JOURNAL OF INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN 1840-4359
Vol. 2(2012), 39-66

Former
BULLETIN OF SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

GALOIS–TYPE CONNECTIONS AND CONTINUITIES OF PAIRS OF RELATIONS

Árpád Száz

Communicated by Siniša Crvenković

Abstract. In a former paper, by using the ideas of J. Schmidt and G. Pataki, we have introduced two kinds of Galois–type connections.

Now, we shall show that these are very particular cases of our upper and mild continuities of pairs of relations on one relator space to another.

Introduction

In [45], by using the ideas of Schmidt [27] and Pataki [22], we have introduced the following two kinds of Galois–type connections.

A function f of one partially ordered set X to another Y is called

(a) increasingly g–normal, for some function g of Y to X, if for any $x \in X$ and $y \in Y$ we have

$$f(x) \leqslant y \quad \iff \quad x \leqslant g(y);$$

(b) increasingly φ–regular, for some function φ of X to itself, if for any $x_1, x_2 \in X$ we have

$$x_1 \leqslant \varphi(x_2) \quad \iff \quad f(x_1) \leqslant f(x_2).$$

2010 *Mathematics Subject Classification.* Primary 06A06, 54E15; Secondary 06A15, 54C60.

Key words and phrases. Partially ordered sets, closure operations, Galois connections (residuated mappings), relator (generalized uniform) spaces, closure operations on relators, upper and mild continuities of pairs of relations.

The work of the author has been supported by the Hungarian Scientific Research Fund (OTKA) Grant NK-81402.

While, in [37] , by using the ideas of [26] and [30] , we have introduced the following two kinds of continuities of pairs of relations between relator (generalized uniform) spaces.

A pair of relations (F, G) on one relator space $(X, Z)(\mathcal{R})$ to another $(Y, W)(\mathcal{S})$ is called

(A) upper $\Box$–continuous, for some operation $\Box$ on relators, if

$$\big(\mathcal{S}^{\Box} \circ F \big)^{\Box} \subset \big(G \circ \mathcal{R}^{\Box} \big)^{\Box} ;$$

(B) mildly $\Box$–continuous, for some operation $\Box$ on relators, if

$$\big(G^{-1} \circ \mathcal{S}^{\Box} \circ F \big)^{\Box} \subset \mathcal{R}^{\Box} .$$

Now, we shall show that properties (a) and (b) are very particular cases of (A) and (B) . Moreover, we shall show that two basic theorems on the relationships between (a) and (b) can be naturally extended to those on the relationships between (A) and (B).

Property (b) is actually a particular case of (a). However, it is frequently more important than the latter one. In particular, it allows of a unified derivation of the various closure operations on relators. Therefore, the study of (A) and (B) has to be preceded by that of (b) .

1. A few basic facts on relations and relators

A subset F of a product set $X \times Y$ is called a relation on X to Y. If in particular $F \subset X^2$, then we may simply say that F is a relation on X. Thus, $\Delta_X = \{ (x, x) : \ x \in X \}$ is a relation on X.

If F is a relation on X to Y, then for any $x \in X$ the set $F(x) = \{ y \in Y : (x, y) \in F \}$ is called the image of x under F. And the set $D_F = \{ x \in X : F(x) \neq \emptyset \}$ is called the domain of F.

In particular, a relation F on X to Y is called a function if for each $x \in D_F$ there exists $y \in Y$ such that $F(x) = \{ y \}$. In this case, by identifying singletons with their elements, we may usually write $F(x) = y$ in place of $F(x) = \{ y \}$.

More generally, if F is a relation on X to Y, then for any $A \subset X$ the set $F[A] = \bigcup_{x \in A} F(x)$ is called the image of A under F. And the set $R_F = F[D_F]$ is called the range of F.

If F is a relation on X to Y such that $D_F = X$, then we say that F is a relation of X to Y. While, if F is a relation on X to Y such that $R_F = Y$, then we say that F is a relation on X onto Y.

If F is a relation on X to Y, then the values $F(x)$, where $x \in X$, uniquely determine F since we have $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the inverse F^{-1} can be defined such that $F^{-1}(y) = \big\{ x \in X : \ y \in F(x) \big\}$ for all $y \in Y$.

Moreover, if in addition G is a relation on Y to Z, then the composition $G \circ F$ can be defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$. Thus, $(G \circ F)[A] = G[F[A]]$ for all $A \subset X$, and $(G \circ F)^{-1} = F^{-1} \circ G^{-1}$.

A relation R on X is called reflexive, symmetric, and transitive if $\Delta_X \subset R$, $R^{-1} \subset R$, and $R \circ R \subset R$, respectively. Moreover, a symmetric (resp. transitive) reflexive relation R on X, is called a tolerance (resp. preorder) relation on X.

Furthermore, a relation R on X is called antisymmetric and directive if $R \cap R^{-1} \subset \Delta_X$ and $X^2 \subset R^{-1} \circ R$, respectively. Moreover, a directive (resp. antisymmetric) preorder R on X is called a direction (resp. partial order) on X.

Note that if d is a metric on X and $r > 0$, then the surrounding $B_r^d = \{ (x, y) : \ d(x, y) < r \}$ is, in general, only a tolerance on X. While, if $A \subset X$, then the Pervin relation $R_A = A^2 \cup A^c \times X$ is, in general, only a preorder on X.

If R is a relation on X, then we write $R^n = R \circ R^{n-1}$ for all $n \in \mathbb{N}$ by agreeing that $R^0 = \Delta_X$. Moreover, we also write $R^\infty = \bigcup_{n=0}^{\infty} R^n$. Thus, R^∞ is the smallest preorder on X such that $R \subset R^\infty$. Therefore, $R^{\infty\infty} = R^\infty$.

A family $\mathcal{R}$ of relations on one nonvoid set X to another Y is called a relator on X to Y. Moreover, the ordered pair $(X, Y)(\mathcal{R}) = \big((X, Y), \mathcal{R}\big)$ is called a relator space. (For the origins, see [30] and the references therein.)

If in particular $\mathcal{R}$ is a relator on X to itself, then we may simply say that $\mathcal{R}$ is a relator on X. Moreover, by identifying singletons with their element, we may naturally write $X(\mathcal{R})$ in place of $(X, X)(\mathcal{R})$. Namely, $(X, X) = \{\{X\}\}$.

A relator $\mathcal{R}$ on X, or a relator space $X(\mathcal{R})$, may be naturally called reflexive, symmetric, and transitive if each member of $\mathcal{R}$ has the corresponding property. Thus, we may also naturally speak of tolerance and preorder relators.

Note that if $\mathcal{D}$ is a family of pseudo-metrics on X, then $\mathcal{R}_{\mathcal{D}} = \{ B_r^d : \ r > 0, d \in \mathcal{D} \}$ is a tolerance relator on X. While, if $\mathcal{A}$ is a family of subsets of X, then $\mathcal{R}_{\mathcal{A}} = \big\{ R_A : \ A \in \mathcal{A} \big\}$ is a preorder relator on X.

A relator $\mathcal{R}$ on X to Y may be called simple if there exists a relation R on X to Y such that $\mathcal{R} = \{R\}$. In this case, by identifying singletons with their elements, we may simply write $(X, Y)(R)$ in place of $(X, Y)(\{R\})$.

If in particular $\leqslant$ is a relation on X, then according to [41] the simple relator space $X(\leqslant)$ will be called a generalized ordered set or an ordered set without axioms. And we may usually write X in place of $X(\leqslant)$.

By our former conventions, a generalized ordered set X may be naturally called preordered, directed, resp. partially ordered if the inequality relation $\leqslant$ in it is has the corresponding properties.

Having in mind the terminology of Birkhoff [1, p. 2], a partially ordered set may called a poset, a preordered set may be called a proset, and a generalized ordered set may be called a goset.

The usual operations on relations can be naturally extended to relators. If $\mathcal{R}$ is a relator on X to Y, then we may naturally write $\mathcal{R}^{-1} = \big\{ R^{-1} : \ \ R \in \mathcal{R} \big\}$. Thus, $(Y, X)(\mathcal{R}^{-1})$ may be called the dual of $(X, Y)(\mathcal{R})$.

Moreover, if in addition $\mathcal{S}$ is a relator on Y to Z, then we may also naturally define $\mathcal{S} \circ \mathcal{R} = \{ S \circ R : \ \ R \in \mathcal{R}, \ \ S \in \mathcal{S} \big\}$. Now, by identifying singletons with their elements, we may naturally write $S \circ \mathcal{R}$ in place of $\{S\} \circ \mathcal{R}$ for all $S \in \mathcal{S}$.

2. Topological structures derived from relators

Relator spaces are natural generalizations of not only ordered sets, formal contexts [10 , p. 17] , and uniform spaces, but also some other classical algebraic and topological structures.

For instance, all reasonable generalizations of proximities, closures, topologies, filters, and convergences can be easily derived from relators with the help of the following definitions. (See [32] .)

If $\mathcal{R}$ is a relator on X to Y, then for any $A \subset X$, $B \subset Y$ and $x \in X$ we write :

(1) $A \in \operatorname{Int}_{\mathcal{R}}(B)$ if $R[A] \subset B$ for some $R \in \mathcal{R}$;

(2) $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ if $R[A] \cap B \neq \emptyset$ for all $R \in \mathcal{R}$;

(3) $x \in \operatorname{int}_{\mathcal{R}}(B)$ if $\{x\} \in \operatorname{Int}_{\mathcal{R}}(B)$; (4) $x \in \operatorname{cl}_{\mathcal{R}}(B)$ if $\{x\} \in \operatorname{Cl}_{\mathcal{R}}(B)$;

(5) $B \in \mathcal{E}_{\mathcal{R}}$ if $\operatorname{int}_{\mathcal{R}}(B) \neq \emptyset$; (6) $B \in \mathcal{D}_{\mathcal{R}}$ if $\operatorname{cl}_{\mathcal{R}}(B) = X$.

Moreover, if in particular $\mathcal{R}$ is a relator on X, then for any $A \subset X$ we also write :

(7) $A \in \tau_{\mathcal{R}}$ if $A \in \operatorname{Int}_{\mathcal{R}}(A)$; (8) $A \in \bar{\tau}_{\mathcal{R}}$ if $A^c \notin \operatorname{Cl}_{\mathcal{R}}(A)$;

(9) $A \in \mathcal{T}_{\mathcal{R}}$ if $A \subset \operatorname{int}_{\mathcal{R}}(A)$; (10) $A \in \mathcal{F}_{\mathcal{R}}$ if $\operatorname{cl}_{\mathcal{R}}(A) \subset A$.

The relations $\operatorname{Int}_{\mathcal{R}}$ and $\operatorname{int}_{\mathcal{R}}$ are called the proximal and the topological interiors induced by the relator $\mathcal{R}$. While, the members of the families $\tau_{\mathcal{R}}$, $\mathcal{T}_{\mathcal{R}}$, and $\mathcal{E}_{\mathcal{R}}$ are called the proximally open, the topologically open, and the fat subsets of the relator spaces $X(\mathcal{R})$ and $(X, Y)(\mathcal{R})$, respectively.

The fat sets are frequently more important tools than the open sets. For instance, if $\leqslant$ is a certain order relation on X, then $\mathcal{T}_{\leqslant}$ and $\mathcal{E}_{\leqslant}$ are just the families of all ascending and residual subsets of the ordered set $X(\leqslant)$, respectively. Moreover, it may occur that $\mathcal{T}_R = \{\emptyset, X\}$, but $\mathcal{E}_R \neq \{X\}$ for some relation R on X .

A function φ of a preordered set Γ to X is called a Γ-net in X . The net φ is said to be eventually (frequently) in a subset A of X if $\varphi^{-1}[A]$ is a fat (dense) subset of Γ. Therefore, Γ could be here an arbitrary relator space. However, preordered nets are usually sufficient.

If $\mathcal{R}$ is a relator on X to Y, then for any Γ-nets φ in X and ψ in Y, and $x \in X$, we write :

(11) $\varphi \in \operatorname{Lim}_{\mathcal{R}}(\psi)$ if the net (φ, ψ) is eventually in each $R \in \mathcal{R}$;

(12) $\varphi \in \operatorname{Adh}_{\mathcal{R}}(\psi)$ if the net (φ, ψ) is frequently in each $R \in \mathcal{R}$;

(13) $x \in \operatorname{lim}_{\mathcal{R}}(\psi)$ if $x_{\Gamma} \in \operatorname{Lim}_{\mathcal{R}}(\psi)$; (14) $x \in \operatorname{adh}_{\mathcal{R}}(\psi)$ if $x_{\Gamma} \in \operatorname{Adh}_{\mathcal{R}}(\psi)$;

where x_{Γ} means now the constant net $(x)_{\gamma \in \Gamma} = \Gamma \times \{x\}$.

To let the reader feel the importance of the fat sets, it is also worth noticing that if $\mathcal{R}$ is a relator on X, then for any $A \subset X$ we may naturally write:

(15) $A \in \mathcal{N}_{\mathcal{R}}$ if $\operatorname{cl}_{\mathcal{R}}(A) \notin \mathcal{E}_{\mathcal{R}}$;

(16) $A \in \mathcal{M}_{\mathcal{R}}$ if $A = \bigcup_{n=1}^{\infty} A_n$ for some sequence $(A_n)_{n=1}^{\infty}$ in $\mathcal{N}_{\mathcal{R}}$.

Thus, the members of the families $\mathcal{N}_{\mathcal{R}}$ and $\mathcal{M}_{\mathcal{R}}$ may be called the rare and meager subsets of the relator space $X(\mathcal{R})$. And the relator $\mathcal{R}$ may be called a Baire relator if $\mathcal{E}_{\mathcal{R}} \cap \mathcal{M}_{\mathcal{R}} = \emptyset$. That is, the fat sets are not meager.

Moreover, it is also worth mentioning that if $\mathcal{R}$ is a relator on X to Y, then beside the relations

(17) $\delta_{\mathcal{R}} = \bigcap \mathcal{R}$; (18) $\sigma_{\mathcal{R}} = \bigcup \mathcal{R}$;

sometimes we also need the sets

(19) $E_{\mathcal{R}} = \bigcap \mathcal{E}_{\mathcal{R}}$; (20) $D_{\mathcal{R}} = \bigcup \big(\mathcal{P}(Y) \setminus \mathcal{D}_{\mathcal{R}} \big)$.

Finally, we note that in an arbitrary relator space $(X, Y)(\mathcal{R})$ we may also naturally introduce some order theoretic notions which are however not completely independent of the above topological ones [38].

For instance, if $\mathcal{R}$ is a relator on X to Y, then for any $A \subset X$ and $B \subset Y$ we may naturally write $A \in \operatorname{Lb}_{\mathcal{R}}(B)$ if $A \times B \subset R$ for some $R \in \mathcal{R}$. Thus, we have $\operatorname{Lb}_{\mathcal{R}} = \big(\operatorname{Cl}_{\mathcal{R}^c} \big)^c$, where $\mathcal{R}^c$ means now the elementwise complement of $\mathcal{R}$.

3. Closure operations and regular structures

Definition 3.1. A function φ of a poset X to itself is called an operation on X. More generally, a function f of X to another poset Y is called a structure on X.

Remark 3.1. The latter terminology has been mainly motivated by the various structures derived from relators.

Note that if X is a nonvoid set, then the mapping $\mathcal{R} \mapsto \mathcal{T}_{\mathcal{R}}$, where $\mathcal{R}$ is a relator on X, is a structure on $\mathcal{P}\big(\mathcal{P}(X^2)\big)$. (For any two sets A and B, we write $A \leqslant B$ if $A \in \mathcal{P}(B)$, i.e., $A \subset B$.)

Definition 3.2. An operation φ on a poset X is called

(1) expansive if $\Delta_X \leqslant \varphi$; (2) quasi-idempotent if $\varphi^2 \leqslant \varphi$.

Moreover, a structure f on X is called increasing if for any $x_1, x_2 \in X$, with $x_1 \leqslant x_2$, we have $f(x_1) \leqslant f(x_2)$.

Remark 3.2. Note that if (1) holds, then we also have $\varphi = \Delta_X \circ \varphi \leqslant \varphi \circ \varphi = \varphi^2$. Therefore, if both (1) and (2) hold, then φ is actually idempotent.

Thus, according to [1, p. 111], we may naturally have the following

Definition 3.3. An increasing and expansive operation is called a precosure operation. And a quasi–idempotent preclosure operation is called a closure operation.

Moreover, an expansive and quasi-idempotent operation is called a semiclosure operation. And an increasing and idempotent operation is called a modification operation.

Remark 3.3. Now, an operation φ on X may be naturally called an interior operation if it is a closure operation on the dual X^* of X.

To feel the importance of modification operations, note that if X is a nonvoid set, then the mapping $\mathcal{R} \mapsto \mathcal{R}^\infty$, where $\mathcal{R}$ is a relator on X, is only a modification operation on $\mathcal{P}\left(\mathcal{P}\left(X^2\right)\right)$.

In [45], having in mind the ideas of [22], we have introduced the following

Definition 3.4. A structure f on a poset X to another Y is called increasingly φ–regular, for some operation φ on X, if for any $x_1, x_2 \in X$ we have

$$x_1 \leqslant \varphi(x_2) \iff f(x_1) \leqslant f(x_2).$$

Remark 3.4. Now, a structure f on X to Y may be naturally called decreasingly φ–regular if it is an increasingly φ–regular structure on X to Y^*.

The above definition closely resembles to a recent definition of Galois connections [3, p. 155]. However, instead of Galois connections, it has been more convenient to use residuated mappings [2, p. 11] in the following relevant form.

Definition 3.5. A structure f on a poset X to another Y is called increasingly g–normal, for some structure g on Y to X, if for any $x \in X$ and $y \in Y$ we have

$$f(x) \leqslant y \iff x \leqslant g(y).$$

Remark 3.5. Now, a structure f on X to Y may be naturally decreasingly g–normal if it is an increasingly g–normal structure on X to Y^*.

In [45], by using the above definitions for prosets instead of posets, we have proved some straightforward extensions of the following theorems.

Theorem 3.1. *If f is an increasingly g–normal structure on X to Y and $\varphi = g \circ f$, then f is increasingly φ–regular.*

Theorem 3.2. *If f is an increasingly φ–regular structure on X onto Y and g is a structure on Y to X such $\varphi = g \circ f$, then f is increasingly g–normal.*

Theorem 3.3. *f is an increasingly g–normal structure on X to Y if and only if g is an increasingly f–normal structure on Y^* to X^*.*

Theorem 3.4. *If f is an increasingly φ–regular structure on X, then*

(1) φ is expansive; *(2) f is increasing;* *(3) $f = f \circ \varphi$.*

THEOREM 3.5. *If* φ *is an operation on* X, *then the following assertions are equivalent:*

(1) φ *is a closure operation;*

(2) φ *is increasingly* φ*–regular;*

(3) *there exists an increasingly* φ*–regular structure* f *on* X.

COROLLARY 3.1. *If* f *is a structure and* φ *is an operation on* X, *then* f *is increasingly* φ*–regular if and only if* φ *is a closure operation and for any* $x_1, x_2 \in X$ *we have* $\varphi(x_1) \leqslant \varphi(x_2)$ *if and only if* $f(x_1) \leqslant f(x_2)$.

THEOREM 3.6. *If* f *is an increasingly* g*–normal structure on* X *to* Y, *then* f *and* g *are increasing. Moreover,* $\varphi = g \circ f$ *is a closure operation on* X *and* $\psi = f \circ g$ *is an interior operation on* Y.

For an easy illustration of the above notions and results, we can note here

EXAMPLE 3.1. Let $\mathcal{P} = \mathcal{P}(X)$ be the poset of all subsets of a poset X. Moreover, define $F(A) = \mathrm{ub}(A)$ and $G(A) = \mathrm{lb}(A)$ for all $A \subset X$.

Then, by the corresponding definitions, it is clear that F is a decreasingly G–normal structure on $\mathcal{P}$ to itself. Hence, by defining $\Phi = G \circ F$ and using the duals of Theorems 3.4 and 3.2, we can easily see that F is a decreasingly Φ–regular structure and Φ is a closure operation on $\mathcal{P}$.

To feel the importance of this example, note that by [3, p. 166] the poset $\Phi[\mathcal{P}]$ is just the Dedekind–MacNeille completion of X by the cuts $\Phi(A)$.

4. Characterizations of increasingly regular structures

In [45], we have also proved some straightforward extensions of the following theorems.

THEOREM 4.1. *For any two structures* f *on* X *to* Y *and* g *on* Y *to* X, *the following assertions are equivalent:*

(1) f *is increasingly* g*–normal;*

(2) f *is increasing and* $f^{-1}[\mathrm{lb}(y)] = \mathrm{lb}\big(g(y)\big)$ *for all* $y \in Y$;

(3) f *is increasing and, for each* $y \in Y$, $g(y)$ *is the largest element of* X *such that* $f\big(g(y)\big) \leqslant y$.

COROLLARY 4.1. *For any structure* f *on* X *to* Y, *there exists at most one structure* g *on* Y *to* X *such that* f *is increasingly* g*–normal.*

THEOREM 4.2. *If* φ *is an operation and* f *is a structure on* X, *then the following assertions are equivalent:*

(1) f *is increasingly* φ*–regular;*

(2) f *is increasing and* $f^{-1}\big[\mathrm{lb}\big(f(x)\big)\big] = \mathrm{lb}\big(\varphi(x)\big)$ *for all* $x \in X$;

(3) f *is increasing and, for each* $x \in X$, $\varphi(x)$ *is the largest element of* X *such that* $f\big(\varphi(x)\big) \leqslant f\big(x\big)$.

Remark 4.1. Note that if (1) holds, then by Theorem 3.4 we also have $f\big(\varphi(x)\big) = f\big(x\big)$ for all $x \in X$.

Moreover, if (1) holds, then by identifying singletons with their elements we simply have $f^{-1} \circ \mathrm{lb} \circ f = \mathrm{lb} \circ \varphi$ by (2).

Corollary 4.2. *For any structure* f *on* X, *there exists at most one operation* φ *on* X *such that* f *is increasingly* φ*–regular.*

Definition 4.1. A structure f on X is called increasingly regular if there exists an operation φ on X such that f is increasingly φ–regular.

Moreover, a structure f on X to Y is called increasingly normal if there exists a structure g on Y to X such that f is increasingly g–normal.

Remark 4.2. By Theorem 3.1, an increasingly normal structure is in particular increasingly regular.

Moreover, in [45] and [44], we have proved some straightforward extensions of the following theorems.

Theorem 4.3. *If* f *is an increasingly regular structure on* X *onto* Y, *then* f *is increasingly normal.*

Theorem 4.4. *If* f *is a structure on a complete poset* X *to* Y, *then the following assertions are equivalent:*

(1) f *is increasingly normal;*

(2) $f[\sup(A)] = \sup\big(f[A]\big)$ *for all* $A \subset X$.

Theorem 4.5. *If* f *is a structure on a complete poset* X *onto* Y, *then the following assertions are equivalent:*

(1) f *is increasingly regular;*

(2) $f[\sup(A)] = \sup\big(f[A]\big)$ *for all* $A \subset X$.

Theorem 4.6. *If* φ *is a closure operation on a complete poset* X, *then*

$$\varphi\big[\sup(A)\big] = \varphi\big(\sup\big(\varphi[A]\big)\big)$$

for all $A \subset X$.

Remark 4.3. This theorem can also be easily proved directly, without using Theorems 3.5 and 4.5.

5. Increasingly regular structures on power sets

In the sequel, we shall assume that X is only a nonvoid set and, by identifying singletons with their elements, we shall consider X to be a subset of $\mathcal{P}(X)$.

Definition 5.1. If F is an increasing structure on $\mathcal{P}(X)$ to Y, then for any $y \in Y$ we define

$$G_F(y) = \{ x \in X : \quad F(x) \leqslant y \}.$$

Moreover, according to Theorem 3.1, we also define $\Phi_F = G_F \circ F$.

Remark 5.1. Note that if F is a decreasing structure on $\mathcal{P}(X)$ to Y, then $G_F(y)$ has to be defined by the reverse inequality.

In [43], we have proved some straightforward extensions of the following theorems.

Theorem 5.1. *If F is an increasing structure on $\mathcal{P}(X)$ to Y, then*

(1) G_F is an increasing structure on Y to $\mathcal{P}(X)$;

(2) $F(A) \leqslant y$ implies $A \subset G_F(y)$ for all $A \subset X$ and $y \in Y$.

Theorem 5.2. *If F is an increasing structure on $\mathcal{P}(X)$ to Y, then*

(1) Φ_F is a preclosure operation on $\mathcal{P}(X)$;

(2) $F(A_1) \leqslant F(A_2)$ implies $A_1 \subset \Phi_F(A_2)$ for all $A_1, A_2 \subset X$.

Theorem 5.3. *If F is a structure on $\mathcal{P}(X)$ to Y, then the following assertions are equivalent:*

(1) F is increasingly normal;

(2) F is increasingly G_F–normal;

(3) F is increasing and $F \circ G_F \leqslant \Delta_Y$;

(4) F is increasing and $A \subset G_F(y)$ implies $F(A) \leqslant y$ for all $A \subset X$ and $y \in Y$.

Theorem 5.4. *If F is a structure on $\mathcal{P}(X)$, then the following assertions are equivalent:*

(1) F is increasingly regular;

(2) F is increasingly Φ_F–regular;

(3) F is increasing and $F \circ \Phi_F \leqslant F$;

(4) F is increasing and $A_1 \subset \Phi_F(A_2)$ implies $F(A_1) \leqslant F(A_2)$ for all $A_1, A_2 \subset X$.

Remark 5.2. Note that if (2) holds, then by Theorem 3.4 we also have $F \circ \Phi_F = F$.

Theorem 5.5. *If F is an increasingly regular structure on $\mathcal{P}(X)$, then Φ_F is a closure operation on $\mathcal{P}(X)$ and for any $A_1, A_2 \subset X$ we have*

$$A_1 \leqslant \Phi_F(A_2) \iff F(A_1) \leqslant F(A_2) \iff \Phi_F(A_1) \leqslant \Phi_F(A_2).$$

Theorem 5.6. *If F is a structure on $\mathcal{P}(X)$ to Y, then the following assertions are equivalent:*

(1) F is increasingly normal;

(2) $F(A) = \sup\big(F[A]\big)$ for all $A \subset X$;

(3) F is increasing and $F\big(G_F(y)\big) \leqslant \sup\big(F[G_F(y)]\big)$ for all $y \in Y$.

Theorem 5.7. *If F is a structure on $\mathcal{P}(X)$ onto Y, then the following assertions are equivalent:*

(1) F is increasingly regular;

(2) $F(A) = \sup\big(F[A]\big)$ for all $A \subset X$;

(3) F is increasing and $F\big(\Phi_F(A)\big) \leqslant \sup\big(F[\Phi_F(A)]\big)$ for all $A \subset X$.

Remark 5.3. Note that if in particular F is a structure on $\mathcal{P}(X)$ to $\mathcal{P}(Y)$, then we simply have $\sup\big(F[A]\big) = \bigcup F[A]$ for all $A \subset X$.

6. Normality of the most important topological structures

Throughout in the sequel, we shall assume that X and Y are nonvoid sets.

Theorem 6.1. *The mappings, defined by*

$$\mathcal{R} \mapsto \mathrm{Int}_{\mathcal{R}}, \qquad \mathcal{R} \mapsto \mathrm{int}_{\mathcal{R}} \qquad \text{and} \qquad \mathcal{R} \mapsto \mathcal{E}_{\mathcal{R}}$$

for any relator $\mathcal{R}$ on X to Y, are increasingly normal structures on $\mathcal{P}\big(\mathcal{P}(X\times Y)\big)$ to $\mathcal{P}\big(\mathcal{P}(Y)\times\mathcal{P}(X)\big)$, $\mathcal{P}\big(\mathcal{P}(Y)\times X\big)$ and $\mathcal{P}\big(\mathcal{P}(Y)\big)$, respectively.

Hint. To prove the first statement, for any relator $\mathcal{R}$ on X to Y, define

$$F(\mathcal{R}) = \mathrm{Int}_{\mathcal{R}}.$$

Moreover, by using the corresponding definitions, note that

$$F(\mathcal{R}) = \bigcup_{R \in \mathcal{R}} F(R) = \bigcup F[\mathcal{R}].$$

Therefore, by Theorem 5.6, F is an increasingly normal structure on $\mathcal{P}\big(\mathcal{P}(X\times Y)\big)$ to $\mathcal{P}\big(\mathcal{P}(Y)\times\mathcal{P}(X)\big)$. □

Remark 6.1. In this respect, it is worth noticing that the mapping defined by $R \mapsto \mathrm{Int}_R$ for any relation R on X to Y is a decreasingly normal structure on $\mathcal{P}(X\times Y)$ to $\mathcal{P}\big(\mathcal{P}(Y)\times\mathcal{P}(X)\big)$.

However, it is now more interesting that in particular we also have the following

THEOREM 6.2. *The mappings, defined by*

$$\mathcal{R} \mapsto \tau_{\mathcal{R}} \qquad \textit{and} \qquad \mathcal{R} \mapsto \overline{\tau}_{\mathcal{R}}$$

for any relator $\mathcal{R}$ *on* X, *are increasingly normal structures on* $\mathcal{P}\big(\mathcal{P}(X^2)\big)$ *to* $\mathcal{P}\big(\mathcal{P}(X)\big)$.

REMARK 6.2. Later, we shall see that the increasing structures, defined by

$$\mathcal{R} \mapsto \mathcal{T}_{\mathcal{R}} \qquad \text{and} \qquad \mathcal{R} \mapsto \mathcal{F}_{\mathcal{R}}$$

for any relator $\mathcal{R}$ on X, are not, in general, even increasingly regular.

In addition to Theorem 6.1, we can also easily establish the following

THEOREM 6.3. *The mappings, defined by*

$$\mathcal{R} \mapsto \mathrm{Cl}_{\mathcal{R}}, \qquad \mathcal{R} \mapsto \mathrm{cl}_{\mathcal{R}} \qquad \textit{and} \qquad \mathcal{R} \mapsto \mathcal{D}_{\mathcal{R}}$$

for any relator $\mathcal{R}$ *on* X *to* Y, *are decreasingly normal structures on* $\mathcal{P}\big(\mathcal{P}(X\times Y)\big)$ *to* $\mathcal{P}\big(\mathcal{P}(Y)\times\mathcal{P}(X)\big)$, $\mathcal{P}\big(\mathcal{P}(Y)\times X\big)$ *and* $\mathcal{P}\big(\mathcal{P}(Y)\big)$, *respectively.*

REMARK 6.3. Unfortunately, the aggregate $\mathcal{N}(X)$ of all nets in X is not a well-defined collection.

Therefore, we can only somewhat incorrectly state here the following

THEOREM 6.4. *The mappings, defined by*

$$\mathcal{R} \mapsto \mathrm{Lim}_{\mathcal{R}}, \qquad \mathcal{R} \mapsto \mathrm{Adh}_{\mathcal{R}}, \qquad \mathcal{R} \mapsto \mathrm{lim}_{\mathcal{R}} \qquad \textit{and} \qquad \mathcal{R} \mapsto \mathrm{adh}_{\mathcal{R}}$$

for any relator $\mathcal{R}$ *on* X *to* Y, *are decreasingly normal structures on* $\mathcal{P}\big(\mathcal{P}(X\times Y)\big)$ *to* $\mathcal{P}\big(\mathcal{N}(Y)\times\mathcal{N}(X)\big)$ *and* $\mathcal{P}\big(\mathcal{N}(Y)\times X\big)$, *respectively.*

REMARK 6.4. This theorem could only be made precise by imposing some inconvenient restrictions on the domains of the corresponding nets.

In addition to the above theorems, it is also worth mentioning the following

THEOREM 6.5. *The modification operation, defined by* $\mathcal{R} \mapsto \mathcal{R}^{\infty}$ *for any relator* $\mathcal{R}$ *on* X, *is an increasingly normal structure on* $\mathcal{P}\big(\mathcal{P}(X^2)\big)$ *to itself.*

REMARK 6.5. In this respect, it is worth noticing that the closure operation, defined by $R \mapsto R^{\infty}$ for any relation on R on X, is, in general, only an increasingly regular structure on $\mathcal{P}\big(\mathcal{P}(X^2)\big)$ to itself.

Actually, it is the unique operation on $\mathcal{P}(X^2)$ for which the mapping, defined by $R \mapsto \mathcal{T}_R$ for any relation R on X, is a decreasingly regular structure on $\mathcal{P}\big(\mathcal{P}(X^2)\big)$ to $\mathcal{P}\big(\mathcal{P}(X)\big)$. (This has also been proved in [43].)

DEFINITION 6.1. For any relator $\mathcal{R}$ on X, we define

$$\mathcal{R}^{\partial} = \big\{ S \subset X^2 : \quad S^{\infty} \in \mathcal{R} \big\}.$$

Remark 6.6. By using this definition and the idempotency of ∞, we can easily see that ∂ is also a modification operation on $\mathcal{P}\big(\mathcal{P}(X^2)\big)$.

Thus, for any relator $\mathcal{R}$ on X, the relators $\mathcal{R}^{\infty}$ and $\mathcal{R}^{\partial}$ may be naturally called the direct and the inverse preorder modifications of $\mathcal{R}$.

However, it is now more interesting that we also have the following

Theorem 6.6. *∂ is the unique operation on $\mathcal{P}\big(\mathcal{P}(X^2)\big)$ such that the operation ∞ considered in Theorem 6.5 is an increasingly ∂–normal structure on $\mathcal{P}\big(\mathcal{P}(X^2)\big)$ to itself.*

Hint. To check this, by Theorems 6.5 and 5.3 and Corollary 4.1, it is enough to note only that if $F(\mathcal{R})=\mathcal{R}^{\infty}$ for any relator $\mathcal{R}$ on X, then according to Definition 5.1 we have $G_F(\mathcal{R})=\mathcal{R}^{\partial}$ for any relator $\mathcal{R}$ on X. □

Remark 6.7. From the above theorem, by Theorem 3.6, we can see that $\infty\,\partial$ is a closure and $\partial\infty$ is an interior operation on $\mathcal{P}\big(\mathcal{P}(X^2)\big)$.

Therefore, in particular, we have $\mathcal{R}^{\partial\,\infty}\subset\mathcal{R}\subset\mathcal{R}^{\infty\,\partial}$, and hence also $\mathcal{R}^{\infty}=\mathcal{R}^{\infty\,\partial\,\infty}$ and $\mathcal{R}^{\partial}=\mathcal{R}^{\partial\,\infty\,\partial}$ for any relator $\mathcal{R}$ on X.

7. The most important closure operations on relators

Definition 7.1. For any relator $\mathcal{R}$ on X to Y, we define

$$\begin{aligned}
\mathcal{R}^{*} &= \big\{ S\subset X\times Y: \quad \exists\ R\in\mathcal{R}: \quad R\subset S\big\},\\
\mathcal{R}^{\#} &= \big\{ S\subset X\times Y: \quad \forall\ A\subset X: \quad A\in \operatorname{Int}_{\mathcal{R}}\big(S(A)\big)\big\},\\
\mathcal{R}^{\wedge} &= \big\{ S\subset X\times Y: \quad \forall\ x\in X: \quad x\in \operatorname{int}_{\mathcal{R}}\big(S(x)\big)\big\},\\
\mathcal{R}^{\vartriangle} &= \big\{ S\subset X\times Y: \quad \forall\ x\in X: \quad S(x)\in\mathcal{E}_{\mathcal{R}}\big\}.
\end{aligned}$$

Remark 7.1. Hence, by the corresponding definitions, it is clear that

$$\mathcal{R}\subset\mathcal{R}^{*}\subset\mathcal{R}^{\#}\subset\mathcal{R}^{\wedge}\subset\mathcal{R}^{\vartriangle}$$

for any relator $\mathcal{R}$ on X to Y.

Moreover, if in particular $\mathcal{R}$ is a relator on X, then we can easily see that

$$\mathcal{R}^{\infty}\subset\mathcal{R}^{*\,\infty}\subset\mathcal{R}^{\infty\,*}\subset\mathcal{R}^{*},$$

and hence also $\mathcal{R}^{*\,\infty}=\mathcal{R}^{\infty\,*\,\infty}$ and $\mathcal{R}^{\infty\,*}=\mathcal{R}^{*\infty\,*}$.

Furthermore, by using Definition 5.1 and Remark 5.1, we can easily establish

Theorem 7.1. *If $\mathcal{R}$ is a relator on X to Y, then*

(1) $\mathcal{R}^{*}=\Phi_{\mathrm{Lim}}(\mathcal{R})=\Phi_{\mathrm{Adh}}(\mathcal{R})$; *(2)* $\mathcal{R}^{\#}=\Phi_{\mathrm{Int}}(\mathcal{R})=\Phi_{\mathrm{Cl}}(\mathcal{R})$;

(3) $\mathcal{R}^{\wedge}=\Phi_{\mathrm{int}}(\mathcal{R})=\Phi_{\mathrm{cl}}(\mathcal{R})$; *(4)* $\mathcal{R}^{\wedge}=\Phi_{\mathrm{lim}}(\mathcal{R})=\Phi_{\mathrm{adh}}(\mathcal{R})$;

(5) $\mathcal{R}^{\vartriangle}=\Phi_{\mathcal{E}}(\mathcal{R})=\Phi_{\mathcal{D}}(\mathcal{R})$.

HINT. To prove the first part of (1), note that Lim is a decreasing structure on $\mathcal{P}\big(\mathcal{P}(X\times Y)\big)$ to $\mathcal{P}\big(\mathcal{N}(Y)\times\mathcal{N}(X)\big)$. Therefore, by Remark 5.1, we now have

$$\Phi_{\text{Lim}}(\mathcal{R})=\big\{\,S\subset X\times Y:\quad \text{Lim}_{\mathcal{R}}\subset\text{Lim}_S\,\big\}\,.$$

Hence, by the corresponding definitions it is clear that $\mathcal{R}^*\subset\Phi_{\text{Lim}}$.

To prove the converse inclusion, assume on the contrary that there exists $S\in\Phi_{\text{Lim}}$ such that $S\notin\mathcal{R}^*$. Then, for each $R\in\mathcal{R}$, there exists

$$(x_R,\,y_R)\in R\qquad\text{such that}\qquad(x_R,\,y_R)\notin S\,.$$

Hence, by partially ordering $\mathcal{R}$ with the reverse set inclusion, we can easily see that

$$\varphi=(x_R)_{R\in\mathcal{R}}\qquad\text{and}\qquad\psi=(y_R)_{R\in\mathcal{R}}$$

are nets in X and Y, respectively, such that

$$\varphi\in\text{Lim}_{\mathcal{R}}(\psi)\,,\qquad\text{but}\qquad\varphi\notin\text{Lim}_S(\psi)\,.$$

Therefore, $\text{Lim}_{\mathcal{R}}\not\subset\text{Lim}_S$, and thus $S\notin\Phi_{\text{Lim}}$. This contradiction proves the required inclusion. □

REMARK 7.2. In this respect, it is noteworthy that to prove $\Phi_{\text{Adh}}\subset\mathcal{R}^*$ any preorder on $\mathcal{R}$ can be applied.

From Theorem 7.1, by using the results of Section 6 and Theorem 5.4 and Corollary 4.2 and their duals, we can immediately derive the following

THEOREM 7.2. $*$, $\#$, $\wedge$ *and* $\vartriangle$ *are the unique operations on* $\mathcal{P}\big(\mathcal{P}(X\times Y)\big)$ *such that*

(1) Lim (*resp.* Adh) *is decreasingly* $*$*–regular;*

(2) Int (*resp.* Cl) *is increasingly (resp. decreasingly)* $\#$*–regular;*

(3) int (*resp.* cl, lim, adh) *is increasingly (resp. decreasingly)* $\wedge$*–regular;*

(4) $\mathcal{E}$ (*resp.* $\mathcal{D}$) *is increasingly (resp. decreasingly)* $\vartriangle$*–regular*

THEOREM 7.3. $*$, $\#$, $\wedge$ *and* $\vartriangle$ *are closure operations on* $\mathcal{P}\big(\mathcal{P}(X\times Y)\big)$ *such that for any two relators* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *on* X *to* Y *we have*

(1) $\mathcal{R}_1\subset\mathcal{R}_2^*\iff\mathcal{R}_1^*\subset\mathcal{R}_2^*\iff\text{Lim}_{\mathcal{R}_2}\subset\text{Lim}_{\mathcal{R}_1}\iff\text{Adh}_{\mathcal{R}_2}\subset\text{Adh}_{\mathcal{R}_1}$;

(2) $\mathcal{R}_1\subset\mathcal{R}_2^{\#}\iff\mathcal{R}_1^{\#}\subset\mathcal{R}_2^{\#}\iff\text{Int}_{\mathcal{R}_1}\subset\text{Int}_{\mathcal{R}_2}\iff\text{Cl}_{\mathcal{R}_2}\subset\text{Cl}_{\mathcal{R}_1}$;

(3) $\mathcal{R}_1\subset\mathcal{R}_2^{\wedge}\iff\mathcal{R}_1^{\wedge}\subset\mathcal{R}_2^{\wedge}\iff\text{int}_{\mathcal{R}_1}\subset\text{int}_{\mathcal{R}_2}\iff\text{cl}_{\mathcal{R}_2}\subset\text{cl}_{\mathcal{R}_1}$;

(4) $\mathcal{R}_1\subset\mathcal{R}_2^{\wedge}\iff\mathcal{R}_1^{\wedge}\subset\mathcal{R}_2^{\wedge}\iff\text{lim}_{\mathcal{R}_2}\subset\text{lim}_{\mathcal{R}_1}\iff\text{adh}_{\mathcal{R}_2}\subset\text{adh}_{\mathcal{R}_1}$;

(5) $\mathcal{R}_1\subset\mathcal{R}_2^{\vartriangle}\iff\mathcal{R}_1^{\vartriangle}\subset\mathcal{R}_2^{\vartriangle}\iff\mathcal{E}_{\mathcal{R}_1}\subset\mathcal{E}_{\mathcal{R}_2}\iff\mathcal{D}_{\mathcal{R}_2}\subset\mathcal{D}_{\mathcal{R}_1}$.

Moreover, as an immediate consequence of Theorem 4.2 and Remark 4.1 and its duals, we can also state

THEOREM 7.4. *For every relator $\mathcal{R}$ on X to Y,*

(1) $\mathcal{R}^*$ is the largest relator on X to Y such that $\operatorname{Lim}_{\mathcal{R}} = \operatorname{Lim}_{\mathcal{R}^*}$, or equivalently $\operatorname{Adh}_{\mathcal{R}} = \operatorname{Adh}_{\mathcal{R}^*}$;

(2) $\mathcal{R}^\#$ is the largest relator on X to Y such that $\operatorname{Int}_{\mathcal{R}} = \operatorname{Int}_{\mathcal{R}^\#}$, or equivalently $\operatorname{Cl}_{\mathcal{R}} = \operatorname{Cl}_{\mathcal{R}^\#}$;

(3) $\mathcal{R}^\wedge$ is the largest relator on X to Y such that $\operatorname{lim}_{\mathcal{R}} = \operatorname{lim}_{\mathcal{R}^\wedge}$, or equivalently $\operatorname{adh}_{\mathcal{R}} = \operatorname{adh}_{\mathcal{R}^\wedge}$;

(4) $\mathcal{R}^\wedge$ is the largest relator on X to Y such that $\operatorname{int}_{\mathcal{R}} = \operatorname{int}_{\mathcal{R}^\wedge}$, or equivalently $\operatorname{cl}_{\mathcal{R}} = \operatorname{cl}_{\mathcal{R}^\wedge}$;

(5) $\mathcal{R}^\triangle$ is the largest relator on X to Y such that $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R}^\triangle}$, or equivalently $\mathcal{D}_{\mathcal{R}} = \mathcal{D}_{\mathcal{R}^\triangle}$.

REMARK 7.3. Because of the above results, the relators $\mathcal{R}^*$, $\mathcal{R}^\#$, $\mathcal{R}^\wedge$ and $\mathcal{R}^\triangle$ are called the uniform, the proximal, the topological and the paratopological closures of $\mathcal{R}$, respectively.

8. The importance of the preorder modifications

The following theorem has been proved in [15] by using the ideas of [13].

THEOREM 8.1. *$\#\infty$ and $\wedge\infty$ are modification operations on $\mathcal{P}\big(\mathcal{P}(X^2)\big)$ such that for any two relators $\mathcal{R}_1$ and $\mathcal{R}_2$ on X we have*

(1) $\mathcal{R}_1^\infty \subset \mathcal{R}_2^\# \iff \mathcal{R}_1^{\#\infty} \subset \mathcal{R}_2^{\#\infty} \iff \tau_{\mathcal{R}_1} \subset \tau_{\mathcal{R}_2} \iff \tau\!\!\!-_{\mathcal{R}_1} \subset \tau\!\!\!-_{\mathcal{R}_2}$;

(2) $\mathcal{R}_1^{\wedge\infty} \subset \mathcal{R}_2^\wedge \iff \mathcal{R}_1^{\wedge\infty} \subset \mathcal{R}_2^{\wedge\infty} \iff \mathcal{T}_{\mathcal{R}_1} \subset \mathcal{T}_{\mathcal{R}_2} \iff \mathcal{F}_{\mathcal{R}_1} \subset \mathcal{F}_{\mathcal{R}_2}$.

HINT. The implication $\mathcal{R}_1^\infty \subset \mathcal{R}_2^\# \implies \tau_{\mathcal{R}_1} \subset \tau_{\mathcal{R}_2}$ follows from the facts that $\tau_{R^\infty} = \tau_R$ and $\tau_{\mathcal{R}^\#} = \tau_{\mathcal{R}}$ for any relation R and relator $\mathcal{R}$ on X.

While, to prove the converse implication, note that if $R \in \mathcal{R}_1$, then for each $A \subset X$ we have

$$R\big[R^\infty[A]\big] \subset R^\infty\big[R^\infty[A]\big] = \big(R^\infty\big)^2[A] \subset R^\infty[A],$$

and thus $R^\infty[A] \in \tau_{\mathcal{R}_1}$. Therefore, if $\tau_{\mathcal{R}_1} \subset \tau_{\mathcal{R}_2}$ holds, then we also have $R^\infty[A] \in \tau_{\mathcal{R}_2}$, and thus $R^\infty[A] \in \operatorname{Int}_{\mathcal{R}_2}\big(R^\infty[A]\big)$. Hence, since $A \subset R^\infty[A]$, it is clear that $A \in \operatorname{Int}_{\mathcal{R}_2}\big(R^\infty[A]\big)$. Therefore, $R^\infty \in \mathcal{R}_2^\#$, and thus $\mathcal{R}_1^\infty \subset \mathcal{R}_2^\#$. □

REMARK 8.1. To prove (2), it is convenient to use that $\tau_{\mathcal{R}^\wedge} = \mathcal{T}_{\mathcal{R}}$ and $\tau\!\!\!-_{\mathcal{R}^\wedge} = \mathcal{F}_{\mathcal{R}}$ for any nonvoid relator $\mathcal{R}$ on X. (For some extensions, see [30, Theorem 6.7].)

Now, by using Theorem 8.1, we can easily establish the following

THEOREM 8.2. *For every relator* $\mathcal{R}$ *on* X,

(1) $\mathcal{R}^{\#\infty}$ *is the largest preorder relator on* X *such that* $\tau_{\mathcal{R}} = \tau_{\mathcal{R}^{\#\infty}}$, *or equivalently* $\bar{\tau}_{\mathcal{R}} = \bar{\tau}_{\mathcal{R}^{\#\infty}}$;

(2) $\mathcal{R}^{\wedge\infty}$ *is the largest preorder relator on* X *such that* $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\wedge\infty}}$, *or equivalently* $\mathcal{F}_{\mathcal{R}} = \mathcal{F}_{\mathcal{R}^{\wedge\infty}}$.

REMARK 8.2. Concerning the modification operation $\vartriangle \infty$, in contrast to [15, Theorem 5.12], we we can only prove that if $\mathcal{R}$ is a total relator on X in the sense that X is the domain of each member of $\mathcal{R}$, then $\mathcal{R}^{\vartriangle\infty}$ is the largest preorder relator on X such that $\mathcal{E}_{\mathcal{R}} = E_{\mathcal{R}^{\vartriangle\infty}}$, or equivalently $\mathcal{D}_{\mathcal{R}} = \mathcal{D}_{\mathcal{R}^{\vartriangle\infty}}$.

Moreover, by using Theorem 8.1, we can also prove the following supplement to Theorem 7.1.

THEOREM 8.3. *If* $\mathcal{R}$ *is a relator on* X, *then*

(1) $\mathcal{R}^{\#\partial} = \Phi_{\tau}(\mathcal{R}) = \Phi_{\bar{\tau}}(\mathcal{R})$; *(2)* $\mathcal{R}^{\wedge\partial} = \Phi_{\mathcal{T}}(\mathcal{R}) = \Phi_{\mathcal{F}}(\mathcal{R})$.

HINT. To prove this, note that by Definition 5.1, Theorem 8.1 and Definition 6.1, for any $S \subset X^2$, we have

$$S \in \Phi_{\tau}(\mathcal{R}) \iff \tau_S \subset \tau_{\mathcal{R}} \iff S^{\infty} \in \mathcal{R}^{\#} \iff S \in \mathcal{R}^{\#\partial}.$$

Hence, by Remark 8.1 and the equalities $\mathcal{T}_S = \tau_S$ and $\mathcal{R}^{\wedge\#} = \mathcal{R}^{\wedge}$, it is clear that we also have

$$S \in \Phi_{\mathcal{T}}(\mathcal{R}) \iff \mathcal{T}_S \subset \mathcal{T}_{\mathcal{R}} \iff \tau_S \subset \tau_{\mathcal{R}^{\wedge}}$$
$$\iff S \in \Phi_{\tau}(\mathcal{R}^{\wedge}) \iff S \in \mathcal{R}^{\wedge\#\partial} \iff S \in \mathcal{R}^{\wedge\partial}.$$

□

From the first statement of the above theorem, by our former results, it is clear that the following three theorems are true.

THEOREM 8.4. $\#\partial$ *is the unique operation on* $\mathcal{P}(\mathcal{P}(X^2))$ *such that* τ *(resp.* $\bar{\tau}$*) is an increasingly* $\#\partial$*–normal structure on* $\mathcal{P}(\mathcal{P}(X^2))$ *to* $\mathcal{P}(\mathcal{P}(X))$.

THEOREM 8.5. $\#\partial$ *is a closure operation on* $\mathcal{P}(\mathcal{P}(X^2))$ *such that for any two relators* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *on* X *we have*

$$\mathcal{R}_1 \subset \mathcal{R}_2^{\#\partial} \iff \mathcal{R}_1^{\#\partial} \subset \mathcal{R}_2^{\#\partial} \iff \tau_{\mathcal{R}_1} \subset \tau_{\mathcal{R}_2} \iff \bar{\tau}_{\mathcal{R}_1} \subset \bar{\tau}_{\mathcal{R}_2}.$$

THEOREM 8.6. *For every relator* $\mathcal{R}$ *on* X, $\mathcal{R}^{\#\partial}$ *is the largest relator on* X *such that* $\tau_{\mathcal{R}} = \tau_{\mathcal{R}^{\#\partial}}$, *or equivalently* $\bar{\tau}_{\mathcal{R}} = \bar{\tau}_{\mathcal{R}^{\#\partial}}$.

Moreover, from the second statement of Theorem 8.3, by Theorem 5.2, it is clear that we also have the following

Theorem 8.7. *$\wedge\partial$ is a preclosure operation on $\mathcal{P}\left(\mathcal{P}\left(X^2\right)\right)$ such that for any two relators $\mathcal{R}_1$ and $\mathcal{R}_2$ on X we have $\mathcal{R}_1 \subset \mathcal{R}_2^{\wedge\partial}$ whenever $\mathcal{T}_{\mathcal{R}_1} \subset \mathcal{T}_{\mathcal{R}_2}$, or equivalently $\mathcal{F}_{\mathcal{R}_1} \subset \mathcal{F}_{\mathcal{R}_2}$.*

Remark 8.3. Now, by our former results, it is clear that

$$\mathcal{R}^{\#\infty} \subset \mathcal{R}^{\#} \subset \mathcal{R}^{\#\partial} \qquad \text{and} \qquad \mathcal{R}^{\wedge\infty} \subset \mathcal{R}^{\wedge} \subset \mathcal{R}^{\wedge\partial},$$

and hence also $\mathcal{R}^{\#\infty} = \mathcal{R}^{\#\partial\infty}$ and $\mathcal{R}^{\wedge\infty} = \mathcal{R}^{\wedge\partial\infty}$.

Moreover, by using the ideas of [13, Example 5.3] and [22, Example 7.2], we can also prove the following statements. (See [43, Example 10.11].)

Example 8.1. If $\operatorname{card}(X) > 2$ and $\mathcal{R} = \{X^2\}$, then $\mathcal{R}$ is a topologically fine equivalence relator on X such that

(1) $\mathcal{R}^{\#\partial} \not\subset \mathcal{R}$; (2) $\mathcal{T}_{\mathcal{R}^{\wedge\partial}} \not\subset \mathcal{T}_{\mathcal{R}}$; (3) $\mathcal{R}^{\wedge\partial\wedge} \not\subset \mathcal{R}^{\wedge\partial}$;

(4) there is no largest relator $\mathcal{S}$ on X such that $\mathcal{T}_{\mathcal{S}} \subset \mathcal{T}_{\mathcal{R}}$ (resp. $\mathcal{T}_{\mathcal{S}} = \mathcal{T}_{\mathcal{R}}$).

Remark 8.4. This example shows that in contrast to $\#$ and $\#\infty$, the operation $\#\partial$ is not, in general, stable.

Moreover, the operation $\wedge\partial$ is not, in general, idempotent. Therefore, by Theorems 5.5 and 8.3, the assertions of Remark 6.2 are true.

9. Composition compatible operations

Definition 9.1. If $\Box$ is an operation on relators, then we say that:

(1) $\Box$ is left composition compatible if $\left(S \circ \mathcal{R}\right)^{\Box} = \left(S \circ \mathcal{R}^{\Box}\right)^{\Box}$ for any three nonvoid sets X, Y and Z and any relator $\mathcal{R}$ on X to Y and any relation S on Y to Z;

(2) $\Box$ is right composition compatible if $\left(\mathcal{S} \circ R\right)^{\Box} = \left(\mathcal{S}^{\Box} \circ R\right)^{\Box}$ for any three nonvoid sets X, Y and Z and any relation R on X to Y and any relator $\mathcal{S}$ on Y to Z.

Remark 9.1. Now, the operation $\Box$ may be naturally called composition compatible if it is both left and right composition compatible.

Note that (1) and (2) are actually very weak composition compatibility properties of $\Box$. However, they will prove to be strong enough to simplify our forthcoming definitions of continuities.

Concerning left composition compatible operations, we can easily establish the following two theorems.

Theorem 9.1. *If $\Box$ is a preclosure operation on relators, then the following assertions are equivalent:*

(1) $\Box$ is left composition compatible;

(2) $\big(S\circ\mathcal{R}^{\Box}\big)^{\Box}\subset\big(S\circ\mathcal{R}\big)^{\Box}$ *for any three nonvoid sets* X, Y *and* Z *and any relator* $\mathcal{R}$ *on* X *to* Y *and any relation* S *on* Y *to* Z.

Proof. If $\mathcal{R}$ and S are as above, then by the expansivity and the increasingness of $\Box$, we have $S\circ\mathcal{R}\subset S\circ\mathcal{R}^{\Box}$, and hence $(S\circ\mathcal{R})^{\Box}\subset\big(S\circ\mathcal{R}^{\Box}\big)^{\Box}$. Therefore, the implication $(2)\Longrightarrow(1)$ is also true. $\Box$

Remark 9.2. Note that if $\Box$ is a preclosure operation on relators, then we also have $(\mathcal{S}\circ\mathcal{R})^{\Box}\subset\big(\mathcal{S}\circ\mathcal{R}^{\Box}\big)^{\Box}$ for any three nonvoid sets X, Y and Z and any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z.

Theorem 9.2. *If* $\Box$ *is a closure operation on relators, then the following assertions are equivalent:*

(1) $\Box$ *is left composition compatible;*

(2) $S\circ\mathcal{R}^{\Box}\subset\big(S\circ\mathcal{R}\big)^{\Box}$ *for any three nonvoid sets* X, Y *and* Z *and any relator* $\mathcal{R}$ *on* X *to* Y *and any relation* S *on* Y *to* Z.

Proof. If (1) holds, and $\mathcal{R}$ and S are as above, then by the expansivity of $\Box$ and Definition 9.1 it is clear that $S\circ\mathcal{R}^{\Box}\subset\big(S\circ\mathcal{R}^{\Box}\big)^{\Box}=\big(S\circ\mathcal{R}\big)^{\Box}$, and thus (2) also holds.

While, if (2) holds, then by the increasingness and the idempotency $\Box$ we also have $\big(S\circ\mathcal{R}^{\Box}\big)^{\Box}\subset\big(S\circ\mathcal{R}\big)^{\Box\Box}=\big(S\circ\mathcal{R}\big)^{\Box}$. Hence, by Theorem 9.1, we can see that (1) also holds. $\Box$

Example 9.1. By using the latter theorem, for instance, we can easily show that the topological closure operation $\wedge$ is left composition compatible.

For this, note that if $\mathcal{R}$ and S are as in Theorem 9.2 and $W\in S\circ\mathcal{R}^{\wedge}$, then there exists $V\in\mathcal{R}^{\wedge}$ such that $W=S\circ V$. Moreover, by the definition of $\wedge$, for each $x\in X$ there exists $R\in\mathcal{R}$ such that $R(x)\subset V(x)$. Now, by using the notation $U=S\circ R$, we can see that $U\in S\circ\mathcal{R}$ such that

$$U(x)=(S\circ R)(x)=S[R(x)]\subset S[V(x)]=(S\circ V)(x)=W(x),$$

and thus $W\in\big(S\circ\mathcal{R}\big)^{\wedge}$. Hence, it is clear that $S\circ\mathcal{R}^{\Box}\subset\big(S\circ\mathcal{R}\big)^{\Box}$, and thus by Theorem 9.2 the required assertion is also true.

Remark 9.3. By using a similar argument, instead of the right composition compatibility of $\wedge$, we can only prove that $\big(\mathcal{S}\circ R\big)^{\wedge}=\big(\mathcal{S}^{\#}\circ R\big)^{\wedge}$ for any three nonvoid sets X, Y and Z and any relation R on X to Y and any relator $\mathcal{S}$ on Y to Z.

In addition to Theorem 9.2, we can also easily prove the following

Theorem 9.3. *If* $\Box$ *is a left composition compatible closure operation on relators, then*

$$\big(\mathcal{S}\circ\mathcal{R}\big)^{\Box}=\big(\mathcal{S}\circ\mathcal{R}^{\Box}\big)^{\Box}$$

for any three nonvoid sets X, Y and Z and any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z.

PROOF. If $\mathcal{R}$ and $\mathcal{S}$ are as above, then by Theorem 4.6 and the left composition compatibility of $\square$ it is clear that

$$\begin{aligned}(\mathcal{S}\circ\mathcal{R})^{\square} &= \Big(\bigcup_{S\in\mathcal{S}} S\circ\mathcal{R}\Big)^{\square} = \Big(\bigcup_{S\in\mathcal{S}} (S\circ\mathcal{R})^{\square}\Big)^{\square} \\ &= \Big(\bigcup_{S\in\mathcal{S}} (S\circ\mathcal{R}^{\square})^{\square}\Big)^{\square} = \Big(\bigcup_{S\in\mathcal{S}} S\circ\mathcal{R}^{\square}\Big)^{\square} = (\mathcal{S}\circ\mathcal{R}^{\square})^{\square}.\end{aligned}$$

$\square$

Now, by using a similar theorem for right composition compatible closure operations, we can also easily establish the following

THEOREM 9.4. *If $\square$ is a composition compatible closure operation on relators, then*

$$(\mathcal{S}\circ\mathcal{R})^{\square} = (\mathcal{S}^{\square}\circ\mathcal{R}^{\square})^{\square}$$

for any three nonvoid sets X, Y and Z and any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z.

PROOF. If $\mathcal{R}$ and $\mathcal{S}$ are as above, then by Theorem 9.3 and its right-sided counterpart, it is clear that $(\mathcal{S}\circ\mathcal{R})^{\square} = (\mathcal{S}\circ\mathcal{R}^{\square})^{\square} = (\mathcal{S}^{\square}\circ\mathcal{R}^{\square})^{\square}$. $\square$

REMARK 9.4. Concerning the operation $\wedge$, because of Remark 9.3, we can only prove that $(\mathcal{S}\circ\mathcal{R})^{\wedge} = (\mathcal{S}^{\#}\circ\mathcal{R}^{\wedge})^{\wedge}$ for any three nonvoid sets X, Y and Z and any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z.

Hence, by writing $\mathcal{S}^{\wedge}$ in place of $\mathcal{S}$ and using that $\mathcal{S}^{\wedge\#} = \mathcal{S}^{\wedge}$, we can immediately derive that $(\mathcal{S}^{\wedge}\circ\mathcal{R})^{\wedge} = (\mathcal{S}^{\wedge\#}\circ\mathcal{R}^{\wedge})^{\wedge} = (\mathcal{S}^{\wedge}\circ\mathcal{R}^{\wedge})^{\wedge}$.

10. Continuities of pairs of relations

If F is a relation on X to Y and G is a relation on Z to W, then we say that (F, G) is a pair of relations on (X, Y) to (Z, W).

Moreover, according to [30], [26] and [37], we introduce the following three basic continuity properties of a pair of relations on one relator space to another.

DEFINITION 10.1. If (F, G) is a pair of relations on one relator space $(X, Z)(\mathcal{R})$ to another $(Y, W)(\mathcal{S})$ and $\square$ is a unary operation on relators, then we say that:

(1) (F, G) is mildly $\square$–continuous if $(G^{-1}\circ\mathcal{S}^{\square}\circ F)^{\square} \subset \mathcal{R}^{\square}$;

(2) (F, G) is upper $\square$–semicontinuous if $(\mathcal{S}^{\square}\circ F)^{\square} \subset (G\circ\mathcal{R}^{\square})^{\square}$;

(3) (F, G) is lower $\square$–semicontinuous if $(G^{-1}\circ S^{\square})^{\square} \subset (\mathcal{R}^{\square}\circ F^{-1})^{\square}$.

REMARK 10.1. Now, the pair (F, G) may be naturally called $\square$–continuous if it is is both upper and lower $\square$–semicontinuous.

Moreover, the pair (F, G) may, for instance, be naturally called properly continuous if it is $\square$–continuous with $\square$ being the identity operation.

And the the pair (F, G) may, for instance, be naturally called uniformly and topologically continuous if it is $\square$–continuous with $\square = *$ and $\square = \wedge$, respectively.

Note that, by using the various operations on relators, some mixed type continuities of (F, G) can also be introduced. However, here we shall only be interested in the properties (1)–(3).

To simplify Definition 10.1, by using Theorem 3.5 and Definition 9.1, we can easily establish the following two theorems.

THEOREM 10.1. *If* (F, G) *is a pair of relations on one relator space* $\big(X, Y\big)(\mathcal{R})$ *to another* $\big(Z, W\big)(\mathcal{S})$, *and* $\square$ *is a closure operation on relators, then*

(1) (F, G) *is mildly* $\square$*–continuous* $\iff$ $G^{-1} \circ \mathcal{S}^{\square} \circ F \subset \mathcal{R}^{\square}$;

(2) (F, G) *is upper* $\square$*–semicontinuous* $\iff$ $\mathcal{S}^{\square} \circ F \subset \big(G \circ \mathcal{R}^{\square}\big)^{\square}$;

(3) (F, G) *is lower* $\square$*–semicontinuous* $\iff$ $G^{-1} \circ \mathcal{S}^{\square} \subset \big(\mathcal{R}^{\square} \circ F^{-1}\big)^{\square}$.

THEOREM 10.2. *If* (F, G) *is a pair of relations on one relator space* $\big(X, Y\big)(\mathcal{R})$ *to another* $\big(Z, W\big)(\mathcal{S})$ *and* $\square$ *is a composition compatible closure operation on relators, then*

(1) (F, G) *is mildly* $\square$*–continuous* $\iff$ $G^{-1} \circ \mathcal{S} \circ F \subset \mathcal{R}^{\square}$;

(2) (F, G) *is upper* $\square$*–semicontinuous* $\iff$ $\mathcal{S} \circ F \subset \big(G \circ \mathcal{R}\big)^{\square}$;

(3) (F, G) *is lower* $\square$*–semicontinuous* $\iff$ $G^{-1} \circ \mathcal{S} \subset \big(\mathcal{R} \circ F^{-1}\big)^{\square}$.

HINT. To prove (1), note that by Definition 9.1 we have

$$\big(G^{-1} \circ \mathcal{S}^{\square} \circ F\big)^{\square} = \Big(G^{-1} \circ \big(\mathcal{S}^{\square} \circ F\big)^{\square}\Big)^{\square} = \Big(G^{-1} \circ \big(\mathcal{S} \circ F\big)^{\square}\Big)^{\square} = \big(G^{-1} \circ \mathcal{S} \circ F\big)^{\square}.$$

Moreover, by Theorem 3.5, we have $\big(G^{-1} \circ \mathcal{S} \circ F\big)^{\square} \subset \mathcal{R}^{\square}$ if and only if $G^{-1} \circ \mathcal{S} \circ F \subset \mathcal{R}^{\square}$. $\square$

REMARK 10.2. Unfortunately, the corresponding theorem is not true for the topological closure operation $\wedge$.

For instance, if F is not a function, then we can only prove that (F, G) is topologically upper semicontinuous if and only if $\mathcal{S}^{\wedge} \circ F \subset \big(G \circ \mathcal{R}\big)^{\wedge}$.

Definition 10.2. If $\Diamond$ and $\Box$ are operations on relators, then we say that $\Box$ is $\Diamond$–compatible if for any two nonvoid sets X and Y and any relator $\mathcal{R}$ on X to Y we have $\mathcal{R}^{\Box\Diamond} = \mathcal{R}^{\Diamond\Box}$.

Remark 10.3. In particular, the operation $\Box$ is called inversion compatible if $\big(\mathcal{R}^{\Box}\big)^{-1} = \big(\mathcal{R}^{-1}\big)^{\Box}$ for any relator $\mathcal{R}$ on X to Y.

Now, to let the reader feel the appropriateness of Definition 10.1, we can note that the following theorem is also true.

Theorem 10.3. *If (F, G) is a pair of relations on one relator space $\big(X, Z\big)(\mathcal{R})$ to another $\big(Y, W\big)(\mathcal{S})$ and $\Box$ is an inversion compatible operation on relators, then*

(1) (F, G) is mildly $\Box$–continuous with respect to the relators $\mathcal{R}$ and $\mathcal{S}$ if and only if (G, F) is mildly $\Box$–continuous with respect to the relators $\mathcal{R}^{-1}$ and $\mathcal{S}^{-1}$;

(2) (F, G) is lower $\Box$–semicontinuous with respect to the relators $\mathcal{R}$ and $\mathcal{S}$ if and only if (G, F) is upper $\Box$–semicontinuous with respect to the relators $\mathcal{R}^{-1}$ and $\mathcal{S}^{-1}$.

Hint. To prove (2), note that

$$\Big(\big(G^{-1}\circ\mathcal{S}^{\Box}\big)^{\Box}\Big)^{-1} = \Big(\big(G^{-1}\circ\mathcal{S}^{\Box}\big)^{-1}\Big)^{\Box} = \Big(\big(\mathcal{S}^{\Box}\big)^{-1}\circ\big(G^{-1}\big)^{-1}\Big)^{\Box} = \Big(\big(\mathcal{S}^{-1}\big)^{\Box}\circ G\Big)^{\Box}$$

and quite similarly $\Big(\big(\mathcal{R}^{\Box}\circ F^{-1}\big)^{\Box}\Big)^{-1} = \Big(F\circ\big(\mathcal{R}^{-1}\big)^{\Box}\Big)^{\Box}$. $\Box$

Remark 10.4. Unfortunately, the topological and the paratopological closure operations $\wedge$ and $\vartriangle$ are not inversion compatible. (See [14].)

Therefore, the above theorem can have only a limited range of applicability. However, it helps us to keep in mind the definition of lower $\Box$–semicontinuity.

11. Relationship with normality and regularity

In addition to Theorems 10.1 and 10.2, it is also worth noticing the following

Theorem 11.1. *If (F, G) is a pair of relations on one simple relator space $\big(X, Y\big)(R)$ to another $\big(Z, W\big)(S)$, then*

(1) (F, G) is properly mildly continuous $\iff$ $G^{-1}\circ S\circ F = R$;

(2) (F, G) is properly upper semicontinuous $\iff$ $S\circ F = G\circ R$;

(3) (F, G) is properly lower semicontinuous $\iff$ $G^{-1}\circ S = R\circ F^{-1}$.

HINT. Namely, if for instance $S \circ F = G \circ R$, then

$$\{ S \} \circ F = \{ S \circ F \} = \{ G \circ R \} = G \circ \{ R \} .$$

Thus, in particular, (F , G) is properly upper semicontinuous.

While, if (F , G) is properly upper semicontinuous, then we have

$$\{ S \circ F \} = \{ S \} \circ F \subset G \circ \{ R \} = \{ G \circ R \} .$$

Hence, it follows that $S \circ F = G \circ R$. Therefore, (2) is true. □

A simple application of the assertion (2) of Theorem 11.1 yields the following

THEOREM 11.2. *If f is a function of one simple relator space $X(R)$ to another $Y(S)$ and g is a function of Y to X, then the following assertions are equivalent:*

(1) (f , g^{-1}) *is properly upper semicontinuous;*

(3) $f(x)\, S\, y \iff x\, R\, g(y)$ *for all* $x \in X$ *and* $y \in Y$.

PROOF. From Theorem 11.1, we can at once see that (1) is equivalent to the equality $S \circ f = g^{-1} \circ R$.

Moreover, by the corresponding definitions, it is clear that for any $x \in X$ and $y \in Y$ we have

$$f(x)\, S\, y \iff y \in S\big(f(x) \big) \iff y \in (S \circ f)(x) \iff (x , y) \in S \circ f$$

and

$$x\, R\, g(y) \iff g(y) \in R(x) \iff y \in g^{-1}[\, R(x) \,]$$
$$\iff y \in \big(g^{-1} \circ R \big)(x) \iff (x , y) \in g^{-1} \circ R .$$

Therefore, (2) is also equivalent to the equality $S \circ f = g^{-1} \circ R$. □

Now, as an immediate consequence of Theorem 11.2 and Definition 3.5, we can also state

COROLLARY 11.1. *If f is a structure on one poset X to another Y and g is a structure on Y to X, then the following assertions are equivalent:*

(1) f *is increasingly* g*–normal;*

(2) (f , g^{-1}) *is properly upper semicontinuous.*

REMARK 11.1. This shows that increasing normality is a very particular case of upper semicontinuity.

Analogously to Theorem 11.2, we can also prove the following

Theorem 11.3. *If f is a function of one simple relator space $X(R)$ to another $Y(S)$ and φ is a function of X to itself, then the following assertions are equivalent:*

(1) $x_1 R \varphi(x_2) \iff f(x_1) S f(x_2)$ *for all* $x_1, x_2 \in X$.

(2) f *is properly mildly continuous with respect* $\varphi^{-1} \circ R$ *and* S.

Proof. From Theorem 11.1, we can at once see that (2) is equivalent to the equality $\varphi^{-1} \circ R = f^{-1} \circ S \circ f$.

Moreover, by the corresponding definitions, it is clear that for any $x_1, x_2 \in X$ we have

$$x_1 R \varphi(x_2) \iff \varphi(x_2) \in R(x_1) \iff x_2 \in \varphi^{-1}\big[R(x_1)\big] \iff x_2 \in \big(\varphi^{-1} \circ R\big)(x_1) \iff (x_1, x_2) \in \varphi^{-1} \circ R$$

and

$$f(x_1) S f(x_2) \iff f(x_2) \in S\big(f(x_1)\big) \iff x_2 \in f^{-1}\big[S\big(f(x_1)\big)\big] \iff x_2 \in \big(f^{-1} \circ S \circ f\big)(x_1) \iff (x_1, x_2) \in f^{-1} \circ S \circ f.$$

Therefore, (2) is also equivalent to the equality $\varphi^{-1} \circ R = f^{-1} \circ S \circ f$. □

Now, as an immediate consequence of Theorem 11.5 and Definition 3.7, we can also state

Corollary 11.2. *If f is a structure on a poset X to another Y and φ is an operation on X, then the following assertions are equivalent:*

(1) f *is increasingly* φ*–regular;*

(2) f *is properly mildly continuous with respect to* $\varphi^{-1} \circ \leqslant$ *and* $\leqslant$.

Remark 11.2. This shows that increasing regularity is a very particular case of mild continuity.

In this respect, it is also worth noticing that a structure f on a poset X to another Y is increasing if and only if it is uniformly mildly continuous.

12. Generalizations of Theorems 3.1 and 3.2

Theorem 12.1. *If (F, G) is a pair relations on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$ and $\square$ is an increasing left composition compatible operation on relators such that (F, G) is upper $\square$–semicontinuous, then by defining $\Phi = F^{-1} \circ G$ we can state that F is mildly $\square$–continuous with respect to the relators $\Phi \circ \mathcal{R}$ and $\mathcal{S}$.*

Proof. By the upper $\square$–semicontinuity of (F, G), we have

$$\big(\mathcal{S}^{\square} \circ F\big)^{\square} \subset \big(G \circ \mathcal{R}^{\square}\big)^{\square}.$$

This implies that

$$F^{-1} \circ \big(\mathcal{S}^{\square} \circ F\big)^{\square} \subset F^{-1} \circ \big(G \circ \mathcal{R}^{\square}\big)^{\square}.$$

Hence, by the increasingness of $\square$, it follows that

$$\Big(F^{-1} \circ \big(\mathcal{S}^{\square} \circ F\big)^{\square}\Big)^{\square} \subset \Big(F^{-1} \circ \big(G \circ \mathcal{R}^{\square}\big)^{\square}\Big)^{\square}.$$

Now, by the left composition compatibility of $\square$ and the definition of Φ, it is clear that

$$\big(F^{-1} \circ \mathcal{S}^{\square} \circ F\big)^{\square} \subset \big(F^{-1} \circ G \circ \mathcal{R}^{\square}\big)^{\square} = \big(\Phi \circ \mathcal{R}^{\square}\big)^{\square} = \big(\Phi \circ \mathcal{R}\big)^{\square}.$$

Thus, F is mildly $\square$–continuous with respect to the relators $\Phi \circ \mathcal{R}$ and $\mathcal{S}$. $\square$

From the above theorem, by using Theorems 11.2 and 11.3, we can easily derive

Corollary 12.1. *If f is a function of one simple relator space $X(R)$ to another $Y(S)$ and g is a function of Y to X such that*

$$f(x)\, S\, y \iff x\, R\, g(y),$$

for all $x \in X$ and $y \in Y$, then by defining $\varphi = g \circ f$ we can state that

$$x_1\, R\, \varphi(x_2) \iff f(x_1)\, S\, f(x_2)$$

for all $x_1, x_2 \in X$.

Proof. Let $\square$ be the identity operation on relators. Then, $\square$ is increasing and left composition compatible. Moreover, by Theorem 11.2, the pair $\big(f, g^{-1}\big)$ is upper $\square$–semicontinuous with respect to the relators $\{R\}$ and $\{S\}$. Furthermore, we can note that

$$\Phi = f^{-1} \circ g^{-1} = (g \circ f)^{-1} = \varphi^{-1} \qquad \text{and} \qquad \Phi \circ \{R\} = \varphi^{-1} \circ \{R\} = \{\varphi^{-1} \circ R\}.$$

Hence, by Theorem 12.1, we can see that the pair $\big(f, g^{-1}\big)$ is mildly $\square$–continuous with respect to the relators $\{\varphi^{-1} \circ R\}$ and $\{S\}$. Therefore, by Theorem 11.3, the required assertion is also true. $\square$

Remark 12.1. Now, as an immediate consequence of Corollary 12.1 and Definitions 3.4 and 3.5, we can also state Theorem 3.1.

As a partial converse to Theorem 12.1, we can also prove the following

Theorem 12.2. *Let f be a function on one relator space $X(\mathcal{R})$ onto another $Y(\mathcal{S})$ and Φ be a relation on X to itself. Suppose that $\square$ is an increasing left composition compatible operation on relators such that f is a mildly $\square$–continuous with respect to the relators $\Phi \circ \mathcal{R}$ and $\mathcal{S}$. Moreover, suppose that G is a relation on X to Y such $\Phi = f^{-1} \circ G$. Then, the pair (f, G) is upper $\square$–semicontinuous with respect to the relators $\mathcal{R}$ and $\mathcal{S}$.*

Proof. By the assumed mild continuity of f, we have

$$\big(f^{-1}\circ\mathcal{S}^{\Box}\circ f\big)^{\Box}\subset\big(\Phi\circ\mathcal{R}\big)^{\Box}.$$

This implies that

$$f\circ\big(f^{-1}\circ\mathcal{S}^{\Box}\circ f\big)^{\Box}\subset f\circ\big(\Phi\circ\mathcal{R}\big)^{\Box}.$$

Hence, by the increasingness of $\Box$, it follows that

$$\Big(f\circ\big(f^{-1}\circ\mathcal{S}^{\Box}\circ f\big)^{\Box}\Big)^{\Box}\subset\Big(f\circ\big(\Phi\circ\mathcal{R}\big)^{\Box}\Big)^{\Box}.$$

Now, by the left composition compatibility of $\Box$ and the equality $\Phi=f^{-1}\circ G$, it is clear that

$$\big(f\circ f^{-1}\circ\mathcal{S}^{\Box}\circ f\big)^{\Box}\subset\big(f\circ\Phi\circ\mathcal{R}\big)^{\Box}=\big(f\circ\Phi\circ\mathcal{R}^{\Box}\big)^{\Box}=\big(f\circ f^{-1}\circ G\circ\mathcal{R}^{\Box}\big)^{\Box}.$$

Moreover, since f is a function on X onto Y, we have $f\circ f^{-1}=\Delta_Y$. Now, by noticing that $f\circ f^{-1}\circ\mathcal{U}=\Delta_Y\circ\mathcal{U}=\mathcal{U}$ for any relator $\mathcal{U}$ on X to Y, we can see that

$$\big(\mathcal{S}^{\Box}\circ f\big)^{\Box}\subset\big(G\circ\mathcal{R}^{\Box}\big)^{\Box}.$$

Therefore, the pair $(f,\,G)$ is upper $\Box$–semicontinuous with respect to the relators $\mathcal{R}$ and $\mathcal{S}$. □

From the above theorem, by using Theorems 11.3 and 11.2, we can easily derive

Corollary 12.2. *Suppose that f is a function of one simple relator space $X(R)$ onto another $Y(S)$ and φ is a function of X to itself such that*

$$x_1\,R\,\varphi(x_2)\quad\Longleftrightarrow\quad f(x_1)\,S\,f(x_2)$$

for all $x_1,\,x_2\in X$. Moreover, suppose that g is a function of Y to X such that $\varphi=g\circ f$. Then

$$f(x)\,S\,y\quad\Longleftrightarrow\quad x\,R\,g(y)$$

for all $x\in X$ and $y\in Y$.

Proof. Let $\Box$ be the identity operation on relators. Then, $\Box$ is increasing and left composition compatible. Moreover, by Theorem 11.3, the function f is mildly $\Box$–continuous with respect to the relators $\{\varphi^{-1}\circ R\}$ and $\{S\}$. Furthermore, we can note that

$$\{\varphi^{-1}\circ R\}=\varphi^{-1}\circ\{R\}\qquad\text{and}\qquad\varphi^{-1}=(g\circ f)^{-1}=f^{-1}\circ g^{-1}.$$

Hence, by Theorem 12.2, we can see that the pair $\big(f,\,g^{-1}\big)$ is upper $\Box$–semicontinouous with respect to the relators $\{R\}$ and $\{S\}$. Therefore, by Theorem 11.2, the required assertion is also true. □

Remark 12.2. Now, as an immediate consequence of Corollary 12.2 and Definitions 3.4 and 3.5, we can also state Theorem 3.2.

13. Relationships between mild and upper continuities

Definition 13.1. If $\lozenge$ and $\square$ are operations on relators, then we say that $\square$ is $\lozenge$–dominating, $\lozenge$–invariant and $\lozenge$–absorbing if for any two nonvoid sets X and Y and any relator $\mathcal{R}$ on X to Y we have $\mathcal{R}^{\lozenge} \subset \mathcal{R}^{\square}$, $\mathcal{R}^{\square} = \mathcal{R}^{\square\lozenge}$ and $\mathcal{R}^{\square} = \mathcal{R}^{\lozenge\square}$, respectively.

Remark 13.1. Note that if $\lozenge$ is an expansive and $\square$ is a $\lozenge$–dominating operation on relators, then $\square$ is also expansive.

Moreover, by using the corresponding definitions, we can also easily prove the following two propositions.

Proposition 13.1. *If $\lozenge$ is an expansive and $\square$ is a $\lozenge$–dominating idempotent operation on relators, then $\square$ is $\lozenge$–invariant.*

Proposition 13.2. *If $\lozenge$ is an expansive and $\square$ is a $\lozenge$–dominating modification operation on relators, then $\square$ is $\lozenge$–absorbing.*

Remark 13.2. Now, from Remark 7.1 and Theorem 7.3, by using the above propositions, we can at once see that, for instance, $\wedge$ is both $*$–invariant and $*$–absorbing.

Moreover, in addition to Theorems 12.2 and 12.1, we can also prove the following two theorems.

Theorem 13.1. *If (F, g) is a pair relations on one relator space $(X, Z)(\mathcal{R})$ to another $(Y, W)(\mathcal{S})$ such that g is a function, and $\square$ is a $*$–dominating modification operation for relators such that (F, g) is mildly $\square$–continuous, then (F, g) is upper $\square$–semicontinuous.*

Proof. Now, by Remark 13.1, $\square$ is also expansive. Therefore, by Definition 10.1, we have

$$g^{-1} \circ \mathcal{S}^{\square} \circ F \subset \mathcal{R}^{\square}.$$

This implies that

$$g \circ g^{-1} \circ \mathcal{S}^{\square} \circ F \subset g \circ \mathcal{R}^{\square}.$$

Hence, by the increasingness of $\square$, it follows that

$$\left(g \circ g^{-1} \circ \mathcal{S}^{\square} \circ F\right)^{\square} \subset \left(g \circ \mathcal{R}^{\square}\right)^{\square}.$$

Moreover, since g is a function on X to Y, we also have $g \circ g^{-1} \subset \Delta_{Y}$. Hence, by using the corresponding definitions, we can easily see that

$$\mathcal{S}^{\square} \circ F = \Delta_{Y} \circ \mathcal{S}^{\square} \circ F \subset \left(g \circ g^{-1} \circ \mathcal{S}^{\square} \circ F\right)^{*} \subset \left(g \circ g^{-1} \circ \mathcal{S}^{\square} \circ F\right)^{\square}.$$

Therefore, we also have

$$\mathcal{S}^{\square} \circ F \subset \left(g \circ \mathcal{R}^{\square}\right)^{\square}.$$

Thus, by Theorem 10.1, the pair (F, g) is upper $\square$–semicontinuous. $\square$

Theorem 13.2. *If (F, G) is a pair relations on one relator space $(X, Z)(\mathcal{R})$ to another $(Y, W)(\mathcal{S})$ such that $D_G = Z$, and $\square$ is an increasing, $*$–absorbing and left composition compatible operation for relators such that (F, G) is upper $\square$–semicontinuous, then (F, G) is also mildly $\square$–continuous.*

Proof. By the upper $\square$–semicontinuity of (F, G), we have

$$\left(\mathcal{S}^{\square} \circ F\right)^{\square} \subset \left(G \circ \mathcal{R}^{\square}\right)^{\square}.$$

This implies that

$$G^{-1} \circ \left(\mathcal{S}^{\square} \circ F\right)^{\square} \subset G^{-1} \circ \left(G \circ \mathcal{R}^{\square}\right)^{\square}.$$

Hence, by the increasingness of $\square$, it follows that

$$\Big(G^{-1} \circ \left(\mathcal{S}^{\square} \circ F\right)^{\square}\Big)^{\square} \subset \Big(G^{-1} \circ \left(G \circ \mathcal{R}^{\square}\right)^{\square}\Big)^{\square}.$$

Hence, by the left composition compatibility of $\square$, it is clear that

$$\left(G^{-1} \circ \mathcal{S}^{\square} \circ F\right)^{\square} \subset \left(G^{-1} \circ G \circ \mathcal{R}^{\square}\right)^{\square} = \left(G^{-1} \circ G \circ \mathcal{R}\right)^{\square}.$$

Moreover, since $D_G = X$, we also have $\Delta_Z \subset G^{-1} \circ G$. Hence, by using the corresponding definitions, we can easily see that

$$G^{-1} \circ G \circ \mathcal{R} \subset \left(\Delta_Z \circ \mathcal{R}\right)^{*} = \mathcal{R}^{*}.$$

Hence, since $\square$ is increasing and $*$–absorbing, it is clear that

$$\left(G^{-1} \circ G \circ \mathcal{R}\right)^{\square} \subset \mathcal{R}^{*\square} = \mathcal{R}^{\square}.$$

Therefore, we also have

$$\left(G^{-1} \circ \mathcal{S}^{\square} \circ F\right)^{\square} \subset \mathcal{R}^{\square}.$$

Thus, (F, G) is mildly $\square$–continuous. $\square$

Now, as an immediate consequence of Theorem 13.1 and 13.2, we can also state

Corollary 13.1. *If (F, g) is a pair relations on one relator space $(X, Z)(\mathcal{R})$ to another $(Y, W)(\mathcal{S})$ such that g is a function and $D_g = Z$, and $\square$ is a $*$–dominating, left composition compatible modification operation for relators, then the following assertions are equivalent:*

(1) (F, g) is mildly $\square$–continuous;

(2) (F, g) is upper $\square$–semicontinuous.

Proof. The implication $(1) \Longrightarrow (2)$ is immediate from Theorem 13.1. While, to prove the converse implication, we can note that now by Proposition 13.2 $\square$ is $*$–absorbing. Therefore, Theorem 13.2 can be applied. $\square$

References

[1] G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloq. Publ. **25**, Providence, Rhode Island, 1967.

[2] T. S. Blyth and M. F. Janowitz, *Residuation Theory*, Pergamon Press, Oxford, 1972.

[3] B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*, Cambridge Univ. Press, Cambridge, 2002.

[4] A. S. Davis, *Indexed systems of neigborhoods for general topological spaces*, Amer. Math. Monthly **68** (1961), 886–893.

[5] I. Del Prete and M. Di Iorio, *Semicontinuous set valued mappings and connected sets*, Matematiche Catania **46** (1991), 609–624.

[6] K. Denecke, M. Erné and S. L. Wismath (Eds.), *Galois Connections and Applications*, Math. Appl. **565**, Kluwer Acad. Publ., Dordrecht, 2004.

[7] J.- C. Derdérian, *Residuated mapppings*, Pacific J. Math. **20** (1967), 35–43.

[8] C. J. Everett, *Closure operators and Galois theory in lattices*, Trans. Amer. Math. Soc. **55** (1944), 514–525.

[9] P. Fletcher and W. F. Lindgren, *Quasi–Uniform Spaces*, Marcel Dekker, New York, 1982.

[10] B. Ganter and R. Wille, *Formal Concept Analysis*, Springer–Verlag, Berlin, 1999.

[11] J. L. Kelley, *General Topology*, Van Nostrand Reinhold, New York, 1955.

[12] J. Lambek, *Some Galois connections in elementary number theory*, J. Number Theory **47** (1994), 371–377.

[13] J. Mala, *Relators generating the same generalized topology*, Acta Math. Hungar. **60** (1992), 291–297.

[14] J. Mala and Á. Száz, *Properly topologically conjugated relators*, Pure Math. Appl. Ser. B **3** (1992), 119–136.

[15] J. Mala and Á. Száz, *Modifications of relators*, Acta Math. Hungar. **77** (1997), 69–81.

[16] B. L. McAllister, *Hyperspaces and multifunctions. The first half century (1900–1950)*, Nieuw Arc. Wisk. **26** (1978), 309–329.

[17] K. Meyer and M. Nieger, *Hüllenoperationen, Galoisverbindungen und Polaritäten*, Boll. Un. Mat. Ital. A (5) **14** 1977), 343–350.

[18] H. Nakano and K. Nakano, *Connector theory*, **56** (1975), 195–213.

[19] L. O'Carroll, *A basis for the theory of residuated groupoids*, J. London Math. Soc. **3** 1971), 7–20.

[20] O. Ore, *Galois connexions*, Trans. Amer. Math. Soc. **55** (1944), 493–513.

[21] G. Pataki, *Supplementary notes to the theory of simple relators*, Radovi Mat. **9** (1999), 101–118.

[22] G. Pataki, *On the extensions, refinements and modifications of relators*, Math. Balk. **15** (2001), 155–186.

[23] G. Pataki and Á. Száz, *A unified treatment of well-chainedness and connectedness properties*, Acta Math. Acad. Paedagog. Nyházi. (N. S.) **19** (2003), 101–165. (electronic)

[24] W. J. Pervin, *Quasi-uniformization of topological spaces*, Math. Ann. **147** (1962), 316–317.

[25] G. Pickert, *Bemerkungen über Galois-Verbindungen*, Arch. Math. **3** (1952), 285–289.

[26] Cs. Rakaczki and Á. Száz, *Semicontinuity and closedness properties of relations in relator spaces*, Mathematica (Cluj) **45** (2003), 73–92.

[27] J. Schmidt, *Bieträge zur Filtertheorie II*, Math. Nachr. **10** (1953), 197–232.

[28] R. E. Smithson, *Multifunctions*, Nieuw Arch. Wisk. **20** (1972), 31–53.

[29] Z. Smuely, *The structure of Galois connections*, Pacific J. Math. **54** (1974), 209–225.

[30] Á. Száz, *Basic tools and mild continuities in relator spaces*, Acta Math. Hungar. **50** (1987), 177–201.

[31] Á. Száz, *Lebesgue relators*, Monatsh. Math. **110** (1990), 315–319.

[32] Á. Száz, *Structures derivable from relators*, Singularité **3** (1992), 14–30.

[33] Á. Száz, *Refinements of relators*, Tech. Rep., Inst. Math., Univ. Debrecen **1993/76**, 19 pp.

[34] Á. Száz, *Uniformly, proximally and topologically compact relators*, Math. Pannon. **8** (1997), 103–116.
[35] Á. Száz, *An extension of Kelley's closed relation theorem to relator spaces*, Filomat (Nis) **14** (2000), 49–71.
[36] Á. Száz, *A Galois connection between distance functions and inequality relations*, Math. Bohem. **127** (2002), 437–448.
[37] Á. Száz, *Somewhat continuity in a unified framework for continuities of relations*, Tatra Mt. Math. Publ. **24** (2002), 41–56.
[38] Á. Száz, *Upper and lower bounds in relator spaces*, Serdica Math. J. **29** (2003), 239–270.
[39] Á. Száz, *An extension of Baire's category theorem to relator spaces*, Math. Morav. **7** (2003), 73–89.
[40] Á. Száz, *Rare and meager sets in relator spaces*, Tatra Mt. Math. Publ. **28** (2004), 75–95.
[41] Á. Száz, *Lower and upper bounds in ordered sets without axioms*, Tech. Rep., Inst. Math., Univ. Debrecen **2004/1**, 11 pp.
[42] Á. Száz, *The importance of reflexivity, transitivity, antisymmetry and totality in generalized ordered sets*, Tech. Rep., Inst. Math., Univ. Debrecen **2004/2**, 15 pp.
[43] Á. Száz, *Galois–type connectios on power sets and their applications to relators*, **2005/2**, 38 pp.
[44] Á. Száz, *Supremum properties of Galois–type connections*, Comment. Math. Univ. Carolin. **47** (2006), 569–583.
[45] Á. Száz, *Galois–type connections and closure operations on preordered sets*, Acta Math. Univ. Comen. **78** (2009), 1–21.
[46] Á. Száz, *Foundations of the theory of vector relators*, Adv. Stud. Contemp. Math. **20** (2010), 139–195.
[47] Á. Száz and J. Túri, *Comparisons and compositions of Galois-type connections*, Miskolc Math. Notes **7** (2006), 189–203.
[48] W. Xia, *Galois connections and formal concept analysis*, Demostratio Math. **27** (1994), 751–767.

Received 14.12.2011; available on internet 12.10.2012

Department of Mathematics, University of Debrecen, H-4010 Debrecen, Pf. 12, Hungary
E-mail address: szaz@science.unideb.hu