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RIEMANN-STIELTJES OPERATORS ON SOME WEIGHTED FUNCTION SPACES

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Abstract. In this paper, Riemann-Stieltjes-type integral operators between weighted logarithmic Bloch spaces are considered. Moreover, we give some criteria for lacunary series of new spaces $\mathcal{B}_\omega^\alpha$ and $\mathcal{B}_{\omega,0}^\alpha$ which have the weight terms in their definitions. Finally, we prove global Besov-type characterizations for the weighted Bloch space and the little weighted Bloch space.

1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Recall that the well known Bloch space (cf. [**15**]) is defined as follows:

$$\mathcal{B} = \{f : f \text{ analytic in } \mathbb{D} \text{ and } \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty\}$$

and the little Bloch space $\mathcal{B}_0$ (cf. [**15**]) is given as follows

$$\lim_{|z|\to 1^-}(1-|z|^2)|f'(z)| = 0.$$

Recently, for given a reasonable function $\omega : [0,1] \to [0,\infty)$, the weighted Bloch space $\mathcal{B}_\omega$ is defined in [**17**] (see also [**9**,**10**,**33**,**34**]) as the set of all analytic functions f on $\mathbb{D}$ satisfying

$$(1.1) \qquad (1-|z|)|f'(z)| \leqslant C\omega(1-|z|), \quad z \in \mathbb{D},$$

for some fixed $C = C_f > 0$. In the special case where $\omega \equiv 1, \mathcal{B}_\omega$ reduces to the classical Bloch space $\mathcal{B}$.

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Also, the little weighted α-Bloch space $\mathcal{B}^{\alpha}_{\omega,0}$, is a subspace of $\mathcal{B}^{\alpha}_{\omega}$ consisting of all $f \in \mathcal{B}^{\alpha}_{\omega}$ such that

$$\lim_{|z|\to 1^-} \frac{(1-|z|)^{\alpha}\big|f'(z)\big|}{\omega(1-|z|)} = 0. \tag{1.2}$$

The Dirichlet space is defined by

$$\mathcal{D} = \{f : f \text{ analytic in } \mathbb{D} \text{ and } \int_{\mathbb{D}} \big|f'(z)\big|^2 dA(z) \ < \infty\},$$

where $dA(z)$ is the Euclidean area element $dxdy$.
Let $0 < q < \infty$. Then the Besov-type spaces consist of analytic functions on $\mathbb{D}$ such that

$$\mathbf{B^q} \ = \ \Big\{f : f \text{ analytic in } \mathbb{D} \text{ and } \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \big|f'(z)\big|^q \big(1-|z|^2\big)^{q-2}(1-|\varphi_a(z)|^2)^2 \, dA(z) < \infty\Big\},$$

these classes are introduced and studied intensively Stroethoff (cf. [**39**]). Here, φ_a stands for the Möbius transformation, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. In 1994, Aulaskari and Lappan [**15**] introduced a class of holomorphic functions, the so called $\mathbf{Q_p}$-spaces as follows:

$$\mathbf{Q_p} = \Big\{f : f \text{ analytic in } \mathbb{D} \text{ and } \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \big|f'(z)\big|^2 g^p(z,a) \, dA(z) < \infty\Big\},$$

where the weight function

$$g(z,a) = \log\left|\frac{1-\bar{a}z}{a-z}\right|$$

is defined as the composition of the Möbius transformation φ_a and the fundamental solution of the two-dimensional real Laplacian. Now, we give the following definitions:
Miao [**31**] studied a gap series with Hadamard condition as in the following theorem:

Theorem 1.1. *Let $0 < p < \infty$. If $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ is analytic on $\mathbf{D}$ and has Hadamard gaps, that is, if*

$$\frac{n_{k+1}}{n_k} \geqslant \lambda > 1, \qquad (k = 1, 2, ...),$$

then the following statements are equivalent:

(I) $f \in \mathbf{B}^p$; $\quad$ *(II)* $f \in \mathbf{B}^p_0$; $\quad$ *(III)* $\sum_{k=1}^{\infty} |a_k|^p < \infty.$

Remark 1.1. The expression $\|f\|_{\mathcal{B}^{\alpha}_{\omega}}$ defines a seminorm while the natural norm is given by

$$\|f\|_{\omega,\alpha} = |f(0)| + \|f\|_{\mathcal{B}^{\alpha}_{\omega}}.$$

With this norm the space $\mathcal{B}^{\alpha}_{\omega}$ is a Banach space.

DEFINITION 1.1. Let $0 < \alpha < \infty$ and $w : [0,1] \to (0,\infty)$. For an analytic function f in $\mathbb{D}$, we define the weighted logarithmic α-Bloch space $\mathcal{B}^{\alpha}_{\omega,\log}$ as follows:

$$\mathcal{B}_{\omega,\log} := \left\{ f : f \text{ analytic in } \mathbb{D} \text{ and } \|f\|_{\mathcal{B}_{\omega,\log}} = \sup_{z\in\mathbb{D}} \frac{(1-|z|)^{\alpha}\left|f'(z)\right|}{\omega(1-|z|)} \log \frac{1}{1-|z|} < \infty \right\}. \tag{1.3}$$

Moreover, the little weighted logarithmic α-Bloch space $\mathcal{B}_{\omega,\log,0}$ is a subspace of $\mathcal{B}_{\omega,\log}$ consisting of all $f \in \mathcal{B}_{\omega,\log}$ such that

$$\lim_{|z|\to 1^-} \frac{(1-|z|)^{\alpha}\left|f'(z)\right|}{\omega(1-|z|)} \log \frac{1}{1-|z|} = 0. \tag{1.4}$$

REMARK 1.2. The expression $\|f\|_{\mathcal{B}^{\alpha}_{\omega,\log}}$ defines a seminorm while the natural norm is given by

$$\|f\|_{\omega,\log,\alpha} = |f(0)| + \|f\|_{\mathcal{B}^{\alpha}_{\omega,\log}}.$$

With this norm the space $\mathcal{B}^{\alpha}_{\omega,\log}$ is a Banach space.

Note that, If $\alpha = 1$ and $\omega \equiv 1$, then logarithmic α-Bloch space $\mathcal{B}^{\alpha}_{\omega,\log}$ reduces to the logarithmic Bloch space $\mathcal{B}_{\log}$ see [**16**]. The logarithmic Bloch space $\mathcal{B}_{\log}$ first appeared in the study of boundedness of the Hankle operators on the Bergman and Hardy spaces followed by many authors. For more details of the logarithmic Bloch space we refer to [**16**, **18**, **38**, **45**] and others.

Let $g : \mathbb{D} \to \mathbb{C}$ be a holomorphic map. Denote by $H(\mathbb{D})$ the space of holomorphic functions on $\mathbb{D}$. For $f \in H(\mathbb{D})$ a class of integral operator introduced by Pommerenke (see [**32**]) as follows:

$$T(f)(z) = T_g f(z) = \frac{1}{z}\int_0^z f(\xi)g'(\xi)d\xi. \tag{1.5}$$

The operator T_g can be viewed as a generalization of Cesàro operator which was called the Riemann-Stieltjes operator (see [**26**, **42**, **43**]).
It has been shown by Pommerenke [**32**] that T_g is a bounded operator on the Hardy space H^2 if and only if $g \in$ BMOA, where BMOA is the space of all analytic functions of bounded mean oscillations. Aleman and Siskakis showed that T_g is bounded (compact) on the Hardy space H^p, $1 \leqslant p < \infty$, if and only if $g \in$ BMOA ($g \in$ VMOA); where VMOA is the space of all analytic functions of vanishing mean oscillations, also that T_g is bounded (compact) on the Bergman space if and only if $g \in \mathcal{B}$ ($g \in \mathcal{B}_0$) (see [**12**, **13**]). Recently, the Riemann-Stieltjes-type integral operator T_g acting on various function spaces, including the Bloch space, the α-Bloch space, the weighted Bergman space, the BMOA and VMOA spaces as well as in mixed norm spaces have been studied (see [**11**–**14**, **19**, **21**, **25**, **28**, **29**, **36**, **37**, **41**, **42**, **44**] and others). It should be mentioned here also that several authors (see e.g. [**20**, **22**, **23**, **26**, **27**, **35**, **40**, **43**] and others) tried to generalize the idea of this operator on some classes of holomorphic function spaces to higher dimensions in the unit ball of $\mathbb{C}^n$.
The purpose of this paper is to investigate the behavior of T_g on the weighted Bloch

spaces $\mathcal{B}^{\alpha}_{\omega}$ and $\mathcal{B}^{\alpha}_{\omega,\log}$. We show that T_g is a bounded operator from $\mathcal{B}_{\omega}$ to $\mathcal{B}^{\alpha}_{\omega}$ if and only if $g \in \mathcal{B}^{\alpha}_{\omega,\log}$, and T_g is a compact operator from $\mathcal{B}_{\omega}$ to $\mathcal{B}^{\alpha}_{\omega,0}$ if and only if $g \in \mathcal{B}^{\alpha}_{\omega,\log,0}$. The rest part of the paper is devoted to study integral criteria for logarithmic α-Bloch space $\mathcal{B}^{\alpha}_{\omega,\log}$ and the little logarithmic α-Bloch space $\mathcal{B}^{\alpha}_{\omega,\log,0}$.

For a point $a \in \mathbb{D}$ and $0 < r < 1$, the pseudo-hyperbolic disk $D(a,r)$ with pseudo-hyperbolic center a and pseudo-hyperbolic radius r is defined by $D(a,r) = \varphi_a(rD)$.
The pseudo-hyperbolic disk $D(a,r)$ is also an Euclidean disk: its Euclidean center and Euclidean radius are $\frac{(1-r^2)a}{1-r^2|a|^2}$ and $\frac{(1-|a|^2)r}{1-r^2|a|^2}$, respectively (see [**39**]). Let A denote the normalized Lebesgue area measure on $\mathbb{D}$, and for a Lebesgue measurable set $K_1 \subset \mathbb{D}$, denote by $|K_1|$ the measure of K_1 with respect to A. It follows immediately that:

$$|D(a,r)| = \frac{(1-|a|^2)^2}{(1-r^2|a|^2)^2} r^2.$$

Now, we give a few facts about the Möbius function φ_a, which will be used in Section 3. First, the function φ_a is easily seen to be its own inverse under composition:

$$(\varphi_a \circ \varphi_a)(z) = z \quad \text{for all} \quad a, z \in \mathbb{D}\,.$$

The following identity can be obtained by straight forward computation (see [**39**]):

$$1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\overline{a}z|^2}, \quad \forall z \in \mathbb{D}.$$

A slightly different form in which we will apply the above identity is (see [**39**]):

$$\frac{1-|\varphi_a(z)|^2}{1-|z|^2} = |\varphi'_a(z).$$

For $a \in \mathbb{D}$, the substitution $z = \varphi_a(w)$ results in the Jacobian change in measure given by

$$dA(w) = |\varphi'_a(z)|^2 dA(z).$$

For a Lebesgue integrable or a non-negative Lebesgue measurable function h on $\mathbb{D}$ we thus have the following change-of-variable formula:

$$\int_{D(0,r)} h(\varphi_a(w))d(z) = \int_{D(a,r)} h(z)\left(\frac{1-|\varphi_a(z)|^2}{1-|z|^2}\right)^2 dA(z)\,.$$

Two quantities A_f and B_f, both depending on an analytic function f on $\mathbb{D}$, are said to be equivalent, written as $A_f \approx B_f$, if there exists a finite positive constant C not depending on f such that for every analytic function f on $\mathbb{D}$ we have:

$$\frac{1}{C} B_f \leqslant A_f \leqslant C B_f.$$

If the quantities A_f and B_f, are equivalent, then in particular we have $A_f < \infty$ if and only if $B_f < \infty$.

Recall that a linear operator $T : X \to Y$ is said to be compact if it takes bounded sets in X to sets in Y which have compact closure. For Banach spaces X

and Y of the space of all analytic functions $H(\mathbb{D})$, we say that T is compact from X to Y if and only if for each bounded sequence $\{x_n\}$ in X, the sequence $\{Tx_n\} \in Y$ contains a subsequence converging to some limit in Y.

2. Boundedness and compactness of T_g

It will be more convenient to work here with the operator $P_g = M_z T_g$, where $M_z f(z) = z f(z)$.
The following Lemma says that P_g and T_g are at the same time bounded or compact from $\mathcal{B}_\omega$ to $\mathcal{B}_\omega^\alpha$.

LEMMA 2.1. *Let $\mathcal{B}_\omega^\alpha$, $\omega : (0,1] \to [0,\infty)$, and let f be an analytic function on Δ. Then $f \in \mathcal{B}_\omega^\alpha$ if and only if $M_z f \in \mathcal{B}_\omega^\alpha$.*

PROOF. It is easy to see that if a function $h \in \mathcal{B}_\omega^\alpha$, then

$$\sup_{z\in\Delta} |h(z)| \frac{(1-|z|)^\alpha}{\omega(1-|z|)} \leqslant k < \infty. \tag{2.1}$$

Let $f \in \mathcal{B}_\omega^\alpha$. Then

$$\|f\|_{\mathcal{B}_\omega^\alpha} = \sup_{z\in\Delta} |f'(z)| \frac{(1-|z|)^\alpha}{\omega(1-|z|)} < \infty.$$

So

$$\begin{aligned}
\|M_z f\|_{\mathcal{B}_\omega^\alpha} &= \sup_{z\in\Delta} |z f'(z) + f(z)| \frac{(1-|z|)^\alpha}{\omega(1-|z|)} \\
&\leqslant \sup_{z\in\Delta} |z f'(z)| \frac{(1-|z|)^\alpha}{\omega(1-|z|)} + \sup_{z\in\Delta} |f(z)| \frac{(1-|z|)^\alpha}{\omega(1-|z|)} \\
&\leqslant \|f\|_{\mathcal{B}_\omega^\alpha} + k < \infty.
\end{aligned}$$

Thus $M_z f \in \mathcal{B}_\omega^\alpha$. On the other hand, if $M_z f \in \mathcal{B}_\omega^\alpha$, then

$$\|M_z f\|_{\mathcal{B}_\omega^\alpha} = \sup_{z\in\Delta} |(M_z f)'(z)| \frac{(1-|z|)^\alpha}{\omega(1-|z|)} < \infty.$$

Because $f(z) = \frac{M_z f(z)}{z}$ is an analytic function on Δ, we have that

$$\sup_{z\in\Delta_{\frac{1}{2}}} \left| f'(z) \right| \frac{(1-|z|)^\alpha}{\omega(1-|z|)} \leqslant k_1 < \infty,$$

where $\Delta_{\frac{1}{2}} = \{z : \ |z| < \frac{1}{2}\}$ and k_1 is a positive constant. Thus by $M_z f \in \mathcal{B}^\alpha_\omega$ and then,

$$\begin{aligned}
\|f\|_{\mathcal{B}^\alpha_\omega} &= \sup_{z\in\Delta} \left|f'(z)\right| \frac{(1-|z|)^\alpha}{\omega(1-|z|)} < \infty. \\
&\leqslant k_1 + \sup_{z\in\Delta\setminus\Delta_{\frac{1}{2}}} \left|f'(z)\right| \frac{(1-|z|)^\alpha}{\omega(1-|z|)} \\
&= k_1 + \sup_{z\in\Delta\setminus\Delta_{\frac{1}{2}}} \left|\frac{z(M_z f)'(z) - (M_z f)(z)}{z^2}\right| \frac{(1-|z|)^\alpha}{\omega(1-|z|)} \\
&\leqslant k_1 + 2 \sup_{z\in\Delta\setminus\Delta_{\frac{1}{2}}} \left|(M_z f)'(z)\right| \frac{(1-|z|)^\alpha}{\omega(1-|z|)} + 4 \sup_{z\in\Delta\setminus\Delta_{\frac{1}{2}}} \left|(M_z f)(z)\right| \frac{(1-|z|)^\alpha}{\omega(1-|z|)} \\
&\leqslant k_1 + 2\|M_z f\|_{\mathcal{B}^\alpha_\omega} + 4k < \infty,
\end{aligned}$$

where k and k_1 are positive constants. Therefore, $f \in \mathcal{B}^\alpha_\omega$.

Lemma 2.2. *Let $0 < \alpha < \infty$ and $\omega : (0,1] \to [0,\infty)$. Then $\mathcal{B}^\alpha_{\omega,\log} \subset \mathcal{B}^\alpha_{\omega,0}$.*

Proof. Let $f \in \mathcal{B}^\alpha_{\omega,\log}$. Then

$$\|f\|_{\mathcal{B}^\alpha_\omega} = \sup_{z\in\Delta} |f'(z)| \frac{(1-|z|)^\alpha}{\omega(1-|z|)} < \infty.$$

Hence,

$$\begin{aligned}
\lim_{|z|\to 1^-} |f'(z)| \frac{(1-|z|)^\alpha}{\omega(1-|z|)} &= \lim_{|z|\to 1^-} |f'(z)| \frac{(1-|z|)^\alpha}{\omega(1-|z|)} \frac{\left(\log\frac{1}{1-|z|}\right)}{\left(\log\frac{1}{1-|z|}\right)} \\
&\leqslant \|f\|_{\mathcal{B}^\alpha_{\omega,\log}} \lim_{|z|\to 1^-} \left(\log \frac{1}{1-|z|}\right)^{-1} = 0
\end{aligned}$$

Thus $f \in \mathcal{B}^\alpha_{\omega,0}$.

Theorem 2.1. *Let $0 < \alpha < \infty$, $\omega : (0,1] \to [0,\infty)$, and let g be an analytic function on Δ. Then the following statements are equivalent.*

(i) $T_g : \mathcal{B}_\omega \to \mathcal{B}^\alpha_\omega$ is bounded.

(ii) $T_g : \mathcal{B}_{\omega,0} \to \mathcal{B}^\alpha_{\omega,0}$ is bounded.

(iii) $T_g : \mathcal{B}_{\omega,0} \to \mathcal{B}^\alpha_\omega$ is bounded.

(iv) $g \in \mathcal{B}^\alpha_{\omega,\log}$.

Proof. Since $\mathcal{B}_{\omega,0} \subset \mathcal{B}_\omega$ and $\mathcal{B}^\alpha_{\omega,0} \subset \mathcal{B}^\alpha_\omega$, it is clear that $(i) \Rightarrow (iii)$ and $(ii) \Rightarrow (iii)$. Thus we need only to prove that $(iv) \Rightarrow (i)$, $(iv) \Rightarrow (ii)$ and $(iii) \Rightarrow (iv)$. From Lemma 2.1, we need only to work with operator $P_g = M_z T_g$. Let $g \in \mathcal{B}^\alpha_{\omega,\log}$. Then

$$\|g\|_{\mathcal{B}^\alpha_{\omega,\log}} = \sup_{z\in\Delta} |g'(z)| \frac{(1-|z|)^\alpha}{\omega(1-|z|)} \log\frac{1}{1-|z|} < \infty.$$

Because for each function $f \in \mathcal{B}_\omega$, we have

$$|f(z)| \leqslant |f(0)| + \|f\|_{\mathcal{B}_\omega} \log \frac{1}{1-|z|}. \tag{2.2}$$

Now, we note that

$$P_g f(z) = \int_0^z f(\xi) g'(\xi) d\xi,$$

then we have

$$\begin{aligned} \|P_g f\|_{\mathcal{B}_\omega^\alpha} &= \sup_{z\in\Delta} |f(z)||g'(z)| \frac{(1-|z|)^\alpha}{\omega(1-|z|)} \\ &\leqslant |f(0)| \sup_{z\in\Delta} |g'(z)| \frac{(1-|z|)^\alpha}{\omega(1-|z|)} + \|f\|_{\mathcal{B}_\omega} \sup_{z\in\Delta} |g'(z)| \frac{(1-|z|)^\alpha}{\omega(1-|z|)} \log \frac{1}{1-|z|} \\ &\leqslant |f(0)| \|g\|_{\mathcal{B}_\omega^\alpha} + \|f\|_{\mathcal{B}_\omega} \|g\|_{\mathcal{B}_{\omega,\log}^\alpha} \end{aligned}$$

By Lemma 2.2, $g \in \mathcal{B}_{\omega,\log}^\alpha$, so $\|g\|_{\mathcal{B}_\omega^\alpha} < \infty$. Therefore $\|P_g f\|_{\mathcal{B}_\omega^\alpha} < \infty$. From the closed Graph theorem we know that $P_g : \mathcal{B}_\omega \to \mathcal{B}_\omega^\alpha$ is bounded. Thus $(iv) \Rightarrow (i)$. Now suppose $f \in \mathcal{B}_{\omega,0}^\alpha$. It is easy to see that, for every $\epsilon > 0$, there is an $r \in (0,1)$ such that for $r < |z| < 1$,

$$|f(z)| \leqslant \epsilon \log \frac{1}{1-|z|}.$$

Thus, for $r < |z| < 1$, we have

$$\begin{aligned} \big|(P_g f)'(z)\big| \tfrac{(1-|z|)^\alpha}{\omega(1-|z|)} &= \big|f(z)\big| \, \big|g'(z)\big| \frac{(1-|z|)^\alpha}{\omega(1-|z|)} \\ &\leqslant \epsilon \big|g'(z)\big| \, \frac{(1-|z|)^\alpha}{\omega(1-|z|)} \log \frac{1}{1-|z|} \\ &\leqslant \epsilon \big\|g\big\|_{\mathcal{B}_{\omega,\log}^\alpha} \end{aligned}$$

Thus $P_g f \in \mathcal{B}_{\omega,0}^\alpha$, and we have proved $(iv) \Rightarrow (ii)$.
Finally, suppose that $P_g : \mathcal{B}_{\omega,0} \to \mathcal{B}_\omega^\alpha$ is bounded. Let $z_0 \in \Delta \smallsetminus \{0\}$ be arbitrary, and let $f(z) = \log \frac{1}{1-z}$. It is easy to check that $f \in \mathcal{B}_{\omega,0}^\alpha$. So

$$\infty > \|P_g f\|_{\mathcal{B}_\omega^\alpha} = \sup_{z\in\Delta} \big|f(z)\big| \, \big|g'(z)\big| \frac{(1-|z|)^\alpha}{\omega(1-|z|)} = \sup_{z\in\Delta} |g'(z)| \frac{(1-|z|)^\alpha}{\omega(1-|z|)} \log \frac{1}{|1-z|}$$

let $z = z_0$, we get

$$\big|g'(z_0)\big| \frac{(1-|z_0|)^\alpha}{\omega(1-|z_0|)} \log \frac{1}{1-|z_0|} \leqslant \|P_g f\|_{\mathcal{B}_\omega^\alpha} < \infty.$$

Because z_0 is arbitrary on $\Delta \smallsetminus \{0\}$, we know

$$\|g\|_{\mathcal{B}_{\omega,\log}^\alpha} \leqslant \|P_g f\|_{\mathcal{B}_\omega^\alpha} < \infty.$$

Hence $g \in \mathcal{B}_{\omega,\log}^\alpha$. Thus $(iii) \Rightarrow (iv)$ and the proof is complete.

We will need the following lemma for the next theorem.

LEMMA 2.3. *A closed set K in $\mathcal{B}^{\alpha}_{\omega,0}$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1^-}\sup_{f\in K}|f'(z)|\frac{(1-|z|)^{\alpha}}{\omega(1-|z|)}=0.$$

PROOF. The proof of this Lemma is similar to the proof of lemma 1 in [**30**], so we will omit it here.

THEOREM 2.2. *Let $0<\alpha<\infty$, $\omega:(0,1]\to[0,\infty)$, and let g be an analytic function on Δ. Then the following statements are equivalent.*
(a) $T_g:\mathcal{B}_{\omega}\to\mathcal{B}^{\alpha}_{\omega,0}$ is compact
(b) $T_g:\mathcal{B}_{\omega,0}\to\mathcal{B}^{\alpha}_{\omega,0}$ is compact;
(c) $g\in\mathcal{B}^{\alpha}_{\omega,\log,0}$.

PROOF. Since $\mathcal{B}^{\alpha}_{\omega,0}\subset\mathcal{B}^{\alpha}_{\omega}$, it is clear that $(i)\Rightarrow(ii)$. Thus we need only to prove that $(iii)\Rightarrow(i)$ and $(ii)\Rightarrow(iii)$.
From Lemma 2.1, we need only to work with the operator $P_g=M_zT_g$. Let $g\in\mathcal{B}^{\alpha}_{\omega,log,0}$.
Then, by Lemma 2.2, $g\in\mathcal{B}^{\alpha}_{\omega,0}$. Thus

$$\lim_{|z|\to 1^-}|g'(z)|\frac{(1-|z|)^{\alpha}}{\omega(1-|z|)}\log\frac{1}{|1-z|}=0\quad\text{and}\quad\lim_{|z|\to 1^-}|g'(z)|\frac{(1-|z|)^{\alpha}}{\omega(1-|z|)}=0.$$

Therefore, by (2.2), for every $f\in\mathcal{B}_{\omega}$,

$$\begin{aligned}&\lim_{|z|\to 1^-}|(P_gf)'(z)|\frac{(1-|z|)^{\alpha}}{\omega(1-|z|)}=\lim_{|z|\to 1^-}|f(z)||g'(z)|\frac{(1-|z|)^{\alpha}}{\omega(1-|z|)}\\&\leqslant|f(0)|\lim_{|z|\to 1^-}|g'(z)|\frac{(1-|z|)^{\alpha}}{\omega(1-|z|)}+\|f\|_{\mathcal{B}^{\alpha}_{\omega}}\lim_{|z|\to 1^-}|g'(z)|\frac{(1-|z|)^{\alpha}}{\omega(1-|z|)}\log\frac{1}{|1-z|}=0.\end{aligned}$$

Thus $P_g:\mathcal{B}_{\omega}\to\mathcal{B}^{\alpha}_{\omega,0}$ is bounded. To see that this operator is moreover compact, let $\{f_n\}\subset\mathcal{B}_{\omega}$ be such that $\|f_n\|_{\mathcal{B}_{\omega}}\leqslant 1$. We must show that $\{P_gf_n\}$ has a subsequence that converges in $\mathcal{B}^{\alpha}_{\omega,0}$. By (2.2), there is a subsequence of $\{f_n\}$ that converges uniformly on compact subsets of Δ to an analytic function f. By passing to this subsequence, we may assume that the sequence $\{f_n\}$ itself converges to f. Also, for any fixed $z\in\Delta$,

$$\big|f'(z)\big|\frac{(1-|z|)^{\alpha}}{\omega(1-|z|)}=\lim_{n\to\infty}\big|f_n'(z)\big|\frac{(1-|z|)^{\alpha}}{\omega(1-|z|)}\leqslant 1,$$

and so $f\in\mathcal{B}^{\alpha}_{\omega}$ with $\|f\|_{\mathcal{B}^{\alpha}_{\omega}}\leqslant 1$. Thus $P_gf\in\mathcal{B}^{\alpha}_{\omega,0}$ and it suffices to show that

$$\lim_{n\to\infty}\|P_gf_n-P_gf\|_{\mathcal{B}^{\alpha}_{\omega}}=0.$$

Since $g \in \mathcal{B}^\alpha_{\omega,\log}$, we get that for every $\epsilon > 0$, there is an $r \in (0,1)$ such that for every $n \geqslant 1$,

$$\begin{aligned} &\sup_{z\in\Delta\setminus\Delta_r} |f_n(z)-f(z)||g'(z)|\frac{(1-|z|)^\alpha}{\omega(1-|z|)} \\ &\leqslant |f_n(0)-f(0)| \sup_{z\in\Delta\setminus\Delta_r} |g'(z)|\frac{(1-|z|)^\alpha}{\omega(1-|z|)} \\ &+(\|f_n(z)\|_{\mathcal{B}^\alpha_\omega}+\|f(z)\|_{\mathcal{B}^\alpha_\omega}) \sup_{z\in\Delta\setminus\Delta_r} |g'(z)|\frac{(1-|z|)^\alpha}{\omega(1-|z|)}\log\frac{1}{|1-z|} < \epsilon. \end{aligned} \tag{2.3}$$

Since $\{f_n\}$ converges to f uniformly on each compact subset of Δ, we get that there in $N > 0$ such that for every $n > N$ and every $z \in \Delta$, $|f_n(z)-f(z)| < \epsilon$. Thus, for $n > N$,

$$\begin{aligned} \sup_{z\in\overline{\Delta_r}} |f_n(z)-f(z)||g'(z)|\tfrac{(1-|z|)^\alpha}{\omega(1-|z|)} &\leqslant \epsilon \sup_{z\in\overline{\Delta}} |g'(z)|\frac{(1-|z|)^\alpha}{\omega(1-|z|)} \\ &\leqslant \epsilon\|g\|_{\mathcal{B}^\alpha_\omega}. \end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4) we get

$$\lim_{n\to\infty}\sup_{z\in\Delta_r} \left|f_n(z)-f(z)\right| \left|g'(z)\right|\frac{(1-|z|)^\alpha}{\omega(1-|z|)} = 0.$$

Thus $\lim\limits_{n\to\infty} \|P_g f_n - P_g f\|_{\mathcal{B}^\alpha_\omega} = 0$, and then $P_g : \mathcal{B}_\omega \to \mathcal{B}^\alpha_{\omega,0}$ is compact. So we have proved $(iii) \Rightarrow (i)$.
To prove $(ii) \Rightarrow (iii)$, let $P_g : \mathcal{B}_{\omega,0} \to \mathcal{B}^\alpha_{\omega,0}$ be compact. Suppose $g \notin \mathcal{B}^\alpha_{\omega,\log}$. Then there is a sequence of points $\{z_n\}$ in Δ, $|z_n| \to 1$, such that

$$|g(z_n)|\frac{(1-|z_n|)^\alpha}{\omega(1-|z_n|)}\log\frac{1}{|1-z|} \longrightarrow C > 0. \tag{2.5}$$

Let $f_n(z) = \log\frac{1}{1-\bar{z}_n z}$. Then $f_n \in \mathcal{B}_{\omega,0}$ and $\|f_n\|_{\mathcal{B}_\omega} \leqslant 1$. Since P_g is compact, $\{P_g f_n\}$ is a compact subset of $\mathcal{B}^\alpha_{\omega,0}$. Thus by Lemma 2.3 we have

$$\lim_{|z|\to 1^-}\sup_n |(P_g f_n)'(z)|\frac{(1-|z|)^\alpha}{\omega(1-|z|)} = 0,$$

or

$$\lim_{|z|\to 1^-}\sup_n |f_n(z)||g'(z)|\frac{(1-|z|)^\alpha}{\omega(1-|z|)} = 0.$$

Thus, for $\epsilon = \frac{C}{2}$, there is an $r > 0$ such that

$$\sup_{|z|>r}\sup_n |f_n(z)||g'(z)|\frac{(1-|z|)^\alpha}{\omega(1-|z|)} < \epsilon = \frac{C}{2}.$$

Let $N > 0$ be such that for $n > N, |z_n| > r$. Then for $n > N$,

$$|g'(z_n)||f_n(z_n)|\frac{(1-|z|)^\alpha}{\omega(1-|z|)} \leqslant \sup_{|z|>r}\sup_n |g'(z)||f_n(z)|\frac{(1-|z|)^\alpha}{\omega(1-|z|)} < \frac{C}{2}\ ,$$

which is contrary to (2.5). The proof is therefore established.

3. Some integral criteria for $\mathcal{B}_{\omega}^{\alpha}$ functions

In this section we give some integral criteria for $\mathcal{B}_{\omega,\log}^{\alpha}$ functions. Our results generalize and extend the corresponding results for $\mathcal{B}^{\alpha}$ functions which can be found in [**39**].
Now, suppose that $\varphi_a(z) = \frac{a-z}{1-az}$ be a Möbius transformation of Δ, let $\Delta(a,r) = \{z \in \Delta : |\varphi_a(z)| < r\}$, and let $g(z,a) = \log \frac{1}{|\varphi_a(z)|}$ be the Green's function on Δ with logarithmic singularity at $a \in \Delta$. Then we will give the following theorem:

Theorem 3.1. *Let $0 < \alpha < \infty$, $0 < r < 1$, $0 < p < \infty$, $1 < q < \infty$ and $\omega : (0,1] \to [0,\infty)$. Also, let f be an analytic function on Δ. Then the following quantities are equivalent:*

(A) $\|f\|^p_{\mathcal{B}^{\alpha}_{\omega,\log}}$.

(B) For $0 < \alpha < \infty$ and $0 < p < \infty$,

$$\sup_{a\in\Delta} \frac{1}{\left|\Delta(a,r)\right|^{1-\frac{p\alpha}{2}}} \int_{\Delta(a,r)} |f'(z)|^p \left(\frac{\log\frac{1}{1-|z|}}{\omega(1-|z|)}\right)^p dA(z).$$

(C) For $0 < \alpha < \infty$ and $0 < p < \infty$,

$$\sup_{a\in\Delta} \int_{\Delta(a,r)} |f'(z)|^p \Big(1-|z|\Big)^{p\alpha-2} \left(\frac{\log\frac{1}{1-|z|}}{\omega(1-|z|)}\right)^p dA(z).$$

(D) For $0 < \alpha < \infty$,$0 < p < \infty$ and $1 < q < \infty$,

$$\sup_{a\in\Delta} \int_{\Delta} |f'(z)|^p \big(1-|z|\big)^{p\alpha-2} \left(\frac{\log\frac{1}{1-|z|}}{\omega(1-|z|)}\right)^p (1-|\varphi_a(z)|)^q \, dA(z).$$

(E) For $0 < \alpha < \infty$ and $0 < p < \infty$,

$$\sup_{a\in\Delta} \int_{\Delta} |f'(z)|^p \left(\log\frac{1}{|z|}\right)^{p\alpha} \left(\frac{\log\frac{1}{1-|z|}}{\omega(1-|z|)}\right)^p |\varphi'_a(z)|^2 \, dA(z).$$

(F)

$$\sup_{a\in\Delta} \int_{\Delta} |f'(z)|^p \, \big(g(z,a)\big)^p \big(1-|z|\big)^{p\alpha-2} \left(\frac{\log\frac{1}{1-|z|}}{\omega(1-|z|)}\right)^p dA(z) < \infty.$$

Proof. Let $0 < \alpha < \infty$, $0 < r < 1$, $0 < p < \infty$ and $\omega : (0,1] \to [0,\infty)$. Because for every analytic function g on Δ, $|g|^p$ is a subharmonic function we have

$$\big|g(0)\big|^p \leqslant \frac{1}{\pi r^2} \int_{\Delta(0,r)} \big|g(w)\big|^p \, dA(w).$$

Set $g = f' \circ \varphi_a$, we obtain that

$$\begin{aligned} \big|f'(a)\big|^p &\leqslant \frac{1}{\pi r^2} \int_{\Delta(0,r)} \big|f' \circ \varphi_a(w)\big|^p dA(w) \\ &= \frac{1}{\pi r^2} \int_{\Delta(a,r)} \big|f'(z)\big|^p \frac{(1-|\varphi_a(z)|^2)^2}{(1-|z|^2)^2} dA(z). \end{aligned}$$

Since (see [**39**]),

$$\frac{1-|\varphi_a(z)|^2}{1-|z|^2} = |\varphi_a'(z)|\ , \quad \text{where} \ \frac{1-|\varphi_a(z)|^2}{1-|z|^2} \leqslant \frac{4}{1-|a|^2} \quad a,z\in\Delta.$$

Then, we obtain that

$$\big|f'(a)\big|^p \quad \leqslant \frac{16}{\pi r^2(1-|a|^2)^2}\int_{\Delta(a,r)}\big|f'(z)\big|^p dA(z).$$

Therefore, by $(1-|a|^2)^2 \sim (1-|z|^2)^2 \sim |\Delta(a,r)|$, for $z\in\Delta(a,r)$, we deduce that

$$|f'(a)|^p\tfrac{(1-|a|)^{p\alpha}}{\omega^p(1-|a|)}\big(\log\tfrac{1}{1-|a|}\big)^p \quad \leqslant \frac{16(1-|a|)^{p\alpha}\big(\log\frac{1}{1-|a|}\big)^p}{\pi r^2(1-|a|^2)^2\omega^p(1-|a|)}\int_{\Delta(a,r)}|f'(z)|^p dA(z).$$

Since $(1-|a|)^2 \sim (1-|a|^2)^2$, then

$$\begin{aligned}|f'(a)|^p\tfrac{(1-|a|)^{p\alpha}}{\omega^p(1-|a|)}\big(\log\tfrac{1}{1-|a|}\big)^p \quad &\leqslant \frac{16\big(\log\frac{1}{1-|a|}\big)^p}{\pi r^2(1-|a|)^{2-p\alpha}\omega^p(1-|a|)}\int_{\Delta(a,r)}|f'(z)|^p dA(z)\\ &\leqslant \frac{16\lambda}{\pi r^2\big|\Delta(a,r)\big|^{1-\frac{p\alpha}{2}}}\int_{\Delta}|f'(z)|^p\bigg(\frac{\log\frac{1}{1-|z|}}{\omega(1-|z|)}\bigg)^p dA(z)\\ &= \frac{M(r)}{\big|\Delta(a,r)\big|^{1-\frac{p\alpha}{2}}}\int_{\Delta}|f'(z)|^p\bigg(\frac{\log\frac{1}{1-|z|}}{\omega(1-|z|)}\bigg)^p dA(z),\end{aligned}$$

where λ is a positive constant and $M(r) = \frac{16\lambda}{\pi r^2}$ is a constant depending on r. Thus the quantity (A) is less than or equal to a constant times the quantity (B).
From $|\Delta(a,r)| \sim (1-|z|^2)^2$ for all $z\in\Delta(a,r)$, it is obvious that $(B)\sim(C)$. By $1-|\varphi_a(z)|^2 > 1-r^2$ and $1-|\varphi_a(z)| > 1-r$ for $z\in\Delta(a,r)$, we thus obtain

$$\begin{aligned}&\int_{\Delta(a,r)}|f'(z)|^p(1-|z|)^{p\alpha-2}\bigg(\frac{\log\frac{1}{1-|z|}}{\omega(1-|z|)}\bigg)^p dA(z)\\ &= \int_{\Delta(a,r)}|f'(z)|^p(1-|z|)^{p\alpha-2}\bigg(\frac{\log\frac{1}{1-|z|}}{\omega(1-|z|)}\bigg)^p \frac{(1-|\varphi_a(z)|^2)^q}{(1-|\varphi_a(z)|^2)^q}dA(z)\\ &\leqslant \frac{1}{(1-r^2)^q}\int_{\Delta(a,r)}|f'(z)|^p(1-|z|)^{p\alpha-2}\bigg(\frac{\log\frac{1}{1-|z|}}{\omega(1-|z|)}\bigg)^p (1-|\varphi_a(z)|^2)^q dA(z).\end{aligned}$$

Hence, the quality (C) is less than or equal to a constant times (D). By $1-|\varphi_a(z)|^2 \leqslant 2g(z,a)$ for all $z,a\in\Delta$, we obtain that the quantity (D) is less than or equal to a constant times (F).

From the following inequality

$$\begin{aligned}
&\int_{\Delta(a,r)} \left|f'(z)\right|^p (1-|z|)^{p\alpha-2} \left(\frac{\log\frac{1}{1-|z|}}{\omega(1-|z|)}\right)^p \big(g(z,a)\big)^p \, dA(z) \\
&= \int_{\Delta(a,r)} \left|f'(\varphi_a(w))\right|^p (1-|\varphi_a(w)|)^{p\alpha} \left(\frac{\log\frac{1}{1-|\varphi_a(w)|}}{\omega(1-|\varphi_a(w)|)}\right)^p \left(\log\frac{1}{|w|}\right)^q \frac{dA(w)}{(1-|w|^2)^2} \\
&\leqslant \|f\|^p_{\mathcal{B}^\alpha_{\omega,\log}} \int_{\Delta(a,r)} \left(\log\frac{1}{|w|}\right)^q \frac{dA(w)}{(1-|w|^2)^2} \, ,
\end{aligned}$$

where

$$C(q,2) = \int_{\Delta(a,r)} (\log\frac{1}{|w|})^q (1-|w|^2)^{-2} d\sigma_w < \infty,$$

then we deduce that the quantity (E) is less than or equal to a constant times (A). Now, from the inequality $1-|z|^2 \leqslant 2\log\frac{1}{|z|}$ for every $z \in \Delta$, putting $q=2$ in (D), we see the quantity (D) is less than or equal to (E). Finally, let

$$\begin{aligned}
I(a) &= \int_{\Delta(a,r)} \left|f'(z)\right|^p \left(\log\frac{1}{|z|}\right)^{p\alpha} \left(\frac{\log\frac{1}{1-|z|}}{\omega(1-|z|)}\right)^p |\varphi'_a(z)|^2 \, dA(z) \\
&= \left(\int_{\Delta_{\frac{1}{4}}} + \int_{\Delta\setminus\Delta_{\frac{1}{4}}}\right) \left|f'(z)\right|^p \left(\log\frac{1}{|z|}\right)^{p\alpha} \left(\frac{\log\frac{1}{1-|z|}}{\omega(1-|z|)}\right)^p |\varphi'_a(z)|^2 \, dA(z) \\
&= I_1(a) + I_2(a),
\end{aligned}$$

where for $z \in \Delta_{\frac{1}{4}} = \{z : |z| < \frac{1}{4}\}$, $|\varphi'_a(z)|^2 = \frac{(1-|a|^2)}{|1-\bar{a}z|^4} \leqslant \frac{1}{(1-|z|)^4} \leqslant (\frac{4}{3})^4$, then we obtain

$$\begin{aligned}
I_1(a) &= \int_{\Delta_{\frac{1}{4}}} \left|f'(z)\right|^p \left(\log\frac{1}{|z|}\right)^{p\alpha} \left(\frac{\log\frac{1}{1-|z|}}{\omega(1-|z|)}\right)^p |\varphi'_a(z)|^2 \, dA(z) \\
&\leqslant \|f\|^p_{\mathcal{B}^\alpha_{\omega,\log}} \int_{\Delta_{\frac{1}{4}}} \left(\frac{\log\frac{1}{|z|}}{(1-|z|)}\right)^{p\alpha} |\varphi'_a(z)|^2 \, dA(z) \\
&\leqslant \|f\|^p_{\mathcal{B}^\alpha_{\omega,\log}} (\frac{4}{3})^{p\alpha+4} \int_{\Delta_{\frac{1}{4}}} \left(\log\frac{1}{|z|}\right)^{p\alpha} dA(z) \\
&= (\frac{4}{3})^{p\alpha+4} C(p,\alpha) \|f\|^p_{\mathcal{B}^\alpha_{\omega,\log}},
\end{aligned}$$

where

$$C(p,\alpha) = \int_{\frac{1}{4}} \left(\log\frac{1}{|z|}\right)^{p\alpha} dA(z) < \infty.$$

Now, for $z \in \Delta \backslash \Delta_{\frac{1}{4}}$, we know that $\log \frac{1}{|z|} \leqslant 4(1-|z|^2) \leqslant 8(1-|z|)$, then

$$\begin{aligned} I_2(a) &\leqslant 8 \int_{\Delta \backslash \Delta_{\frac{1}{4}}} \big|f'(z)\big|^p \bigg(\log \frac{1}{|z|}\bigg)^{p\alpha} \bigg(\frac{\log \frac{1}{1-|z|}}{\omega(1-|z|)}\bigg)^p |\varphi_a'(z)|^2 \, dA(z) \\ &\leqslant 8^{p\alpha} \|f\|^p_{\mathcal{B}^\alpha_{\omega,\log}} \int_{\Delta \backslash \Delta_{\frac{1}{4}}} |\varphi_a'(z)|^2 dA(z) \leqslant \lambda_1 \|f\|^p_{\mathcal{B}^\alpha_\omega} \end{aligned}$$

where λ_1 is a positive constant. Hence, the quantity (E) is less than or equal to a constant times (A). The proof is complete.

For $\mathcal{B}^\alpha_{\omega,\log,0}$, we have the corresponding result with Theorem 3.1.

Theorem 3.2. *Let $0 < \alpha < \infty$, $0 < r < 1$, $0 < p < \infty$, $1 < q < \infty$ and $\omega : (0,1] \to [0,\infty)$. Also, let f be an analytic function on Δ. Then the following quantities are equivalent:*

(A) $\|f\|^p_{\mathcal{B}^\alpha_{\omega,\log,0}}$.

(B) For $0 < \alpha < \infty$ and $0 < p < \infty$, we have

$$\lim_{|a| \to 1^-} \frac{1}{\big|\Delta(a,r)\big|^{1-\frac{p\alpha}{2}}} \int_{\Delta(a,r)} |f'(z)|^p \bigg(\frac{\log \frac{1}{1-|z|}}{\omega(1-|z|)}\bigg)^p dA(z) = 0.$$

(C) For $0 < \alpha < \infty$ and $0 < p < \infty$,

$$\lim_{|a| \to 1^-} \int_{\Delta(a,r)} |f'(z)|^p \Big(1-|z|\Big)^{p\alpha-2} \bigg(\frac{\log \frac{1}{1-|z|}}{\omega(1-|z|)}\bigg)^p dA(z) = 0.$$

(D) For $0 < \alpha < \infty$,$0 < p < \infty$ and $1 < q < \infty$,

$$\lim_{|a| \to 1^-} \int_\Delta |f'(z)|^p \big(1-|z|\big)^{p\alpha-2} \bigg(\frac{\log \frac{1}{1-|z|}}{\omega(1-|z|)}\bigg)^p (1-|\varphi_a(z)|)^q \, dA(z) = 0.$$

(E) For $0 < \alpha < \infty$ and $0 < p < \infty$,

$$\lim_{|a| \to 1^-} \int_\Delta |f'(z)|^p \left(\log \frac{1}{|z|}\right)^{p\alpha} \bigg(\frac{\log \frac{1}{1-|z|}}{\omega(1-|z|)}\bigg)^p |\varphi_a'(z)|^2 \, dA(z) = 0.$$

(F)

$$\lim_{|a| \to 1^-} \int_\Delta |f'(z)|^p \, \big(g(z,a)\big)^p \big(1-|z|\big)^{p\alpha-2} \bigg(\frac{\log \frac{1}{1-|z|}}{\omega(1-|z|)}\bigg)^p \, dA(z) = 0.$$

Proof. The proof it is very similarly to the Theorem 3.1.

Remark 3.1. It should be remarked that it is still open problems to extend Theorems 3.1 and 3.2 to Clifford analysis. For some characterizations connecting Bloch type spaces and Beov spaces in Clifford analysis, we refer to [**1**–**5**, **7**, **8**, **24**].

Remark 3.2. It is still an open problem to study Riemann-Stieltjes-type integral operators on the defined spaces in [**6**].

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References

[1] A. El-Sayed Ahmed, *On some classes and spaces of holomorphic and hyperholomorphic functions*, Dissertationes, Bauhaus Uinversity at Weimar-Germany.(2003), 1-127.

[2] A. El-Sayed Ahmed, *On weighted α-Besov spaces and α-Bloch spaces of quaternion-valued functions*, Numerical Functional Analysis and Optimization, Vol 29(9-10)(2008), 1064-1081.

[3] A. El-Sayed Ahmed, *Lacunary series in quaternion* $\mathbf{B^{p,q}}$ *spaces*, Complex variables and elliptic equations, 54(7)(2009), 705-723.

[4] A. El-Sayed Ahmed, *Lacunary series in weighted hyperholomorphic $B^{p,q}(G)$ spaces*, Numerical Functional Analysis and Optimization, 32(1)(2010), 41-58.

[5] A. El-Sayed Ahmed and A. Ahmadi, On weighted Bloch spaces of quaternion-valued functions, AIP Conference Proceedings. Volume 1389. (2011), 272-275.

[6] A. El-Sayed Ahmed and S. Omran, *Some analytic classes of Banach function spaces*, Global Journal of Science and Frontier Research, 10(8)(2010), 30-39.

[7] A. El-Sayed Ahmed and S. Omran, *Weighted classes of quaternion-valued functions*, Banach Journal of Mathematical Analysis, Vol 6(2)(2012), 180-191.

[8] A. El-Sayed Ahmed, K. Gürlebeck, L. F. Reséndis and L.M. Tovar, *Characterizations for the Bloch space by* $\mathbf{B^{p,q}}$ *spaces in Clifford analysis*, Complex variables and elliptic equations, Vol 51(2)(2006), 119-136.

[9] A. El-Sayed Ahmed and Alaa Kamal, *Generalized composition operators on $Q_{K,\omega}(p,q)$ type spaces*, Mathematical Sciences, 6(2012), Article 14, doi:10.1186/2251-7456-6-14.

[10] A. El-Sayed Ahmed and Alaa Kamal, $Q_{K,\omega,\log}(p,q)$*-type spaces of analytic and meromorphic functions*, Mathematica (Cluj), Tome 54(74)(2012), No.1, 26-37.

[11] A. Aleman and J. Cima, *An integral operator on H^p and Hardy's inequality*, J. Anal. Math. 85 (2001), 157-176.

[12] A. Aleman and A. G. Siskakis, *An integral operator on H^p*, Vomplex Variables, Theory and Application, An International Journal, 28(2)(1995), 149-158.

[13] A. Aleman and A. G. Siskakis, *Integration operators on Bergman spaces*, Indiana Univ. Math. J. 46 (2)(1997), 337-356.

[14] A. G. Arvanitidis and A. G. Siskakis, *Cesàro operators on the Hardy spaces of the half-plane*, Can. Math. Bull. 56(2)(2013), 229-240.

[15] R. Aulaskari and P. Lappan, *Criteria for an analytic function to be Bloch and a harmonic or meromorphic function to be normal*, In: Yang, Chung-Chun; Wen, Guo-Chun; Li, Kin-Yin; Chiang, Yik-Man (eds.): *Complex Analysis and its Applications*, Pitman Research Notes in Mathematics 305, Longman (1994), (pp. 136-146)

[16] J. Cima and D. Stegenga, *Hankel operators on H^p*, In: Earl R. Berkson, N. T. Peck and J. Ulh (Eds), Analysis at Urbana, London Math. Soc., Lect. Note Ser. 137, Cambridge University Press 1989, (pp. 133-150)

[17] K. M. Dyakonov, *Weighted Bloch spaces, H^p, and BMOA*, J. Lond. Math. Soc. 65(2)(2002), 411-417.

[18] P. Galanopoulos, *On $\mathcal{B}_{\log}$ to $\mathcal{Q}^p_{\log}$ pullbacks*, J. Math. Anal. Appl. Vol 337(2008), 712-725.

[19] Z. Hu, *Extended Cesàro operators on mixed norm spaces*, Proc. Am. Math. Soc. 131(7)(2003), 2171-2179.

[20] Z. Hu, *Extended Cesàro operators on the Bloch space in the unit ball of $\mathbb{C}^n$*, Acta Math. Sci. Ser. B 23(4)(2003), 561-566.

[21] Z. Hu, *Extended Cesàro operators on Bergman spaces*, J. Math. Anal. Appl. 296(2)(2004), 435-454.

[22] Z. Hu, *Equivalent norms on Fock spaces with some application to extended Cesàro operators*, Proc. Am. Math. Soc. 141(8)(2013), 2829-2840.

[23] X. Guo and G. Ren, *Cesàro operators on Hardy spaces in the unit ball*, J. Math. Anal. Appl. 339(1)(2008), 1-9 .

[24] K. Gürlebeck and A. El-Sayed Ahmed, *On B^q spaces of hyperholomorphic functions and the Bloch space in $\mathbb{R}^3$*, In: Le Hung Son; Tutschke, Wolfgang; Chung-Chun Yang (Eds.) *Finite or Infinite Dimensional Complex Analysis and Applications*, Advanced complex Analysis and Applications, Kluwer Academic Publishers, 2004, (pp. 269-286)

[25] S. Li, *Riemann-Stieltjes operators between Bergman-type spaces and α-Bloch spaces*, Int. J. Math. Math. Sci. No. 15, (2006), article ID 86259.

[26] S. Li and S. Stević, *Riemann-Stieltjes-type integral operators on the unit ball in $\mathbb{C}^n$*, Complex Var. Elliptic Equ. 52(6)(2007), 495-517 .

[27] S. Li and S. Stević, *Riemann-Stieltjes operators on Hardy spaces in the unit ball of $\mathbb{C}^n$*, Bull. Belg. Math. Soc.- Simon Stevin 14(4)(2007), 621-628.

[28] S. Li and S. Stević, *Riemann-Stieltjes operators between mixed norm spaces*, Indian J. Math. 50(1)(2008), 177-188.

[29] J. Miao, *The Cesàro operator is bounded on H^p for $0 < p < 1$*, Proc. Amer. Math. Soc. 116(4)(1992), 1077-1079.

[30] K. Madigan and A. Matheson, *Compact composition operators on the Bloch space*, Transactions of the American Mathematical Society, 347(7)(1995), 2679-2687.

[31] J. Miao, *A property of analytic functions with Hadamard gaps*, Bull. Austral. Math. Soc. 45(1992), 105-112.

[32] Ch. Pommerenke, Schlichte funktionen und analytische funktionen von beschrankter mittlerer oszillation, Comment. Math. Helv. 52(1)(1977), 591-602.

[33] R. A. Rashwan, A. El-Sayed Ahmed and A. Kamal, *Integral characterizations of weighted Bloch spaces and $Q_{K,\omega}(p,q)$ spaces*, Mathematica (Cluj), 51(74)(2009), No.1, 63-76.

[34] R. A. Rashwan, A. El-Sayed Ahmed and A. Kamal, Some characterizations of weighted Bloch space, European Journal of Pure and Applied Mathematics, 2(2)(2009), 250-267.

[35] S. Stević, *On integral operator on the unit ball in $\mathbb{C}^n$*, J. Inequal. Appl. 2005, 434806 doi:10.1155/JIA.2005.81, (pp. 81-88).

[36] A. G. Siskakis, *Composition semigroups and the Cesàro operator on H^p*, J. London Math. Soc. 36(2)(1987), 153-164.

[37] A. G. Siskakis, *On the Bergman space norm of the Cesàro operator*, Arch. Math. 67 (1996), 312-318.

[38] A. Siskakis and R. Zhao, *A Volterra type operator on spaces of analytic functions*, In: Krzysztof Jarosz (Ed.), *Function spaces.* Proceedings of the 3rd conference, Edwardsville, IL, USA, May 19-23, 1998. Providence, RI: American Mathematical Society. Contemp. Math. 232 (1999), pp. 299-311)

[39] K. Stroethoff, *Besov-type characterisations for the Bloch space*, Bull. Austral. Math. Soc. 39(1989), 405-420.

[40] X. Tang, *Extended Cesàro operators between Bloch-type spaces in the unit ball of $\mathbb{C}^n$*, J. Math. Anal. Appl. 326(2)(2007), 1199-1211.

[41] J. Wagner, *On the Cesàro operator in weighted l^2-sequence spaces and the generalized concept of normality*, Ann. Funct. Anal. AFA, 4(2)(2013), 1-11.

[42] J. Xiao, *Cesàro operators on Hardy, BMOA and Bloch spaces*, Arch. Math. 68 (1997), 398-406.

[43] J. Xiao, *Riemann-Stieltjes operators on weighted Bloch and Bergman spaces of the unit ball*, J. London Math. Soc. 70(2)(2004), 199-214.

[44] S. Ye, *Extended Cesàro operators from BMOA spaces to Bloch-type*, J. Math. Res. Expo. 27(3)(2007), 484-488.

[45] R. Yoneda, *The composition operators on weighted Bloch space.* Arch. Math. Vol 78(2002), 310-317.

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