JOURNAL OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4866, ISSN (o) 2303-4947
www.imvibl.org /JOURNALS / JOURNAL
J. Int. Math. Virtual Inst., **13**(1)(2023), 95-136
DOI: 10.7251/JIMVI2301093R

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

SETS WITH APARTNESS ORDERED UNDER CO-QUASIORDER. REVIEW AND SOME NEW REFLECTIONS

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Abstract. The logical and working environment of this text is the Intuitionistic Logic - a logic without the principle of 'Tertium non datur' and with respect for the principled-philosophical orientation of Bishop's constructive mathematics. This orientation allows us to construct mathematical concepts on the relational structures of the type $(X, =, \neq)$ as the basic carriers where $' \neq '$ is a diversity relation / apartness significantly different than in the classic case. Studying algebraic structures with apartness in the period 1985-1988, this author reached completely natural to the concept of co-equality relations on sets with apartness. By varying the requirements for which these relations were determined, he constructed the concept of co-quasiorder relations (the concept of co-order relations) as the dual of the quasi-order relation (the relation of the partial ordering). In addition to the recapitulation of the properties of these relations, this article will also discuss co-ideals and co-filters in ordered sets under co-quasiorder. Also, some new results are presented in this article. Developed ideas and results are the specificity of the orientation of the Bishop's mathematics and often do not have their counterparts in the classical theory of ordered sets.

1. Introduction

1.1. Logical Environment. Our setting is Bishop's constructive mathematics **Bish** ([**3**], [**4**], [**7**], [**9**], [**22**] and [**64**]), mathematics developed with Constructive logic (or Intuitionistic logic **IL** [**64**]) - logic without the Law of Excluded Middle

2010 *Mathematics Subject Classification.* Primary 03F65; Secondary 06A99, 06F05.
Key words and phrases. Bishop constructive mathematics, set with apartness, co-equality relation, co-quasiorder relation and co-order relation, strongly extensional mapping between co-quasiordered (co-ordered) sets.
Communicated by Dusko Bogdanic.

$P \vee \neg P$ [TND]. We have to note that so-called 'the crazy axiom' $\neg P \Longrightarrow (P \Longrightarrow Q)$ is included in the Constructive logic. Precisely, in Constructive logic the 'Double Negation Law' $P \Longleftrightarrow \neg\neg P$ does not hold, but the following implication $P \Longrightarrow \neg\neg P$ holds even in Minimal logic. In Constructive logic 'Weak Law of Excluded Middle' $\neg P \vee \neg\neg P$ does not hold as well. It is interesting, in Constructive logic the following deduction principle $A \vee B, \neg A \vdash B$ holds, but this is impossible to prove it without 'the crazy axiom'.

Bishop's constructive mathematics includes the following two aspects:
(1) The Intuitionistic logic and
(2) The principled-philosophical orientations of constructivism.

1.2. Set with apartness. Intuitionistic logic does not accept the TND principle as an axiom. In addition, Intuitionistic logic does not accept the validity of the 'double negation' principle. This makes it possible to have a difference relation in sets which is not a negation of equality relation. Dual of the equality relations '=' in a set A is diversity relation '$\neq$'. Therefore, we accept that in Bishop's constructive mathematics we consider set A as one relational system $(A, =, \neq)$. Now, we look at the carrier A as a relational system $(A, =, \neq)$, where $'='$ is the standard equality, and $'\neq'$ is an *apartness* [**3, 7**]:
$(\forall x, y \in A)(x \neq y \Longrightarrow \neg(x = y))$ (consistency);
$(\forall x, y \in A)(x \neq y \Longrightarrow y \neq x)$ (symmetry);
$(\forall x, y, z \in A)(x \neq z \Longrightarrow (x \neq y \vee y \neq z))$ (co-transitivity).
This last relation is extensional in term of the equality in the following sense:

$$= \circ \neq \subseteq \neq \text{ and } \neq \circ = \subseteq \neq$$

where $'\circ'$ is the standard mark for the composition of the relations. It is obvious that the following connection between these relations is valid:

$$= \subseteq \neg \neq .$$

In this case for relations $=$ and $\neq$ we say that they are associate. So, it's quite natural to ask the question:

QUESTION 1.1. Let in the set X we have the equation relation '='. With $\neq_=$, for single use, we denote the family of apartness relations associated with the given relation $=$. How to check if $\neq_=$ is an empty set or it is inhabited? Is there the maximal relation '$\neq$' such that it is associated with equality '='?

NOTE 1.1. According to the above, it seems justified to consider that for one relation equality on a given set X there can be several apartness relations that associated with it. Accepting this possibility, there is a need to design a hierarchy among apartness relations associated with the given equality.

Generally speaking: Let S be a subset of set $(A, =, \neq)$ determined by a predicate $\mathfrak{P}$. The first task is to construct a dual T of the set S so that the subsets $\neg T = \{a \in A : \neg(a \in T)\}$ and its strong compliment $T^{\triangleleft} = \{a \in A : a \triangleleft T\}$ have property $\mathfrak{P}$ where $a \triangleleft T$ means $(\forall t \in T)(t \neq a)$. In addition, $T^{\triangleleft} \subseteq \neg T$ holds.

If the relation $' \neq '$ satisfies only the first two conditions, then it is said to be a '*diversity relation*'. Let X and Y be subsets of A. Let us determine $X \neq Y$ if is valid

$$(\exists x \in X)\neg(x \in Y) \vee (\exists y \in Y)\neg(y \in X).$$

Obviously, this relation is not an apartness relation in the family $\mathfrak{P}(A)$ of all subsets of A. In this collection, an analogous relation to an apartness relation in a set can be introduced on axiomatic way as it is, for example, done in the [**8**] or, in the way it is shown in article [**25**]. We will write $X \bowtie Y$ if it is

$$(\forall x \in X)(x \lhd Y) \wedge (\forall y \in Y)(y \lhd X).$$

This relation is a diversity relation in the family $\mathfrak{P}(A)$ of all subsets of A and it is not an apartness, in the general case.

For example, if $(A, =_A, \neq_A)$ and $(B, =_B, \neq_B)$ are sets with apartnesses, then at the $A \times B$ this relation is determined as follows

$$(\forall x, x' \in A)(\forall y, y' \in B)((x, y) \neq (x', y') \iff (x \neq_A x' \vee y \neq_B y')).$$

For a natural number n, we put $\overline{n} := \{1, 2, ..., n\}$ Let $\{(X_t, =_t, \neq_t)\}_{t\in\overline{n}}$ be a family of sets with apartness. For the pair $(x_1, ..., x_n), (y_1, ..., y_n) \in \prod_{t\in\overline{n}}$, it is defined

$$(x_1, ..., x_n) = (y_1, ..., y_n) \iff (\forall\, t \in \overline{n})(x_t =_t y_t)$$

$$(x_1, ..., x_n) \neq (y_1, ..., y_n) \iff (\exists\, t \in \overline{n})(x_t \neq_t y_t).$$

Generally, if $\{(X_y, =_t, \neq_y)\}_{t\in T}$ is a family of sets with apartness, then in the Cartesian product $\prod_{t\in T} X_t \quad (\neq \emptyset)$ the equality and the apartness are determined by the following:

$$(\forall\, a, b \in \prod_{t\in T} X_t)(a = b \iff (\forall\, t \in T)(a(t) =_t b(t))),$$

$$(\forall\, a, b \in \prod_{t\in T} X_t)(a \neq b \iff (\exists\, t \in T)(a(t) \neq_t b(t))).$$

Remark 1.1. In the previous analysis, accepting the validity of the condition $\prod_{t\in T} X_t \neq \emptyset$ in advance is necessary because, in Constructive Mathematics, which does not take into account the Axiom of choice, it is not possible to verify this requirement. For constructive set theory see [**1**].

For function $f : A \longrightarrow B$, it is said to be *strongly extensional* [**59**] if the following

$$(\forall x, y \in A)(f(x) \neq_B f(y) \implies x \neq_A y)$$

holds. For function $f : A \longrightarrow B$, it is said to be an *embedding* [**59**] if the following

$$(\forall x, y \in A)(x \neq y \implies f(x) \neq_B f(y))$$

holds.

In what follows, in the writing of relations of equality and the relation of apartnesses, we will always omit the indexes whenever it is possible and when it will not allow a different understanding of what the author imagined.

1.3. Our intention with this article. By choosing the Intuitionistic logic instead of the Classic logic as a logic background, when looking at and developing possible algebraic structures, the first opens much more possibilities than is the case with the Classic algebra. The obligation of mathematicians is to recognize them, describe them and correctly point out their properties and, if possible, prove them in a logically acceptable way. The designed material on logically possible ideas and concepts and their relationships with each other should be acceptable to those who affect the perception of the logic of the possible structures that it allows by prior orientation. The vast majority of mathematicians adhere to the orientation that Constructive Mathematics has the meaning in the following sense: The procedures and algorithms used in proving within its aspects must be constructive in the colloquial sense of the word. This attitude greatly narrows the scope of this domain. Speaking in a colloquial language, it is rather unusual (it can also be said astoundingly) that the academic community for a long time was reluctant to accept the possibilities that offer the choice of the Intuicionistic logic for background in the perception, understanding and development of algebraic structures.

By choosing Intuitionistic logic instead of classical logic for the work environment, the possibility of perceiving and analyzing the algebraic world parallel to the classical algebraic world is opened. This author deeply believes that such a world exists and that it should be of interest to both the academic community of mathematicians and the academic community of researchers in the philosophy of mathematics. By accepting the existence of an independent entity 'apartness relation $\neq$ on set $(S, =)$', which has a strong connection with the equation $=$, and the construction of algebraic structures on the relational system $(S, =, \neq)$ allows to mathematicians to accept the existence of two intertwined algebraic worlds. Such a commitment would enable them not only to see the newly discovered algebraic world, but also to better understand the classical world by recognizing the properties of the intertwining of these two algebraic worlds. For example, in inverse semigroups ([**12, 58**]), designed on the relational system $(S, =, \neq)$, the existence of an interconnected pair of natural order relations on them can be shown: the natural order $\leqslant$ and the natural co-order $\nleqslant$. The observed environment enables the recognition of pairs of interconnected substructures such as, for example, ideals and co-ideals, and filters and co-filters on such semigroups. Also, connections between elements in such semigroups often occur in pairs: for example, one such pair is the concepts of congruences and co-congruences on semigroups with apartness.

It is now quite natural to try to answer the question:

Are inverse semigroups and inverse semigroups with apartness one and the same class of semigroups?

If we look at these algebraic structures through the eyes of a traditional mathematician, then it is obvious that they are two classes of algebraic structures built on different supports. If we look at these algebraic structures through the eyes of an open-minded mathematician through non-traditional glasses, then, of course, it is only one class of algebraic structures. The essence is that looking through the first glasses, one cannot notice the complexities of algebraic structures, but perceiving these complexities in constructed algebraic structures allow looking through other spectacles.

In this text, some of the received results that do not have their counterparts in the Classical theory are presented, such as, for example, the concept of co-equality relations on sets with apartness (Section 2). Some of the information in subsection 2.4 (Theorem 2.7 and Corollary 2.3) and subsection 2.6 (Theorem 2.12) appears for the first time. In Section 3 are presented the concept of co-quasiorder relations on sets with apartness and described some of the obtained results related to ordered sets under these relations. Part of the material on display (Definition 3.5 and Definition 3.6, Proposition 3.5 and Proposition 3.6) in Subsection 3.2 is new and it appears for the first time. In section 4 we analyze the mappings between co-quasiordered relational systems. The material exposed in subsection 4.5 is new an it is appears for the first time.

2. Co-equality relations

2.1. Insight into the historical development of the concept. Let $(R, =, \neq, +, 0, \cdot, 1)$ be a commutative ring in the sense of books [**7, 26, 59, 29, 64**] and papers [**32**]. A subset K of R is a co-ideal of R ([**32**], Definition2) if the following holds

$0 \lhd K$,
$-x \in K \implies x \in K$,
$x + y \in K \implies (x \in K \vee y \in K)$ and
$xy \in K \implies (x \in K \wedge y \in K)$.

Co-ideals of commutative ring with apartness was first defined and studied by Wim Ruitenburg 1982 in his dissertation [**59**]. After that, co-ideals (anti-ideals) studied by A. S. Troelstra and D. van Dalen in their monograph [**64**]. Co-ideals of commutative rings with apartness studied by this author in his dissertation [**29**] and several his papers (see, for example, [**32**]). He proved, in [**32**], if K is a co-ideal of R, then the relation q on R, defined by

$$(x, y) \in q \iff x - y \in K,$$

satisfies the following properties

(1) $(\forall a \in R)((a, a) \lhd q)$, (consistency)
(2) $(\forall a, b \in R)((a, b) \in q \Longrightarrow (b, a) \in q)$, (symmetric)
(3) $(\forall a, b, c \in R)((a, c) \in q \implies ((a, b) \in q \vee (b, c) \in q))$, (co-transitivity)
(4) $(\forall a, b, c, d \in R)((a + c, b + d) \in q \implies ((a, b) \in q \vee (c, d) \in q)$,
(5) $(\forall a, b, c, d \in R)((ac, bd) \in q \implies ((a, b) \in q \vee (c, d) \in q))$.

The relation q on R, which satisfies the above properties, is called co-congruence on R ([**32**]). Inversely, if q is a co-congruence on a ring R, then the set $Q = \{a \in R : (a, o) \in q\}$ is a co-ideal of R ([**32**], Proposition 2.5). Let J be an ideal of R and let K be a co-ideal of R. Ruitenburg, in his dissertation [**59**], first stated a demand that

$$J \subseteq \neg K.$$

This condition is equivalent with the following condition

$$(\forall a, b \in R)(a \in J \wedge b \in K \Longrightarrow a + b \in K).$$

In this case, we say that J and K are associated. A relation q on a set R with apartness, which satisfies the conditions (1)-(3), is called a co-equality relation. Coequality relation was first defined and studies by D. A. Romano in his dissertation [**29**] and in his early works (see, for example, [**30, 5, 31, 33, 34, 35, 36, 37, 43, 44, 45, 46**]).

2.2. Concept of co-equality relations. Let ρ be an equivalence relation on the set A. For the relation q we say that it is a *co-equality* relation on A [**29, 38**] if and only if the following is valid

$q \subseteq \neq$, (consistency), $q^{-1} = q$, (symmetric) and $q \subseteq q * q$. (co-transitivity).

Here, '$*$' is the filed product between relations defined by the following way: If α and β are relations on set A, then filed product $\beta * \alpha$ of relation α and β is the relation given by $\{(x, z) \in A \times A : (\forall y \in A)((x, y) \in \alpha \vee (y, z) \in \beta)\}$.

If the relations ρ and q are associated, i.e., if the following $\rho \circ q \subseteq q$ and $q \circ \rho \subseteq q$ holds, it is possible to design the factor-set

$$A/(\rho, q) := \{x\rho : x \in A\}$$

where the equality '$=_1$' and the apartness '$\neq_1$ on it are determined in the following way:

$$(\forall x, y \in A)(a\rho =_1 y\rho \iff (x, y) \in \rho)$$

and

$$(\forall x, y \in A)(x\rho \neq_1 y\rho \iff (x, y) \in q).$$

It is easy to check that the relations of equality '$=_1$' and apartness '$\neq_1$', determined in this way, are associated: Let $x, y, z \in A$ be arbitrary elements such that $x\rho =_1 y\rho$ and $y\rho \neq_1 z\rho$. Then $(x, y) \in \rho$ and $(y, z) \in q$. From $(y, z) \in q$ it follows $(y, x) \in q \vee (x, z) \in q$ by co-transitivity of q. Thus $(x, z) \in q$ since the first option is impossible according to associativity of ρ and q. This means $x\rho \neq_1 z\rho$. That the second inclusion is valid can be shown in an analogous way to the previous one.

In order to emphasize the differences, in this analysis we used indices with the relations equality and co-equality on the factor set $(A/(\rho, q), =_1, \neq_1)$ in relation to the initial set $(A, =, \neq)$.

Of course, the strong compliment $q^{\lhd}$ of the relation q is an equivalence in A and the following

$$q^{\lhd} \subseteq \neg q, \quad q \circ q^{\lhd} \subseteq q \text{ and } q^{\lhd} \circ q \subseteq q$$

are valid. Although the evidence of this claim is known, we will again show it here that the reader can gain an impression of the proof technique that is applied.

PROPOSITION 2.1. *The strong compliment $q^{\lhd}$ of the relation q is an equivalence in A and the following $q^{\lhd} \subseteq \neg q, \quad q \circ q^{\lhd} \subseteq q$ and $q^{\lhd} \circ q \subseteq q$ holds.*

PROOF. Obviously, It is true that $= \subseteq q^{\lhd}$ and that $q^{\lhd}$ is a symmetric relation. We need to prove that $q^{\lhd}$ is a transitive relation. Let $x, y, z, u, v \in A$ arbitrary elements such that $(x.y) \lhd q$, $(y, z) \lhd q$ and $(u, v) \in q$. Then

$$(u,x) \in q \vee (x,y) \in q \vee (y,z) \in q \vee (z,v) \in q$$

by co-transitivity of q. From here follows $(u,x) \in q \subseteq \neq$ or $(z,v) \in q \subseteq \neq$ by consistency of q and by taking into account the hypothesis of this deduction. So, $u \neq x$ and $z \neq v$ therefore $(x,z) \neq (u,v) \in q$. Finally, we have $(x,z) \lhd q$. On this way, the transitivity of the relation q is proven.

For the sake of illustration, we will prove inclusion $q \circ q^{\lhd} \subseteq q$. The second inclusion can be proven by an analogous way. Let x,z be arbitrary elements of A such that $(x,z) \in q \circ q^{\lhd}$. Then there exists an element $y \in A$ such that $(x,y) \in q^{\lhd}$ and $(y,z) \in q$. Thus $(y,x) \in q$ or $(x,z) \in q$ by co-transitivity of q. Since, the first option is impossible because $q^{\lhd}$ is a symmetric relation on A and $(y,x) \lhd q$, we have $(x,z) \in q$. □

As corollary of above Proposition 2.1 we can construct the quotient-set

$$A/(q^{\lhd}, q) = \{aq^{\lhd} : a \in A\}$$

with

$$aq^{\lhd} = bq^{\lhd} \iff (a,b) \lhd q \text{ and } aq^{\lhd} \neq bq^{\lhd} \iff (a,b) \in q.$$

For the total surjective function

$$\pi : A \longrightarrow A/(q^{\lhd}, q),$$

defined by the $\pi(a) = aq^{\lhd}$ $(a \in A)$, it is said that *the canonical mapping* from A onto $A/(q^{\lhd}, q)$.

Proposition 2.2. *The canonical mapping* $\pi : A \longrightarrow A(q, q^{\lhd})$ *is strongly extensional.*

Proof. Indeed: Let $xq^{\lhd}$ and $yq^{\lhd}$ two arbitrary elements of $A/(q^{\lhd}, q)$ such that $\pi(x) = xq^{\lhd} \neq yq^{\lhd} = \pi(y)$. Then $(x,y) \in q$. Thus $x \neq y$ by consistency of q. □

The following theorem describes some basic properties of the classes of the relation q.

Theorem 2.1 ([**5**]). *Let q be a co-equality relation on a set $(A,=,\neq)$. For the family $\{qx\}_{x\in A}$, where $qx = \{y \in A : (x,y) \in q\}$, the following holds:*

(i) $x \lhd xq$; (ii) $xq = qx$; (iii) $(x,y) \in q \Longrightarrow xq \cup yq = A$ *for any* $x,y \in A$.

Proof. Let $x \in A$ be an arbitrary elements and let u be an arbitrary element of xq. Then $(u,x) \in q \subseteq \neq$. Thus $x \neq u \in xq$. So, $x \lhd xq$. It is clear that $xq = qx$ since q is symmetric. Let (x,y) be an arbitrary element of q and let t be an arbitrary element of A. Then $(x,t) \in q \vee (t,y) \in q$ by co-transitivity of q. Thus $t \in xq \vee t \in qy = yq$. So, $t \in xq \cup yq$. Finally, $(x,y) \in q \Longrightarrow A = xq \cup yq$. □

Example 2.1. (1) The relation $\neg(=)$ is an apartness relation on the set $\mathbb{Z}$ of integers.

(2) The relation q, defined on the set $\mathbb{Q}^{\mathbb{N}}$ by

$$(f,g) \in q \iff (\exists k \in \mathbb{N})(\exists n \in \mathbb{N})(m \geqslant n \Longrightarrow |f(m) - g(m)| > k^{-1}),$$

is a coequality relation.

(3) A ring R is a local ring if for each $r \in R$, either r or $1 - r$ is a unit, and let M be a module over R. Then the relation q on M, defined by $(x, y) \in q$ if there exists a homomorphism $f : R \longrightarrow M$ such that $f(x - y)$ is a unit, is a coequality relation on M.

EXAMPLE 2.2. ([**33**], Theorem 4) Let T be a set and J be a subfamily of $\mathfrak{P}(T)$ such that

$$\emptyset \in J,\ A \subseteq B \wedge B \in J \Longrightarrow A \in J,\ A \cap B \in J \Longrightarrow A \in J \vee B \in J.$$

If $\{X_t\}_{t\in T}$ is a family of sets, then the relation q on $\prod_{t\in T} X_t$ $(\neq \emptyset)$, defined by

$$(f, g) \in q \iff \{s \in T : f(s) = g(s)\} \in J,$$

is a coequality relation on the Cartesian product $\prod_{t\in T} X_t$.

EXAMPLE 2.3. ([**31**]) Let T be a set and K be a subfamily of $\mathfrak{P}(T)$ such that

$$T \in K,\ A \subseteq B \wedge A \in K \Longrightarrow B \in K,\ A \cup B \in K \Longrightarrow A \in K \vee B \in K.$$

If $\{X_t\}_{t\in T}$ is a family of sets, then the relation q on $\prod_{t\in T} X_t$ $(\neq \emptyset)$, defined by

$$(f, g) \in q \iff \{s \in T : f(s) \neq g(s)\} \in K,$$

is a coequality relation on the Cartesian product $\prod_{t\in T} X_t$.

2.3. Concept of co-partitions. Before proceeding to a further analysis, we recall the term 'strongly extensional subset of the set': for a subset X of the set $(A, =, \neq)$ it is said that a strongly extensional subset of the set A if

$$(\forall x, y \in A)(x \in X \Longrightarrow (x \neq y \vee y \in X))$$

is valid. A coequality relation q on A is strongly extensional subset of $A \times A$: Let $(x, y) \in q$ and $(u, v) \in A \times A$ be arbitrary elements. Then:

$$\begin{aligned}(x, y) \in q &\Longrightarrow ((x, u) \in q \vee (u, v) \in q \vee (v, y) \in q)\\ &\Longrightarrow (x \neq u \vee (u, v) \in q \vee v \neq y)\\ &\Longrightarrow ((x, y) \neq (u, v) \vee (u, v) \in q).\end{aligned}$$

In addition, it has shown that the classes of any co-equality relation q are strongly extensional subsets of A. Indeed. Let x, y and u be arbitrary elements of A such that $u \in xq$. Then $(x, u) \in q$. This $(x, y) \in q \in \vee (y, u) \in q$ by co-transitivity of q. It is follows $y \in xq \vee y \neq u$. So, the subset xq is a *strongly extensional subset* of A.

Now, suppose that a family $\{X_t\}_{t\in A}$ of strongly extensional proper subsets of A satisfies the following two conditions:

(a) For any $t \in A$ there exists a strongly extensional subset X_t such that that $t \lhd X_t$;

(b) $X_t \neq X_s \Longrightarrow X_t \cup X_s = A$ for any $t, s \in A$.

Then the relation R on A defined by

$$(x, y) \in R \iff (\exists u \in A)(x \in X_u \wedge y \lhd X_u)$$

is a co-equality in A ([**38, 52**]).

For a set $(A, =, \neq)$ and a co-equality relation q on A, the family $\{aq : a \in A\}$ we will indicate with $[A : q]$. The family $V = \{Y_u\}_{u \in U}$ of subsets of A is a *co-partition* of A if V satisfies the conditions (a) and (b).

THEOREM 2.2 ([**5**]). *Let V be a subfamily of $\mathcal{P}(A)$ such that it satisfies the conditions (a) and (b). Then the relation*

$$q(V) = \{(y, z) \in A \times A : (\exists Y \in V)(y \lhd Y \wedge z \in Y)\}$$

is a coequality relation on A.

PROOF. (1) Let (y, z) be an element of $q(V)$, i.e. let Y be an element of the family V such that $y \lhd Y$ and $z \in Y$. From $z \in Y \subseteq A$ follows that there exists a subset Z of X such that $z \lhd Z$. So $Y \neq Z$ and $Y \cup Z = A$. Therefore $y \in Z$ and $(z, y) \in q$. So, the relation $q(V)$ is a symmetric relation on A.

(2) Let u be an arbitrary element of A. Then there exists a subset U of A such that $u \lhd U$. Let (y, z) be an arbitrary element of $q(V)$. Then

$$(\exists Y \in V)(y \lhd Y \wedge z \in Y) \text{ and } (\exists Z \in V)(z \lhd Z \wedge y \in Z).$$

From $u \in X = Y \cup Z$ follows $u \in Y$ or $u \in Z$. So, $(y, u) \in q(V)$ or $(z, u) \in q(V)$. Therefore, the relation $q(V)$ is a co-transitive relation on A.

(3) Let u be an element of A and let $(y, z) \in q(V)$. Then

$$(\exists Y \in V)(y \lhd Y \wedge z \in Y) \vee (\exists Z \in V)(z \lhd Z \wedge y \in Z)$$

and $Y \cup Z = A$. From $u \in A = Y \cup Z$ follows $u \in Y$ or $u \in Z$. If $u \in Y$, then $u \neq y$. If $u \in Z$, yhen $u \neq z$. So, $(u, u) \neq (y, z)$. So, the relation $q(V)$ is a consistent relation on A. $\square$

REMARK 2.1. When this result was first time published in 1996 [**38**], some of the members of the academic public refused to accept that there were scientific needs for researching of such families of subsets of a set with apartness. The difficulties encountered by researchers of algebraic structures based on sets with apartness relations were nicely described by Bauer in his article [**2**]. Even today, many working mathematicians are extremely reluctant to accept the possibility of existence an academic interest in researching and developing of structures in a logical environment that is not the Classic logic. Unfortunately, the existence of interest in publishing the results of the research of algebraic structures based on sets with apartness is still being classify in activity with the prefix of exoticism.

The next theorem shows that if we generate a co-partition $[A : q']$ by means of a co-equality relation q', then the co-equality relation $q([A : q'])$ generated by the co-partition $[A : q']$ is simply q' again; and similarly if we begin with the co-equality relation $q(V)$ generated by a co-partition V, this relation generates the given co-partition $[A : q(V)] = V$. Since the article with this results is published in a journal that is not available to the mathematical public now, the proof of this theorem is repeated.

THEOREM 2.3 ([**5, 38**]). *Let* $(A, =, \neq)$ *be a set with apartness. Let* $Q(A) = \{q \in A \times A : q\,\textit{is a co-equality relation on}\,A\}$ *and let* $V(A) = \{V \subseteq P(A) : V\textit{ is a co-partition on } A\}$. *Then*

(1) $q([A : q']) = q'$ *for every* $q \in Q(X)$*;*

(2) $V(q(V)) = V$ *for every* $V \in V(A)$.

PROOF. (1) Let q' be a co-equality relation on A. Then

$$
\begin{aligned}
((x,y) \in q' &\iff y \in qx = Y_x \\
&\Longrightarrow (x \lhd Y_x \wedge y \in Y_x) \\
&\Longrightarrow (x,y) \in q(A : q']).
\end{aligned}
$$

$$
\begin{aligned}
(x,y) \in q([A : q']) &\iff (\exists Y_u \in [A : q'])(x \lhd Y_u \wedge y \in Y_u) \\
&\iff (\exists Y_u \in [A : q'])(x \lhd Y_u \wedge (y,u) \in q') \\
&\Longrightarrow (\exists Y_u \in [A : q'])(x \lhd Y_u \wedge ((y,x) \in q' \vee (x,u) \in q')) \\
&\Longrightarrow (\exists Y_u \in [A : q'])(x \lhd Y_u \wedge (y,x) \in q') \\
&\Longrightarrow (y,x) \in q'.
\end{aligned}
$$

(2) Let $P \subseteq \mathcal{P}(A)$. If $Y \in P$, then $Y \subset A$ and $(\exists x \in A)(x \lhd Y)$. So, for every $y \in Y$, we have $(x,y) \in q(P)$. Therefore $y \in Y_x$. Thus $Y \subseteq Y_x$. At the other hand, $u \in Y_x$ implies $(x,u) \in q(P)$, i.e. $x \lhd Y$ and $u \in Y$. So, $Y = Y_x$. We have $P \subset [A : q(P)]$. Let $xq = Y_x \in [A : q(P)]$. Then for every element y of Y_x we have $(x,y) \in q(P)$. Thus we conclude that there exists $Y \in P$ such that $x \lhd Y$ and $y \in Y$. So $Y_x \subseteq Y$. Therefore $Y_x \in P$, and $[A : q(P)] \subseteq P$. □

In the second part of proof of Theorem 2.3, we have the following statement:

COROLLARY 2.1. *Let* q *be a co-equality relation on a set* A *with apartness,* x *be an arbitrary element of* A *and let* Y *be an arbitrary element of* $[A : q]$ *such that* $x \lhd Y$. *Then* $Y_x = Y$.

2.4. Some important theorems. Let $f : A \longrightarrow B$ be a strongly extensional function between sets with apartness. Our first theorem links the function f with the function π.

THEOREM 2.4. *Let* $f : A \longrightarrow B$ *be a strongly extensional mapping between sets with apartness. The the relation*

$$Coker(f) = \{(x,y) \in A \times A : f(x) \neq f(y)\}$$

is a co-equality relation on A *and there exists the bijection*

$$g : A/(Ker(f), Coker(f)) \longrightarrow f(A) \subseteq B$$

such that $f = g \circ \pi$.

Without major difficulties, it can be verified that there exists a strongly extensional surjective function $\vartheta : A \longrightarrow [A : q]$, determined by $\vartheta(a) = aq$, and the bijection (= strongly extensional surjective, injective and embedding function) $h : A/(q^{\lhd}, q) \longrightarrow [A : q]$ such that $\vartheta = h \circ \pi$ and $\pi = h^{-1} \circ \vartheta$, where $\pi : A \longrightarrow A/(q^{\lhd}, q)$ is the canonical surjective function.

PROPOSITION 2.3. *Let q be a co-equality relation on a set A. Then:*
- The correspondence $\vartheta : A \longrightarrow [A : q]$ is a strongly extensional surjective mapping;
- There exists the bijective mapping $h : A/(q^{\triangleleft}, q) \longrightarrow [A : q]$ such that

$$\vartheta = h \circ \pi \text{ and } \pi = h^{-1} \circ \vartheta.$$

PROOF. (1) Let ϑ be defined by $\vartheta(x) = xq$ for any $x \in A$. It is clear that ϑ is a correct defined mapping from A onto $[A : q]$. If x, y be arbitrary elements of A such that $\vartheta(x) = xq \neq yq = \vartheta$, then $(x, y) \in q$. Thus $x \neq y$. So, ϑ is a strongly extensional surjective mapping from A onto $[A : q]$.

(2) Let us define a correspondence $h : A/(q^{\triangleleft}, q) \longrightarrow [A : q]$ by $h(xq^{\triangleleft}) = xq$ for any $x \in A$. Let x and y be arbitrary elements of A such that $xq^{\triangleleft} = yq^{\triangleleft}$. Then $(x, y) \triangleleft q$ and $h(xq^{\triangleleft}) = xq = yq = h(yq^{\triangleleft})$. So, h is a correct defined mapping from $A/(q^{\triangleleft}, q)$ to $[A : q]$. Obviously, the reverse implication

$$(\forall x, y \in A)(h(xq^{\triangleleft}) = xq = yq = h(yq^{\triangleleft}) \implies xq^{\triangleleft} = yq^{\triangleleft})$$

is valid too. Therefore, h is an injective mapping.

Let $x, y \in A$ be arbitrary elements. Then

$$xq^{\triangleleft} \neq yq^{\triangleleft} \iff (x, y) \in q \iff h(xq^{\triangleleft}) = xq \neq yq = h(xq^{\triangleleft}).$$

So, h is a strongly extensional and embedding mapping from $A/(q^{\triangleleft}, q)$ onto $[A : q]$.

Finally, since that

$$\vartheta(x) = xq = h(xq^{\triangleleft}) = h(\pi(x)) = (h \circ \pi)(x)$$

holds for any $x \in A$, we have $\vartheta = h \circ \pi$. □

The follows theorem is one specificity of this aspect of observing sets (and algebraic structures) and there is no a counterpart in corresponding the classical theory.

THEOREM 2.5. *Let $f : A \longrightarrow B$ be a strongly extensional mapping between sets with apartness. Then there exists the bijection*

$$k : [A : Coker(f)] \longrightarrow f(A) \subseteq B$$

such that $k = g \circ h^{-1}$ and $f = k \circ \vartheta$.

PROOF. If $f : A \longrightarrow B$ be a strongly extensional mapping, then there exists the bijection

$$g : A/(Ker(f), Coker(f)) \longrightarrow f(A)$$

by Theorem 2.1 and the bijection

$$h : A/(Ker(f), Coker(f)) \longrightarrow [A : Coker(f)]$$

by Proposition 2.3 such that

$$f = g \circ \pi = g \circ (h^{-1} \circ \vartheta) = (g \circ h^{-1}) \circ \vartheta.$$

Therefore, $k = g \circ h^{-1}$ is the required bijection. □

Let us recall (Subsection 2.2) that for the couple ρ and q of a an equality relation and a co-equality relation we say they are associated if holds $\rho \circ q \subseteq q$ and $q \circ \rho \subseteq q$. In this case, the factor-set $A/(\rho, q)$ can be constructed with the equality and apartness determined by

$$(\forall x, y \in A)(x\rho = y\rho \iff (x, y) \in \rho),\ (\forall x, y \in A)(x\rho \neq y\rho \iff (x, y) \in q).$$

For example, if q is a co-equality relation on A, then $q^{\triangleleft}$ and q are associated. If $f : A \longrightarrow B$ is a strongly extensional function, then the relation $Ker(f)$ and $Coker(f)$ are associated, too.

THEOREM 2.6 ([**34**]). *Let ρ and q be an equality and a co-equality relations on a set A. Then ρ and q are associated if and only if*

$$(\forall x, z \in A)(x \neq z \wedge x\rho \cap zq \neq \emptyset \implies x\rho \subseteq zq).$$

Since the article [**34**] is not available to the public in electronic form, the proof of this theorem will be shown again.

PROOF. (1) Let ρ and q be associated an equality and a co-equality relations on set A and suppose that $x\rho \cap zq \neq \emptyset$ for each $x, z \in A$ such that $x \neq z$. Then there exists an element $y \in A$ such that $y \in x\rho$ and $y \in zq$. These means $(x, y) \in \rho$ and $(y, z) \in q$. Thus $(x, z) \in q$. Let u be an arbitrary element of $x\rho$. Then $(u, x) \in \rho$. Further, from $(u, x) \in \rho$ and $x, z) \in q$ follows $(u, z) \in q$ since ρ and q are associated relations. So, $u \in zq$.

(2) Let the formula $(\forall x, z \in A)(x \neq z \wedge x\rho \cap zq \neq \emptyset \implies x\rho \subseteq zq)$ be valid and let $x, y, z \in$ arbitrary elements such that $(x, y) \in \rho$ and $(y, z) \in q$. First, from $(y, z) \in q$ follows $(x, y) \in q$ or $(x, z) \in q$. Let $(x, y) \in q$ be holds. Then $x \neq y$ and $x \in x\rho \cap yq \neq \emptyset$. Thus $y \in x\rho \subseteq yq$. We got a contradiction. So, have to be $(x, z) \in q$. Therefore, relations ρ and q are associated. $\square$

On the other hand, the validity of the following claim can be checked without difficulty:

COROLLARY 2.2. *Let $(A, =, \neq)$ be a set with apartness and ρ and q ne an equality and a co-equality relations A. Then ρ and q associatd relations if and only if $\rho \cap q = \emptyset$.*

PROOF. Let ρ and q be associated an equality relation and a co-equality relation on A. Suppose $\rho \cap q \neq \emptyset$. Then there exists an element $(u, v) \in \rho \cap q$. This meas $(u, v) \in \rho$ and $(u, v) \in q$. Then from $(v, u) \in \rho \wedge (u, v) \in q$ it follows $(v, v) \in q$ by associativity of relations ρ and q which is impossible by consistency of q. The obtained contradiction refutes the assumption $\rho \cap q \neq \emptyset$. Therefore, it must be $\rho \cap q = \emptyset$.

Conversely, let $\rho \cap q = \emptyset$ be valid. Let $u, z \in A$ be elements such that $(u, z) \in \rho \circ q$. Then there exists an element $v \in A$ such that $(u, v) \in q$ and $(v, z) \in \rho$. Thus from $(u, v) \in q$ it follows $(u, z) \in q \vee (z, v) \in q$ by co-transitivity of q. Since the second option is impossible because $(z, v) \in q \cap \rho = \emptyset$, we have $(y, z) \in q$. The second inclusion can be proved analogously to the previous. $\square$

The question naturally arises:

QUESTION 2.1. Let ρ be an equality relation on a set with apartness A. Let q_ρ, for single use, be the family of all coequality relations on A associated with ρ. Is this family empty or inhabited? For the given equivalence ρ is there the maximal coequality relation q associated with ρ?

In the Subsection 2.6, one reflection will be offered as a partial answer to this question.

In what follows in this subsection we will pay attention to the family $[A : q]$. Let q_1 and q_2 be co-equality relations on A such that $q_2 \subseteq q_1$. We can construct co-partitions $[A : q_1]$ and $[A : q_2]$. In the following theorem we will give a construction of relation $[q_2 : q_1]$ on $[A : q_1]$ by relations q_1 and q_2.

THEOREM 2.7. *Let q_1 and q_2 be coequality relations on set with apartness A such that $q_2 \subseteq q_1$. Then the relation $[q_2 : q_1] \subseteq [A : q_1] \times [A : q_1]$, defined by*

$$(\forall xq_1, yq_1 \in [A : q_1])((xq_1, yq_1) \in [q_2 : q_1] \iff (x, y) \in q_2)$$

is a co-equality relation on $[A : q_1]$.

PROOF.

(i) $$\begin{aligned}(aq_1, bq_1) \in [q_2 : q_1] &\iff (a, b) \in q_2\\ &\implies (\forall u \in A)((a, u) \in q_2 \vee (u, b) \in q_2)\\ &\implies (\forall uq_1 \in [A : q_1])((aq_1, uq_1) \in [q_2 : q_1] \vee (uq_1, bq_1) \in [q_2 : q_1]).\end{aligned}$$

(ii) $$\begin{aligned}(aq_1, bq_1) \in [q_2 : q_1] &\iff (a, b) \in q_2\\ &\iff (b, a) \in q_2\\ &\iff (bq_1, aq_1) \in [q_2 : q_1].\end{aligned}$$

(iii) $$\begin{aligned}(aq_1, bq_1) \in [q_2 : q_1] &\implies ((a, b) \in q_2 \subseteq q_1\\ &\implies (\forall u \in A)((a, u) \in q_1 \vee (u, b) \in q_1)\\ &\implies (\forall uq_1 \in [A : q_1])(aq_1 \neq uq_1 \vee uq_1 \neq bq_1).\\ &\implies (\forall uq_1 \in [A : q_1])((aq_1, bq_1) \neq (uq_1, uq_1)).\end{aligned}$$ □

COROLLARY 2.3. *Let q_1 and q_2 be co-equality relations on set with apartness A such that $q_2 \subseteq q_1$. Then there exists a strongly extensional bijective and embedding function*

$$\varphi : [[A : q_1] : [q_2 : q_1]] \longrightarrow [A : q_2].$$

PROOF. Let notations be as in the previous theorem and let $\phi : [A : q_1] \longrightarrow [A : q_2]$ be mapping defined by $\phi(xq_1) = xq_2$ for any $x \in A$.

(i) Let $xq_1 = yq_1$ and let $u \in xq_2$ where $x, y \in A$. Then $(u, x) \in q_2$ and $(u, y) \in q_2 \vee (y, x) \in q_2 \subseteq q_1$. Thus, from the second option we have $x \in yq_1 = xq_1$. It is impossible because $x \lhd xq_1$. So, gave to be $u \in yq_2$. Therefore, $xq_2 \subseteq yq_2$. Reverse inclusion $yq_2 \subseteq xq_2$ can be proven by analogously to the previous one. Finally. $xq_2 = yq_2$ and the mapping ϕ is correctly defined.

(ii) Let $\phi(xq_1) = xq_2 \neq yq_2 = \phi(yq_1)$ for some $x, y \in A$. Then $(x, y) \in q_2 \subseteq q_1$ and $xq_1 \neq yq_1$. So, the mapping ϕ is a strongly extensional function.

(iii) Let xq_2 be an element of $[A : q_2]$. Then xq_1 is an element of $[A : q_1]$ such that $\phi(xq_1) = xq_2$. So, the function ϕ is a surjective mapping.

(iv) First, from $xq_2 = \phi(xq_1) \neq \phi(yq_1) = yq$ we have $(x, y) \in q_2$. Thus follows, the relation

$$\{(xq_1, yq_1) \in [A : q_1] \times [A : q_1] : (x, y) \in q_2\} = [q_2 : q_1]$$

is a co-equality relation on $[A : q_1]$ and the mapping

$$\vartheta : [A : q_1] \longrightarrow [[A : q_1] : [q_2 : q_1]]$$

is a surjective by Proposition 2.3. Therefore, there exists strongly extensional and embedding surjective function

$$\varphi : [[A : q_1] : [q_2 : q_1]] \longrightarrow [A : q_2]$$

such that $\phi = \varphi \circ \vartheta$. $\square$

Let q be a coequality relation on a set A and let $f : A \times A \longrightarrow A$ be a strongly extensional mapping. We say that f is *compatible* with the coequality relation q if

$$(\forall x, y, u, v \in A)((f(x, y), f(u, v)) \in q \Longrightarrow (x, u) \in q \vee (y, v) \in q)$$

holds. In the following theorem we give a result on compatibility of function $f : A^2 \longrightarrow A$ with the given coequality relation q on the set A.

THEOREM 2.8 ([**52**]). *If the strongly extensional mapping $f : A^2 \longrightarrow A$ is compatible with the coequality relation q on A, then there is a strongly extensional mapping*

$$F : [A : q] \times [A : q] \longrightarrow [A : q]$$

such that

$$\vartheta \circ f = F \circ (\vartheta, \vartheta).$$

PROOF. Let us define mapping F by $F(uq, vq) = f(u, v)q$. Then:

(1) Let $(xq, yq) = (uq, vq)$. It means $xq = uq$ and $yq = vq$. Suppose that $s \in f(x, y)q$, i.e. suppose that $(f(x, y), s) \in q$. Thus, by co-transitivity of q, we have $(f(x, y), f(u, v)) \in q$ or $(f(u, v), s) \in q$. Hence, by compatibility f and q follows $(x, u) \in q$ or $(y, v) \in q$ or $s \in f(u, v)q$. So, $s \in f(u, v)q$ because $(x, u) \triangleright q$ and $(y, v) \triangleright q$. Finally, we have $f(x, y)q \subseteq f(u, v)q$. We also have $f(u, v)q \subseteq f(x, y)q$ by analogy. Finally, we have $f(u, v)q = f(x, y)q$. Therefore, the correspondence F is a mapping.

(2) Let $F(uq, vq) \neq F(xq, yq)$ be holds for $uq, vq, xq, yq \in [A : q]$. It means $f(u, v)q \neq f(x, y)q$ and $(f(u, v), f(x, y)) \in q$. Since the mapping f is compatible with q, follows $(u, x) \in q$ or $(v, y) \in q$. Finally, we have $uq \neq xq$ or $vq \neq yq$. So, the mapping F is a strongly extensional.

(3) Let (x, y) be an arbitrary pair of elements of $A \times A$. We have

$$\begin{aligned}(\vartheta \circ f)(x, y) &= \vartheta(f(x, y)) = f(x, y)q = F(xq, yq)\\ &= F(\vartheta(x), \vartheta(y)) = F((\vartheta, \vartheta)(x, y)) = (F \circ (\vartheta, \vartheta))(x, y).\end{aligned}$$

Therefore, seeking equality is valid. $\square$

Accepting the validity of the previous theorem makes it possible to design an algebraic structure that has no counterpart in the classical theory of algebraic structures. Let $(A, =, \neq)$ be set with apartness. A total strongly extensional function $w : A \times A \longrightarrow A$ is a binary internal operation in A and it constructs an algebraic structure $((A, =, \neq), w)$ with

$$(\forall x, y, u, v \in A)((x, y) = (y, v) \implies w(x, y) = w(u, v))$$

$$(\forall x, y, u, v \in A)(w(x, y) \neq (u, v) \implies (x \neq u \vee y \neq v)).$$

If w is an associative operation, then (A, w) is a semigroup with apartness. An interested reader can find in [**12, 14, 15, 54**] something about semigroups with appartness.

Let q be a relation coequality on the semigroup $(A, =, \neq), w)$ which satisfies the following condition

$$(\forall x, y, u, v \in A)((w(x, y), w(y, u)) \in q \implies ((x, u) \in q \vee (y, v) \in q)).$$

In this case, it is said that q is compatible with w. Speaking in the language of classical algebra, q and w are compatible if w is cancellative with respect to q. The compatibility of q and w allows us to construct internal binary operations on the sets $A/(q^{\lhd}, q)$ and $[A : q]$ and thus design two new semigroups. The second of these two does not have its dual in classical semigroup theory.

Example 2.4. Let $(S, =_S, \neq_S)$ and $(\Gamma, =_\Gamma, \neq_\Gamma)$ be two non-empty sets with apartness. Then S is called a Γ-*semigroup with apartness* if there exist a strongly extensional mapping

$$w_S : S \times \Gamma \times S \ni (x, a, y) \longmapsto w_S(x, a, y) \in S$$

satisfying the condition

$$(\forall x, y, z \in S)(\forall a, b \in \Gamma)(w_S(w_S(x, a, y), b, z) =_S w_S(x, a, w_S(y, b, z))).$$

Let S be a Γ-semigroup with apartness. A co-equality relation $q \subseteq S \times S$ is called a Γ-cocongruence on S if the following holds

$$((w_S(x, a, u), w_S(y, b, v)) \in q \implies ((x, y) \in q \vee a \neq b \vee (u, v) \in q))$$

for any $x, y, u, v \in S$ and all $a, b \in \Gamma$. The operation $w_{[S:q]}$ on $[S : q] \times \Gamma \times [S : q]$ is defined as

$$(\forall\, xq, yq \in [S : q])(\forall a \in \Gamma)(w_{[S:q]}(xq, a, yq) := (w_S(x, a, y))q).$$

The structure $([S : q], =_2, \neq_2, w_{[S:q]})$ is a Γ-semigroup and there is a unique strongly extensional mapping $(\vartheta, i) : S \longrightarrow [S : q]$ such that the following holds

$$w_{[S:q]} \circ (\vartheta, i, \vartheta) = (\vartheta, i) \circ w_S.$$

About Γ-semigroups with apartness, the reader can look at the paper [**55**].

Example 2.5. Let $(S, =_S, \neq_S)$ be a set with apartness. Then S is called a *semigroup with apartness* if there exist a strongly extensional mapping

$$w_S : S \times S \ni (x, y) \longmapsto w_S(x, y) \in S$$

satisfying the condition

$$(\forall x,y,z\in S)(w_S(w_S(x,y),z) =_S w_S(x,w_S(y,z))).$$

EXAMPLE 2.6. We call ([**12**], Definition 4) I-semigroup with apartness an inhabited set with apartness $(S,=,\neq)$ equipped with an associative, strongly extensional binary operation on S, denoted by '$\cdot$', and with a strongly extensional unary operation on S denoted by '$^{-1}$' such that

$$(\forall x\in S)(x\cdot x^{-1}\cdot x = x \,\wedge\, (x^{-1})^{-1} = x).$$

In other words, an I-semigroup with apartness is a tuple $(S,=,\neq,\cdot,^{-1})$ where
(I1) $(S,=,\neq)$ is an inhabited set with apartness;
(I2) '$\cdot$' is a binary operation on S such that:
(a) for all $x,y,z\in S$, it holds $x\cdot(y\cdot z) = (x\cdot y)\cdot z$,
(b) for all $x,y,u,v\in S$, $x\cdot u\neq y\cdot v$ implies $x\neq y$ or $u\neq v$;
(I3) '$^{-1}$' is a unary operation such that:
(c) for all $x\in S$ it holds $(x^{-1})^{-1} = x$,
(d) for all $x,y\in S$, $x^{-1}\neq y^{-1}$ implies $x\neq y$;
(I4) for all $x\in S$ it holds $x\cdot x^{-1}\cdot x = x$.

As usual, we will write xy instead of $x\cdot y$. Since we assumed that all properties we are dealing with are extensional, we immediately derive that $\cdot$ and $^{-1}$ are well defined, i.e., for all $x,y,u,v\in S$, $x=u\wedge y=v$ implies $xy=uv$ and $x=y$ implies $x^{-1}=y^{-1}$. Moreover, by extensionality and (I3)(a), we also derive that for all $x,y\in S$, $x\neq y$ implies $x^{-1}\neq y^{-1}$. Then, in the definition of I-semigroup with apartness, condition (I3)(b) can be written as

(I3)(d') for all $x,y\in S$ it holds $x^{-1}\neq y^{-1} \iff x\neq y$.

Moreover, condition (I3)(a) implies that $x^{-1}xx^{-1} = x^{-1}$ for all $x\in S$. Lastly, (I3)(a) and extensionality of '$^{-1}$' give that for all $x,y\in S$ it holds

$$x^{-1} = y^{-1} \iff x = y \tag{2.1}$$

An inverse semigroup with apartness ([**12**], Definition 5) is an I-semigroup with apartness $(S,=,\neq,\cdot,^{-1})$ such that

(I5) for all $x,y\in S$ it holds

$$xx^{-1}yy^{-1} = yy^{-1}xx^{-1}.$$

Observe that property (I5) implies, as usual, that

$$(xy)^{-1} = y^{-1}x^{-1}.$$

2.5. Class preserving mappings of co-equality relational systems. The concept of a relational system was introduced by A. I. Maltsev [**24**]. This notion was investigated by I. Chajda (for example in [**10**]). However, R. D. Maddux [**23**] suggests that text [**62**] written by A. Tarski in 1941 is probably one of the first articles which relates to 'The calculus of relations' ([**23**], page 438). The approach outlined in [**62**] is worked out in more detail in [**63**]. According to R. D. Madduox already mentioned, the first definition of relation algebras appears in [**20**] (cited by [**23**], page 441).

We will restrict our consideration to relational systems with only one binary relation. Hence, for a relational system we will take a pair $\mathfrak{A} = ((A, =, \neq), \alpha)$, where $(A, =, \neq)$ is a set with apartness and $\alpha \subseteq A \times A$, i. e., α is a binary relation on A. Relational systems play an important role both in mathematics and in applications since every formal description of a real system can be done by means of relations. In this subsection, we are mostly interested in relational systems $\mathfrak{A} = ((A, =, \neq), \alpha)$ where α is consistent, symmetric or co-transitive. In cumulative case, the relation α is a co-equality relation and the system $\mathfrak{A}$ is called a *co-equality relational system.*

A mapping $\varphi : ((A, =, \neq), \alpha) \longrightarrow ((B, =, \neq), \beta)$ of co-equality relational systems is *class preserving mapping* if $\varphi(\alpha a) = (\varphi(a))\beta$ holds for each $a \in A$. The following result 'A mapping φ is a class preserving mapping if and only if α and $Ker\varphi$ permute.' is the main result of this subsection.

Let $f : ((A, =, \neq), \alpha) \longrightarrow ((B, =, \neq), \beta)$ be a strongly extensional mapping.

- f is called *isotone* if

$$(\forall x, y \in A)((x, y) \in \alpha \Longrightarrow (f(x), f(y)) \in \beta);$$

- f is called *reverse isotone* if

$$(\forall x, y \in A)((f(x), f(y)) \in \beta \Longrightarrow (x, y) \in \alpha).$$

If we introduce the relation $\varphi^{-1}(\beta)$ on set A by the following way

$$(a, b) \in \varphi^{-1}(\beta) \iff (\varphi(a), \varphi(b)) \in \beta,$$

then φ is isotone if and only if $\alpha \subseteq \varphi^{-1}(\beta)$. Also, φ is a reverse isotone if and only if $\varphi^{-1}(\beta) \subseteq \alpha$ holds.

Further on,

(i) a strongly extensional reverse isotone mapping $\varphi : A \longrightarrow B$ is called a *reverse isotone strong mapping* of A to B if

$$\varphi^{-1}(\beta) \subseteq \alpha \subseteq Ker\varphi \circ \varphi^{-1}(\beta) \circ Ker\varphi;$$

(ii) a strongly extensional isotone mapping $\varphi : A \longrightarrow B$ is called *isotone strong mapping* ([**10**]) of A to B if:

$$\alpha \subseteq \varphi^{-1}(\beta) \subseteq Ker\varphi \circ \alpha \circ Ker\varphi.$$

Firstly, we will start with two definitions: Let $\mathfrak{A} = ((A, =, \neq), \alpha)$ be a relational system and q a coequality relation on A. Define a binary relation $[\alpha : q]$ on the set $[A : q]$ as follows:

$$(\forall a, b \in A)((aq, bq) \in [\alpha : q] \iff (a, b) \in q^{\triangleleft} \circ \alpha \circ q^{\triangleleft}).$$

Then $[\mathfrak{A} : q] = (([A : q], =_1, \neq_1), [\alpha : q])$ will be called a *quotient relational system* of $\mathfrak{A}$ by q.

It is evident that we need for the following assertions:

LEMMA 2.1 ([**16**], Theorem 1). *Let (A, α) be a relational system and q be a co-equality on A.*

- *If $\alpha \subseteq q$, then $[\alpha : q]$ is a consistent relation, too.*

- *$[\alpha : q]$ is a symmetric relation if and only if α is a symmetric relation.*

- If α is co-transitive relation, then $[\alpha : q]$ is a co-transitive relation, too.

THEOREM 2.9 ([**16**], Theorem 2). *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two co-equality relational systems and $\varphi : A \longrightarrow B$ be a surjective mapping. Then*

- If φ is a class preserving isotone mapping, then φ is an isotone strong mapping of $\mathfrak{A}$ onto $\mathfrak{B}$.

- If φ is a reverse isotone strong mapping, then φ is a class preserving mapping.

EXAMPLE 2.7. The converse of the previus theorem does not hold in general. Consider $\mathfrak{A} = ((A, =, \neq), \alpha)$ and $\mathfrak{B} = ((B, =, \neq), \beta)$, where $A = \{x_1, x_2, y_1, y_2, z_1, z_2, w\}$, $B = \{a, b, c, d\}$, and

$$\alpha = A \times A \setminus (\{x_1, x_2\}^2 \cup \{y_1, y_2\}^2 \cup \{z_1, z_2\}^2 \cup \{w\}^2),$$
$$\beta = \{(a, d), (b, d), (c, d), (d, a), (d, b), (d, c)\}$$

are coequalities relations respectively. Let $\varphi : A \longrightarrow B$ be defined as follows: $\varphi(x_1) = \varphi(y_1) = a$, $\varphi(x_2) = \varphi(z_1) = b$, $\varphi(y_2) = \varphi(z_2) = c$, $\varphi(w) = d$. Then φ is a surjective and strong mapping of $\mathfrak{A}$ onto $\mathfrak{B}$ but it is not a class preserving mapping. For example, for x_1 we have: $\varphi(x_1\alpha) = (\{y_1, y_2, z_1, z_2, w\}) = \{\varphi(y_1), \varphi(y_2), \varphi(z_1), \varphi(z_2), \varphi(w)\} = \{a, b, c, d\} \neq \{d\} = a\beta = \varphi(x_1)\beta$. $\diamond$

In the following theorem we present a condition for class preserving mapping.

THEOREM 2.10 ([**16**], Theorem 3). *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two co-equality relational systems and $\varphi : A \longrightarrow B$ be a strong surjective mapping. Then φ is a class preserving mapping if and only if α and $Ker\varphi$ permute.*

COROLLARY 2.4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two co-equality relational systems and $\varphi : A \longrightarrow B$ be a class preserving reverse isotone surjective mapping. Then φ is a strong reverse isotone mapping if and only if β and $Ker\varphi^{-1}$ permute.*

At the end of this consideration we analyze the natural mapping $\vartheta : A \longrightarrow [A : q]$.

THEOREM 2.11 ([**16**], Theorem 4). *Let $\mathfrak{A} = ((A, =, \neq), \alpha)$ be a coequality system and q be a co-equality on A such that $\alpha \subseteq q$. Suppose that α and $q^{\triangleleft}$ permute. Then the natural mapping $\vartheta : A \longrightarrow [A : q]$ is a class preserving mapping.*

2.6. The important question. As mentioned earlier, this subsection offers a partial answer to the Question 2.1. For this purpose we need some prior thinking:

PROPOSITION 2.4. *Let $\{q_k\}_{k\in K}$ be a family of co-equality relations on a set with apartness $(A, =, \neq)$. Then $\bigcup_{k\in K} q_k$ is a co-equality relation on A, too.*

PROOF. It is clear that $\bigcup_{k\in K} q_k$ is a consistent and symmetric relation since each of the components q_k is a consistent and symmetric relation on A. It remains to show that the union is a co-transitive relation. Let $x, y, z \in A$ be arbitrary elements such that $(x, z) \in \bigcup_{k\in K} q_k$. Then there exists an index $j \in K$ such that $(x, z) \in q_j$. Thus $(x, y) \in q_j \subseteq \bigcup_{k\in K} q_k \vee (y, z) \in q_j \subseteq \bigcup_{k\in K} q_k$. So, the relation $\bigcup_{k\in K} q_k$ is a co-transitive relation on A. Therefore, $\bigcup_{k\in K} q_k$ is a co-equality relation on A $\square$

As an important consequence of the previous proposition, we have the following corollary.

COROLLARY 2.5. *Let $(A,=,\neq)$ be a set with apartness. Then the family $\mathfrak{Q}(A)$ od all co-equality relations on A form a complete lattice.*

PROOF. Let $\{q_k\}_{k\in K}$ a family of co-equality relations in A and let $\mathfrak{T}$ be a family of all co-equality relations on A included in $\bigcap_{k\in K} q_k$. Then $\bigcup\mathfrak{T}$ is the maximal co-equality relation on A included in $\bigcap_{k\in K} q_k$. If we put $\sqcup_{k\in K} q_k = \bigcup_{k\in K} q_k$ and $\sqcap_{k\in K} q_k = \bigcup\mathfrak{T}$, then $(\mathfrak{Q}(A),\sqcup,\sqcap)$ is a completely lattice. □

The next theorem is a partial answer to the above question.

THEOREM 2.12. *Let ρ be an equality relation on a set $(A,=,\neq)$. If $\mathfrak{T}$ is the family of all co-equality relations included in $\neg\rho$, then $q_{max}(\rho) = \bigcup\mathfrak{T}$ is the maximal co-equality relation on A associated with ρ.*

PROOF. From the analysis that preceded this theorem, we conclude that $\bigcup\mathfrak{T}$ is the maximal co-equality relation included in $\neg\rho$. Let us show that $q_{max}(\rho)$ is associate with ρ. Let $x,z\in A$ arbitrary elements such that $(x,z)\in\rho\circ q_{max}(\rho)$. Then there exists an element y such that $(x,y)\in q_{max}(\rho)$ and $(y,z)\in\rho$. Thus there exists a co-equality relation q included in $\neg\rho$ such that $(x,y)\in q$. From this follows $(x,z)\in q\subseteq q_{max}(\rho)$ or $(z,y)\in q\subseteq\neg\rho$ by co-transitivity of q. Since the second option is impossible due to $(z,y)\in\rho^{-1}=\rho$, it must be $(x,z)\in q_{max}(\rho)$. The second inclusion $q_{max}(\rho)\circ\rho\subseteq q_{max}(\rho)$ can be show in an analogous way. Therefore, the relation $q_{max}(\rho)$ is associated with ρ. □

REMARK 2.2. We have proved the hypothetical existence of the maximal co-equality relation that associated with given an equality relation. There remains an open question how to construct this relation effectively. As one of the possible answers to this problem, the offered is a construction described with more detailed in the article [**40**] which some members of the academic community treat as unacceptable.

3. Co-quasiorder relations

3.1. Concept of co-quasiorder relations. If in the definition of the concept of co-equality relations we omit the requirement of symmetry, a new notion that meets the axiom of consistency and the axiom of co-transitivity occurs. Thus, if τ is one such relation in a set $(A,=,\neq)$, then $q=\tau\cup\tau^{-1}$ is a relation of co-equality on A. This analysis justifies introducing a new concept in set with apartness.

DEFINITION 3.1. A relation $\tau\subseteq A\times A$ is a *co-quasiorder* on A if holds

$$\tau\subseteq\neq \text{ (consistency)} \quad \text{and} \quad \tau\subseteq\tau*\tau \text{ (co-transitivity)}.$$

It is clear that each coequality relation q on set A is a co-quasiorder relation on A, and the apartness is the trivial co-quasiorder relation on A. The notion of co-quasiorder the first time is defined in article [**39**] (See the articles [**6, 13**] also).

Remark 3.1. Sometimes in the definition of the concept of co-quasiorder relations we add the request $\tau \cap \tau^{-1} = \emptyset$. In this case, we are talking about the *strict* co-quasiorder relation.

If in the previous determination we add a new request, the linearity request, a new notion appears. In the following definition, we precisely describe this new concept.

Definition 3.2. Let A be a set with apartness and $\sigma \subseteq A \times A$ be a relation on A. The relation σ on A is a *co-order* on A if the following is valid
$\sigma \subseteq \neq$ (consistency), $\sigma \subseteq \sigma * \sigma$ (co-transitivity) and $\neq \subseteq \sigma \cup \sigma^{-1}$ (linearity).

If τ is a co-quasiorder (co-order, res.) relation on a set $(A, =, \neq)$, it is said that A is a *ordered set under relation* τ, or that A is *co-quasiordered set* (res. *co-ordered set*). The notion of co-order relation the first time is defined in articles [**38, 41**].

Example 3.1. ([**41**]) Let $(K, =, 0, +, \cdot, 1)$ be a Heyting field with apartness. The subset D of K is a co-subring of K if and only if
$1 \lhd D$, $0 \lhd D$, $-a \in D \implies a \in D$,
$a + b \in D \implies (a \in D \vee b \in D)$,
$ab \in D \implies (a \in D \vee b \in D)$.
The set $S = \{a \in K : a \in D \vee a^{-1} \in D\}$ is a strongly extensional co-subgroup of the multiplicative group $K^* = \{a \in K : a \neq 0\}$ compatible with the subgroup $S^{\lhd} = \{a \in K^* : a \lhd S\}$ of K^* such that we can construct the factor-group $G = K^*/(S^{\lhd}, S)$. On the group G we define a relation τ by

$$(aS^{\lhd}, bS^{\lhd}) \in \tau \iff a^{-1}b \in D.$$

The relation τ is a co-order relation on G compatible with the group operation.

Example 3.2. Let $((S, =_S, \neq_S), w_S)$ be a Γ-semigroup with apartness (Example 2.4). A co-order relation $\nleqslant_S$ on S is *compatible* with the semigroup operations w_S in S if the following holds

$$((w_S(x, a, z) \nleqslant_S w_S(y, a, z) \vee w_S(z, a, x) \nleqslant_S w_S(z, a, y)) \implies x \nleqslant_S y$$

for all $x, y, z \in S$ and $a \in \Gamma$. In this case it is said that S is an *ordered* Γ-semigroup *under co-order* $\nleqslant$ or it is *co-ordered* Γ-semigroup.

About co-ordered Γ-semigroups with apartness, the reader can look at the articles [**56, 57**].

Example 3.3. Let $((S, =_S, \neq_S), w_S)$ be a semigroup with apartness (Example 2.5) and let $\nleqslant_S$ be a co-order relation on S satisfy the following

$$(\forall x, y, z \in S)((w_S(x, z) \nleqslant_S w_s(y, z) \vee w_S(z, x) \nleqslant_S w_S(z, y)) \implies x \nleqslant_S y).$$

S is a semigroup with apartness ordered under the co-order $\nleqslant_S$.

Example 3.4. ([**27**]) Following to classical definition, for algebraic structure $((S, =, \neq, \cdot, 1), \otimes)$ is called a (*strong*) *semilattice-ordered semigroup* if :
(i) $(S, =, \neq, \cdot, 1)$ is a semigroup, where the semigroup operation is strongly extensional in the following way

$$(\forall a, b, c \in S)((ac \neq bc \vee ca \neq cb) \Longrightarrow a \neq b);$$

(ii) $(S, =, \neq, \otimes)$ is a semilattice, i.e. $(S, \otimes)$ is a commutative semigroup with $(\forall x \in S)(x \otimes x = x)$ where the semigroup operation is strongly extensional:

$$(\forall a, b, c \in S)((a \otimes c \neq b \otimes c \vee c \otimes a \neq c \otimes b) \Longrightarrow a \neq b);$$

(iii) $(\forall a, b, c \in S)((a(b \otimes c) = ab \otimes ac) \wedge ((a \otimes b)c = ac \otimes bc))$; and

(iv) $(\forall x \in S)(x \otimes 1 = 1)$.

In the following we show that semilattice-ordered semigroup is equipped with the natural defined co-order relation:

If $(S, =, \neq, , \otimes)$ is a semilattice-ordered semigroup and we define, for any a, b of S,

$$(a, b) \in \alpha \Longleftrightarrow a \otimes b \neq a,$$

then the structure $(S, =, \neq, \cdot, \otimes)$ is an ordered semigroup under xo-order α.

Proof. (i) It is clear that the relation α is consistent.

(ii) Let a, b, c be arbitrary elements of S such that $(a, c) \in \alpha$, i.e. such that $a \otimes c \neq a$. Then,

$$a \otimes c \neq a \Longrightarrow a \otimes c \neq b \otimes a \vee b \otimes a \neq a.$$

If $b \otimes a \neq a$, then $(a, b) \in \alpha$. Suppose that $a \otimes c \neq b \otimes a$. Then, $a \otimes c \neq a \otimes b \otimes c$ or $a \otimes b \otimes c \neq b \otimes a$. In the first case, we conclude:

$$\begin{aligned} a \otimes c \neq a \otimes b \otimes c &\Longrightarrow a \neq a \otimes b \vee c \neq c \\ &\Longrightarrow (a, b) \in \alpha. \end{aligned}$$

In the second case, we have

$$\begin{aligned} a \otimes b \otimes c \neq b \otimes a &\Longrightarrow b \otimes c \neq b \vee a \neq a \\ &\Longrightarrow (b, c) \in \alpha\ . \end{aligned}$$

Therefore, the relation α is co-transitive.

(iii) Let a and b be arbitrary element of S such that $a \neq b$. Thus, $a \neq a \otimes b$ or $a \otimes b \neq b$. So, we have $a \neq b \Longrightarrow (a, b) \in \alpha \vee (b, a) \in \alpha$, and the relation α is linear.

(iv) Let a, b, c be arbitrary elements of semigroup $(S, =, \neq, \cdot, \otimes)$ such that $(ac, bc) \in \alpha$. Then,

$$\begin{aligned} ac \otimes bc \neq ac &\Longleftrightarrow (a \otimes b)c \neq ac \\ &\Longrightarrow a \otimes b \neq a \\ &\Longrightarrow (a, b) \in \alpha. \end{aligned}$$

Analogously, we derive the implication $(ca, cb) \in \alpha \Longrightarrow (a, b) \in \alpha$.

(v) Let a, b, c be elements of S such that $(a \otimes c, b \otimes c) \in \alpha$, i.e. such that $a \otimes c \otimes b \otimes c \neq a \otimes c$. Thus, $a \otimes b \otimes c \neq a \otimes c$ and $a \otimes b \neq a$. Hence, $(a, b) \in \alpha$.

Finally, the relation α is a co-order relation on semigroup $(S, =, \neq,)$ and the structure $(S, =, \neq, \alpha)$ is a semigroup ordered under co-order α. □

Example 3.5. On an inverse semigroup with apartness S (Example 2.6), the relation '$\nleqslant$' we define ([**58**]) as follows

$$(\forall a, b \in S)(a \nleqslant b \iff a \neq ba^{-1}a).$$

We may now establish the main properties of the relation '$\nleqslant$'.

In the mentioned text it is shown that the condition that is determined by this co-order is equivalent to the following conditions:

$$ab^{-1} \neq aa^{-1}, \;\; a^{-1}b \neq a^{-1}a \;\; a \neq aa^{-1}b.$$

- The relation $\nleqslant$ is a co-order relation on the set S.
- $(\forall a, v \in S)(a \nleqslant b \implies a^{-1} \nleqslant b^{-1})$.
- $(\forall a, b, u \in S)(au \nleqslant bu \implies a \nleqslant b)$ and
- $(\forall a, b, u \in S)(ua \nleqslant ub \implies a \nleqslant b)$.

This co-order relation on inverse semigroups with apartness is the constructive dual of natural order on classical inverse semigroups.

Proof. (1) The relation $\nleqslant$ is a co-order relation on the set $(S, =, \neq)$:

(1.1) The relation $\nleqslant$ is consistent. Let $a, b \in S$ be such that $x \nleqslant y$. This means $a \neq ba^{-1}a$. On the other hand, since $a = aa^{-1}a$, we have $aa^{-1}a \neq ba^{-1}a$. Then $a \neq b$ by (I2)(b) thus showing that the relation $\nleqslant$ is consistent.

(1.2) The relation $\nleqslant$ is co-transitive. Let $a, b, c \in S$ be arbitrary elements such that $a \nleqslant c$. This means $a \neq ca^{-1}a$. Then

$$a \neq ba^{-1}a \vee ba^{-1}a \neq ca^{-1}a$$

by co-transitivity od the apartness.

(i) The first option gives $a \nleqslant b$.

(ii) Suppose the second option is valid Since $a = aa^{-1}a$, we have

$$ba^{-1}(aa^{-1}a) \neq ca^{-1}(aa^{-1}a)$$

which we can write in form

$$b(a^{-1}a)(a^{-1}a) \neq c(a^{-1}a)(a^{-1}a).$$

Hence follows

$$b(a^{-1}a)(a^{-1}a) \neq (cb^{-1}b)(a^{-1}a)(a^{-1}a) \vee (cb^{-1}b)(a^{-1}a)(a^{-1}a) \neq c(a^{-1}a)(a^{-1}a)$$

due to the co-transitivity of the apartness relation.

(ii-a) From the first option

$$b(a^{-1}a)(a^{-1}a) \neq (cb^{-1}b)(a^{-1}a)(a^{-1}a),$$

we get $b \neq cb^{-1}b$ by (I2)(b). So, $b \nleqslant c$.

(ii-b) Assume that the second option

$$(cb^{-1}b)(a^{-1}a)(a^{-1}a) \neq c(a^{-1}a)(a^{-1}a)$$

is valid. We can write it in form

$$c(b^{-1}b)(a^{-1}a)(a^{-1}a) \neq c(a^{-1}a)(a^{-1}a).$$

From here, we have

$$(b^{-1}b)(a^{-1}a)(a^{-1}a) \neq (a^{-1}a)(a^{-1}a)$$

by (I2)(b). The next three transformations of this formula are

$$(a^{-1}a)(b^{-1}b)(a^{-1}a) \neq (a^{-1}a)(a^{-1}a),$$

$$(a^{-1}ab^{-1})(ba^{-1}a) \neq (a^{-1}aa^{-1})a$$

and

$$(ba^{-1}a)^{-1}(ba^{-1}a) \neq a^{-1}a,$$

respectively. Hence

$$(ba^{-1}a)^{-1} \neq a^{-1} \lor ba^{-1}a \neq a.$$

Both previous cases give $a \nleqslant b$.

This completes the proof of the co-transitivity of the relation $\nleqslant$.

(1.3) The relation $\nleqslant$ is linear. let $a, b \in S$ be elements such that $a \neq b$. Then

$$a \neq b(a^{-1}a) \lor b(a^{-1}a) \neq b$$

by co-transitivity of the apartness. If the first option $a \neq b(a^{-1}a)$ is valid, then we have $a \nleqslant b$. Assume that the second option

$$b(a^{-1}a) \neq b$$

is valid. The previous formula can be written as follows $(bb^{-1}b)(a^{-1}a) \neq b$ and, further on, in the following way $b(b^{-1}b)(a^{-1}a) \neq b$. The next allowed transformation of the previous valid formula is $b(a^{-1}a)(b^{-1}b) \neq b$ according to (I5). From here it follows

$$(b(a^{-1}a))(b^{-1}b) \neq a(b^{-1}b) \lor a(b^{-1}b) \neq b$$

according to the co-transitivity of the apartness. The second option gives immediately $b \nleqslant a$. Suppose the $(b(a^{-1}a))(b^{-1}b) \neq a(b^{-1}b)$ option is valid. From here, according to (I2)(b), it follows $b(a^{-1}a) \neq a$. This means $a \nleqslant b$.

(2) Let $a, b \in S$ be elements such that $a \nleqslant b$. This means $a \neq b(a^{-1}a)$ and $a \neq aa^{-1}b$. From here, according to (I3)(d'), it follows $a^{-1} \neq b^{-1}(aa^{-1}) = b^{-1}((a^{-1})^{-1}a^{-1})$. This means $a^{-1} \nleqslant b^{-1}$.

(3) Let $a, b, u \in S$ be arbitrary elements such that $au \nleqslant bu$. Then $(au)^{-1}(bu) \neq (au)^{-1}(au)$. Thus $u^{-1}a^{-1}bu \neq u^{-1}a^{-1}au$. Hence $a^{-1}b \neq a^{-1}a$ by (I2)(b). This means $a \nleqslant b$.

(4) Let $a, b, u \in S$ be arbitrary elements such that $ua \nleqslant ub$. Then $(ua)(ub)^{-1} \neq (ua)(ua)^{-1}$. Thus $uab^{-1}u^{-1} \neq uaa^{-1}u^{-1}$. Hence $ab^{-1} \neq aa^{-1}$ by (I2)(b). This means $a \nleqslant b$. □

In order for an interested reader to gain an idea of what kind of relation is a co-quasiorder relation, in our next proposition we show the connection between the concept of quasi-order relations and the concept of co-quasiorder relations.

Proposition 3.1. *Let τ be a co-quasiorder relation on a set $(A, =, \neq)$. Then the relation $\tau^{\lhd}$ is a quasi-order relation on A associate with τ in the following sense*

$$(\tau^{\lhd})^{-1} \circ \tau \subseteq \tau \text{ and } \tau \circ (\tau^{\lhd})^{-1} \subseteq \tau.$$

Proof. Let $x, u, v \in A$ arbitrary elements such that $(u, v) \in \tau$. Then $(u, x) \in \tau$ or $(x, v) \in \tau$ by co-transitivity of τ. Thus $u \neq x \lor x \neq v$ by consistency of τ. So, we have $(x, x) \neq (u, v) \in \tau$ and $(x, x) \in \tau^{\lhd}$. Therefore, the relation $\tau^{\lhd}$ is a reflexive relation in A.

Let x, y, z, u, v be arbitrary elements of A such that $(x, y) \in \tau^{\lhd}$, $(y, z) \in \tau^{\lhd}$ and $(u, v) \in \tau$. Then $(u, x) \in \tau \vee (x, y) \in \tau \vee (y, z) \in \tau \vee (z, v) \in \tau$ by co-transitivity of τ. Thus $u \neq x$ or $z \neq v$ because the options $(x, y) \in \tau$ and $(y, z) \in \tau$ are impossible by hypothesis. So, $(x, z) \neq (u, v) \in \tau$ and $(x, z) \in \tau^{\lhd}$. Therefore, the relation $\tau^{\lhd}$ is a transitive relation in A.

Let $x, z \in A$ such that $(x, z) \in \tau \circ (\tau^{\lhd})^{-1}$. Then there exists an element $y \in A$ such that $(y, x) \lhd \tau$ and $(y, z) \in \tau$. Thus $(y, x) \in \tau \vee (x, z) \in \tau$. Since the first option is impossible, we have $(x, z) \in \tau$. The second inclusion is proven analogously to the previous one. $\square$

Our second propositions link the idea of a co-quasiorder relation to the concept of co-equality relations.

Proposition 3.2 ([**42**], Lemma 1). *If τ is a co-quasiorder relation on a set A, then the relation $q = \tau \cup \tau^{-1}$ is a co-equality relation on A.*

From this claim, follow are very interesting consequences.

Corollary 3.1 ([**42**], Lemma 2). *Let τ be a co-quasiorder relation on A, $q = \tau \cup \tau^{-1}$¿. Then the relation α on the set $[A : q]$, defined by*

$$(\forall\, aq, bq \in [A : q])((xq, yq) \in \alpha \iff (x, y) \in \tau),$$

is a co-order relation on $[X : q]$.

The relationship that exists between the relations α and $\alpha^{\lhd}$ if α is a co-order relation on a set $(A, =, \neq)$, is much more complex than in the previous proposition.

Proposition 3.3. *Let α be a co-order relation on a set $(A, =, \neq)$. Then the relation $\alpha^{\lhd}$ is an order relation on the set $(A, \neq^{\lhd}, \neq)$ associated with α in the following sense*

$$\alpha \circ (\alpha^{\lhd})^{-1} \subseteq \alpha \text{ and } (\alpha^{\lhd})^{-1} \circ \alpha \subseteq \alpha.$$

Proof. Reflexivity and transitivity can be proved as in Proposition 3.1. Antisymmetry should be demonstrated. Let $x, y, u, v \in A$ be arbitrary elements such that $(x, y) \in \alpha^{\lhd}$, $(y, x) \in \alpha^{\lhd}$ and $u \neq v$. Then $(u, v) \in \alpha \cup \alpha^{-1}$ by linearity of α and $(u, v) \in \alpha$ or $(v, u) \in \alpha$. As the following implications

$$\begin{aligned}(u, v) \in \alpha &\implies (u, x) \in \alpha \vee (x, y) \in \alpha \vee (y, v) \in \alpha \\ &\implies (u \neq x \vee y \neq v)\end{aligned}$$

and

$$\begin{aligned}(v, u) \in \alpha &\implies (v, y) \in \alpha \vee (y, x) \in \alpha \vee (x, u) \in \alpha \\ &\implies (u \neq x \vee y \neq v)\end{aligned}$$

hold, we conclude that $x \neq^{\lhd} y$ is valid. $\square$

Remark 3.2. If $\neq$ is not a tight relation on A, i.e. if the following

$$(\forall x, y \in A)(\neg(x \neq y) \implies x = y)$$

not valid, then the sets $(A, =, \neq)$ and $(A, \neq^{\lhd}, \neq)$ are different. The presence of this difference significantly complicates the functional connections between co-ordered sets.

Let $(A, =, \neq, \tau)$ and $(B, =, \neq, \sigma)$ be ordered sets under co-quasiorders τ and σ respectively and let $f : A \longrightarrow B$ be a strongly extensional function. Let's introduce a notation $f^{-1}(\sigma)$ by the following way

$$(\forall x, x' \in A)((x, x') \in f^{-1}(\sigma) \iff (f(x), f(x')) \in \sigma).$$

The following notions are naturally appear.

DEFINITION 3.3. Let $f : (A, =, \neq, \tau) \longrightarrow (B, =, \neq, \sigma)$ be a strongly extensional function between ordered sets under co-quasiorders.
(a) f is said to be an *isotone function* if $\tau \subseteq f^{-1}(\sigma)$ is valid;
(b) For f, it is said that it is *reverse isotone* function if $f^{-1}(\sigma) \subseteq \tau$ holds.

COROLLARY 3.2. *Let τ be a co-quasiorder relation on a set A and $q = \tau \cup \tau^{-1}$ and α as in Corollary 3.1. The mapping $\vartheta : A \longrightarrow [A : q]$ is a strongly extensional surjective reverse isotone mapping and holds $\alpha \circ \vartheta = \tau$, in which case α is equal to $\tau \circ \vartheta^{-1}$.*

PROOF. ϑ is a strongly extensional surjective mapping by Proposition 2.3. Let us prove that it is a reverse isotone mapping. In that direction, let us take $(a, b) \in vartheta^{-1}(\alpha)$. Then $(\vartheta(a), \vartheta(b)) \in \alpha$. Thus $(a, b) \in \tau$ by definition of α. So, $\vartheta^{-1}(\alpha) \subseteq \tau$. This means that the mapping ϑ is a reverse isotone mapping. □

If τ is a co-quasiorder relation on a set $(A, =, \neq)$ and let a, b be elements of A, then the set $a\tau = \{y \in A : (a, y) \in \tau\}$ is a *left class* of τ generated by the element a, and the set $\tau b = \{x \in A : (x, b) \in \tau\}$ is a *right class* of τ generated by the element b. Our third proposition describes in more detail the properties of these classes.

PROPOSITION 3.4. *Let τ be a co-quasiorder relation in a set $(A, =, \neq)$. Then*
(0) *Subsets $a\tau$ and τb are strongly extended subsets in A for any $a, b \in A$.*
(1) $(\forall a, b \in A)((a, b) \in \tau \implies A = a\tau \cup \tau b)$.
(2) $(\forall a, b \in A)(\neg((a, b) \in \tau) \implies (a\tau \subseteq b\tau \wedge \tau b \subseteq \tau a))$.

PROOF. (0) Let $x, y \in A$ be arbitrary elements such that $y \in a\tau$. Then from $(a, y) \in \tau$ follows $(a, x) \in \tau \vee (x, y) \in \tau$. Thus $x \in a\tau \vee x \neq y$. So, the set $a\tau$ is a strongly extensional subset of A. The second claim can be proven by analogy with the first one.

(1) Let $a, b \in A$ be arbitrary elements such that $(a, b) \in \tau$. Then the following $(\forall x \in A)((a, x) \in \tau \vee (x, b) \in \tau)$ holds by co-transitivity of τ. Thus $(\forall x \in A)(x \in a\tau \vee x \in \tau b)$. So, $A \subseteq a\tau \cup \tau b$.

(2) Let $a, b \in A$ elements such that $\neg((a, b) \in \tau)$. Suppose $x \in a\tau$. Then $(a, x) \in \tau$. Thus $(a, b) \in \tau$ or $(b, x) \in \tau$. Since the first option is impossible, we have $x \in b\tau$. So, $a\tau \subseteq b\tau$. The second claim can be proven analogous to the previous one. □

THEOREM 3.1. *Let $(A, =, \neq)$ be a non-empty set with apartness. Then the family $\mathfrak{C}(A)$ of all co-quasiorders in A forms a complete lattice.*

PROOF. Let $\{\tau_k\}_{k\in K}$ be a family of co-quasiorder relations on a set A. It is clear that $\bigcup_{k\in K}\tau_k$ is a consistent relation since each of the components in the union is a consistent relation. Let $x, y, z \in A$ be arbitrary elements such that $(x,z) \in \bigcup_{k\in K}\tau_k$. Then there exists an index $k \in K$ such that $(x,z) \in \tau_k$. Thus $(x,y) \in \tau$ or $(y,z) \in \tau$ by co-transitivity of τ_l. So, we have $(x,y) \in \bigcup_{k\in K}\tau_k$ or $(y,z) \in \bigcup_{k\in K}\tau_k$. Therefore, the relation $\bigcup_{k\in K}\tau_k$ is a co-transitive relation n X.

Let $\mathfrak{Y}$ be the family of all co-quasiorders included in $\bigcap_{k_K}\tau_k$. Then $\bigcup\mathfrak{Y}$ is a co-quasiorder relation on X included in $\bigcap_{k_K}\tau_k$.

If we put $\sqcup_{k\in K}\tau_k = \bigcup_{k\in K}\tau_k$ and $\sqcap_{k\in K}\tau_k = \bigcup\mathfrak{Y}$, then we have that $(\mathfrak{C}(X), \sqcup, \sqcap)$ is a complete lattice. $\square$

Let τ be a co-quasiorder relation on a set A. Then for every pair (x,z) of τ there exists a pair (A_x, B_z) of strongly extensional subsets of A such that $x \lhd A_x$, $z \lhd B_z$, $A = A_x \cup B_z$ and $x \in B_z \wedge z \in A_x$. Indeed, if $(x,z) \in \tau$ is a pair of elements, we can put $A_x = x\tau$ and $B_z = \tau z$. Then the following holds $x \lhd A_z$ and $z \lhd B_z$ and $A = A_z \cup B_z$. Since $x \in A = A_x \cup B_z$ and $x \lhd A_x$, it must be $x \in B_z$. The second claim, $z \in A_x$, can be proven analogous to the previous one.

EXAMPLE 3.6. If B is a strongly extensional subset of A, then the relation τ on X, defined by

$$(x,y) \in \tau \iff (x \in A \wedge x \neq y),$$

is a co-quasiorder on A.

PROOF. It is clear that τ is a consistent relation on A. Let $(x,z) \in \tau$ and let y be an arbitrary element of A. Then $x \in B \wedge x \neq z$. Thus $x \neq y \vee y \neq z$. If $x \neq y$ and $x \in B$, then $(x,y) \in \tau$. If $y \neq z$ and $x \in B$, we have $y \neq z$ and $x \in B$ and $x \neq y \vee y \in B$ by strongly extensionality of B. In the case $y \neq z \wedge x \in B\, x \neq y$ we have again $(x,y) \in \tau$; in the case $y \neq z$ and $x \in B$ and $y \in B$ we have $(y,z) \in \tau$. So, the relation τ is a co-transitive relation on A. Therefore, relation τ is a co-quasiorder relation on A. Further, we have:
$x \in B \implies x\tau = \{t \in A : t \neq x\}$ and $\neg(x \in B) \implies x\tau = \emptyset$ and
$y \in B \implies \tau y = \{t \in A : t \neq x\} \cap B$ and $y \lhd B \implies \tau y = B$. $\square$

In the following theorem we give a connection between the family $\mathfrak{C}(A)$ of all co-quasiorders on set A and the family $\mathfrak{Q}(A)$ of all coequality relations on A. The following theorem is similar to Theorem 7 in the article [**47**]. Since in that article is about regular co-equality relations, and in this theorem this requirement is omitted, we will demonstrate the proof.

THEOREM 3.2. *The mapping $f : \mathfrak{C}(A) \longrightarrow \mathfrak{Q}(A)$, defined by $f(\tau) = \tau \cup \tau^{-1}$, is a strongly extensional function. Relations*
$\varepsilon = Ker(f) = \{(\tau,\sigma) \in \mathfrak{C}(A) \times \mathfrak{C}(A) | \tau \cup \tau^{-1} = \sigma \cup \sigma^{-1}\}$ *and*
$\omega = Coker(f) = \{(\tau,\sigma) \in \mathfrak{C}(A) \times \mathfrak{C}(A) | \tau \cup \tau^{-1} \neq \sigma \cup \sigma^{-1}\}$
are compatible an equality and a diversity relation on $\mathfrak{C}(A)$ and there is the strongly extensional injective and embedding mapping

$$g : \mathfrak{C}(A)/(Ker(f), Coker(f)) \longrightarrow \mathfrak{Q}(A).$$

PROOF. The mapping f is a well-defined strongly extensional function: If τ is a co-quasiorder relation on A, then $f(\tau) = \tau \cup \tau^{-1}$ is a coequality relation on A.

Let σ and τ be elements of $\mathfrak{C}(A)$ such that $\tau\varepsilon = \sigma\varepsilon$. Then $(\tau, \sigma) \in \varepsilon$ and $g(\tau\varepsilon) = f(\tau) = \tau \cup \tau^{-1} = \sigma \cup \sigma^{-1} = f(\sigma) = g(\sigma\varepsilon)$. Suppose that $g(\tau\varepsilon) = f(\tau) = \tau \cup \tau^{-1} \neq \sigma \cup \sigma^{-1} = f(\sigma) = g(\sigma\varepsilon)$ for some $\sigma, \tau \in \mathfrak{C}(A)$. Then there exists an element $(x, y) \in A \times A$ such that $((x, y) \in \tau \cup \tau^{-1}$ and $(x, y) \lhd \sigma \cup \sigma^{-1})$ or $((x, y) \in \sigma \cup \sigma^{-1}$ and $(x, y) \lhd \tau \cup \tau^{-1})$. In the first case, we have:

$((x, y) \in \tau \vee (x, y) \in \tau^{-1}) \wedge (x, y) \lhd \sigma \wedge (x, y) \lhd \sigma^{-1} \Longrightarrow$
$((x, y) \in \tau \wedge (x, y) \lhd \sigma) \vee ((x, y) \in \tau^{-1} \wedge (x, y) \lhd \sigma^{-1}) \Longleftrightarrow$
$((x, y) \in \tau \wedge (x, y) \lhd \sigma) \vee ((y, x) \in \tau \wedge (y, x) \lhd \sigma) \Longrightarrow \tau \neq \sigma.$

In the second case we derive similar implication analogously.

g is an injective function. In fact: let τ and σ be elements of $\mathfrak{C}(A)$ such that $g(\tau\varepsilon) = f(\tau) = \tau \cup \tau^{-1} = \sigma \cup \sigma^{-1} = f(\sigma) = g(\sigma\varepsilon)$. Then, $(\tau, \sigma) \in \varepsilon$ and $\tau\varepsilon = \sigma\varepsilon$.

g is an embedding. Indeed, let τ and σ be elements of $\mathfrak{C}(A)$ such that $\tau\varepsilon \neq \sigma\varepsilon$, i.e. such that $(\tau, \sigma) \in \omega$. It means

$$g(\tau\varepsilon) = f(\tau) = \tau \cup \tau^{-1} \neq \sigma \cup \sigma^{-1} = f(\sigma) = g(\sigma\varepsilon).$$

□

Let $\prec$ and $\not\prec$ be a pair of a quasi-order and a co-quasiorder relation on set $(A, =, \neq)$. For pair $(\prec, \not\prec)$ it is said that they are associated if the following holds

$$\prec^{-1} \circ \not\prec \subseteq \not\prec \quad \text{and} \quad \not\prec \circ \prec^{-1} \subseteq \not\prec .$$

As proved in Proposition 3.1, for every co-quasiorder relation $\not\prec$ there exists a quasi-order $\not\prec^{\lhd}$ associated with $\not\prec$. It is quite justified to ask the question:

QUESTION 3.1. Does a co-quasiorder relation exist for a given quasi-order relation $\prec$ associated with it?

One partial affirmative answer to the posed question is given by Theorem 3.1. Let $\mathfrak{T}$ be the family of all co-quasiorder relations included in $\neg \prec$. Then $\tau := \cup\mathfrak{T}$ is the maximal co-quasiorder relation included in $\neg \prec$. Without much difficulty it can be shown that $\prec$ and τ are associated relations.

3.2. Concept of regular co-equality relations. The proof of the previous theorem is the motivation for the introduction of a particular link between the relation of co-equality q and the relation of co-quasiorder τ in the intention that the function g be surjective mapping.

For a given ordered set $(A, =, \neq, \alpha)$ under a co-order α it is essential to know if there exists a coequality relation q on X such that $[A : q]$ is an ordered set. This plays an important role in the investigation of ordered sets under co-orders. The following question is natural:

QUESTION 3.2. If $(A, =, \neq, \alpha)$ is an ordered set under a co-order α and q a coequality on A, is the set $[A : q]$ an ordered set?

A possible co-order on $[A : q]$ could be the relation Θ on $[A : q]$ defined by the co-order α on A, by $\Theta = \{(xq, yq) \in [A : q] \times X : q|(x, y) \in \alpha\}$. But it is not a co-order, in general. The following question arises:

QUESTION 3.3. Is there a coequality q on A for which $[A:q]$ is ordered set?

According to Proposition 3.2, if $(A,=,\neq,\alpha)$ is an ordered set and $\sigma\,(\subseteq\ \alpha)$ is a co-quasiorder on A, then the relation q on A, defined by $q=\sigma\cup\sigma^{-1}$ is a coequality relation on A and the set $[A:q]$ is ordered set under co-order Θ defined by $(xq,yq)\in\Theta\Longleftrightarrow(x,y)\in\sigma$. This was motivation for a new notion.

DEFINITION 3.4. A co-equality relation q on a ordered set $(A,=,\neq,\alpha)$ under a co-order α is called *regular co-equality relation* if there is a co-order θ on $[A:q]$ satisfying the following conditions:

(1) $([A:q],=_1,\neq_1,\Theta)$ is a co-ordered set under co-order Θ;

(2) The mapping $\vartheta:A\ni a\longmapsto aq\in[A:q]$ is a reverse isotone surjection with respect to α and Θ.

We call the co-order θ on $[X:q]$ a *regular co-order with respect* to a regular co-equality q on A and the co-order α.

About these concepts a reader can look at the texts [**43, 44, 47, 61**].

In the following theorem we giving an answer on the Question 3.2, we find necessary and sufficient conditions that the relation $\Theta=\vartheta\circ\alpha\circ\vartheta^{-1}$ is a co-order relation on $[A:q]$.

THEOREM 3.3 ([**43**], Theorem 4). *Let q be a co-equality relation on an ordered set $(A,=,\neq)$ under a co-order α. Then the relation $\Theta=\vartheta\circ\alpha\circ\vartheta^{-1}$ is a co-order relation on the family $[A:q]$ if and only if the relation $\tau=q^{\lhd}\circ\alpha\circ q^{\lhd}$ is a co-quasiorder relation on A such that $\tau\cup\tau^{-1}=q$.*

We will show the proof of this theorem again because an interested reader can recognize in it the specifics of the technique used in this principled-logical orientation.

PROOF. (1) Let q be a coequality relation on A and let $\Theta:=\vartheta\circ\alpha\circ\vartheta^{--1}$ be an co-order relation on $[X:q]$. Then the relation $\vartheta^{--1}(\Theta)=\{(a,b)\in A\times A:(aq,bq)\in\Theta\}$ is a co-quasiorder relation on X under α such that $q=(\vartheta^{-1}(\Theta))\cup(\vartheta^{--1}(\Theta))^{--1}$ where the mapping $\vartheta:X\longrightarrow[X:q]$ is the canonical reverse isotone surjective strongly extensional function. At the other hand, we have

$(a,b)\in\vartheta^{-1}(\Theta)\iff$

$(aq,bq)\in\Theta=\vartheta\circ\alpha\circ\vartheta^{-1}\iff$

$(\exists x,y\in A)((aq,x)\in\vartheta^{-1}\wedge(x,y)\in\alpha\wedge(y,bq)\in\vartheta)\implies$

$(\exists x,y\in A)((a,aq)\in\vartheta\wedge(aq,x)\in\vartheta^{-1}\wedge(x,y)\in\alpha\wedge(y,bq)\in\vartheta\wedge(bq,b)\in\vartheta^{-1})$

$\implies(a,b)\in(\vartheta^{-1}\circ\vartheta)\circ\alpha\circ(\vartheta^{-1}\circ\vartheta)$

$\implies(a,b)\in q^{\lhd}\circ\alpha\circ q^{\lhd}$.

Opposite, let (a,b) be an arbitrary element of $q^{\lhd}\circ\alpha\circ q^{\lhd}$. Then there exist elements $x,y\in A$ such that $(a,x)\in q^{\lhd}$, $(x,y)\in\alpha$ and $(y,b)\in q^{\lhd}$. Thus, $aq=_2 xq=_2\vartheta(x)$, $\vartheta(y)=_2 by=_2 yq$ and $(x,y)\in\alpha$. Since $(aq,x)\in\vartheta^{-1}$, $(x,y)\in\alpha$ and $(y,bq)\in\vartheta$ we have the following $(aq,bq)\in\vartheta\circ\alpha\circ\vartheta^{-1}=\Theta$. Hence, $(a,b)\in\vartheta^{-1}(\Theta)$. Therefore, the relation $q^{\lhd}\circ\alpha\circ q^{\lhd}=\tau$ is a co-quasiorder relation on A such that

$\tau \cup \tau^{-1} = q$ where ϑ is the canonical reverse isotone strongly extensional function from A onto $[A : q]$.

Finally, let us show that $q = \vartheta^{-1} \circ \vartheta$ holds. Let $a, b \in A$ such that $(a, b) \in \vartheta^{-1} \circ \vartheta$. Then there exists an element $xq \in [A : q]$ such that $(a, xq) \in \vartheta$ and $(xq, b) \in \vartheta^{-1}$. Thus $\vartheta(a) =_2 xq =_2 \vartheta(b)$. Hence $(a, b) \in q^{\lhd}$. The reverse implication can be proved as follows. If $a, b \in A$ be such that $(a, b) \in q^{\lhd}$, then $aq = \vartheta(a) =_2 \vartheta(b) = bq$. Thus $(a, aq) \in \vartheta \wedge aq =_2 bq \wedge (bq, b) \in \vartheta^{-1}$. So, $(a, b) \in \vartheta^{-1} \circ \vartheta$.

(2) Let $\tau := q^{\lhd} \circ \alpha \circ q^{\lhd}$ is a co-quasiorder relation on A such that $\tau \cup \tau^{-1} = q$. Then, the relation $\Theta = \{(aq, bq) \in [A : q] \times [A : q] : (a, b) \in \tau\}$ is a co-order relation on $[A : q]$. The inclusion of $\vartheta \circ \alpha \circ \vartheta^{-1} \subseteq \Theta$ can be proved by analogy with the procedure in point (1) of this proof. Conversely, if $a, b \in A$ are such that $(aq, bq) \in \Theta$, then $(a, b) \in \tau \subseteq \alpha$. Thus

$$(aq, a) \in \vartheta^{-1} \wedge (a, b) \in \alpha \wedge (b, bq) \in \vartheta^{-1}.$$

Hence $(aq, bq) \in \vartheta \circ \alpha \circ \vartheta^{-1}$. □

In what follows we introduce the concept of quotient co-ordered mappings and the concept of quotient co-quasiorders:

Definition 3.5. Let $(A, =, \neq, \alpha)$ be a co-ordered set. A co-quasiorder τ on A is called a *quotient co-quasiorder* on A if the following holds

$$\alpha \subseteq q^{\lhd} \circ \tau \circ q^{\lhd}.$$

Let

$$\varphi : (A, =, \neq, \alpha) \longrightarrow (B, =, \neq, \beta)$$

be a strongly extensional reverse isotone mapping. Then, the relation $\varphi^{-1}(\beta)$ is a co-quasiorder on A with $\varphi^{-1}(\varphi) \cup (\varphi^{-1}(\beta)) = Coker\varphi$, and $[A : Coker\varphi]$ is a co-ordered sets. Besides, holds $\varphi^{-1}(\beta) \subseteq \alpha$ because φ is a reverse isotone mapping. A little generalization of notion introduced in the Definition 3.5 is the following notion:

Definition 3.6. Let $(A, =, \neq, \alpha)$ and $(B, =, \neq, \beta)$ be co-ordered sets. A reverse isotone strongly extensional mapping $\varphi : A \longrightarrow B$ is called a *quotient co-ordered mapping* from A to B if the following holds

$$\alpha \subseteq q^{\lhd} \circ \varphi^{-1}(\beta) \circ q^{\lhd}.$$

In the case when φ is onto, B is called a quotient co-ordered set of A.

Proposition 3.5. *Let $(A, =, \neq, \alpha)$ be a co-ordered set and τ be a quotient co-quasiorder on A. Then $\vartheta : X \longrightarrow [A : \tau \cup \tau^{-1}]$ is a quotient co-ordered mapping from A onto $[A : \tau \cup \tau^{-1}]$. Thus, $[A : \tau \cup \tau^{-1}]$ is a quotient co-ordered set of A.*

Proof. Let τ is a quotient co-quasiorder relation on A. Then $q = \tau \cup \tau^{-1}$ is a coequality relation on A and Θ, defined by $(aq, bq) \in \Theta \Longleftrightarrow (a, b) \in \tau$, is a co-order on $[A : q]$ and the mapping $\vartheta : A \longrightarrow [A : q]$ is a strongly extensional reverse isotone mapping from A onto $[A : q]$. Since τ is a quotient co-quasiorder relation on A, then the inclusion $\alpha \subseteq q^{\lhd} \circ \tau \circ q^{\lhd}$ holds. Besides, since $\tau = \vartheta^{-1}(\Theta)$, we have $\alpha \subseteq q^{\triangleleft} \circ \vartheta^{-1}(\Theta) \circ q^{\triangleleft}$. Therefore, ϑ is a quotient co-ordered mapping from A onto $[A : q]$. □

In the next assertion we give a connection between quotient co-ordered mapping and quotient co-quasiorder on co-ordered set.

Proposition 3.6. *Let $(A, =, \neq, \alpha)$ and $(B, =, \neq, \beta)$ be co-ordered sets and $\varphi : A \longrightarrow B$ be a strongly extensional reverse isotone quotient co-ordered mapping. Then, $\varphi^{-1}(\beta)$ is a quotient co-quasiorder on A with $\varphi^{-1}(\beta) \cup (\varphi^{-1}(\beta))^{-1} = Coker\varphi$.*

Proof. Let $\varphi : A \longrightarrow B$ be a strongly extensional reverse isotone quotient co-ordered mapping. Then $\varphi^{-1}(\beta)$ is a co-quasiorder on A such that $\varphi^{-1}(\beta) \cup (\varphi^{-1}(\beta)) = Coker\varphi$. Since we have $\alpha \subseteq Ker\varphi \circ \varphi^{-1}(\beta) \circ Ker\varphi$, we conclude that $\varphi^{-1}(\beta)$ is a quotient co-quasiorder relation on A. □

At the end of this subsection we introduce and study a coequality relation q on an ordered set $(A, =, \neq)$ under a co-order α when the following inclusion $q^{\triangleleft} \circ \alpha \subseteq \alpha \circ q^{\triangleleft}$ holds. For such co-equality we say that it is a weakly regular coequality relation on A. The following result can be proved by a similar technique to the previous theorem:

Theorem 3.4 ([**61**], Theorem 2.1). *If a co-equality relation q is a weakly regular, then the relation $\tau := \alpha \circ q^{\triangleleft}$ is a co-quasiorder relation on A and the set $[A : q]$ is an ordered set under co-quasiorder $\Theta := \vartheta \circ \alpha \circ \vartheta^{-1}$ where the mapping $\vartheta : A \longrightarrow [A : q]$ is the canonical strongly extensional reverse isotone surjective function.*

Proof. (1) We have:

$$\begin{aligned}\alpha \circ q^{\triangleleft} &\subseteq q^{\triangleleft} \circ \alpha \circ q^{\triangleleft} \subseteq q^{\triangleleft} \circ (\alpha * \alpha) \circ q^{\triangleleft} \subseteq (q^{\triangleleft} \circ \alpha) * (\alpha \circ q^{\triangleleft}) \\ &\subseteq (\alpha \circ q^{\triangleleft}) * (\alpha \circ q^{\triangleleft}).\end{aligned}$$

(2) Let us prove that the implication

$$q^{\triangleleft} \circ \alpha \subseteq \alpha \circ q^{\triangleleft} \Longrightarrow \alpha \circ q^{\triangleleft} = q^{\triangleleft} \circ \alpha \circ q^{\triangleleft}$$

is valid. In fact:

(i) $\alpha \circ q^{\triangleleft} = Id_X \circ \alpha \circ q^{\triangleleft} \subseteq q^{\triangleleft} \circ \alpha \circ q^{\triangleleft}$

(ii) $q^{\triangleleft} \circ \alpha \circ q^{\triangleleft} \subseteq \alpha \circ q^{\triangleleft} \circ q^{\triangleleft} \subseteq \alpha \circ q^{\triangleleft}$.

Therefore, if the relation q is a weakly regular coequality relation on set $(A, =, \neq, \alpha)$, then holds $\alpha \circ q^{\triangleleft} = q^{\triangleleft} \circ \alpha \circ q^{\triangleleft}$. From this, we conclude that the following relation

$$\vartheta \circ (\alpha \circ q^{\triangleleft}) \circ \vartheta^{-1} = \vartheta \circ (\alpha \circ (\vartheta^{-1} \circ \vartheta)) \circ \vartheta^{-1} = (\vartheta \circ \alpha \circ \vartheta^{-1}) \circ (\vartheta \circ \vartheta^{-1}) = (\vartheta \circ \alpha \circ \vartheta^{-1}) = \Theta$$

is a co-quasiorder on $[A : q]$. □

3.3. Concepts of co-ideals and co-filters. We will start this subsection with the following statement.

Proposition 3.7 ([**51**], Proposition 3.1). *Let τ be a co-quasiorder on a set A. Then classes $a\tau$ and τb are strongly extensional subsets of A such that $a \triangleleft a\tau$ and $b \triangleleft \tau b$, for any $a, b \in A$. Moreover, the following implications holds*

(1) $(\forall x, y \in A)(y \in a\tau \wedge x \in A \Longrightarrow x \in a\tau \vee (x, y) \in \tau)$;
(2) $(\forall x, y \in A)(y \in \tau b \wedge x \in A \Longrightarrow x \in \tau b \vee (y, x) \in \tau)$.

Generalizing the example (1) and (2) in Proposition 3.7, we can introduce the concept of special subsets in ordered set under a co-quasiorder.

Definition 3.7. Let A be ordered set under co-quasiorder τ. For subset G of A we say that it is a *co-filter* in A if

$$(\forall x, y \in A)(y \in G \implies (x \in G \vee (x, y) \in \tau)).$$

So, the subset $a\tau$ is a *principal co-filter* of A generated by the element a. In addition, the sets $\emptyset$ and A are trivial co-filters of A.

Definition 3.8. For subset K of ordered subset A under a co-quasiorder τ we say that it is a *co-ideal* in A if

$$(\forall x, y \in A)(y \in K \implies (x \in K \vee (y, x) \in \tau)).$$

So, the subset τb is a *principal co-ideal* of A generated by the element b. In addition, the sets $\emptyset$ and A are trivial co-ideals of A.

Theorem 3.5 ([**51**], Theorem 3.1). *If G is a co-filter of ordered set A under co-quasiorder τ, then $G^{\lhd}$ is a filter in ordered set A under quasiorder $\tau^{\lhd}$ such that $G \cap G^{\lhd} = \emptyset$.*

If K is a co-ideal of ordered set A under co-quasiorder τ, then $K^{\lhd}$ is an ideal in ordered set A under quasiorder $\tau^{\lhd}$ such that $G \cap G^{\lhd} = \emptyset$.

It is quite justified to ask the following question:

Question 3.4. Let $\not\preceq$ be a co-quasiorder in a set with apartness $(A, =, \neq)$. If H is an ideal (a filter) in an ordered set $(A, =, \neq)$ under a co-quasiorderd $\not\preceq^{\lhd}$, is there a maximal co-ideal K (co- filter G, res.) u $(A, =, \neq, \preceq)$ such that $H \cap K = \emptyset$ (res. $G \cap H = \emptyset$)?

To answer this question we need the following theorem and its immediate consequences:

Theorem 3.6 ([**51**], Theorem 3.2). *If $\{K_j\}_{j\in J}$ be a family of co-filters (co-ideals) in ordered set A under co-quasiorder τ, then $\bigcup_{j\in J} K_j$ is a co-filter (co-ideal respectively) too.*

If G_1 and G_2 are co-filters (co-ideals), then the intersection $G_1 \cap G_2$ is also co-filter (co-ideal respectively) in A.

Corollary 3.3. *Let A is a ordered set with apartness under co-quasiorder τ. Then the family $\mathfrak{G}(A)$ of all co-filters (the family $\mathfrak{K}(A)$ of all co-ideals) in A forms a complete lattice. The greatest element in this lattice is A.*

Proof. Let $\{K_j\}_{j\in J}$ be a family of co-filters (co-ideals) in ordered set A under co-quasiorder τ. If $\mathfrak{T}$ is the family of all co-filters (co-ideals, res.) in A contained in $\bigcap_{j\in J} K_j$, then $\cup\mathfrak{T}$ is the maximal co-filter (the maximal co-ideal) contained in $\bigcap_{j\in J} K_j$.

If we put $\sqcup_{j\in J} K_j$ and $\sqcap_{j\in J} K_j = \cup\mathfrak{T}$, then $(\mathfrak{G}(A), \sqcup, \sqcap)$ $((\mathfrak{K}(A), \sqcup, \sqcap)$, res.) is a complete lattice. $\square$

THEOREM 3.7. *Let H be an ideal (a filter) in ordered set $(A, =, \neq)$ under a co-quasiorder $\nleq$. Then there is a maximal co-ideal (maximal co-filter, res.) M u A such that $M \cap H = \emptyset$.*

PROOF. Let $\mathfrak{T}$ be the family of all co-ideals (co-filters) included in $\neg H$. Then $M := \cup\mathfrak{T}$ is the maximal co-ideal (co-filter, res.) in A such that $H \cap M = \emptyset$ according to previous corollary. $\square$

About other forms of substructures in the ordered set under a co-quasiorder relation, an interested reader can look at the article [**51**]. About co-quasiordered residuated relational systems, a reader can find in the text [**53**].

4. Some mappings between co-quasiordered sets

The notion of a mapping between ordered sets is one of the fundamental notions in the study of the structure of ordered sets. In literature, for example, in [**11**], [**18**], [**21**], [**28**] and [**60**] there are several definitions of (isotone) mappings on ordered sets. Following [**11**] and [**18**] there are:
(i) isotone mappings;
(ii) strong isotone mappings;
(iii) U-mappings and L-mappings.
This paper presents a new approach to mappings between co-quasiordered relational systems in the constructive setting. The concepts of relational systems with co-order and co-quasiorder relations were analyzed in article [**19, 48, 49, 50**].

4.1. Strong mappings. Halaš and Hort ([**18**]) observed that there are several different definitions of mappings on partially ordered sets. Following ideas exposed in articles [**19, 48**], we describe some mappings between co-quasiordered relational systems. First, we modify the notion of a strong mapping in Chajda and Hoškova sense ([**11**]), as in article [**48**].

DEFINITION 4.1. ([**48**]) Let $((A, =, \neq), \sigma)$ and $((B, =, \neq), \tau)$ be co-quasiordered relational systems and let $\varphi : A \longrightarrow B$ be a strongly extensional mapping. Isotone mapping φ is called an *isotone strong mapping* from A to B if the following holds:
$$\sigma \subseteq \varphi^{-1}(\tau) \subseteq Ker\varphi \circ \sigma \circ Ker\varphi.$$

DEFINITION 4.2. ([**48**]) Let $((A, =, \neq), \sigma)$ and $((B, =, \neq), \tau)$ be co-quasiordered relational systems and let $\varphi : A \longrightarrow B$ be a strongly extensional mapping. A reverse isotone mapping φ is called a *reverse isotone strong mapping* from A to B if the following holds
$$\varphi^{-1}(\tau) \subseteq \sigma \subseteq Ker\varphi \circ \varphi^{-1}(\tau) \circ Ker\varphi.$$

Results in the following proposition are very important in our understanding of these notions:

PROPOSITION 4.1 ([**49**], lemma 3.1). *Let $((A, =, \neq), \sigma)$ and $((B, =, \neq), \tau)$ be co-quasiordered relational systems and let $\varphi : A \longrightarrow B$ be a strongly extensional mapping. Then:*
$$\sigma \subseteq \varphi^{-1}(\tau) \subseteq Ker\varphi \circ \sigma \circ Ker\varphi \Longleftrightarrow \varphi^{-1}(\tau) = Ker\varphi \circ \sigma \circ Ker\varphi.$$

$$\varphi^{-1}(\tau) \subseteq \sigma \subseteq Ker\varphi \circ \varphi^{-1}(\tau) \circ Ker\varphi \Longleftrightarrow \sigma = Ker\varphi \circ \varphi^{-1}(\tau) \circ Ker\varphi.$$

4.2. U **- mappings.** Following ideas in the articles [**19**] and [**48**] we describe U-mappings between co-quasiorder relational systems in this section.

Let (A, α) and (B, β) be relational systems. For a binary relation α on A and $a \in A$ denote $U_\alpha(a) = \{t \in A : (a, t) \in \alpha\}$. The set $U_\alpha(a)$ (the left class of α generated by a) is called *the upper class* generated by a. A mapping $\varphi : A \longrightarrow B$ is called U*-mapping* if

$$\varphi(U_\alpha(a)) = U_\beta(\varphi(a)) \text{ for each } a \in A.$$

REMARK 4.1. **(1)**. If φ is a strongly extensional and reverse isotone surjective mapping, then $U_\beta(\varphi(x)) \subseteq \varphi(U_\alpha(x))$. Indeed, let $z \in U_\beta(\varphi(x))$, i.e. let $(\varphi(x), z) \in \beta$. Since φ is a surjective mapping, then there exists an element t of A such that $z = \varphi(t)$ and $(\varphi(x), \varphi(t)) \in \beta$. Since φ is a reverse isotone mapping, we have $(x, t) \in \alpha$. Thus, $t \in U_\alpha(x)$ and $z = \varphi(t) \in \varphi(U_\alpha(x))$.
(2). If φ is a strongly extensional and isotone surjective mapping, then $U_\beta(\varphi(x)) \supseteq \varphi(U_\alpha(x))$. Indeed, let $z \in \varphi(U_\alpha(x))$. Then there exists an element t of A such that $z = \varphi(t)$ and $(x, t) \in \alpha$. Since φ is an isotone mapping, we have $(\varphi(x), \varphi(t)) \in \beta$. Thus, $z = \varphi(t) \in U_\beta(\varphi(x))$.

In the following theorem we prove that every strongly extensional reverse isotone strong mapping between two co-quasiorder relational systems is a U-mapping.

THEOREM 4.1 ([**49**], Theorem 3.1). *Let $((A, =, \neq), \sigma)$ and $((B, =, \neq), \tau)$ be co-quasiordered relational systems and let $\varphi : A \longrightarrow B$ be a strongly extensional reverse isotone surjective strong mapping. Then, the mapping φ is an isotone and reverse isotone U-mapping.*

It is easy to verify that the converse assertion is not valid in general. The following theorem is the main result of [**48**]:

THEOREM 4.2 ([**48**]). *Let (A, σ) and (B, τ) be co-quasiordered relational systems and let $\varphi : A \longrightarrow B$ be a strongly extensional surjective mapping. Then:*
φ is a U-mapping if and only if $\sigma \subseteq \varphi^{-1}(\tau) \subseteq Ker\varphi \circ \sigma$ holds;
φ is a U-mapping if and only if $\varphi^{-1}(\tau) \subseteq \sigma \subseteq Ker\varphi \circ \varphi^{-1}(\tau)$ holds.

Now, in the following theorem we show a necessary condition for a U-mapping to be a strong mapping:

THEOREM 4.3 ([**49**], Theorem 3.3). *Let (A, σ) and (B, τ) be quasi-antiordered relational systems and let $\varphi : A \longrightarrow B$ be a strongly extensional surjective U-mapping and*

$$Ker\varphi \circ \varphi^{-1}(\tau) \subseteq \varphi^{-1}(\tau) \circ Ker\varphi$$

holds. Then, the mapping φ is a strongly extensional reverse isotone strong mapping.

4.3. L **- mappings.** Let (A, α) and (B, β) be relational systems. For a binary relation α on A and $a \in A$ we set $L_\alpha(a) = \{t \in A : (t, a) \in \alpha\}$. The set $L_\alpha(a)$ (the right class of α generated by a) is called *the lower class* generated by a. A mapping $\varphi : A \longrightarrow B$ is called *L-mapping* if

$$\varphi(L_\alpha(a)) = L_\beta(\varphi(a)) \text{ for each } a \in A.$$

REMARK 4.2. **(1)**. Let φ be a (strongly extensional) reverse isotone surjective mapping and let $z \in L_\beta(\varphi(a))$, i.e. let $(z, \varphi(a)) \in \beta$. Since φ is a surjective mapping, there exists an element t of A such that $z = \varphi(t)$ and $(\varphi(t), \varphi(a)) \in \beta$. Since φ is a reverse isotone mapping, we have $(t, a) \in \alpha$. Thus, $t \in L_\alpha(a)$ and $z = \varphi(t) \in \varphi(L_\alpha(a))$. So,

$$\varphi^{-1}(\beta) \subseteq \alpha \Longrightarrow L_\beta(\varphi(a)) \subseteq \varphi(L_\alpha(a)).$$

(2). If φ is a strongly extensional and isotone surjective mapping, then $L_\beta(\varphi(a)) \subseteq \varphi(L_\alpha(a))$. Indeed, let $z \in \varphi(L_\alpha(x))$. Then there exists an element t of A such that $z = \varphi(t)$ and $(t, a) \in \alpha$. Since φ is an isotone mapping, we have $(\varphi(t), \varphi(a)) \in \beta$. Thus, $z = \varphi(t) \in L_\beta(\varphi(a))$. Therefore, the following implication is valid

$$\alpha \subseteq \varphi^{-1}(\beta) \Longrightarrow \varphi(L_\alpha(a)) \subseteq L_\beta(\varphi(a)).$$

THEOREM 4.4 ([**49**], Theorem 3.4). *Let (A, σ), (B, τ) be co-quasiordered relational systems and let $\varphi : A \longrightarrow B$ be a strongly extensional reverse isotone surjective mapping. Then, if φ is an L-mapping then the following holds $\varphi^{-1}(\tau) \subseteq \sigma \subseteq \varphi^{-1}(\tau) \circ Ker\varphi$.*

Some more details about co-quasiordered systems and mappings between co-quasiordered relational systems can be found in articles [**19, 53**] and [**49, 50**].

4.4. Strongly isotone and weakly reverse isotone mapping. Strongly isotone mappings, in classical mathematics, were introduced by Esakia in his well-known article [**17**]. Recall that a mapping φ of an ordered set $(A, \leqslant_A)$ into an ordered set $(B, \leqslant_B)$ is said to be *strongly isotone in classical sense* if $\varphi(x) \leqslant_B y$ holds for $(x, y) \in A \times B$ if and only if there exists $x' \in A$ such that $x \leqslant_A x'$ and $\varphi(x') = y$.

For our needs, a strongly isotone (a weakly reverse isotone) mapping of a relational system into another one is a special case of a strong homomorphism of relational systems in the sense of papers [**17, 48, 49**] modified for our needs.

DEFINITION 4.3. ([**50**]) Let $\varphi : A \longrightarrow B$ be a mapping of a relational system (A, α) into a relational system (B, β). It is said that φ to be:

- *strongly isotone* if $(\varphi(x), y) \in \beta$ holds for $(x, y) \in A \times B$ if there exists $x' \in A$ such that $(x, x') \in \alpha$ and $\varphi(x') = y$;

- *weakly reverse isotone* if $(\varphi(x), y) \in \beta$ holds for $(x, y) \in A \times B$ than there exists an element $x' \in A$ such that $(x, x') \in \alpha$ and $\varphi(x') = y$.

Let us note that in our special case, a mapping $\varphi : (A, \neq_A) \longrightarrow (B, \neq_B)$ is:

- a strongly isotone if $\varphi(x) \neq_B y$ holds if there exists an element $x' \in A$ such that $x \neq_A x'$ and $\varphi(x') = y$;

- a weakly reverse isotone if the following implication holds

$$\varphi(x) \neq_B y \Longrightarrow (\exists x' \in A)(x \neq_A x' \wedge \varphi(x') = y).$$

In the following proposition we show connection between isotone (reverse isotone) and strongly isotone (weakly reverse isotone) mappings.

THEOREM 4.5 ([**50**], Theorem 2.2). *Let $\varphi : A \longrightarrow B$ a mapping from relational system (A, α) into relational system (B, β). Then:*
(1) *If φ is strongly isotone mapping, then φ is an isotone mapping.*
(2) *If φ is reverse isotone mapping, then φ is a weakly reverse isotone mapping.*

So, since the notion of strongly isotone mapping is stronger then notion of isotone mapping, the notion of weakly reverse isotone mapping is weaker then notion of reverse isotone mapping.

In the next theorems we analyze some characteristics of weakly isotone mappings between relational systems.

Our first characterization of weakly reverse isotone mapping is given in the following proposition:

THEOREM 4.6 ([**50**], Theorem 3.1). *Let $\varphi : A \longrightarrow B$ be a mapping between two relational systems. Then, φ is a weakly reverse isotone mapping if and only if $\varphi^{-1}(\beta) \subseteq Ker\varphi \circ \alpha$ holds.*

For strongly isotone mapping we have the following assertion:

THEOREM 4.7 ([**50**], Theorem 3.2). *A mapping $\varphi : (A, \alpha) \longrightarrow (B, \beta)$ between relational systems is a strongly isotone mapping if and only if $\alpha \subseteq \varphi^{-1}(\beta) \circ Ker\varphi$ holds.*

4.5. Some important results. Some of the above considerations about functions between co-ordered sets with apartness can be sublimated into the following two theorems of which the second has no a counterpart in the classical theory:

THEOREM 4.8. *Let $(A, =_A, \neq_A, \nleqslant_A)$ and $(B, =_B, \neq_B, \nleqslant_B)$ be co-ordered sets and let $f : A \longrightarrow B$ is a strongly extensional reverse isotone mapping. Then $(\not< :=)\, f^{-1}(\nleqslant_B)$ is a co-quasiorder relation on A such that $\not< \subseteq \nleqslant_A$ and the following holds*

$$(q :=)\, Coker(f) = (f^{-1}(\nleqslant_B))^{-1} \cup f^{-1}(\nleqslant_B) = \not<^{-1} \cup \not< .$$

The set $(A/(q^{\lhd}, q), =_1, \neq_1, \npreceq)$ is an ordered set under a co-order $\npreceq$, defined by

$$(\forall x, y \in A)(xq^{\lhd} \npreceq yq^{\lhd} \iff x \not< y)$$

and $\pi : A \longrightarrow A/(q^{\lhd}, q)$ is a reverse isotone strongly extensional surjective mapping. In addition, there exists a unique injective, embedding and surjective strongly extensional mapping

$$g : A/(q^{\lhd}, q) \longrightarrow (f(A), =_B, \neq_B, \nleqslant_B)$$

such that

$$f = g \circ \pi.$$

THEOREM 4.9. *Let* $(A, =_A, \neq_A, \nleqslant_A)$ *and* $(B, =_B, \neq_B, \nleqslant_B)$ *be co-ordered sets and let* $f : A \longrightarrow B$ *is a strongly extensional reverse isotone mapping. Then* $(\not< :=)\, f^{-1}(\nleqslant_B)$ *is a co-quasiorder relation on A such that* $\not< \subseteq \nleqslant_A$ *and the following holds*

$$(q :=)\, Coker(f) = (f^{-1}(\nleqslant_B))^{-1} \cup f^{-1}(\nleqslant_B) \ = \not<^{-1} \cup \not< .$$

The set $([A : q], =_2, \neq_2, \not\preceq)$ *is an ordered set under a co-order* $\not\preceq$*, defined by*

$$(\forall x, y \in A)(xq \not\preceq yq \iff x \not< y)$$

and $\vartheta : A \longrightarrow [A : q]$ *is a reverse isotone strongly extensional surjective mapping. In addition, there exists a unique injective, embedding and surjective strongly extensional mapping*

$$h : [A : q] \longrightarrow (f(A), =_B, \neq_B, \nleqslant_B)$$

such that

$$f = h \circ \vartheta.$$

The following theorem is one of the specifics of the Bishop aspect on sets with apartness ordered under a co-order relation and does not have its counterpart in the classical theory.

Let us consider ordered set with apartness $(S, =_S, \neq_S, \nleqslant_S)$ under a co-order $\nleqslant_S$ and let σ and τ be co-quasiorder relations on S such that $\sigma \subseteq \tau \subseteq \nleqslant_S$. It is known that co-congruences $q_\sigma = \sigma \cup \sigma^{-1}$ and $q_\tau = \tau \cup \tau^{-1}$ on S such that $q_\sigma \subseteq q_\tau$ can be designed. Further, by Theorem 4.9, this allows us to construct sets with apartness $([S : q_\sigma], =_\sigma, \neq_\sigma, w_\sigma)$ and $([S : q_\tau], =_\tau, \neq_\tau, w_\tau)$ which are ordered by co-order relations $\nleq_\sigma$ and $\nleq_\tau$ respectfully as follows

$$(\forall x, y \in S)(xq_\sigma \nleq_\sigma yq_\sigma \iff (x, y) \in \sigma)$$

and

$$(\forall x, y \in S)(xq_\tau \nleq_\tau yq_\tau \iff (x, y) \in \tau).$$

Let us define the relation $[\sigma : \tau]$ on co-ordered set with apartness $[S : q_\tau]$ as follows

$$(\forall x, y \in S)((xq_\tau, yq_\tau) \in [\sigma : \tau] \iff (x, y) \in \sigma).$$

LEMMA 4.1. *Let* σ *and* τ *be co-quasiorder on a co-ordered set with apartness* $(S, =_S, \neq_S, \nleqslant_S)$ *such that* $\sigma \subseteq \tau \subseteq \nleqslant_S$*. Then* $[\sigma : \tau]$ *is a co-quasiorder relation on* $[S : q_\tau]$*.*

PROOF. Ler $x, y, z \in S$ be arbitrary elements. Then:

$$\begin{aligned}
(xq_\tau, yq_\tau) \in [\sigma : \tau] &\implies (x, y) \in \sigma \subseteq \tau \subseteq q_\tau \\
&\implies xq_\tau \neq_\tau yq_\tau; \\
(xq_\tau, zq_\tau) \in [\sigma : \tau] &\iff (x, z) \in \sigma \\
&\implies (x, y) \in \sigma \vee (y, z) \in \sigma \\
&\implies (xq_\tau, yq_\tau) \in [\sigma : \tau] \vee (yq_\tau, zq_\tau) \in [\sigma : \tau].
\end{aligned}$$

□

For ease of writing, let's put

$$q := q_{[\sigma:\tau]} = [\sigma : \tau] \cup [\sigma : \tau]^{-1}.$$

Without major difficulties it can be verified that q is a co-congruence on the co-ordered set with apartness $([S, q_\tau], =_\tau, \neq_\tau, \nleqslant_\tau)$. Set with apartness

$$([[S : q_\tau] : q], =_3, \neq_3, \nleqslant_3)$$

can be designed, where is

$$(\forall x, y \in S)((xq_\tau)q =_3 (yq_\tau)q \iff (xq_\tau, yq_\tau) \lhd q),$$

$$(\forall x, y \in S)((xq_\tau)q \neq_3 (yq_\tau)q \iff (xq_\tau, yq_\tau) \in q).$$

The co-order relation $\nleqslant_3$ in $[[S : q_\tau] : q]$ is determined as follows

$$(\forall x, y \in S)((xq_\tau)q \nleqslant_3 (yq_\tau)q \iff (xq_\tau, yq_\tau) \in [\sigma : \tau]).$$

We can now design and prove the following theorem. Of course, this form of this theorem does not have its counterpart in the classical theory.

Theorem 4.10. *Let σ and τ be co-quasiorder relations on co-ordered set with apartness $(S, =_S, \neq_S, \nleqslant_S)$ such that $\sigma \subseteq \tau \subseteq \nleqslant_S$. Then there is a unique injective, embedding and surjective strongly extensional mapping*

$$\gamma : [[S; q_\tau] : q] \longrightarrow [S; q_\sigma].$$

Proof. Let us define γ by

$$(\forall (xq_\tau)q \in [[S : q_\tau] : q])(\gamma((xq_\tau)q) := xq_\sigma).$$

First, let us show that γ is a well-defined mapping. Assume $x, y, u, v \in S$ are such that $(xq_\tau)q =_3 (yq_\tau)q$ and $(u, v) \in q_\sigma$. then $(xq_\tau, yq_\tau) \lhd q$. On the other hand, from $(u, v) \in q_\sigma$ we get $(u, x) \in q_\sigma \vee (x, y) \in q_\sigma \vee (y, v) \in q_\sigma$. If we assume that $(x, y) \in q_\sigma$ is valid, then we would have the following $(x, y) \in \sigma$ or $(y, x) \in \sigma$. It would follow from here

$$(x, y) \in \sigma \vee (y, x) \in \sigma \implies (xq_\tau, yq_\tau) \in [\sigma : \tau] \subseteq q \vee (yq_t, xq_t) \in [\sigma : \tau] \subseteq q$$

which would contradict the hypothesis $(xq_\tau, yq_\tau) \lhd q$. So, it has to be $(u, x) \in q_\sigma$ or $(y, v) \in q_\sigma$. Thus $x \neq_S u$ or $y \neq_S v$. Therefore, $(x, y) \neq (u, v) \in q_\sigma$. This means $(x, y) \lhd q_\sigma$. Hence

$$\gamma((xq_\tau)q) := xq_\sigma =_\sigma yq_\sigma := \gamma((yq_\tau)q).$$

Second, let us show that γ is a strongly extensional mapping. Let $x, y \in S$ be such that

$$xq_\sigma = \gamma((xq_\tau)q) \neq_3 \gamma((yq_\tau)q) = yq_\sigma.$$

Then $(x, y) \in q_\sigma = \sigma \cup \sigma^{-1}$. Thus $((xq_\tau, yq_\tau) \in [\sigma : \tau] \cup [\sigma : \tau]^{-1} = q$. Hence

$$(xq_\tau)q \neq_3 (yq_\tau)q.$$

Let $x, y, u, v \in S$ be such that $xq_\sigma =_\sigma yq_\sigma$ and $(uq_\tau, vq_\tau) \in q$. Then $(x, y) \lhd q_\sigma = \sigma \cup \sigma^{-1}$ and $(uq_\tau, vq_\tau) \in q$. Thus $(uq_\tau, xq_\tau) \in q$ or $(xq_\tau, yq_\tau) \in q$ or $(yq_\tau, vq_{tau}) \in q$. The option $(xq_\tau, yq_\tau) \in q$ gives $(x, y) \in \sigma \cup \sigma^{-1}$ which is in contradiction with the hypothesis. So, it has to be $(uq_\tau, xq_\tau) \in q$ or $(yq_\tau, vq_\tau) \in q$. Thus $xq_\tau \neq_\tau uq_\tau$

or $yq_\tau \neq_\tau vq_\tau$. Hence $(xq_\tau, yq_\tau) \neq (uq_\tau, vq_\tau) \in q$. This means $(xq_\tau, yq_\tau) \lhd q$, i.e. $(xq_\tau)q =_3 (yq_\tau)q$. This shows that γ is an injective mapping.

It remains to show that γ is an embedding. Let $x, y \in S$ be such that $(xq_\tau)q \neq_3 (yq_\tau)q$. Then $(xq_\tau, yq_\tau) \in q = [\sigma : \tau] \cup [\sigma : \tau]^{-1}$. Thus $(x, y) \in \sigma \cup \sigma^{-1} = q_\sigma$. Hence $xq_\sigma \neq_\sigma yq_\sigma$.

Finally, it is obvious that γ is a surjective mapping. □

Let A be an ordered set with apartness under a co-order α and let S be a family of co-quasiorders on A. We say that S separates the elements of A if for each $x, y \in A$ such that $(x, y) \in \alpha$ there exist $\sigma \in S$ such that $(x, y) \in \sigma$.

In what follows, we need the following lemmas:

Lemma 4.2. *Let $(A, =, \neq)$ be an ordered set with apartness under a co-order α and let S be a family of co-quasiorders on A. If every $\sigma \in S$ is contained in α and if S separates the elements of A, then $\alpha = \bigcup\{\sigma : \sigma \in S\}$. Conversely, if $\bigcup\{\sigma : \sigma \in S\} \supseteq \alpha$, then S separates the elements of A.*

Proof. Since S is a family of co-quasiorders, then $\bigcup\{\sigma : \sigma \in S\}$ is a co-quasiorder relation on A such that $\bigcup\{\sigma : \sigma \in S\} \subseteq \alpha$. Let $x, y \in A$ be arbitrary elements such that $(x, y) \in \alpha$. Since S separates the elements of A, then there exists $\sigma \in S$ such that $(x, y) \in \sigma$. Therefore $(x, y) \in \bigcup\{\sigma : \sigma \in S\}$. This shows that $\alpha = \bigcup\{\sigma : \sigma \in S\}$ is valid.

Suppose that $\bigcup\{\sigma : \sigma \in S\} \supseteq \alpha$. Then for any pair $(x, y) \in \alpha$ there exists and a co-quasiorder $\sigma \in S$ such that $(x, y) \in \sigma$. Thus, the family S separates the elements of set A. □

Lemma 4.3. *Let $\{(B_i, =_i, \neq_i, \alpha_i) : i \in I\}$ be an inhabited family of ordered sets under co-orders α_i respectively, where I is a discrete set. Then the inhabited Cartesian product $B := \prod_{i \in I} B_i$ is an ordered set under the co-order β determined by*

$$(x, y) \in \beta \iff (\exists i \in I)(((x(i), y(i)) \in \alpha_i).$$

Proof. Let $x, y \in B$ such that $(x, y) \in \beta$. then there exist an index $i \in I$ such that $((x(i), y(i)) \in \alpha_i \subseteq \neq_i$. This means $x \neq_B y$.

Let $x, y, z \in B$ be elements such that $(x, z) \in \beta$. Then there exists an index $i \in I$ such that $((x(i), y(i)) \in \alpha_i$. Thus $(x(i), y(i)) \in \alpha_i \vee (y(i), z(i)) \in \alpha_i$. Hence $(x, y) \in \beta \vee (y, z) \in \beta$.

For elements $x, y \in B$ such that $x \neq_B y$ there exists an index $i \in I$ such that $x(i) \neq_i y(i)$. Then $(x(i), y(i)) \in \alpha_i \vee (y(i), x(i)) \in \alpha_i$. This means $(x, y) \in \beta$ or $(y, x) \in \beta$. □

Let $\{B_i : i \in I\}$ be a family of ordered set with apartness under co-orders α_i respectively. An ordered set with apartness $(A, =_A, \neq_A)$ under a co-order α_A is a subdirect product of the family $\{B_i : i \in I\}$ if and only if:

(1) There exists a subset D of $\prod_{i \in I} B_i$ and a strongly extensional injective embedding mapping Ψ from D onto A; and

(2) $(\forall i \in I)(proj_i(D) = B_i)$.

THEOREM 4.11. *Let $(A, =_A, \neq_A)$ be an ordered set with apartness under a co-order α_A. If A is a subdirect product of the co-ordered sets $\{(B_i, =_i, \neq_i, \alpha_i) : i \in I\}$, then there exists a family $\{\sigma_i \subseteq A \times A : i \in I\}$ of co-quasiorders on A which separates the elements of A. Conversely, if $\{\sigma_i : i \in I\}$ is a family of co-quasiorders on A which separates the elements if A, then A is a subdirect product of the co-ordered sets $\{[A : \sigma_i \cup (\sigma_i)^{-1}] : i \in I\}$.*

PROOF. (1) Let $\Psi : A \longrightarrow \prod_{i\in I} B_i$ be an isotone and reverse isotone strongly extensional mapping such that $proj_i(\Psi(A)) = B_i$ for each $i \in I$. For each $j \in I$, we consider the mapping $\psi_j : A \longrightarrow B_j$ by $\psi_j(x) = proj_j(\Psi(x)) = \Psi(x)(j)$.

(1.1) The mapping ψ_j is a strongly extensional function because it is a component of a strongly extensional mapping;

(1.2) Let $x, y \in A$ be elements such that $(x, y) \in \alpha_A$. Since Ψ is an isotone mapping we have $(\Psi(x), \Psi(y)) \in \beta$. This means that there exists an index $k \in I$ such that $(\Psi(x)(k), \Psi(y)(k)) \in \alpha_k$. Thus $(proj_k\Psi(x), proj_k\Psi(y)) \in \alpha_k$ that is $(\psi_k(x), \psi_k(y)) \in \alpha_k$. Since ψ_k is a strongly extensional mapping, the relation $\psi^{-1}(\sigma_k)$, is a co-quasiorder relation on A.

By Lemma 4.2, it is enough to prove that $\bigcup\{\sigma_k : k \in I\} \supseteq \alpha$. Let $(x, y) \in \alpha$. Then $(\Psi(x), \Psi(y)) \in \beta$ because Ψ is an isotone mapping. This means that there exist an index $k \in I$ such that $(\Psi(x)(k), \Psi(y)(k)) \in \alpha_k$. Thus $(x, y) \in \sigma_k$. Hence $(x, y) \in \bigcup\{\sigma_k : k \in I\}$. Thus, the family $\{\sigma_k\}_{k\in I}$ of co-quasiorders in A separates the elements of A.

(2) Converse, let $\{\sigma_i : i \in I\}$ be a family of co-quasiorder relations on a co-ordered set $(S, =_A, \neq_A, \alpha_A)$ which separates the elements of A. We can construct the co-equality $q_k = \sigma_k \cup (\sigma_k)^{-1}$, and the ordered set $[S : q_k]$ under a co-order α_k, defined by $(xq_k, yq_k) \in \alpha_k \iff (x, y) \in \sigma_k$, for every $k \in I$. Now, we can construct the Cartesian product $\prod_{k\in I}([A : q_k], =_k, \neq_k, \alpha_k)$ with

$$a = b \iff (\forall k \in I)(a(k), b(k) \in [A : q_k] \wedge a(k) =_k b(k)),$$

$$a \neq b \iff (\exists k \in I)(a(k), b(k) \in [A : q_k] \wedge a(k) \neq_k b(k))$$

and with xo-order β, defined by

$$(a, b) \in \beta \iff (\exists k \in I)((a(k), b(k)) \in \alpha_k).$$

We consider the mapping $\Psi : A \longrightarrow \prod_{k\in I}[A : q_k]$.

(2.1) Ψ is well-defined mapping:

If $x \in A$, then $xq_k \in [A : q_k]$ for any $k \in I$. So, we have $\Psi(x) \in \prod_{l\in I}[S : q_k]$.

Let $x, y \in A$ be arbitrary elements. If $x =_A y$, then $(x, y) \lhd q_i$ for every $i \in I$. So, for every $i \in I$ we have $xq_i =_i yq_i$, i.e. $(\forall i \in I)(\Psi(x)(i) =_i \Psi(y)(i))$, i.e. $\Psi(x) = \Psi(y)$. So, Ψ is a mapping.

Let $x, y \in A$ be elements such that $f(x) \neq f(y)$. Then there exists an index $j \in I$ such that $\Psi(x)(j) \neq_j \Psi(y)(j)$. This means $xq_j \neq_j yq_j$. Thus $(x, y) \in q_j = \sigma_j \cup (\sigma_j)^{-1}$. Hence $(x, y) \in \sigma_j \vee (y, x) \in \sigma_j$. Therefore, $x \neq y$. So, the mapping Ψ is a strongly extensional function from A into $\prod_{j\in I}[A : q_j]$.

(2.2) Ψ is an isotone mapping: Let $x, y \in A$ ve sych that $(x, y) \in \alpha$. Then $(x, y) \in \bigcup\{\sigma_k : k \in I\}$ because the family $\{\sigma_k : k \in I\}$ separates the elements of

A. Then there exists an index $k \in I$ such that $(x, y) \in \sigma_k$. i.e. $(xq_k, yq_k) \in \alpha_k$. Therefore, $(\Psi(x), \Psi(y)) \in \beta$.

(2.3) Ψ is a reverse isotone mapping:: Let $x, y \in A$ be such that $(\Psi(x), \Psi(y)) \in \beta$. Then there exists an index $k \in I$ such that $(\Psi(x)(k), \Psi(y)(k)) \in \alpha_k$, i.e. $(xq_k, yq_k) \in \alpha_k$. Thus $(x, y) \in \sigma_k \subseteq \alpha$.

(2.4) $(\forall k \in I)(proj_k(\Psi(A)) = [A : q_k])$: For $x \in A$ we have $\Psi(x) \in \prod_{k\in I}[A : q_k]$. Thus for any index $k \in I$ we have $proj_k\Psi(x) = xq_k \in [A : q_k]$. This means $(\forall k \in I)(proj_k(\Psi(S))\ [A : q_k])$. Let $b \in \prod_{kinI}[A : q_k]$, i.e. let $(\forall k \in I)(b(k) \in [A : q_k])$. Then there exists an element x in A such that $\Psi(x) = b$, determined by $(\forall k \in I)(b(k) = \Psi(x)(k))$. So, $(\forall k \in I)(proj_k(\Psi(x)) = b(k) \in [A : q_k])$. Therefore, $(\forall k \in I)(proj_k(\Psi(S)) \supseteq [A : q_k])$. $\square$

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Received by editors 5.5.2023; Revised version 17.7.2023; Available online 25.9.2023.

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