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STRONG CONVERGENCE THEOREMS OF MULTI-STEP ITERATION WITH ERRORS FOR UNIFORMLY EQUI-CONTINUOUS AND ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we study modified multi-step iteration scheme with errors for a finite family of uniformly equi-continuous and asymptotically quasi-nonexpansive mappings in the framework of uniformly convex Banach spaces. By employing the above said scheme we establish some strong convergence theorems to converge to common fixed point for a finite family of uniformly equi-continuous and asymptotically quasi-nonexpansive mappings. The results presented in this paper extend and improve the corresponding results of Khan and Fukhar-ud-din [**6**], Khan and Takahashi [**7**], Qin et al. [**12**], Shahzad and Udomene [**17**], Xu and Noor [**20**] and some others.

1. Introduction and Preliminaries

Let E be a real Banach space, K be a nonempty subset of E. Throughout the paper, $\mathbb{N}$ denotes the set of positive integers and $F(T) = \{x : Tx = x\}$ the set of fixed points of a mapping T. A mapping $T\colon K \to K$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leqslant k_n \|x - y\|,$$

for all $x, y \in K$ and $n \in \mathbb{N}$.

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This class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [**4**] in 1972. They proved that, if K is a nonempty bounded closed convex subset of a uniformly convex Banach space E, then every asymptotically nonexpansive self-mapping of K has a fixed point. Moreover, the set $F(T)$ of fixed points of T is closed and convex. Since 1972, many authors have studied weak and strong convergence problem of the iterative sequences (with errors) for asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces (see [**4**, **7**, **13**, **14**, **20**] and references therein).

The mapping $T\colon K \to K$ is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - p\| \leqslant k_n \|x - p\|$$

for all $x \in K$, $p \in F(T)$ and $n \geqslant 1$.

The mapping $T\colon K \to K$ is said to be uniformly L-Lipschitzian if there exists a positive constant L such that

$$\|T^n x - T^n y\| \leqslant L \|x - y\|$$

for all $x, y \in K$ and $n \geqslant 1$.

The mapping $T\colon K \to K$ is said to be uniformly Holder continuous [**12**] if there exist positive constants L and α such that

$$\|T^n x - T^n y\| \leqslant L \|x - y\|^{\alpha}$$

for all $x, y \in K$ and $n \geqslant 1$.

The mapping $T\colon K \to K$ is said to be uniformly equi-continuous [**12**] if, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|T^n x - T^n y\| \leqslant \epsilon$$

whenever $\|x - y\| < \delta$ for all $x, y \in K$ and $n \geqslant 1$ or, equivalently, T is uniformly equi-continuous if and only if $\|T^n x_n - T^n y_n\| \to 0$ whenever $\|x_n - y_n\| \to 0$ as $n \to \infty$.

Remark 1.1. (1) It is easy to see that, if T is asymptotically nonexpansive, then it is uniformly L-Lipschitzian.

(2) If T is uniformly L-Lipschitzian, then it is uniformly Holder continuous.

(3) If T is uniformly Holder continuous, then it is uniformly equi-continuous.

In recent years, Mann iterative scheme [**10**], Ishikawa iterative scheme [**5**] and Noor iterative scheme [**20**] have been studied extensively by many authors. In 1995, Liu [**8**] introduced iterative schemes with errors as follows:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T x_n + u_n, \end{aligned} \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\{u_n\}$ a sequence in E satisfying $\sum_{n=1}^{\infty}\|u_n\| < \infty$ is known as Mann iterative scheme with errors.

The sequence $\{x_n\}$ defined by

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n + u_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n + v_n, \end{aligned} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$, $\{u_n\}$ and $\{v_n\}$ are sequences in E satisfying $\sum_{n=1}^{\infty}\|u_n\| < \infty$ and $\sum_{n=1}^{\infty}\|v_n\| < \infty$ is known as Ishikawa iterative scheme with errors.

While it is clear that consideration of errors terms in iterative scheme is an important part of the theory, it is also clear that the iterative scheme with errors introduced by Liu [**8**], as in (1.1), (1.2) above, are not satisfactory. The errors can occur in a random way. The conditions imposed on the error terms in (1.1), (1.2) which say that they tend to zero as n tends to infinity are, therefore, unreasonable. Xu [**21**] introduced a more satisfactory error term in the following iterative schemes.

The sequence $\{x_n\}$ defined by

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= \alpha_n T x_n + \beta_n x_n + \gamma_n u_n, \end{aligned} \tag{1.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$ and $\{u_n\}$ is a bounded sequence in K, is known as Mann iterative scheme with errors. This scheme reduces to Mann iterative scheme if $\gamma_n = 0$.

The sequence $\{x_n\}$ defined by

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= \alpha_n T y_n + \beta_n x_n + \gamma_n u_n, \\ y_n &= \alpha'_n T x_n + \beta'_n x_n + \gamma'_n v_n, \end{aligned} \tag{1.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$ and $\{\gamma'_n\}$ are sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1$, $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K, is known as Ishikawa iterative scheme with errors. This scheme becomes Ishikawa iterative scheme if $\gamma_n = \gamma'_n = 0$. Chidume and Moore [**2**] and Takahashi and Tamura [**19**] studied the above schemes, respectively.

The sequence $\{x_n\}$ defined by

$$\begin{aligned} z_n &= \alpha''_n T x_n + \beta''_n x_n + \gamma''_n w_n, \\ y_n &= \alpha'_n T z_n + \beta'_n x_n + \gamma'_n v_n, \\ x_{n+1} &= \alpha_n T y_n + \beta_n x_n + \gamma_n u_n, \end{aligned} \tag{1.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, $\{\alpha''_n\}$, $\{\beta''_n\}$ and $\{\gamma''_n\}$ are sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in K, is known as Noor iterative scheme with errors. This scheme reduces to Noor iterative schemes if $\gamma_n = \gamma'_n = \gamma''_n = 0$.

Many authors starting from Das and Debata [**3**] and including Takahashi and Tamura [**19**], Khan and Takahashi [**7**] and Shahzad and Udomene [**17**] have studied the two mappings case of iterative schemes for different types of mappings. We now give and study an iterative scheme with errors for a finite family of equi-continuous and asymptotically quasi-nonexpansive mappings. It worth mentioning that our scheme can be viewed as an extension of all above schemes.

In this paper, we generalize scheme (1.5) to a finite family of mappings with errors as follows:

$$\begin{aligned} x_{n+1} = x_n^{(N)} &= \alpha_n^{(N)} T_N^n x_n^{(N-1)} + \beta_n^{(N)} x_n + \gamma_n^{(N)} u_n^{(N)} \\ x_n^{(N-1)} &= \alpha_n^{(N-1)} T_{N-1}^n x_n^{(N-2)} + \beta_n^{(N-1)} x_n + \gamma_n^{(N-1)} u_n^{(N-1)} \\ \dots &= \dots \\ \dots &= \dots \\ x_n^{(3)} &= \alpha_n^{(3)} T_3^n x_n^{(2)} + \beta_n^{(3)} x_n + \gamma_n^{(3)} u_n^{(3)} \\ x_n^{(2)} &= \alpha_n^{(2)} T_2^n x_n^{(1)} + \beta_n^{(2)} x_n + \gamma_n^{(2)} u_n^{(2)} \\ x_n^{(1)} &= \alpha_n^{(1)} T_1^n x_n + \beta_n^{(1)} x_n + \gamma_n^{(1)} u_n^{(1)} \end{aligned} \tag{1.6}$$

where $\{u_n^{(1)}\}$, $\{u_n^{(2)}\}$, $\dots$, $\{u_n^{(N)}\}$ are bounded sequences in K and $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$, $\{\gamma_n^{(i)}\}$ are appropriate sequences in $[0,1]$ such that $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for each $i \in \{1, 2, \dots, N\}$.

The purpose of this paper is to establish some strong convergence theorems of the above said iteration scheme to converge to common fixed point for a finite family of equi-continuous and asymptotically quasi-nonexpansive mappings in the setting of uniformly convex Banach spaces. Our results improve and extend the corresponding results of Khan and Fukhar-ud-din [**6**], Khan and Takahashi [**7**], Qin et al. [**12**], Rhoades [**13**], Schu [**14**], Shahzad and Udomene [**17**], Xu and Noor [**20**] and some others.

In the sequel we need the following lemmas and definitions to prove our main results:

LEMMA 1.1. (*See* [**14**]) *Let E be a uniformly convex Banach space and $0 < \alpha \leqslant t_n \leqslant \beta < 1$ for all $n \in \mathbb{N}$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that*

$$\limsup_{n\to\infty} \|x_n\| \leqslant a, \ \limsup_{n\to\infty} \|y_n\| \leqslant a$$

and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = a$ hold for some $a \geqslant 0$, then

$$\lim_{n\to\infty} \|x_n - y_n\| = 0$$

.

LEMMA 1.2. (*See* [**18**]) *Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be sequences of nonnegative numbers satisfying the inequality*

$$\alpha_{n+1} \leqslant (1+\beta_n)\alpha_n + r_n, \quad \forall n \geqslant 1.$$

If $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$, then $\lim_{n\to\infty} \alpha_n$ exists. In particular, $\{\alpha_n\}_{n=1}^{\infty}$ has a subsequence which converges to zero, then $\lim_{n\to\infty} \alpha_n = 0$.

Recall that a mapping $T \colon K \to K$ where K is a subset of E, is said to satisfy Condition (A) [**16**] if there exists a nondecreasing function $f \colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that $\|x - Tx\| \geqslant f(d(x, F(T)))$ for all $x \in K$ where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.

Senter and Dotson [**16**] approximated fixed points of a nonexpansive mapping T by Mann iterates. Later on, Maiti and Ghosh [**9**] and Tan and Xu [**18**] studied the approximation of fixed points of a nonexpansive mapping T by Ishikawa iterates under the same Condition (A) which is weaker than the requirement that T is demicompact. We modify this condition for N mappings $T_1, T_2, \ldots, T_N \colon K \to K$ as follows.

A finite family $\{T_1, T_2, \ldots, T_N\}$ of N self mappings of K where K is a subset of E, is said to satisfy Condition (B) if there exists a nondecreasing function $f \colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that $a_1 \|x - T_1 x\| + a_2 \|x - T_2 x\| + \cdots + a_N \|x - T_N x\| \geqslant f(d(x, F))$ for all $x \in K$, where $d(x, F) = \inf\{\|x - p\| : p \in F = \cap_{i=1}^{N} F(T_i)\}$ and $a_1, a_2, \ldots, a_N$ are N nonnegative real numbers such that $a_1 + a_2 + \cdots + a_N = 1$.

REMARK 1.2. Condition (B) reduces to condition (A) when $T_1 = T_2 = \cdots = T_N = T$.

2. Main Results

In this section, we shall prove the strong convergence theorems of the iteration scheme (1.6) to converge to common fixed point for a finite family of equicontinuous and asymptotically quasi-nonexpansive mappings in the framework of uniformly convex Banach spaces. We first prove the following lemmas.

Lemma 2.1. *Let E be a normed space and K be a nonempty convex subset of E. Let $T_1, T_2, \dots, T_N \colon K \to K$ be N asymptotically quasi-nonexpansive mappings with sequence $\{k_n^{(i)}\}$ for $1 \leqslant i \leqslant N$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ where $k_n = \max\{k_n^{(i)} : i = 1, 2, \dots, N\}$. Let $\{x_n\}$ be the sequence as defined in (1.6) with the restriction $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$, $1 \leqslant i \leqslant N$. If $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$, then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$.*

Proof. Let $p \in F$. Since $\{u_n^{(1)}\}$, $\{u_n^{(2)}\}$, …, $\{u_n^{(N)}\}$ are bounded sequences in K. So we can set

$$M = \max\{\sup_{n\geqslant 1} \left\|u_n^{(i)} - p\right\| : i = 1, 2, \dots, N\}.$$

It follows from (1.6), that

$$\begin{aligned}
\left\|x_n^{(1)} - p\right\| &= \left\|\alpha_n^{(1)} T_1^n x_n + \beta_n^{(1)} x_n + \gamma_n^{(1)} u_n^{(1)} - p\right\| \\
&\leqslant \alpha_n^{(1)} \|T_1^n x_n - p\| + \beta_n^{(1)} \|x_n - p\| + \gamma_n^{(1)} \left\|u_n^{(1)} - p\right\| \\
&\leqslant \alpha_n^{(1)} k_n \|x_n - p\| + \beta_n^{(1)} \|x_n - p\| + \gamma_n^{(1)} \left\|u_n^{(1)} - p\right\| \\
&\leqslant [\alpha_n^{(1)} + \beta_n^{(1)}] k_n \|x_n - p\| + \gamma_n^{(1)} \left\|u_n^{(1)} - p\right\| \\
&= [1 - \gamma_n^{(1)}] k_n \|x_n - p\| + \gamma_n^{(1)} \left\|u_n^{(1)} - p\right\| \\
&\leqslant k_n \|x_n - p\| + \gamma_n^{(1)} M \\
&\leqslant k_n \|x_n - p\| + d_n^{(1)}, \qquad (2.1)
\end{aligned}$$

where $d_n^{(1)} = \gamma_n^{(1)} M$. Since $\sum_{n=1}^{\infty} \gamma_n^{(1)} < \infty$, we can see that $\sum_{n=1}^{\infty} d_n^{(1)} < \infty$. It follows from (2.1) that

$$\begin{aligned}
\left\|x_n^{(2)} - p\right\| &\leqslant \alpha_n^{(2)} \left\|T_2^n x_n^{(1)} - p\right\| + \beta_n^{(2)} \|x_n - p\| + \gamma_n^{(2)} \left\|u_n^{(2)} - p\right\| \\
&\leqslant \alpha_n^{(2)} k_n \left\|x_n^{(1)} - p\right\| + \beta_n^{(2)} \|x_n - p\| + \gamma_n^{(2)} \left\|u_n^{(2)} - p\right\| \\
&\leqslant \alpha_n^{(2)} k_n [k_n \|x_n - p\| + d_n^{(1)}] + \beta_n^{(2)} \|x_n - p\| + \gamma_n^{(2)} \left\|u_n^{(2)} - p\right\| \\
&\leqslant [\alpha_n^{(2)} + \beta_n^{(2)}] k_n^2 \|x_n - p\| + \alpha_n^{(2)} k_n d_n^{(1)} + \gamma_n^{(2)} \left\|u_n^{(2)} - p\right\| \\
&= [1 - \gamma_n^{(2)}] k_n^2 \|x_n - p\| + \alpha_n^{(2)} k_n d_n^{(1)} + \gamma_n^{(2)} M \\
&\leqslant k_n^2 \|x_n - p\| + k_n d_n^{(1)} + \gamma_n^{(2)} M \\
&\leqslant k_n^2 \|x_n - p\| + d_n^{(2)}, \qquad (2.2)
\end{aligned}$$

where $d_n^{(2)} = k_n d_n^{(1)} + \gamma_n^{(2)} M$. Since $\sum_{n=1}^{\infty} \gamma_n^{(2)} < \infty$ and $\sum_{n=1}^{\infty} d_n^{(1)} < \infty$, we can see that $\sum_{n=1}^{\infty} d_n^{(2)} < \infty$. It follows from (2.2), that

$$\begin{aligned}
\left\|x_n^{(3)} - p\right\| &\leqslant \alpha_n^{(3)} \left\|T_3^n x_n^{(2)} - p\right\| + \beta_n^{(3)} \|x_n - p\| + \gamma_n^{(3)} \left\|u_n^{(3)} - p\right\| \\
&\leqslant \alpha_n^{(3)} k_n \left\|x_n^{(2)} - p\right\| + \beta_n^{(3)} \|x_n - p\| + \gamma_n^{(3)} \left\|u_n^{(3)} - p\right\| \\
&\leqslant \alpha_n^{(3)} k_n [k_n^2 \|x_n - p\| + d_n^{(2)}] + \beta_n^{(3)} \|x_n - p\| + \gamma_n^{(3)} \left\|u_n^{(3)} - p\right\| \\
&\leqslant [\alpha_n^{(3)} + \beta_n^{(3)}] k_n^3 \|x_n - p\| + \alpha_n^{(3)} k_n d_n^{(2)} + \gamma_n^{(3)} \left\|u_n^{(3)} - p\right\| \\
&= [1 - \gamma_n^{(3)}] k_n^3 \|x_n - p\| + \alpha_n^{(3)} k_n d_n^{(2)} + \gamma_n^{(3)} M \\
&\leqslant k_n^3 \|x_n - p\| + k_n d_n^{(2)} + \gamma_n^{(3)} M \\
&\leqslant k_n^3 \|x_n - p\| + d_n^{(3)}, \qquad (2.3)
\end{aligned}$$

where $d_n^{(3)} = k_n d_n^{(2)} + \gamma_n^{(3)} M$. Since $\sum_{n=1}^{\infty} \gamma_n^{(3)} < \infty$ and $\sum_{n=1}^{\infty} d_n^{(2)} < \infty$, we can see that $\sum_{n=1}^{\infty} d_n^{(3)} < \infty$. Continuing the above process, we get that

$$\begin{aligned}
\|x_{n+1} - p\| &= \left\|x_n^{(N)} - p\right\| \\
&\leqslant k_n^N \|x_n - p\| + d_n^{(N)} \\
&= [1 + (k_n^N - 1)] \|x_n - p\| + d_n^{(N)}, \qquad (2.4)
\end{aligned}$$

since $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ is equivalent to $\sum_{n=1}^{\infty}(k_n^N - 1) < \infty$ and $\sum_{n=1}^{\infty} d_n^{(N)} < \infty$. Thus from Lemma 1.2, we know that $\lim_{n\to\infty} \|x_n - p\|$ exists. This completes the proof. □

LEMMA 2.2. *Let E be a uniformly convex Banach space and K be a nonempty convex subset of E. Let $T_1, T_2, \ldots, T_N \colon K \to K$ be N uniformly equi-continuous and asymptotically quasi-nonexpansive mappings with sequence $\{k_n^{(i)}\}$ for $1 \leqslant i \leqslant N$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ where $k_n = \max\{k_n^{(i)} : i = 1, 2, \ldots, N\}$. Let $\{x_n\}$ be the sequence as defined in (1.6) and for some $\delta_1, \delta_2 \in (0, 1)$ with the following restrictions:*

(i) $0 < \delta_1 \leqslant \alpha_n^{(i)} \leqslant \delta_2 < 1$, $\forall n \geqslant n_0$ for some $n_0 \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$, $1 \leqslant i \leqslant N$.

If $F = \cap_{i=1}^N F(T_i) \neq \emptyset$, then $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$.

PROOF. For any $p \in F$, it follows from Lemma 2.1 that $\lim_{n\to\infty} \|x_n - p\|$ exists. Let $\lim_{n\to\infty} \|x_n - p\| = a$ for some $a \geqslant 0$. We note that

$$\left\|x_n^{(N-1)} - p\right\| \leqslant k_n^{N-1} \|x_n - p\| + d_n^{(N-1)}, \qquad \forall n \geqslant 1,$$

where $d_n^{(N-1)} = k_n d_n^{(N-2)} + \gamma_n^{(N-1)} M$ such that $\sum_{n=1}^{\infty} d_n^{(N-1)} < \infty$. It follows that

$$\limsup_{n\to\infty} \left\| x_n^{(N-1)} - p \right\| \leqslant \limsup_{n\to\infty} [k_n^{N-1} \left\| x_n - p \right\| + d_n^{(N-1)}] = \lim_{n\to\infty} \left\| x_n - p \right\| = a$$

and so

$$\limsup_{n\to\infty} \left\| T_N^n x_n^{(N-1)} - p \right\| \leqslant \limsup_{n\to\infty} k_n \left\| x_n^{(N-1)} - p \right\| = \limsup_{n\to\infty} \left\| x_n^{(N-1)} - p \right\| \leqslant a.$$

Next, consider

$$\left\| T_N^n x_n^{(N-1)} - p + \gamma_n^{(N)} (u_n^{(N)} - x_n) \right\| \leqslant \left\| T_N^n x_n^{(N-1)} - p \right\| + \gamma_n^{(N)} \left\| u_n^{(N)} - x_n \right\|.$$

Thus

$$\limsup_{n\to\infty} \left\| T_N^n x_n^{(N-1)} - p + \gamma_n^{(N)} (u_n^{(N)} - x_n) \right\| \quad \leqslant \quad a. \tag{2.5}$$

Also

$$\left\| x_n - p + \gamma_n^{(N)} (u_n^{(N)} - x_n) \right\| \leqslant \left\| x_n - p \right\| + \gamma_n^{(N)} \left\| u_n^{(N)} - x_n \right\|$$

gives that

$$\limsup_{n\to\infty} \left\| x_n - p + \gamma_n^{(N)} (u_n^{(N)} - x_n) \right\| \quad \leqslant \quad a, \tag{2.6}$$

and we observe that

$$x_n^{(N)} - p = \alpha_n^{(N)} (T_N^n x_n^{(N-1)} - p + \gamma_n^{(N)} (u_n^{(N)} - x_n)) + (1 - \alpha_n^{(N)}) (x_n - p + \gamma_n^{(N)} (u_n^{(N)} - x_n)).$$

Therefore

$$\begin{aligned} a &= \lim_{n\to\infty} \| x_n^{(N)} - p \| \\ &= \lim_{n\to\infty} \| \alpha_n^{(N)} (T_N^n x_n^{(N-1)} - p + \gamma_n^{(N)} (u_n^{(N)} - x_n)) \\ &\quad + (1 - \alpha_n^{(N)}) (x_n - p + \gamma_n^{(N)} (u_n^{(N)} - x_n)) \|. \end{aligned}$$

By (2.5), (2.6) and Lemma 1.1, we have

$$\lim_{n\to\infty} \left\| T_N^n x_n^{(N-1)} - x_n \right\| \quad = \quad 0. \tag{2.7}$$

Now, we shall show that $\lim_{n\to\infty} \left\| T_{N-1}^n x_n^{(N-2)} - x_n \right\| = 0$. For each $n \geqslant 1$, we have

$$\begin{aligned} \| x_n - p \| \quad &\leqslant \quad \left\| T_N^n x_n^{(N-1)} - x_n \right\| + \left\| T_N^n x_n^{(N-1)} - p \right\| \\ &\leqslant \quad \left\| T_N^n x_n^{(N-1)} - x_n \right\| + k_n \left\| x_n^{(N-1)} - p \right\|. \end{aligned}$$

Using (2.7), we have

$$a = \lim_{n\to\infty} \| x_n - p \| \leqslant \liminf_{n\to\infty} \left\| x_n^{(N-1)} - p \right\|.$$

It follows that

$$a \leqslant \liminf_{n\to\infty} \left\| x_n^{(N-1)} - p \right\| \leqslant \limsup_{n\to\infty} \left\| x_n^{(N-1)} - p \right\| \leqslant a.$$

This implies that

$$\lim_{n\to\infty} \left\| x_n^{(N-1)} - p \right\| = a.$$

On the other hand, we have

$$\left\| x_n^{(N-2)} - p \right\| \leqslant k_n^{N-2} \left\| x_n - p \right\| + d_n^{(N-2)}, \qquad \forall n \geqslant 1,$$

where $\sum_{n=1}^{\infty} d_n^{(N-2)} < \infty$. Therefore

$$\limsup_{n\to\infty} \left\| x_n^{(N-2)} - p \right\| \leqslant \limsup_{n\to\infty} (k_n^{N-2} \left\| x_n - p \right\| + d_n^{(N-2)}) = a,$$

and hence

$$\limsup_{n\to\infty} \left\| T_{N-1}^n x_n^{(N-2)} - p \right\| \leqslant \limsup_{n\to\infty} k_n \left\| x_n^{(N-2)} - p \right\| \leqslant a.$$

Next, consider

$$\left\| T_{N-1}^n x_n^{(N-2)} - p + \gamma_n^{(N-1)} (u_n^{(N-1)} - x_n) \right\| \leqslant \left\| T_{N-1}^n x_n^{(N-2)} - p \right\| + \gamma_n^{(N-1)} \left\| u_n^{(N-1)} - x_n \right\|.$$

Thus

$$\limsup_{n\to\infty} \left\| T_{N-1}^n x_n^{(N-2)} - p + \gamma_n^{(N-1)} (u_n^{(N-1)} - x_n) \right\| \quad \leqslant \quad a. \tag{2.8}$$

Also

$$\left\| x_n - p + \gamma_n^{(N-1)} (u_n^{(N-1)} - x_n) \right\| \leqslant \left\| x_n - p \right\| + \gamma_n^{(N-1)} \left\| u_n^{(N-1)} - x_n \right\|$$

gives that

$$\limsup_{n\to\infty} \left\| x_n - p + \gamma_n^{(N-1)} (u_n^{(N-1)} - x_n) \right\| \quad \leqslant \quad a, \tag{2.9}$$

and we observe that

$$\begin{aligned} x_n^{(N-1)} - p &= \alpha_n^{(N-1)} (T_{N-1}^n x_n^{(N-2)} - p + \gamma_n^{(N-1)} (u_n^{(N-1)} - x_n)) \\ &\quad + (1 - \alpha_n^{(N-1)}) (x_n - p + \gamma_n^{(N-1)} (u_n^{(N-1)} - x_n)), \end{aligned}$$

and hence

$$\begin{aligned} a &= \lim_{n\to\infty} \| x_n^{(N-1)} - p \| \\ &= \lim_{n\to\infty} \| \alpha_n^{(N-1)} (T_{N-1}^n x_n^{(N-2)} - p + \gamma_n^{(N-1)} (u_n^{(N-1)} - x_n)) \\ &\qquad + (1 - \alpha_n^{(N-1)}) (x_n - p + \gamma_n^{(N-1)} (u_n^{(N-1)} - x_n)) \|. \end{aligned}$$

By (2.8), (2.9) and Lemma 1.1, we have

$$\lim_{n\to\infty} \left\| T_{N-1}^n x_n^{(N-2)} - x_n \right\| \quad = \quad 0. \tag{2.10}$$

Similarly, using the same argument as in the proof above, we have

$$\lim_{n\to\infty} \left\| T_{N-2}^n x_n^{(N-3)} - x_n \right\| \quad = \quad 0. \tag{2.11}$$

Continuing the similar process, we have

$$\lim_{n\to\infty}\left\|T^n_{N-i}x_n^{(N-i-1)}-x_n\right\| \;=\; 0, \qquad 0\leqslant i\leqslant (N-2). \tag{2.12}$$

Now

$$\left\|T_1^n x_n - p + \gamma_n^{(1)}(u_n^{(1)}-x_n)\right\| \leqslant \|T_1^n x_n - p\| + \gamma_n^{(1)}\left\|u_n^{(1)}-x_n\right\|.$$

Thus

$$\limsup_{n\to\infty}\left\|T_1^n x_n - p + \gamma_n^{(1)}(u_n^{(1)}-x_n)\right\| \leqslant a. \tag{2.13}$$

Also

$$\left\|x_n - p + \gamma_n^{(1)}(u_n^{(1)}-x_n)\right\| \leqslant \|x_n - p\| + \gamma_n^{(1)}\left\|u_n^{(1)}-x_n\right\|$$

gives that

$$\limsup_{n\to\infty}\left\|x_n - p + \gamma_n^{(1)}(u_n^{(1)}-x_n)\right\| \leqslant a, \tag{2.14}$$

and hence

$$\begin{aligned} a &= \lim_{n\to\infty}\|x_n^{(1)}-p\| \\ &= \lim_{n\to\infty}\|\alpha_n^{(1)}(T_1^n x_n - p + \gamma_n^{(1)}(u_n^{(1)}-x_n)) \\ &\quad + (1-\alpha_n^{(1)})(x_n - p + \gamma_n^{(1)}(u_n^{(1)}-x_n))\|. \end{aligned}$$

By (2.13), (2.14) and Lemma 1.1, we have

$$\lim_{n\to\infty}\|T_1^n x_n - x_n\| \;=\; 0. \tag{2.15}$$

On the other hand, we also have

$$\begin{aligned} \left\|x_n^{(N-1)}-x_n\right\| &= \left\|\alpha_n^{(N-1)}T^n_{N-1}x_n^{(N-2)} + \beta_n^{(N-1)}x_n + \gamma_n^{(N-1)}u_n^{(N-1)} - x_n\right\| \\ &= \left\|\alpha_n^{(N-1)}(T^n_{N-1}x_n^{(N-2)} - x_n) + \gamma_n^{(N-1)}(u_n^{(N-1)} - x_n)\right\| \\ &\leqslant \alpha_n^{(N-1)}\left\|T^n_{N-1}x_n^{(N-2)} - x_n\right\| + \gamma_n^{(N-1)}\left\|u_n^{(N-1)} - x_n\right\|. \end{aligned}$$

Therefore, it follows from (2.10) and condition $\sum_{n=1}^{\infty}\gamma_n^{(N-1)} < \infty$ that

$$\left\|x_n^{(N-1)}-x_n\right\| \to 0, \quad \text{as} \quad n\to\infty.$$

It follows from the uniform equi-continuity of T_N that

$$\left\|T_N^n x_n^{(N-1)} - T_N^n x_n\right\| \to 0, \text{ as } n\to\infty. \tag{2.16}$$

Now observe that

$$\|x_n - T_N^n x_n\| \;\leqslant\; \left\|T_N^n x_n - T_N^n x_n^{(N-1)}\right\| + \left\|T_N^n x_n^{(N-1)} - x_n\right\|. \tag{2.17}$$

From (2.7), (2.16) and (2.17), we can obtain

$$\lim_{n\to\infty}\|T_N^n x_n - x_n\| \;=\; 0. \tag{2.18}$$

Now, we have

$$\begin{aligned} \|x_{n+1}-x_n\| &= \left\|\alpha_n^{(N)}T_N^n x_n^{(N-1)} + (1-\alpha_n^{(N)}-\gamma_n^{(N)})x_n + \gamma_n^{(N)}u_n^{(N)} - x_n\right\| \\ &\leqslant \alpha_n^{(N)}\left\|T_N^n x_n^{(N-1)} - x_n\right\| + \gamma_n^{(N)}\left\|u_n^{(N)} - x_n\right\| \qquad (2.19) \\ &\to 0 \ \text{ as } \ n\to\infty. \end{aligned}$$

It follows from the uniform equi-continuity of T_N that $\left\|T_N^{n+1}x_{n+1} - T_N^{n+1}x_n\right\| \to 0$ as $n\to\infty$. Again it follows from (2.17) that

$$\lim_{n\to\infty}\left\|T_N^{n+1}x_n - T_N x_n\right\| = 0.$$

Therefore we have

$$\begin{aligned} \|T_N x_n - x_n\| \leqslant\ & \|x_{n+1}-x_n\| + \left\|x_{n+1} - T_N^{n+1}x_{n+1}\right\| \\ & + \left\|T_N^{n+1}x_{n+1} - T_N^{n+1}x_n\right\| + \left\|T_N^{n+1}x_n - T_N x_n\right\| \end{aligned} \qquad (2.20)$$

It follows from (2.18), (2.19) and above inequality that

$$\lim_{n\to\infty}\|T_N x_n - x_n\| = 0. \qquad (2.21)$$

Similarly, by using the same argument as in the proof above, we have

$$\lim_{n\to\infty}\|T_{N-1} x_n - x_n\| = 0. \qquad (2.22)$$

Continuing similar process, we have

$$\lim_{n\to\infty}\|T_{N-i} x_n - x_n\| = 0, \quad 0\leqslant i\leqslant (N-2). \qquad (2.23)$$

Now

$$\begin{aligned} \|T_1 x_n - x_n\| \leqslant\ & \|x_{n+1}-x_n\| + \left\|x_{n+1} - T_1^{n+1}x_{n+1}\right\| \\ & + \left\|T_1^{n+1}x_{n+1} - T_1^{n+1}x_n\right\| + \left\|T_1^{n+1}x_n - T_1 x_n\right\| \qquad (2.24) \end{aligned}$$

Since T_1 is uniformly equi-continuous, it follows from (2.15) and (2.19) that $\left\|T_1^{n+1}x_n - T_1 x_n\right\| \to 0$ and $\left\|T_1^{n+1}x_{n+1} - T_1^{n+1}x_n\right\| \to 0$ as $n\to\infty$.

Therefore from (2.15), (2.19), (2.24) and above that

$$\lim_{n\to\infty}\|T_1 x_n - x_n\| = 0, \qquad (2.25)$$

and hence

$$\lim_{n\to\infty}\|T_{N-i} x_n - x_n\| = 0, \quad 0\leqslant i\leqslant (N-1). \qquad (2.26)$$

This completes the proof. □

THEOREM 2.1. *Let E be a uniformly convex Banach space and K be a nonempty convex subset of E. Let $T_1, T_2, \ldots, T_N\colon K\to K$ be N uniformly equi-continuous and asymptotically quasi-nonexpansive mappings with sequences $\{k_n^{(i)}\}$ for $1\leqslant i\leqslant N$ such that $\sum_{n=1}^{\infty}(k_n-1)<\infty$ where $k_n = \max\{k_n^{(i)} : i=1,2,\ldots,N\}$. Let $\{x_n\}$ be the sequence as defined in (1.6) and for some $\delta_1,\delta_2\in(0,1)$ with the following* *restrictions:*

(i) $0 < \delta_1 \leqslant \alpha_n^{(i)} \leqslant \delta_2 < 1$, $\forall n \geqslant n_0$ *for some* $n_0 \in \mathbb{N}$,

(ii) $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$, $1 \leqslant i \leqslant N$.

If $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. *Suppose* $\{T_1, T_2, \ldots, T_N\}$ *satisfies condition* (B). *Then* $\{x_n\}$ *converges strongly to a common fixed point of the mappings* $\{T_1, T_2, \ldots, T_N\}$.

Proof. By Lemma 2.1, we know that $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$. Let $\lim_{n\to\infty} \|x_n - p\| = a$ for some $a \geqslant 0$. Without loss of generality, if $a = 0$, there is nothing to prove. Assume that $a > 0$. As proved in Lemma 2.1, we have

$$\|x_{n+1} - p\| = \left\| x_n^{(N)} - p \right\| \leqslant k_n^N \|x_n - p\| + d_n^{(N)}, \quad \forall n \geqslant 1,$$

where $d_n^{(N)} = k_n d_n^{(N-1)} + \gamma_n^{(N)} M$ such that $\sum_{n=1}^{\infty} d_n^{(N)} < \infty$. This gives that

$$d(x_{n+1}, F) \leqslant [1 + (k_n^N - 1)] d(x_n, F) + d_n^{(N)}, \quad \forall n \geqslant 1,$$

since $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ is equivalent to $\sum_{n=1}^{\infty} (k_n^N - 1) < \infty$ and since $\sum_{n=1}^{\infty} d_n^{(N)} < \infty$. From Lemma 1.2, we know that $\lim_{n\to\infty} d(x_n, F)$ exists. Also by Lemma 2.2, $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for all $i = 1, 2, \ldots, N$. Since $\{T_1, T_2, \ldots, T_N\}$ satisfies Condition (B), we conclude that $\lim_{n\to\infty} d(x_n, F) = 0$.

Next we show that $\{x_n\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} d(x_n, F) = 0$, given any $\varepsilon > 0$, there exists a natural number n_0 such that $d(x_n, F) < \frac{\varepsilon}{3}$ for all $n \geqslant n_0$. So, we can find $p^* \in F$ such that $\|x_{n_0} - p^*\| < \frac{\varepsilon}{2}$. For all $n \geqslant n_0$ and $m \geqslant 1$, we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leqslant \|x_{n+m} - p^*\| + \|x_n - p^*\| \\ &\leqslant \|x_{n_0} - p^*\| + \|x_{n_0} - p^*\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence and so is convergent since E is complete. Let $\lim_{n\to\infty} x_n = q^*$. Then $q^* \in K$. It remains to show that $q^* \in F$. Let $\varepsilon_1 > 0$ be given. Then there exists a natural number n_1 such that $\|x_n - q^*\| < \frac{\varepsilon_1}{4}$ for all $n \geqslant n_1$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, there exists a natural number $n_2 \geqslant n_1$ such that for all $n \geqslant n_2$ we have $d(x_n, F) < \frac{\varepsilon_1}{5}$ and in particular we have $d(x_{n_2}, F) \leqslant \frac{\varepsilon_1}{5}$. Therefore, there exists $w^* \in F$ such that $\|x_{n_2} - w^*\| < \frac{\varepsilon_1}{4}$. For any $i \in I$ and $n \geqslant n_2$, we have

$$\begin{aligned} \|T_i q^* - q^*\| &\leqslant \|T_i q^* - w^*\| + \|w^* - q^*\| \\ &\leqslant 2\|q^* - w^*\| \\ &\leqslant 2[\|q^* - x_{n_2}\| + \|x_{n_2} - w^*\|] \\ &< 2[\frac{\varepsilon_1}{4} + \frac{\varepsilon_1}{4}] \\ &< \varepsilon_1. \end{aligned}$$

This implies that $T_i q^* = q^*$. Hence $q^* \in F(T_i)$ for all $i \in I$ and so $q^* \in F = \cap_{i=1}^N F(T_i)$. Thus $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N\}$. This completes the proof. $\square$

For our next result, we shall need the following definition:

Definition 2.1. Let K be a nonempty closed subset of a Banach space E. A mapping $T\colon K \to K$ is said to be semi-compact, if for any bounded sequence $\{x_n\}$ in K such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\lim_{n\to\infty} x_{n_j} = x \in K$.

Theorem 2.2. *Let E be a uniformly convex Banach space and K be a nonempty convex subset of E. Let $T_1, T_2, \ldots, T_N\colon K \to K$ be N uniformly equi-continuous and asymptotically quasi-nonexpansive mappings with sequence $\{k_n^{(i)}\}$ for $1 \leqslant i \leqslant N$ such that $\sum_{n=1}^\infty (k_n - 1) < \infty$ where $k_n = \max\{k_n^{(i)} : i = 1, 2, \ldots, N\}$. Let $\{x_n\}$ be the sequence as defined in (1.6) and for some $\delta_1, \delta_2 \in (0,1)$ with the following restrictions:*

(i) $0 < \delta_1 \leqslant \alpha_n^{(i)} \leqslant \delta_2 < 1$, $\forall n \geqslant n_0$ for some $n_0 \in \mathbb{N}$,

(ii) $\sum_{n=1}^\infty \gamma_n^{(i)} < \infty$, $1 \leqslant i \leqslant N$.

If $F = \cap_{i=1}^N F(T_i) \neq \emptyset$. Suppose one of the mappings in $\{T_1, T_2, \ldots, T_N\}$ is semi-compact. Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N\}$.

Proof. Suppose T_{i_0} is semi-compact for some $i_0 \in \{1, 2, \ldots, N\}$. By Lemma 2.2, we have

$$\lim_{n\to\infty} \|x_n - T_{i_0} x_n\| = 0.$$

So there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} x_{n_j} = x^* \in K$. Now Lemma 2.2 guarantees that $\lim_{n_j\to\infty} \left\|x_{n_j} - T_i x_{n_j}\right\| = 0$ for all $i = 1, 2, \ldots, N$ and so $\|x^* - T_i x^*\| = 0$ for all $i = 1, 2, \ldots, N$. This implies that $x^* \in F = \cap_{i=1}^N F(T_i)$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, it follows, as in the proof of Theorem 2.1, that $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N\}$. This completes the proof. $\square$

Remark 2.1. Theorem 2.1 extends Theorem 2 and 3 of Rhoades [**13**], Theorem 1.5 of Schu [**14**], Khan and Fukhar-ud-din [**6**], Khan and Takahashi [**7**] to the case of finite family of more general class of nonexpansive and asymptotically nonexpansive mappings and multi-step iteration scheme with errors considered here and no boundedness condition imposed on K.

Remark 2.2. Theorem 2.1 also extends the corresponding results of Xu and Noor [**20**] to the case of finite family of more general class of asymptotically nonexpansive mappings and multi-step iteration scheme with errors considered here and no boundedness condition imposed on K.

Remark 2.3. Theorem 2.1 also extends the corresponding result of Shahzad and Udomene [**17**] to the case of finite family of uniformly equi-continuous asymptotically quasi-nonexpansive mappings and multi-step iteration scheme with errors considered in this paper.

Remark 2.4. Theorem 2.1 also extends the corresponding result of Cho et al. [**1**] to the case of finite family of more general class of asymptotically nonexpansive mappings and multi-step iteration scheme with errors considered in this paper.

Remark 2.5. Theorem 2.2 extends Theorem 2 of Osilike and Aniagbosor [**11**] and Theorem 2.2 of Schu [**15**] to the case of finite family of more general class of asymptotically nonexpansive mappings and multi-step iteration scheme with errors considered here and no boundedness condition imposed on K.

Remark 2.6. Theorem 2.1 also extends Theorem 2.3 of Qin et al. [**12**] to the case of finite family of mappings and multi-step iteration scheme considered in this paper.

Example 2.1. Let $E = [-\pi, \pi]$ and let T be defined by

$$Tx = x\, cos x$$

for each $x \in E$. Clearly $F(T) = \{0\}$. T is a quasi-nonexpansive mapping since if $x \in E$ and $z = 0$, then

$$|Tx - z| = |Tx - 0| = |x||cos\, x| \leqslant |x| = |x - z|,$$

and hence T is asymptotically quasi-nonexpansive mapping with constant sequence $\{1\}$. But it is not a nonexpansive mapping and hence asymptotically nonexpansive mapping. In fact, if we take $x = \frac{\pi}{2}$ and $y = \pi$, then

$$|Tx - Ty| = |\frac{\pi}{2}\, cos\, \frac{\pi}{2} - \pi\, cos\, \pi| = \pi,$$

whereas

$$|x - y| = |\frac{\pi}{2} - \pi| = \frac{\pi}{2}.$$

Example 2.2. Let $E = \mathbb{R}$ and let T be defined by

$$T(x) = \begin{cases} \frac{x}{2}\, sin\, \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

If $x \neq 0$ and $Tx = x$, then $x = \frac{x}{2}\, sin\, \frac{1}{x}$. Thus $2 = sin\, \frac{1}{x}$ which is impossible. T is a quasi-nonexpansive mapping since if $x \in E$ and $z = 0$, then

$$|Tx - z| = |Tx - 0| = |\frac{x}{2}||sin\, \frac{1}{x}| \leqslant \frac{|x|}{2} < |x| = |x - z|,$$

and hence T is asymptotically quasi-nonexpansive mapping with constant sequence $\{1\}$. But it is not a nonexpansive mapping and hence asymptotically nonexpansive mapping. In fact, if we take $x = \frac{2}{9\pi}$ and $y = \frac{3}{19\pi}$, then

$$|Tx - Ty| \leqslant |x - y|$$

is not satisfied.

3. Conclusion

The class of mappings considered in this paper is more general than the class of nonexpansive, quasi-nonexpansive and asymptotically nonexpansive mappings. Hence the results presented in this paper are good improvement and generalization of many known results from the literature (see e.g. [**1**,**6**,**7**,**11**–**15**,**17**,**20**]) and many others.

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