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CIRIC TYPE P-CYCLIC CONTRACTION RESULTS FOR DISCONTINUOUS MAPPINGS

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Abstract. Cyclic mappings constitute a recently introduced class of mappings which have appeared in a number of papers in the contexts of fixed point theory and in the theory of optimization. In this paper we consider p-cyclic contraction type mappings which are generalized cyclic contractions through p number of subsets of a probabilistic metric space. Moreover the contractive mappings are discontinuous. We use Hadzic type t-norms in our theorems which is characterized by the equicontinuities of its iterates. We deduce two fixed point theorems. In one of the theorems we use a control function which is the counterpart of a function widely used in fixed point theory in metric spaces. The theorems are supported with examples. The work is in the line of research developing probabilistic fixed point theory.

1. Introduction

In 1942 K. Menger [**24**] introduced the concept of probabilistic metric spaces. The idea was to use distribution functions instead of non-negative real numbers as values of the metric. Thus probabilistic metric spaces have the notion of uncertainty built within the structure of the space. Different aspect of probabilistic metric space theory has developed over the years. Descriptions of several of its aspects have been given in the book by Schweizer and Sklar [**32**]. Fixed point theory in probabilistic metric spaces was initiated by Sehgal and Bharucha-Reid who had established a probabilistic version of the Banach's contraction mapping principle in such spaces [**33**]. After that fixed point theory developed greatly over the years. A comprehensive survey of research in this line is given in [**19**] by Hadzic and Pap. Some more recent references are noted in [**1, 2, 9, 10, 17, 23, 27, 28, 29**].

In metric fixed point theory a new direction was opened by Khan, Swaleh

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and Sessa [**21**]. They introduced a new contraction principle and proved a fixed point result through a control function which they called altering distance function. These functions have been used in a number of works in fixed point theory, as, for instances, in [**6, 26, 30, 31**]. It has been extended to Menger spaces in [**3**] where a generalization of Sehgal's contraction has been defined with the help of such functions and a unique fixed point result has been established for such contractions. This extension of altering distance function has been called ϕ-function.

We have used the following important definitions and mathematical preliminaries in our main results.

Definitioin 1.1. [**19, 32**] A mapping $F : R \to R^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in R} F(t) = 0$ and $\sup_{t\in R} F(t) = 1$, where R is the set of real numbers and R^+ denotes the set of non-negative real numbers.

Definitioin 1.2. t-norm [**19, 32**]
A t-norm is a function $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ which satisfies the following conditions for all $a, b, c, d \in [0, 1]$:

(i) $\Delta(1, a) = a$,
(ii) $\Delta(a, b) = \Delta(b, a)$,
(iii) $\Delta(c, d) \geqslant \Delta(a, b)$ whenever $c \geqslant a$ and $d \geqslant b$,
(iv) $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$.

Definitioin 1.3. Hadzic type t-norm [**19**]
A t-norm Δ is said to be Hadzic type t-norm if the family $\{\Delta^p\}_{p\in N}$ of its iterates, defined for each $s \in (0, 1)$ as

$$\Delta^0(s) = 1,\ \Delta^{p+1}(s) = \Delta(\Delta^p(s), s) \text{ for all } p \geqslant 0,$$

is equi-continuous at $s = 1$, that is, given $\lambda > 0$ there exits $\eta(\lambda) \in (0, 1)$ such that

$$1 \geqslant s > \eta(\lambda) \Rightarrow \Delta^p(s) \geqslant 1 - \lambda \text{ for all } p \geqslant 0.$$

The following is the definition of Menger space which is a probabilistic metric space of a specific type where the triangular inequality is postulated with the help of a t-norm.

Definitioin 1.4. Menger space [**19, 32**]
A Menger space is a triplet (X, F, Δ), where X is a non empty set, F is a function defined on $X \times X$ to the set of distribution functions and Δ is a t-norm, such that the following are satisfied:

(i) $F_{x,y}(0) = 0$ for all $x, y \in X$,
(ii) $F_{x,y}(s) = 1$ for all $s > 0$ if and only if $x = y$,
(iii) $F_{x,y}(s) = F_{y,x}(s)$ for all $x, y \in X, s > 0$ and
(iv) $F_{x,y}(u + v) \geqslant \Delta(F_{x,z}(u), F_{z,y}(v))$ for all $u, v \geqslant 0$ and $x, y, z \in X$.

An interpretation of $F_{x,y}(t)$ is that it is the probability of the event that the distance between the points x and y is less than t. A metric space becomes a Menger space if we write $F_{x,y}(t) = H(t - d(x,y))$ where H is the Heaviside function given by

$$H(t) = 1, \text{ if } t > 0,$$
$$= 0, \text{ if } t \leqslant 0.$$

Definitioin 1.5.[**19, 32**] A sequence $\{x_n\} \subset X$ is said to be converge to some point $x \in X$ if given $\epsilon > 0, \lambda > 0$ we can find a positive integer $N_{\epsilon,\lambda}$ such that for all $n \geqslant N_{\epsilon,\lambda}$

$$F_{x_n,x}(\epsilon) \geqslant 1 - \lambda. \tag{1.1}$$

Definitioin 1.6. [**19, 32**] A sequence $\{x_n\}$ is said to be a Cauchy sequence in X if given $\epsilon > 0, \lambda > 0$ there exists a positive integer $N_{\epsilon,\lambda}$ such that

$$F_{x_n,x_m}(\epsilon) \geqslant 1 - \lambda \, for\, all\, m, n \geqslant N_{\epsilon,\lambda}. \tag{1.2}$$

Definitions 1.5 and 1.6 can be equivalently written by replacing '$\geqslant$' with '$>$' in (1.1) and (1.2) respectively. More often than not, they are written in that way. We have given them in the present form for our convenience in the proofs of our theorems.

Definitioin 1.7. [**19, 32**] A Menger space (X, F, Δ) is said to be complete if every Cauchy sequence is convergent in X.

In recent years cyclic contraction and cyclic contractive type mapping have appeared in several works.

Definitioin 1.8. Let A and B be two non-empty sets. A cyclic mapping is a mapping $T : A \bigcup B \to A \bigcup B$ which satisfies:

$TA \subseteq B$ and $TB \subseteq A$.

This line of research was initiated by Kirk, Srinivasan and Veeramani [**22**], where they, amongst other results, established the following generalization of the contraction mapping principle.

Theorem 1.1. [**22**] *Let A and B be two non-empty closed subsets of a complete metric space X and suppose $f : A \bigcup B \to A \bigcup B$ satisfies:*

(1) $fA \subseteq B$ and $fB \subseteq A$,

(2) $d(fx, fy) \leqslant kd(x, y)$ for all $x \in A$ and $y \in B$ where $k \in (0, 1)$.

Then f has a unique fixed point in $A \bigcap B$.

The problems of cyclic contractions have been strongly associated with proximity point problems. Some other results dealing with cyclic contractions and proximity point problems in probabilistic metric and 2-probabilistic metric spaces may be noted in [**8, 11, 15, 18, 20, 34, 35**] and [**36**]. A cyclic contraction result in generalized menger space was established by the recent result of Choudhury, Das and Bhandari [**12**].

A generalization of cyclic mapping is p-cyclic mapping. The definition is the following:

Definitioin 1.9. Let $\{A_i\}_{i=1}^{p}$ be non-empty sets. A p-cyclic mapping is a mapping $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ which satisfies the following conditions:

(i) $TA_i \subseteq A_{i+1}$ for $1 \leqslant i < p$, $TA_p \subseteq A_1$.

In this case where $p = 2$, this reduces to cyclic mappings. Some fixed point results of p-cyclic maps have been obtained in [**13, 34**].

The following is the extension of the control function introduced by Khan et. al [**21**] to probabilistic metric spaces. It has been shown that a ϕ-function can generate an altering distance function [**3**].

Definitioin 1.10. Φ-function [**3**]
A function $\phi : R \to R^+$ is said to be a Φ-function if it satisfies the following conditions:

(i) $\phi(t) = 0$ if and only if $t = 0$,
(ii) $\phi(t)$ is strictly monotone increasing and $\phi(t) \to \infty$ as $t \to \infty$,
(iii) ϕ is left continuous in $(0, \infty)$,
(iv) ϕ is continuous at 0.

Several uses of the ϕ-function in probabilistic fixed point and coincidence point problems have been made as, for instances, in [**4, 5, 7, 16, 17, 25**].

We will make use of the following function in our theorems.

Definitioin 1.11. Ψ-function
A function $\psi : [0, 1] \times [0, 1] \times [0, 1] \to [0, 1]$ is said to be a Ψ-function if

(i) ψ-is monotone increasing and continuous,
(ii) $\psi(x, x, x) > x$ for all $0 < x < 1$,
(iii) $\psi(1, 1, 1) = 1, \psi(0, 0, 0) = 0$.

An example of Ψ-function is $\frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{3}$.

The purpose of this paper is to prove some p-cyclic fixed point theorems in Menger spaces. The inequalities we have used are motivated by a recent result of Ciric [**14**]. We have also two examples illustrating the basic features of our theorems. Our results are derived without any assumption of continuity.

2. Main Results

In this section we have established two theorems and two examples. The first theorem is motivated by the form of the inequality used by Ciric [**14**].

Theorem 2.1. *Let (X, F, Δ) be a complete Menger space with a Hadzic type t-norm Δ such that whenever $x_n \to x$ and $y_n \to y$, $F_{x_n,y_n}(t) \to F_{x,y}(t)$. Let $\{A_i\}_{i=1}^{p}$ be non-empty closed subsets of X and the mapping T is a p-cyclic mapping such* *that T satisfies the following conditions:*

$$(i)\quad TA_i \subseteq A_{i+1}\; for\; 1 \leqslant i < p, TA_p \subseteq A_1 \tag{2.1}$$

$$\begin{aligned}&(ii)\; F_{Tx,Ty}(t) + q(1 - \max\{F_{x,Ty}(t), F_{y,Tx}(t)\})\\ &\geqslant \psi(F_{x,y}(\frac{t}{k}), F_{x,Tx}(\frac{t}{k}), F_{y,Ty}(\frac{t}{k}))\end{aligned} \tag{2.2}$$

whenever $x \in A_i$ and $y \in A_{i+1}$ for $1 \leqslant i < p$

and

$$\begin{aligned}&(iii)\; F_{Tz,Tw}(t) + q(1 - \max\{F_{z,Tw}(t), F_{w,Tz}(t)\})\\ &\geqslant \psi(F_{z,w}(\frac{t}{k}), F_{z,Tz}(\frac{t}{k}), F_{w,Tw}(\frac{t}{k}))\end{aligned} \tag{2.3}$$

whenever $z \in A_p$ and $w \in A_1$

for $t > 0$, where $0 < k < 1$, $q \geqslant 0$ and ψ is a Ψ-function. Then $\bigcap_{i=1}^p A_i$ is non-empty and T has a fixed point in $\bigcap_{i=1}^p A_i$. The fixed point is unique if $q = 0$.

Proof. Let x_0 be any arbitrary point in A_1. Now we define the sequence $\{x_n\}_{n=0}^{\infty}$ in X by $x_n = Tx_{n-1}, n \in N$ where N is the set of natural numbers.

By (2.1), we have
$x_o \in A_1$, $x_1 \in A_2$, $x_2 \in A_3$,......,$x_{p-1} \in A_p$ and in general $x_{np} \in A_1$, $x_{np+1} \in A_2$,......., $x_{np+(p-1)} \in A_p$ for all $n \geqslant 0$.
Putting $x = x_{n-1}, y = x_n$ in (2.2), for all $t > 0$, we have

$$\begin{aligned}&F_{Tx_{n-1},Tx_n}(t) + q(1 - \max\{F_{x_{n-1},Tx_n}(t), F_{x_n,Tx_{n-1}}(t)\})\\ &\qquad \geqslant \psi(F_{x_{n-1},x_n}(\tfrac{t}{k}), F_{x_{n-1},Tx_{n-1}}(\tfrac{t}{k}), F_{x_n,Tx_n}(\tfrac{t}{k})),\end{aligned}$$

that is,

$$\begin{aligned}&F_{x_n,x_{n+1}}(t) + q(1 - \max\{F_{x_{n-1},x_{n+1}}(t), F_{x_n,x_n}(t)\})\\ &\qquad \geqslant \psi(F_{x_{n-1},x_n}(\tfrac{t}{k}), F_{x_{n-1},x_n}(\tfrac{t}{k}), F_{x_n,x_{n+1}}(\tfrac{t}{k})).\end{aligned}$$

Now, for all $t > 0$ and $n \geqslant 1$,

$$\max\{F_{x_{n-1},x_{n+1}}(t), F_{x_n,x_n}(t)\} = \max\{F_{x_{n-1},x_{n+1}}(t), 1\} = 1.$$

Therefore, for all $t > 0$, $n \geqslant 1$, we have

$$F_{x_n,x_{n+1}}(t) \geqslant \psi(F_{x_{n-1},x_n}(\frac{t}{k}), F_{x_{n-1},x_n}(\frac{t}{k}), F_{x_n,x_{n+1}}(\frac{t}{k})). \tag{2.4}$$

We now claim that for all $t > 0$, $n \geqslant 1$,

$$F_{x_n,x_{n+1}}(\frac{t}{k}) \geqslant F_{x_{n-1},x_n}(\frac{t}{k}). \tag{2.5}$$

If possible, let for some $s > 0$ and some $n \geqslant 1$,

$$F_{x_n,x_{n+1}}(\tfrac{s}{k}) < F_{x_{n-1},x_n}(\tfrac{s}{k}).$$

Then, from (2.4), using the properties of ψ, we have

$$\begin{aligned}F_{x_n,x_{n+1}}(s) &\geqslant \psi(F_{x_{n-1},x_n}(\tfrac{s}{k}), F_{x_{n-1},x_n}(\tfrac{s}{k}), F_{x_n,x_{n+1}}(\tfrac{s}{k}))\\ &\geqslant \psi(F_{x_n,x_{n+1}}(\tfrac{s}{k}), F_{x_n,x_{n+1}}(\tfrac{s}{k}), F_{x_n,x_{n+1}}(\tfrac{s}{k}))\\ &> F_{x_n,x_{n+1}}(\tfrac{s}{k})\end{aligned}$$

$\geqslant F_{x_n,x_{n+1}}(s)$,
which is a contradiction.
Therefore (2.5) holds for all $t > 0$ and $n \geqslant 1$.
Using (2.5) in (2.4), and by the properties of ψ, for all $t > 0$, $n \geqslant 1$, we have

$$\begin{aligned} F_{x_n,x_{n+1}}(t) &\geqslant \psi(F_{x_{n-1},x_n}(\tfrac{t}{k}), F_{x_{n-1},x_n}(\tfrac{t}{k}), F_{x_n,x_{n+1}}(\tfrac{t}{k})) \\ &\geqslant \psi(F_{x_{n-1},x_n}(\tfrac{t}{k}), F_{x_{n-1},x_n}(\tfrac{t}{k}), F_{x_{n-1},x_n}(\tfrac{t}{k})) \\ &> F_{x_{n-1},x_n}(\tfrac{t}{k}), \end{aligned}$$

that is,

$$F_{x_n,x_{n+1}}(t) > F_{x_{n-1},x_n}(\frac{t}{k}). \tag{2.6}$$

By repeated applications of this inequality, for all $t > 0$, $n \geqslant 1$, we obtain

$$F_{x_n,x_{n+1}}(t) > F_{x_0,x_1}(\frac{t}{k^n}). \tag{2.7}$$

Taking limit as $n \to \infty$ on both sides, for all $t > 0$, we have

$$\lim_{n\to\infty} F_{x_n,x_{n+1}}(t) = 1. \tag{2.8}$$

Again, by repeated applications of (2.6), it follows that for all $t > 0$, $n \geqslant 0$ and each $i \geqslant 1$,

$$F_{x_{n+i},x_{n+i+1}}(t) > F_{x_n,x_{n+1}}(\frac{t}{k^i}). \tag{2.9}$$

We next prove that $\{x_n\}$ is a Cauchy sequence (Definition 1.6), that is, we prove that for arbitrary $\epsilon > 0$ and $0 < \lambda < 1$, there exists $N(\epsilon, \lambda)$ such that

$$F_{x_n,x_m}(\epsilon) \geqslant 1 - \lambda \text{ for all } m, n \geqslant N(\epsilon, \lambda).$$

Without loss of generality we can assume that $m > n$.

Now,

$$\epsilon = \epsilon\tfrac{1-k}{1-k} > \epsilon(1-k)(1 + k + k^2 + \dots\dots\dots + k^{m-n-1}).$$

Then, by the monotone increasing property of F, we have

$$F_{x_n,x_m}(\epsilon) \geqslant F_{x_n,x_m}(\epsilon(1-k)(1 + k + k^2 + \dots\dots\dots + k^{m-n-1})),$$

that is,

$$F_{x_n,x_m}(\epsilon) \geqslant \Delta(F_{x_n,x_{n+1}}(\epsilon(1-k)), \Delta(F_{x_{n+1},x_{n+2}}(\epsilon k(1-k)), \Delta(\dots\dots\dots,$$

$$\Delta(F_{x_{m-2},x_{m-1}}(\epsilon k^{m-n-2}(1-k)), F_{x_{m-1},x_m}(\epsilon k^{m-n-1}(1-k)))\dots\dots))). \tag{2.10}$$

Putting $t = (1-k)\epsilon k^i$ in (2.9), we get

$$F_{x_{n+i},x_{n+i+1}}((1-k)\epsilon k^i) > F_{x_n,x_{n+1}}((1-k)\epsilon).$$

Then, by (2.10), we have

$$\begin{aligned} F_{x_n,x_m}(\epsilon) \geqslant \Delta(&F_{x_n,x_{n+1}}(\epsilon(1-k)), \Delta(F_{x_n,x_{n+1}}(\epsilon(1-k)), \Delta(\dots\dots\dots, \\ &\Delta(F_{x_n,x_{n+1}}(\epsilon(1-k)), F_{x_n,x_{n+1}}(\epsilon(1-k)))\dots\dots))), \end{aligned}$$

that is,

$$F_{x_n,x_m}(\epsilon) \geqslant \Delta^{(m-n)} F_{x_n,x_{n+1}}(\epsilon(1-k)). \tag{2.11}$$

Since the t-norm Δ is a Hadzic type t-norm, the family $\{\Delta^p\}$ of its iterates is equicontinuous at the point $s=1$, that is, there exists $\eta(\lambda) \in (0,1)$ such that for all $m > n$,

$$\Delta^{(m-n)}(s) \geqslant 1-\lambda \textit{ whenever } \eta(\lambda) < s \leqslant 1. \tag{2.12}$$

Since, $F_{x_0,x_1}(t) \to 1$ as $t \to \infty$ and $0 < k < 1$, there exists an positive integer $N(\epsilon, \lambda)$ such that

$$F_{x_0,x_1}(\frac{(1-k)\epsilon}{k^n}) > \eta(\lambda) \textit{ for all } n \geqslant N(\epsilon,\lambda). \tag{2.13}$$

From (2.13) and (2.9), with $n = 0$, $i = n$ and $t = (1-k)\epsilon$, we get

$$F_{x_n,x_{n+1}}(\epsilon(1-k)) > F_{x_0,x_1}(\tfrac{(1-k)\epsilon}{k^n}) > \eta(\lambda) \text{ for all } n \geqslant N(\epsilon,\lambda).$$

Then, from (2.12) with $s = F_{x_n,x_{n+1}}(\epsilon(1-k))$, we have

$$\Delta^{(m-n)}(F_{x_n,x_{n+1}}(\epsilon(1-k))) \geqslant 1-\lambda.$$

It then follows from (2.11), that

$$F_{x_n,x_m}(\epsilon) \geqslant 1-\lambda \text{ for all } m,n \geqslant N(\epsilon,\lambda).$$

Thus $\{x_n\}$ is a Cauchy sequence.

Since X is complete, we have

$$\lim_{n\to\infty} x_n = z. \tag{2.14}$$

By the construction of the sequence $\{x_n\}$, we have $x_p \in A_1$, $x_{2p} \in A_1$, ...$x_{np} \in A_1$. Therefore the subsequence $\{x_{np}\}$ of $\{x_n\}$ which belongs to A_1 also converges to z in A_1, since A_1 is closed. Similarly subsequence $\{x_{np+1}\}$ belongs to A_2 also converges to z in A_2. Since A_3, A_4, ...,A_p are closed sets, similarly we get $z \in A_3$, A_4,........,A_p. Therefore $z \in A_1 \bigcap A_2 \bigcap A_3 ...... \bigcap A_p$.

Now, we prove that $Tz = z$.

If $F_{z,Tz}(t) = 1$, then $z = Tz$ is obviously true.

If $F_{z,Tz}(t) \neq 1$ then $0 < F_{z,Tz}(t) < 1$ for all $t > 0$.

Putting $x = x_n$, $y = z$ in the inequality (2.2), for all $t > 0$, we have

$$\begin{aligned} &F_{Tx_n,Tz}(t) + q(1 - \max\{F_{x_n,Tz}(t), F_{z,Tx_n}(t)\}) \\ &\qquad \geqslant \psi(F_{x_n,z}(\tfrac{t}{k}), F_{x_n,Tx_n}(\tfrac{t}{k}), F_{z,Tz}(\tfrac{t}{k})), \\ &(x_n \in A_{n+1}, z \in A_{n+2}), \end{aligned}$$

that is,

$$\begin{aligned} &F_{x_{n+1},Tz}(t) + q(1 - \max\{F_{x_n,Tz}(t), F_{z,x_{n+1}}(t)\}) \\ &\qquad \geqslant \psi(F_{x_n,z}(\tfrac{t}{k}), F_{x_n,x_{n+1}}(\tfrac{t}{k}), F_{z,Tz}(\tfrac{t}{k})). \end{aligned}$$

Taking limit as $n \to \infty$ on both sides for all $t > 0$, we have

$$F_{z,Tz}(t) + q(1 - \max\{F_{z,Tz}(t), F_{z,z}(t)\})$$

$$\geqslant \psi(F_{z,z}(\frac{t}{k}), F_{z,z}(\frac{t}{k}), F_{z,Tz}(\frac{t}{k})), \tag{2.15}$$

(since by our assumption $x_n \to x, y_n \to y$ implies $F_{x_n,y_n} \to F_{x,y}$)

that is,

$$F_{z,Tz}(t) + q(1 - \max\{F_{z,Tz}(t), 1\})$$

$$\geqslant \psi(F_{z,z}(\tfrac{t}{k}), F_{z,z}(\tfrac{t}{k}), F_{z,Tz}(\tfrac{t}{k})),$$

that is,

$$F_{z,Tz}(t) \geqslant \psi(1, 1, F_{z,Tz}(\tfrac{t}{k}))$$
$$\geqslant \psi(F_{z,Tz}(\tfrac{t}{k}), F_{z,Tz}(\tfrac{t}{k}), F_{z,Tz}(\tfrac{t}{k}))\ (0 < F_{z,Tz}(\tfrac{t}{k}) < 1)$$

$$> F_{z,Tz}(\frac{t}{k}). \tag{2.16}$$

(by the properties of ψ)

By repeated applications of (2.16), for all $t > 0$, we obtain

$$F_{z,Tz}(t) > F_{z,Tz}(\tfrac{t}{k^n}).$$

Taking limit as $n \to \infty$ on both sides, for all $t > 0$,

$$F_{z,Tz}(t) \geqslant \lim_{n\to\infty} F_{z,Tz}(\frac{t}{k^n}) = 1,$$

which implies

$$F_{z,Tz}(t) = 1.$$

Thus $z = Tz$.

Now, we prove the uniqueness of the fixed point for $q = 0$.

Let $Tz = z$ and $Ty = y$ be two distinct fixed point of T in $A_1 \bigcap A_2 \bigcap ......A_p$.

Then, by (2.2), we get

$F_{Tz,Ty}(t) \geqslant \psi\{F_{z,y}(\frac{t}{k}), F_{z,Tz}(\frac{t}{k}), F_{y,Ty}(\frac{t}{k})\}$, (We can consider $z \in A_i, y \in A_{i+1}$.)

which implies

$$F_{z,y}(t) \geqslant \psi\{F_{z,y}(\tfrac{t}{k}), F_{z,z}(\tfrac{t}{k}), F_{y,y}(\tfrac{t}{k})\},$$

that is,

$F_{z,y}(t) \geqslant \psi\{F_{z,y}(\frac{t}{k}), F_{z,y}(\frac{t}{k}), F_{z,y}(\frac{t}{k})\} > F_{z,y}(\frac{t}{k}) \geqslant F_{z,y}(t)$,

(since ψ is monotone increasing)

which is a contradiction.

Hence $z = y$, that is, the fixed point is unique.

This completes the proof of the theorem.

Now, we give the following example to illustrate Theorem 2.1 for $p = 2$ and $q = 0$. □

Example 2.1. Let $X = \{x_1, x_2, x_3\}$, $A_1 = \{x_1, x_2\}$, $A_2 = \{x_1, x_3\}$, $\Delta(a, b) = min(a, b)$ and $F_{x,\,y}(t)$ be defined as:

$$F_{x_1,\,x_2}(t) = \begin{cases} 0, & \text{if} \quad t \leqslant 0, \\ 0.75, & \text{if} \quad 0 < t \leqslant 2, \\ 1, & \text{if} \quad t > 2. \end{cases}$$

$$F_{x_1,\,x_3}(t) = \begin{cases} 0, & \text{if} \quad t \leqslant 0, \\ 0.92, & \text{if} \quad 0 < t \leqslant 1, \\ 1, & \text{if} \quad t > 1. \end{cases}$$

$$F_{x_2,\,x_3}(t) = \begin{cases} 0, & \text{if} \quad t \leqslant 0, \\ 0.75, & \text{if} \quad 0 < t \leqslant 2, \\ 1, & \text{if} \quad t > 2. \end{cases}$$

Then (X, F, Δ) be a complete Menger space. If we define a mapping $T : A_1 \bigcup A_2 \to A_1 \bigcup A_2$ satisfies all the conditions of Theorem 2.1 by taking $Tx_1 = x_1, Tx_2 = x_3, Tx_3 = x_1$ with $\psi(x, y, z) = \frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{3}$, $q = 0$ and $k = \frac{4}{5}$. Here x_1 is the unique fixed point of T in $A_1 \bigcap A_2$.

In our next theorem we use the control function 'ϕ' (Definition 1.10) in the inequality (2.2) with $q = 0$. Then the result of the previous theorem can not be established by following the same argument. Instead, we have a fixed point result in this case where the t-norm is the minimum t-norm.

Theorem 2.2. *Let (X, F, Δ) be a complete Menger space where Δ is the minimum t- norm. Let $\{A_i\}_{i=1}^{p}$ be non-empty closed subsets of X such that the mapping $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ satisfies the following conditions:*

$$(i) \quad TA_i \subseteq A_{i+1} \ for \ 1 \leqslant i < p, TA_p \subseteq A_1 \tag{2.17}$$

$$(ii) F_{Tx,Ty}(\phi(t)) \geqslant \psi(F_{x,y}(\phi(\frac{t}{c})), F_{x,Tx}(\phi(\frac{t}{c})), F_{y,Ty}(\phi(\frac{t}{c}))) \tag{2.18}$$

whenever $x \in A_i$, $y \in A_j$ and $1 \leqslant i, j \leqslant p$, $i \neq j$, ψ is a Ψ-function and ϕ is a Φ-function. Then $\bigcap_{i=1}^{p} A_i$ is non-empty and T has a unique fixed point in $\bigcap_{i=1}^{p} A_i$.

Proof. Let x_0 be any arbitrary point in A_1. Now we define the sequence $\{x_n\}_{n=0}^{\infty}$ in X by $x_n = Tx_{n-1}, n \in N$ where N is the set of natural numbers.

By (2.17), we have
$x_o \in A_1$, $x_1 \in A_2$, $x_2 \in A_3$,......,$x_{p-1} \in A_p$ and in general $x_{np} \in A_1$, $x_{np+1} \in A_2$,......., $x_{np+(p-1)} \in A_p$ for all $n \geqslant 0$.
Putting $x = x_{n-1}, y = x_n$ in (2.18), for all $t > 0$, we have

$$F_{Tx_{n-1},Tx_n}(\phi(t)) \geqslant \psi(F_{x_{n-1},x_n}(\phi(\tfrac{t}{c})), F_{x_{n-1},Tx_{n-1}}(\phi(\tfrac{t}{c})), F_{x_n,Tx_n}(\phi(\tfrac{t}{c}))),$$

that is,

$$F_{x_n,x_{n+1}}(\phi(t)) \geqslant \psi(F_{x_{n-1},x_n}(\phi(\frac{t}{c})), F_{x_{n-1},x_n}(\phi(\frac{t}{c})), F_{x_n,x_{n+1}}(\phi(\frac{t}{c}))). \tag{2.19}$$

We now claim that for all $t > 0$, $n \geqslant 1$,

$$F_{x_n,x_{n+1}}(\phi(\frac{t}{c})) \geqslant F_{x_{n-1},x_n}(\phi(\frac{t}{c})). \tag{2.20}$$

If possible, let for some $s > 0$ and some $n \geqslant 1$,

$$F_{x_n,x_{n+1}}(\phi(\tfrac{s}{c})) < F_{x_{n-1},x_n}(\phi(\tfrac{s}{c})).$$

Then, from (2.19), using the properties of ψ, we have for $s > 0$ and $n \geqslant 1$,

$$\begin{aligned} F_{x_n,x_{n+1}}(\phi(s)) &\geqslant \psi(F_{x_{n-1},x_n}(\phi(\tfrac{s}{c})), F_{x_{n-1},x_n}(\phi(\tfrac{s}{c})), F_{x_n,x_{n+1}}(\phi(\tfrac{s}{c}))) \\ &\geqslant \psi(F_{x_n,x_{n+1}}(\phi(\tfrac{s}{c})), F_{x_n,x_{n+1}}(\phi(\tfrac{s}{c})), F_{x_n,x_{n+1}}(\phi(\tfrac{s}{c}))) \\ &> F_{x_n,x_{n+1}}(\phi(\tfrac{s}{c})) \\ &\geqslant F_{x_n,x_{n+1}}(\phi(s)), \end{aligned}$$

which is a contradiction.

Therefore (2.20) holds for all $t > 0$ and $n \geqslant 1$.
Using (2.20) in (2.19), and by the properties of ψ, for all $t > 0$, $n \geqslant 1$, we have

$$F_{x_n,x_{n+1}}(\phi(t)) \geqslant \psi(F_{x_{n-1},x_n}(\phi(\tfrac{t}{c})), F_{x_{n-1},x_n}(\phi(\tfrac{t}{c})), F_{x_n,x_{n+1}}(\phi(\tfrac{t}{c})))$$
$$\geqslant \psi(F_{x_{n-1},x_n}(\phi(\tfrac{t}{c})), F_{x_{n-1},x_n}(\phi(\tfrac{t}{c})), F_{x_{n-1},x_n}(\phi(\tfrac{t}{c})))$$

$$> F_{x_{n-1},x_n}(\phi(\frac{t}{c})). \tag{2.21}$$

By repeated applications of this inequality, for all $t > 0$, $n \geqslant 1$, we have

$$F_{x_n,x_{n+1}}(\phi(t)) > F_{x_0,x_1}(\phi(\frac{t}{c^n})). \tag{2.22}$$

Taking limit as $n \to \infty$ on both sides of (2.22), for all $t > 0$, we obtain

$$\lim_{n\to\infty} F_{x_n,x_{n+1}}(\phi(t)) = 1. \tag{2.23}$$

Again, by virtue of a property of ϕ, given $s > 0$ we can find $t > 0$ such that $s > \phi(t)$. Thus the above limit implies that for all $s > 0$,

$$\lim_{n\to\infty} F_{x_n,x_{n+1}}(s) = 1. \tag{2.24}$$

We next prove that $\{x_n\}$ is a Cauchy sequence. If possible, let $\{x_n\}$ be not a Cauchy sequence. Then there exist $\epsilon > 0$ and $0 < \lambda < 1$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that

$$F_{x_{m(k)},x_{n(k)}}(\epsilon) < 1 - \lambda. \tag{2.25}$$

We take $n(k)$ corresponding to $m(k)$ to be the smallest integer satisfying (2.25) so that

$$F_{x_{m(k)},x_{n(k)-1}}(\epsilon) \geqslant 1 - \lambda. \tag{2.26}$$

If $\epsilon_1 < \epsilon$, then we have

$$F_{x_{m(k)},x_{n(k)}}(\epsilon_1) \leqslant F_{x_{m(k)},x_{n(k)}}(\epsilon).$$

We conclude that it is possible to construct $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ with $n(k) > m(k) > k$ and satisfying (2.25) and (2.26) whenever ϵ is replaced by a smaller positive value. As ϕ is continuous at 0 and strictly monotone increasing with $\phi(0) = 0$, it is possible to obtain $\epsilon_2 > 0$ such that $\phi(\epsilon_2) < \epsilon$.
Then, by the above argument, it is possible to obtain an increasing sequence of integers $\{m(k)\}$ and $\{n(k)\}$ with $n(k) > m(k) > k$ such that

$$F_{x_{m(k)},x_{n(k)}}(\phi(\epsilon_2)) < 1 - \lambda. \tag{2.27}$$

and

$$F_{x_{m(k)},x_{n(k)-1}}(\phi(\epsilon_2)) \geqslant 1-\lambda. \tag{2.28}$$

By (2.27), we have
$1-\lambda > F_{x_{m(k)},x_{n(k)}}(\phi(\epsilon_2))$
$= F_{Tx_{m(k)-1},Tx_{n(k)-1}}(\phi(\epsilon_2))$ ($x_{m(k)-1} \in A_{m(k)}$, $x_{n(k)-1} \in A_{n(k)}$ where $m(k) \neq n(k)$)
$\geqslant \psi(F_{x_{m(k)-1},x_{n(k)-1}}(\phi(\frac{\epsilon_2}{c})), F_{x_{m(k)-1},Tx_{m(k)-1}}(\phi(\frac{\epsilon_2}{c})),\ F_{x_{n(k)-1},Tx_{n(k)-1}}(\phi(\frac{\epsilon_2}{c}))),$
that is,

$$1-\lambda > \psi(F_{x_{m(k)-1},x_{n(k)-1}}(\phi(\frac{\epsilon_2}{c})), F_{x_{m(k)-1},x_{m(k)}}(\phi(\frac{\epsilon_2}{c})), F_{x_{n(k)-1},x_{n(k)}}(\phi(\frac{\epsilon_2}{c}))). \tag{2.29}$$

Since ϕ is strictly increasing and $0 < c < 1$, we can choose $\eta > 0$ such that $\phi(\frac{\epsilon_2}{c}) = \phi(\epsilon_2) + \eta$.
Therefore,

$$F_{x_{m(k)-1},x_{n(k)-1}}(\phi(\frac{\epsilon_2}{c})) \geqslant \Delta(F_{x_{m(k)-1},x_{m(k)}}(\eta), F_{x_{m(k)},x_{n(k)-1}}(\phi(\epsilon_2))). \tag{2.30}$$

Again, by (2.24), we have for sufficiently large k and by the property of ϕ,

$$F_{x_{m(k)-1},x_{m(k)}}(\eta) \geqslant 1-\lambda. \tag{2.31}$$

Using (2.28) and (2.31) in (2.30), we have

$$\begin{aligned} F_{x_{m(k)-1},x_{n(k)-1}}(\phi(\tfrac{\epsilon_2}{c})) &\geqslant \Delta(F_{x_{m(k)-1},x_{m(k)}}(\eta), F_{x_{m(k)},x_{n(k)-1}}(\phi(\epsilon_2))) \\ &\geqslant \Delta(1-\lambda, 1-\lambda) \\ &= 1-\lambda. \end{aligned} \tag{2.32}$$

Again, by (2.24) we have for sufficiently large k,

$$F_{x_{m(k)-1},x_{m(k)}}(\phi(\frac{\epsilon_2}{c})) \geqslant 1-\lambda \tag{2.33}$$

and

$$F_{x_{n(k)-1},x_{n(k)}}(\phi(\frac{\epsilon_2}{c})) \geqslant 1-\lambda. \tag{2.34}$$

Using (2.32), (2.33) and (2.34) in (2.29) for $\epsilon_2 > 0$, $0 < c < 1$ and by the property of ψ, we have

$$\begin{aligned} 1-\lambda &> \psi(F_{x_{m(k)-1},x_{n(k)-1}}(\phi(\tfrac{\epsilon_2}{c})), F_{x_{m(k)-1},x_{m(k)}}(\phi(\tfrac{\epsilon_2}{c})), F_{x_{n(k)-1},x_{n(k)}}(\phi(\tfrac{\epsilon_2}{c}))) \\ &\geqslant \psi(1-\lambda, 1-\lambda, 1-\lambda) > 1-\lambda, \end{aligned}$$

which is a contradiction.
Thus $\{x_n\}$ is a Cauchy sequence.

Since X is complete, we have

$$\lim_{n\to\infty} x_n = z. \tag{2.35}$$

By the construction of the sequence $\{x_n\}$, we have $x_p \in A_1$, $x_{2p} \in A_1$, ...$x_{np} \in A_1$. Therefore the subsequence $\{x_{np}\}$ of $\{x_n\}$ which belongs to A_1 also converges to z in A_1, since A_1 is closed. Similarly subsequence $\{x_{np+1}\}$ belongs to A_2 also converges to z in A_2. Since A_3, A_4, ...,A_p are closed sets, similarly we get $z \in A_3$, A_4,........,A_p. Therefore $z \in A_1 \bigcap A_2 \bigcap A_3 ...... \bigcap A_p$.
Now we prove that $Tz = z$.
Let us choose c_0 such that

$$0 < c < c_0 < 1. \tag{2.36}$$

Now, for all $t > 0$, we have

$$F_{z,Tz}(\phi(t)) \geqslant \Delta(F_{z,Tx_{2n}}(\phi(t) - \phi(c_0 t)), F_{Tx_{2n},Tz}(\phi(c_0 t))). \tag{2.37}$$

As Δ is continuous, taking liminf as $n \to \infty$ on both sides of the above inequality, for all $t > 0$, we have

$$F_{z,Tz}(\phi(t)) \geqslant \Delta(\liminf_{n\to\infty} F_{z,Tx_{2n}}(\phi(t) - \phi(c_0 t)), \liminf_{n\to\infty} F_{Tx_{2n},Tz}(\phi(c_0 t)))$$

$$= \Delta(1, \liminf_{n\to\infty} F_{Tx_{2n},Tz}(\phi(c_0 t))). \tag{2.38}$$

(by (2.35))

Now, for all $t > 0$ and $n \geqslant 0$, from (2.18), we have

$$F_{Tx_{2n},Tz}(\phi(c_0 t)) \geqslant \psi(F_{x_{2n},z}(\phi(\frac{c_0 t}{c})), F_{x_{2n},Tx_{2n}}(\phi(\frac{c_0 t}{c})), F_{z,Tz}(\phi(\frac{c_0 t}{c}))). \tag{2.39}$$

Taking liminf as $n \to \infty$ on both sides of (2.39) we have for all $t > 0$,

$\liminf_{n\to\infty} F_{Tx_{2n},Tz}(\phi(c_0 t)) \geqslant \psi(\liminf_{n\to\infty} F_{x_{2n},z}(\phi(\frac{c_0 t}{c})), \liminf_{n\to\infty} F_{x_{2n},Tx_{2n}}(\phi(\frac{c_0 t}{c})),$
$F_{z,Tz}(\phi(\frac{c_0 t}{c}))),$
that is,
$\liminf_{n\to\infty} F_{Tx_{2n},Tz}(\phi(c_0 t)) \geqslant \psi(1, 1, F_{z,Tz}(\phi(\frac{c_0 t}{c}))),$
(by (2.35))
that is,
$\liminf_{n\to\infty} F_{Tx_{2n},Tz}(\phi(c_0 t)) \geqslant \psi(F_{z,Tz}(\phi(\frac{c_0 t}{c})), F_{z,Tz}(\phi(\frac{c_0 t}{c})), F_{z,Tz}(\phi(\frac{c_0 t}{c}))),$
$> F_{z,Tz}(\phi(\frac{c_0 t}{c})),$
(by the properties of ψ)
that is,

$$\liminf_{n\to\infty} F_{Tx_{2n},Tz}(\phi(c_0 t)) > F_{z,Tz}(\phi(\frac{t}{(\frac{c}{c_0})})). \tag{2.40}$$

(since ψ is monotone increasing)
We now take $\frac{c}{c_0} = p$. Then by (2.36), $0 < p < 1$. Hence we get from (2.40) that for all $t > 0$,

$$\liminf_{n\to\infty} F_{Tx_{2n},Tz}(\phi(c_0 t)) > F_{z,Tz}(\phi(\frac{t}{p})). \tag{2.41}$$

Combining (2.38) and (2.41) for all $t > 0$, we get
$F_{z,Tz}(\phi(t)) \geqslant F_{z,Tz}(\phi(\frac{t}{p}))$. ($0 < p < 1$)
By repeated applications of this inequality, for all $t > 0$, we obtain

$$F_{z,Tz}(\phi(t)) \geqslant F_{z,Tz}(\phi(\frac{t}{p^n})). \tag{2.42}$$

Taking limit as $n \to \infty$ on both sides of (2.42), for all $t > 0$, we get
$$F_{z,Tz}(\phi(t)) \geqslant \lim_{n\to\infty} F_{z,Tz}(\phi(\frac{t}{p^n})) = 1.$$
Therefore by a property of ϕ we get $z = Tz$.

Now, we prove the uniqueness of the fixed point. Let z and w be two distinct fixed points of T in $A_1 \bigcap A_2 \bigcap ..... \bigcap A_p$. Then the properties of ϕ imply $0 < F_{z,w}(\phi(t)) < 1$ for $t > 0$.
Then, by the inequality (2.18), for all $t > 0$, we get
$F_{Tz,Tw}(\phi(t)) \geqslant \psi(F_{z,w}(\phi(\frac{t}{c})), F_{z,Tz}(\phi(\frac{t}{c})), F_{w,Tw}(\phi(\frac{t}{c})))$, $(z \in A_i, w \in A_j, i \neq j)$
that is,
$$\begin{aligned} F_{z,w}(\phi(t)) &\geqslant \psi(F_{z,w}(\phi(\tfrac{t}{c})), F_{z,z}(\phi(\tfrac{t}{c})), F_{w,w}(\phi(\tfrac{t}{c}))) \\ &= \psi(F_{z,w}(\phi(\tfrac{t}{c})), 1, 1) \\ &\geqslant \psi(F_{z,w}(\phi(\tfrac{t}{c})), F_{z,w}(\phi(\tfrac{t}{c})), F_{z,w}(\phi(\tfrac{t}{c}))) \\ &> F_{z,w}(\phi(\tfrac{t}{c})) \ (0 < F_{z,w}(\phi(\tfrac{t}{c})) < 1) \\ &\geqslant F_{z,w}(\phi(t)), \end{aligned}$$
which is a contradiction.
Hence $z = w$.

Remark: In Theorem 2.1 the '$\geqslant$' sign in equality (2.2) can be replaced by '$>$' provided that in the definition of ψ are taken $\psi(t,t,t) \geqslant t$ for all $0 < t < 1$. The structure of the proof remains unaltered. The same is the case with Theorem 2.2. If we take $p = 2$, then we get the following Corollary. This type of contraction is known as Ciric type cyclic contraction result. □

Corollary 2.1. Let (X, F, Δ) be a complete Menger space where Δ is the minimum t-norm and let there exist two non-empty closed subsets A and B of X. The mapping $T : A \bigcup B \to A \bigcup B$ is a cyclic mapping with respect to A and B such that

(i) $TA \subseteq B$ and $TB \subseteq A$

and T satisfies the following inequality

$$F_{Tx,Ty}(\phi(t)) \geqslant \psi(F_{x,y}(\phi(\tfrac{t}{c})), F_{x,Tx}(\phi(\tfrac{t}{c})), F_{y,Ty}(\phi(\tfrac{t}{c})))$$

for all $x \in A$ and $y \in B$ where $c \in (0,1)$, ϕ is a Φ-function, ψ is a Ψ-function and $t > 0$. Then $A \bigcap B$ is non-empty and T has a unique fixed point in $A \bigcap B$.

Example 2.2.

Let $X = \{x_1, x_2, x_3, x_4\}$, $p = 3$ and $A_1 = \{x_2, x_3\}$, $A_2 = \{x_2, x_1\}$, $A_3 = \{x_2, x_4\}$. Also we take the t-norm $\Delta(a,b) = min(a,b)$ and $F_{x,\,y}(t)$ be defined as:

$$F_{x_1,\,x_2}(t) = F_{x_1,\,x_3}(t) = F_{x_1,\,x_4}(t) =$$

$$F_{x_2,\,x_4}(t) = F_{x_3,\,x_4}(t) = \begin{cases} 0, & \text{if } \ t \leqslant 0, \\ 0.60, & \text{if } \ 0 < t < 8, \\ 1, & \text{if } \ t \geqslant 8. \end{cases}$$

$$F_{x_2,\,x_3}(t) = \begin{cases} 0, & \text{if } \ t \leqslant 0, \\ 1, & \text{if } \ t > 0. \end{cases}$$

Then (X, F, Δ) be a complete Menger space. If we define $T : \bigcup_{i=1}^{3} A_i \to \bigcup_{i=1}^{3} A_i$ satisfies all the conditions of Theorem 2.2 by taking $Tx_1 = x_2, Tx_2 = x_2, Tx_3 = x_2, Tx_4 = x_3$ with $\phi(t) = t$, $\psi(x,y,z) = \frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{3}$. Here x_2 is the unique fixed point of T in $\bigcap_{i=1}^{3} A_i$.

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