poles of f can only occur at the poles of A_j $(j=0,1,\cdots,k-1)$ and F. Since A_j $(j=0,1,\cdots,k-1)$ and F are meromorphic functions having only finitely many poles, then $f(z)$ must have only finitely many poles. Therefore, by Hadamard factorization theorem, we can write f as $f(z)=\frac{g(z)}{d(z)}$, where $d(z)$ is a polynomial and $g(z)$ is an entire function with $\rho(g)=\rho(f)$. By the first assertion in the proof of Theorem 1.2 we get $\rho(g)\geqslant n$. From (3.20), we have

$$\left|\frac{g(z)}{d(z)}\right| \leqslant e^{Mr^{\sigma+2\varepsilon}}$$

on the ray $\arg(z)=\theta$. Then

$$|g(z)| \leqslant |d(z)|\, e^{Mr^{\sigma+2\varepsilon}} \leqslant Ar^{\beta} e^{Mr^{\sigma+2\varepsilon}}$$

on the ray $\arg(z)=\theta$, where $A>0$ is a constant and $\beta=\deg d\geqslant 1$ is an integer. Hence

$$|g(z)| \leqslant e^{Mr^{\sigma+3\varepsilon}} \tag{3.21}$$

on the ray $\arg(z)=\theta$. Therefore, for any given $\theta\in[0,2\pi)\setminus(E_1\cup E_2\cup E_3)$, where $(E_1\cup E_2\cup E_3)\subset[0,2\pi)$ is a set of linear measure zero, we have (3.21), for sufficiently large $|z|=r$. Then, by Lemma 2.4 we have $\rho(g)\leqslant\rho+3\varepsilon<n$ for a small positive ε, a contradiction with $\rho(g)\geqslant n$. Hence, every meromorphic solution $f\not\equiv 0$ of (1.2) must be of infinite order.

Now, by using Lemma 2.9, we obtain

$$\rho_2(f)\leqslant \max\{\rho(A_j)\ \ (j=0,1,\cdots,k-1),\ \rho(F)\}=n.$$

Suppose that $F\not\equiv 0$. Then, by Lemma 2.9 and Lemma 2.11, we obtain

$$\overline{\lambda}(f)=\lambda(f)=\rho(f)=\infty \quad \text{and} \quad \overline{\lambda}_2(f)=\lambda_2(f)=\rho_2(f)\leqslant n.$$

□

Proof of Theorem 1.3. Let f be a nontrivial meromorphic solution of equation (1.2). Then, by Theorem 1.2, we have $\rho(f)=\infty$.

Step 1. We consider the fixed points of $f(z)$. Set $g_0(z)=f(z)-z$. Then z is a fixed point of $f(z)$ if and only if $g_0(z)=0$. We have $g_0(z)$ is a meromorphic function and $\rho(g_0)=\rho(f)=\infty$. Substituting $f(z)=g_0(z)+z$ into equation (1.2), we obtain

$$g_0^{(k)}+h_{k-1}e^{P_{k-1}(z)}g_0^{(k-1)}+\cdots+h_s e^{P(z)}g_0^{(s)}+\cdots+h_1 e^{P_1(z)}g_0'$$

$$+h_0 e^{P_0(z)} g_0 = F - h_1 e^{P_1(z)} - z h_0 e^{P_0(z)}. \tag{3.22}$$

We can rewrite (3.22) in the form

$$g_0^{(k)} + A_{0,k-1} g_0^{(k-1)} + \cdots + A_{0,1} g_0' + A_{0,0} g_0 = F - A_{0,1} - z A_{0,0} = A_0. \tag{3.23}$$

For the equation (3.23), we consider just meromorphic solutions of infinite order satisfying $g_0(z) = f(z) - z$. We have

$$A_0 = F - A_{0,1} - z\, A_{0,0} = -z h_0 e^{P_0(z)} - h_1 e^{P_1(z)} + F.$$

Since $\deg P_j$ $(j = 0, 1)$ are distinct integer numbers and

$$\sigma = \max\{\rho(zh_0),\ \rho(h_1),\ \rho(F)\} < \deg P_j(z) \ \ (j = 0, 1),$$

then $zh_0 e^{P_0(z)}$, $h_1 e^{P_1(z)}$, F are linearly independent terms with $h_0 \not\equiv 0$. Hence $A_0 \not\equiv 0$. By applying Lemma 2.11 to equation (3.23) above, we obtain

$$\overline{\lambda}(g_0) = \overline{\tau}(f) = \rho(g_0) = \infty.$$

Step 2. We consider the fixed points of $f'(z)$**.** Set $g_1(z) = f'(z) - z$. Then z is a fixed point of $f'(z)$ if and only if $g_1(z) = 0$. We have $g_1(z)$ is a meromorphic function with $\rho(g_1) = \rho(f') = \rho(f) = \infty$. By differentiating the both sides of equation (1.2), we obtain

$$f^{(k+1)} + h_{k-1} e^{P_{k-1}(z)} f^{(k)} + \left[\left(h_{k-1} e^{P_{k-1}(z)}\right)' + h_{k-2} e^{P_{k-2}(z)}\right] f^{(k-1)}$$
$$+ \cdots + \left[\left(h_s e^{P_s(z)}\right)' + h_{s-1} e^{P_{s-1}(z)}\right] f^{(s)}$$
$$+ \cdots + \left[\left(h_2 e^{P_2(z)}\right)' + h_1 e^{P_1(z)}\right] f''$$
$$+ \left[\left(h_1 e^{P_1(z)}\right)' + h_0 e^{P_0(z)}\right] f' + \left(h_0 e^{P_0(z)}\right)' f = F'. \tag{3.24}$$

By equation (1.2), we have

$$f = -\frac{1}{h_0 e^{P_0(z)}} \left[f^{(k)} + h_{k-1} e^{P_{k-1}(z)} f^{(k-1)} + \cdots + h_s e^{P_s(z)} f^{(s)}\right.$$
$$\left. + \cdots + h_1 e^{P_1(z)} f' - F\right]. \tag{3.25}$$

Substituting (3.25) into (3.24), we obtain

$$f^{(k+1)} + \left[h_{k-1} e^{P_{k-1}(z)} - \frac{\left(h_0 e^{P_0(z)}\right)'}{h_0 e^{P_0(z)}}\right] f^{(k)}$$
$$+ \left[\left(h_{k-1} e^{P_{k-1}(z)}\right)' + h_{k-2} e^{P_{k-2}(z)} - \frac{\left(h_0 e^{P_0(z)}\right)'}{h_0 e^{P_0(z)}} h_{k-1} e^{P_{k-1}(z)}\right] f^{(k-1)}$$

$$+\cdots+\left[\left(h_se^{P_s(z)}\right)'+h_{s-1}e^{P_{s-1}(z)}-\frac{\left(h_0e^{P_0(z)}\right)'}{h_0e^{P_0(z)}}h_se^{P_s(z)}\right]f^{(s)}$$

$$+\cdots+\left[\left(h_2e^{P_2(z)}\right)'+h_1e^{P_1(z)}-\frac{\left(h_0e^{P_0(z)}\right)'}{h_0e^{P_0(z)}}h_2e^{P_2(z)}\right]f''$$

$$+\left[\left(h_1e^{P_1(z)}\right)'+h_0e^{P_0(z)}-\frac{\left(h_0e^{P_0(z)}\right)'}{h_0e^{P_0(z)}}h_1e^{P_1(z)}\right]f'+\frac{\left(h_0e^{P_0(z)}\right)'}{h_0e^{P_0(z)}}F=F'. \tag{3.26}$$

We can write equation (3.26) in the following form

$$f^{(k+1)}+A_{1,k-1}f^{(k)}+A_{1,k-2}f^{(k-1)}+\cdots+A_{1,s}f^{(s+1)}+A_{1,s-1}f^{(s)}$$

$$+\cdots+A_{1,1}f''+A_{1,0}f'+\frac{\left(h_0e^{P_0(z)}\right)'}{h_0e^{P_0(z)}}F=F', \tag{3.27}$$

where $A_{1,j}\,(j=0,1,\cdots,k-1)$ are meromorphic functions defined by the equation (3.26). Substituting $f'(z)=g_1(z)+z,\ f''(z)=g_1'+1,\ f^{(j+1)}=g_1^{(j)}\ (j=2,3,\cdots,k)$ into equation (3.27), we obtain

$$g_1^{(k)}+A_{1,k-1}g_1^{(k-1)}+A_{1,k-2}g_1^{(k-2)}+\cdots+A_{1,s+1}g_1^{(s+1)}+A_{1,s}g_1^{(s)}$$

$$+\cdots+A_{1,1}g_1'+A_{1,0}g_1=-A_{1,1}-z\ A_{1,0}-\frac{\left(h_0e^{P_0(z)}\right)'}{h_0e^{P_0(z)}}F+F'=A_1, \tag{3.28}$$

where

$$A_1=-\left[\left(h_2e^{P_2(z)}\right)'+h_1e^{P_1(z)}-\frac{\left(h_0e^{P_0(z)}\right)'}{h_0e^{P_0(z)}}h_2e^{P_2(z)}\right]$$

$$-z\left[\left(h_1e^{P_1(z)}\right)'+h_0e^{P_0(z)}-\frac{\left(h_0e^{P_0(z)}\right)'}{h_0e^{P_0(z)}}h_1e^{P_1(z)}\right]-\frac{\left(h_0e^{P_0(z)}\right)'}{h_0e^{P_0(z)}}F+F'$$

$$=-\frac{1}{h_0e^{P_0(z)}}\left(zh_0^2e^{2P_0(z)}+B_1e^{P_0}+B_2e^{P_0+P_1}+B_3e^{P_0+P_2}\right)$$

$$=-\frac{1}{h_0}\left(zh_0^2e^{P_0(z)}+B_1+B_2e^{P_1}+B_3e^{P_2}\right),$$

where $B_0=zh_0^2$ and $B_j\ (j=1,2,3)$ are meromorphic functions of finite order which is less than n, written on the form of a sum of terms of kinds of multiplications of the functions $z,\ h_i,\ h_i',\ P_i',\ F,\ F'$. Since $\deg P_j\ (j=0,1,2)$ are distinct integer numbers and $\sigma=\max\{\rho(B_j)\,(j=0,2,1,3)\}<\deg P_j(z)\ (j=0,1,2)$, then $zh_0^2e^{P_0(z)},\ B_1,\ B_2e^{P_1},\ B_3e^{P_2}$ are linearly independent terms with $h_0\not\equiv 0$. Hence $A_1\not\equiv 0$. By applying Lemma 2.11 to equation (3.28) above, we obtain

$$\overline{\lambda}(g_1)=\overline{\lambda}(f'-z)=\overline{\tau}(f')=\rho(g_1)=\rho(f)=\infty.$$

Step 3. We prove that $\overline{\tau}(f'')=\overline{\lambda}(f''-z)=\infty$**.** Set $g_2(z)=f''(z)-z$. Then z is a fixed point of $f''(z)$ if and only if $g_2(z)=0$. We have $g_2(z)$ is a meromorphic function with $\rho(g_2) = \rho(f'') = \rho(f) = \infty$. We just prove that $\overline{\lambda}(g_2) = \infty$. By differentiating the both sides of equation (3.27), we obtain

$$f^{(k+2)} + A_{1,k-1}f^{(k+1)} + \left(A'_{1,k-1} + A_{1,k-2}\right) f^{(k)} + \cdots + \left(A'_{1,s-1} + A_{1,s-2}\right) f^{(s)}$$

$$+ \cdots + \left(A'_{1,1} + A_{1,0}\right) f'' + A'_{1,0}f' = H', \tag{3.29}$$

where H is a meromorphic function with order $\rho(H) < n$ and

$$H = -\frac{\left(h_0 e^{P_0(z)}\right)'}{h_0 e^{P_0(z)}} F + F'.$$

By equation (3.27) we have

$$f' = -\frac{1}{A_{1,0}} \Big[f^{(k+1)} + A_{1,k-1}f^{(k)} + A_{1,k-2}f^{(k-1)}$$

$$+ \cdots + A_{1,s-1}f^{(s)} + \cdots + A_{1,1}f'' - H \Big]. \tag{3.30}$$

We remark that $A_{1,0} \not\equiv 0$, because $h_0 \not\equiv 0$ (for the proof, we can apply Lemma 2.10). Substituting (3.30) into (3.29), we obtain

$$f^{(k+2)} + \left[A_{1,k-1} - \frac{A'_{1,0}}{A_{1,0}}\right] f^{(k+1)} + \left[A'_{1,k-1} + A_{1,k-2} - \frac{A'_{1,0}}{A_{1,0}} A_{1,k-1}\right] f^{(k)} + \cdots$$

$$+ \left[A'_{1,s-1} + A_{1,s-2} - \frac{A'_{1,0}}{A_{1,0}} A_{1,s-1}\right] f^{(s)} + \cdots + \left[A'_{1,2} + A_{1,1} - \frac{A'_{1,0}}{A_{1,0}} A_{1,2}\right] f^{(3)}$$

$$+ \left[A'_{1,1} + A_{1,0} - \frac{A'_{1,0}}{A_{1,0}} A_{1,1}\right] f'' + \frac{A'_{1,0}}{A_{1,0}} H = H'. \tag{3.31}$$

We can write equation (3.31) in the form

$$f^{(k+2)} + A_{2,k-1}f^{(k+1)} + A_{2,k-2}f^{(k)} + \cdots + A_{2,1}f^{(3)} + A_{2,0}f'' = -\frac{A'_{1,0}}{A_{1,0}} H + H', \tag{3.32}$$

where $A_{2,j}$ $(j = 0, 1, \cdots, k-1)$ are meromorphic functions defined by equation (3.31) above. We have

$$A_{2,0} = A'_{1,1} + A_{1,0} - \frac{A'_{1,0}}{A_{1,0}} A_{1,1},$$

$$A_{2,1} = A'_{1,2} + A_{1,1} - \frac{A'_{1,0}}{A_{1,0}} A_{1,2}.$$

Substituting $f''(z) = g_2(z) + z$, $f^{(3)}(z) = g'_2 + 1$, $f^{(j+2)} = g_2^{(j)}$ $(j = 2, 3, \cdots, k)$ into equation (3.32), we obtain

$$g_2^{(k)} + A_{2,k-1}g_2^{(k-1)} + A_{2,k-2}g_2^{(k-2)} + \cdots + A_{2,s}g_2^{(s)} + \cdots + A_{2,1}g'_2 + A_{2,0}g_2 = A_2, \tag{3.33}$$

where

$$A_2 = -A_{2,1} - zA_{2,0} - \frac{A'_{1,0}}{A_{1,0}} H + H'$$

$$= -\left[A'_{1,2} + A_{1,1} - \frac{A'_{1,0}}{A_{1,0}}A_{1,2}\right] - z\left[A'_{1,1} + A_{1,0} - \frac{A'_{1,0}}{A_{1,0}}A_{1,1}\right] - \frac{A'_{1,0}}{A_{1,0}}H + H'$$

$$= -\frac{1}{A_{1,0}}\left[A'_{1,2}A_{1,0} + A_{1,1}A_{1,0} - A'_{1,0}A_{1,2} + zA'_{1,1}A_{1,0}\right.$$

$$\left. + zA^2_{1,0} - zA'_{1,0}A_{1,1} + A'_{1,0}H - A_{1,0}H'\right]. \tag{3.34}$$

We have

$$A_{1,0} = \left(h_1 e^{P_1(z)}\right)' + h_0 e^{P_0(z)} - \frac{\left(h_0 e^{P_0(z)}\right)'}{h_0 e^{P_0(z)}} h_1 e^{P_1(z)},$$

$$A_{1,1} = \left(h_2 e^{P_2(z)}\right)' + h_1 e^{P_1(z)} - \frac{\left(h_0 e^{P_0(z)}\right)'}{h_0 e^{P_0(z)}} h_2 e^{P_2(z)},$$

$$A_{1,2} = \left(h_3 e^{P_3(z)}\right)' + h_2 e^{P_2(z)} - \frac{\left(h_0 e^{P_0(z)}\right)'}{h_0 e^{P_0(z)}} h_3 e^{P_3(z)}.$$

Therefore

$$A_{1,0} = \frac{1}{h_0 e^{P_0(z)}}\left(h_0^2 e^{2P_0} + \alpha_{1,0}^{(1)} e^{P_0+P_1}\right),$$

$$A_{1,1} = \frac{1}{h_0 e^{P_0(z)}}\left(\alpha_{1,1}^{(0)} e^{P_0+P_2} + \alpha_{1,1}^{(1)} e^{P_0+P_1}\right),$$

$$A_{1,2} = \frac{1}{h_0 e^{P_0(z)}}\left(\alpha_{1,2}^{(0)} e^{P_0+P_2} + \alpha_{1,2}^{(1)} e^{P_0+P_3}\right)$$

and

$$A'_{1,0} = \frac{1}{\left(h_0 e^{P_0(z)}\right)^2}\left(\beta_{1,0}^{(0)} e^{3P_0} + \beta_{1,0}^{(1)} e^{2P_0+P_1}\right),$$

$$A'_{1,1} = \frac{1}{\left(h_0 e^{P_0(z)}\right)^2}\left(\beta_{1,1}^{(0)} e^{2P_0+P_2} + \beta_{1,1}^{(1)} e^{2P_0+P_1}\right),$$

$$A'_{1,2} = \frac{1}{\left(h_0 e^{P_0(z)}\right)^2}\left(\beta_{1,2}^{(1)} e^{2P_0+P_2} + \beta_{1,2}^{(2)} e^{2P_0+P_3}\right),$$

where $\alpha_{i,j}^{(l)}$, $\beta_{i,j}^{(l)}$ are meromorphic functions of finite order which is less than n, written on the form of a sum of terms of kinds of multiplications of the functions h_i, h'_i, P'_i, $(i = 0, 1, 2, 3)$. From (3.34) we have

$$A_2 = -\frac{1}{A_{1,0}\left(h_0 e^{P_0(z)}\right)^3}\left[zh_0^5 e^{5P_0} + B_1 e^{4P_0} + B_2 e^{4P_0+P_1} + B_3 e^{4P_0+P_2}\right.$$

$$\left. + B_4 e^{4P_0+P_3} + B_5 e^{3P_0+P_1} + B_6 e^{3P_0+2P_1} + B_7 e^{3P_0+P_1+P_2} + B_8 e^{3P_0+P_1+P_3}\right]$$

$$= -\frac{1}{A_{1,0}h_0^3}\left[zh_0^5 e^{2P_0} + B_1 e^{P_0} + B_2 e^{P_0+P_1} + B_3 e^{P_0+P_2} + B_4 e^{P_0+P_3}\right.$$

$$\left. + B_5 e^{P_1} + B_6 e^{2P_1} + B_7 e^{P_1+P_2} + B_8 e^{P_1+P_3}\right] = -\frac{1}{A_{1,0}h_0^3}\left[\sum_{j=0}^{8} B_j e^{G_j}\right],$$

where G_j are polynomials defined as above, $G_0 = 2P_0$, $B_0 = zh_0^5$ and B_j $(j = 1, 2, \cdots, 8)$ are meromorphic functions of finite order which is less than n, written on the form of a sum of terms of kinds of multiplications of the functions z, h_i, h_i', P_i', H, H'. We discuss four cases:

Case 1. If $\deg P_0 > \deg P_i$ $(i = 1, 2, 3)$, then we have $\deg (G_0 - G_i) = \deg P_0$ $(i = 1, 2, \cdots, 8)$. According to Lemma 2.10) and the fact $B_0 = zh_0^5 \not\equiv 0$ we have $A_2 \not\equiv 0$.

Case 2. If $\deg P_1 > \deg P_i$ $(i = 0, 2, 3)$, then we rewrite A_2 in the form

$$\begin{aligned} A_2 = &-\frac{1}{A_{1,0}h_0^3}\left[e^{P_0}\left(zh_0^5e^{P_0} + B_1 + B_3e^{P_2} + B_4e^{P_3}\right)\right. \\ &\left.+e^{P_1}\left(B_2e^{P_0} + B_5 + B_6e^{P_1} + B_7e^{P_2} + B_8e^{P_3}\right)\right] \\ = &-\frac{1}{A_{1,0}h_0^3}\left[e^{P_0}\left(zh_0^5e^{P_0} + B_1 + B_3e^{P_2} + B_4e^{P_3}\right) + Be^{P_1}\right], \end{aligned}$$

where $B = B_2e^{P_0} + B_5 + B_6e^{P_1} + B_7e^{P_2} + B_8e^{P_3}$. Since $\deg P_j$ $(j = 0, 1, 2, 3)$ are distinct integer numbers and $\sigma = \max\{\rho(B_j)\ (j = 0, 1, \cdots, 8)\} < \deg P_j(z)$ $(j = 0, 1, 2, 3)$, then $zh_0^5e^{P_0},\ B_1,\ B_3e^{P_2},\ B_4e^{P_3}$ are linearly independent terms with $h_0 \not\equiv 0$. Hence $K_1 = (zh_0^5e^{P_0} + B_1 + B_3e^{P_2} + B_4e^{P_3}) \not\equiv 0$. We have $e^{P_0}K_1 = e^{P_0}(zh_0^5e^{P_0} + B_1 + B_3e^{P_2} + B_4e^{P_3}) \not\equiv 0$, $K_2 = Be^{P_1}$ $\left(\rho\left(Be^{P_1}\right) = \deg P_1 \text{ or } B \equiv 0\right)$ have not the same order of growth, then $e^{P_0}K_1,\ K_2$ are linearly independent functions, hence

$$A_2 = -\frac{1}{A_{1,0}h_0^3}\left[e^{P_0}K_1 + K_2\right] \not\equiv 0.$$

Case 3. If $\deg P_2 > \deg P_i$ $(i = 0, 1, 3)$, then we rewrite A_2 in the form

$$\begin{aligned} A_2 = &-\frac{1}{A_{1,0}h_0^3}\left[e^{P_0}\left(zh_0^5e^{P_0} + B_1 + B_2e^{P_1} + B_4e^{P_3}\right)\right. \\ &+\left(B_5e^{P_1} + B_6e^{2P_1} + B_8e^{P_1+P_3}\right) \\ &\left.+\, e^{P_2}\left(B_3e^{P_0} + B_7e^{P_1}\right)\right], \text{ where } \deg P_0 > \deg P_i\ (i = 1, 3) \end{aligned}$$

or

$$\begin{aligned} A_2 = &-\frac{1}{A_{1,0}h_0^3}\left[e^{P_0}\left(zh_0^5e^{P_0} + B_1 + B_4e^{P_3}\right)\right. \\ &+e^{P_1}\left(B_2e^{P_0} + B_5 + B_6e^{P_1} + B_8e^{P_3}\right) \\ &\left.+\, e^{P_2}\left(B_3e^{P_0} + B_7e^{P_1}\right)\right], \text{ where } \deg P_1 > \deg P_0 > \deg P_3 \end{aligned}$$

or

$$\begin{aligned} A_2 = &-\frac{1}{A_{1,0}h_0^3}\left[e^{P_0}\left(zh_0^5e^{P_0} + B_1\right) + B_4e^{P_3+P_0}\right. \\ &+e^{P_1}\left(B_2e^{P_0} + B_5 + B_6e^{P_1} + B_8e^{P_3}\right) \\ &\left.+e^{P_2}\left(B_3e^{P_0} + B_7e^{P_1}\right)\right], \text{ where } \deg P_1 > \deg P_3 > \deg P_0 \end{aligned}$$

or

$$\begin{aligned} A_2 = &-\frac{1}{A_{1,0}h_0^3}\left[e^{P_0}\left(zh_0^5e^{P_0} + B_1 + B_2e^{P_1}\right)\right. \\ &+e^{P_3}\left(B_4e^{P_0} + B_8e^{P_1}\right) + \left(B_5e^{P_1} + B_6e^{2P_1}\right) \\ &\left.+\, e^{P_2}\left(B_3e^{P_0} + B_7e^{P_1}\right)\right], \text{ where } \deg P_3 > \deg P_0 > \deg P_1 \end{aligned}$$

or

$$A_2 = -\frac{1}{A_{1,0}h_0^3}\left[e^{P_0}\left(zh_0^5e^{P_0}+B_1\right)+e^{P_3}\left(B_4e^{P_0}+B_8e^{P_1}\right)\right.$$
$$+e^{P_1}\left(B_2e^{P_0}+B_5+B_6e^{P_1}\right)$$
$$\left.+e^{P_2}\left(B_3e^{P_0}+B_7e^{P_1}\right)\right], \text{ where } \deg P_3 > \deg P_1 > \deg P_0.$$

Then we can write A_2 in the form

$$A_2 = -\frac{1}{A_{1,0}h_0^3}\left[e^{P_0}K_1+K_2\right].$$

By the same reasoning as in the proof of **Case 2** above, we conclude that $e^{P_0}K_1 \not\equiv 0$ and K_2 are linearly independent functions. Hence $A_2 \not\equiv 0$.

Case 4. If $\deg P_3 > \deg P_i \ (i = 0,1,2)$, then by the same reasoning as in the proof of **Case 2** and **Case 3** above, we conclude that $A_2 \not\equiv 0$.

In all cases, we have $A_2 \not\equiv 0$. By applying Lemma 2.11 to equation (3.33) above, we obtain

$$\overline{\lambda}(g_2) = \overline{\lambda}(f''-z) = \overline{\tau}(f'') = \rho(g_2) = \rho(f) = \infty.$$

□

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